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## From triangles to curves

Monique Teillaud

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# From triangles to curves

Monique Teillaud

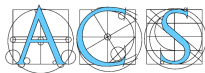
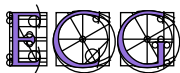


22nd European Workshop on Computational Geometry  
March 2006 - Δελφοί

# Warning

- focus on practical methods
- non exhaustive, biased

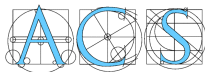
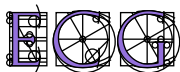
mostly (not only)



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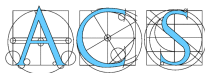
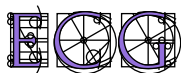


“Commercial”: ECG book coming out soon...

# Warning

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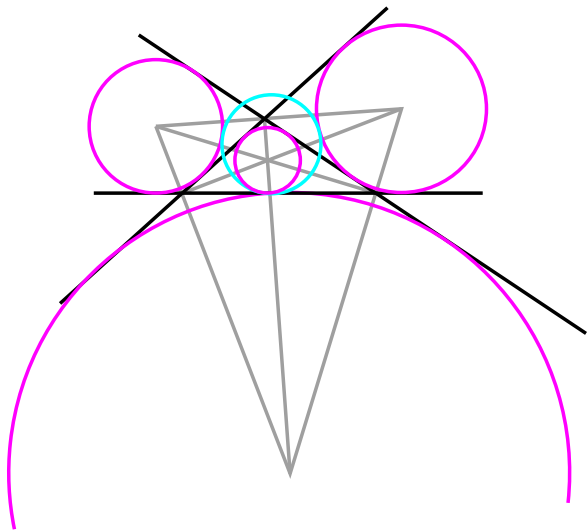
mostly (not only)



**Advice** to people having some knowledge of Computer Algebra:  
you may leave the room

- non technical, superficial. . .

# Circles are never far from triangles



# Construction of curves from lines

## Parabola: smooth connection between line segments

Du point Q on peut mener deux tangentes.

II. — Tracer, parallèlement à une droite donnée RR', une tangente à la parabole (fig. 107).

Abaissier du foyer F une perpendiculaire FM sur RR', puis élever une perpendiculaire QS au milieu de MF.

§ 93. **Raccord obtenu par un arc de parabole.**

— Décrire une parabole, et mener deux tangentes QC et QC' (fig. 108); tracer la corde des contacts CC', soit ED une tangente quelconque. Porter CD en QD<sub>1</sub>, et joindre D<sub>1</sub>E; constater que D<sub>1</sub>E est parallèle à CC'.

De cette constatation nous tirons la conclusion suivante :

$$\frac{QD_1}{QC} = \frac{QE}{QC'}$$

que nous pouvons énoncer ainsi :

*Les points d'intersection D et E partagent les tangentes QC et QC' en segments proportionnels inversement placés par rapport au point Q.*

Proposons-nous de raccorder par un arc de parabole les deux directrices concourantes QC et QC' (fig. 109) :

Partager les distances QC et QC' en un même nombre de parties égales, cinq par exemple, et numéroter les points de division de

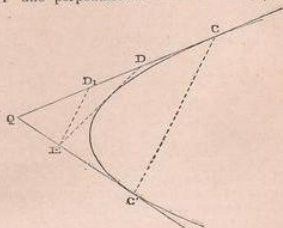


Fig. 108.

# Construction of curves from lines

## Parabola: smooth connection between line segments

248

CHAPITRE II.

l'une à partir du sommet, et de l'autre à partir du raccord; joindre les points portant le même numéro.

Toutes les droites ainsi tracées seront tangentes à l'arc de parabole, et chaque contact se trouvera au milieu de la portion de tangente comprise entre les tangentes voisines. On aura ainsi autant de points de la courbe qu'on voudra, et de plus, en chacun de ces points une tangente qui servira de limite, il sera donc très facile de tracer la courbe.

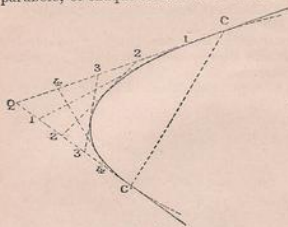


Fig. 109.

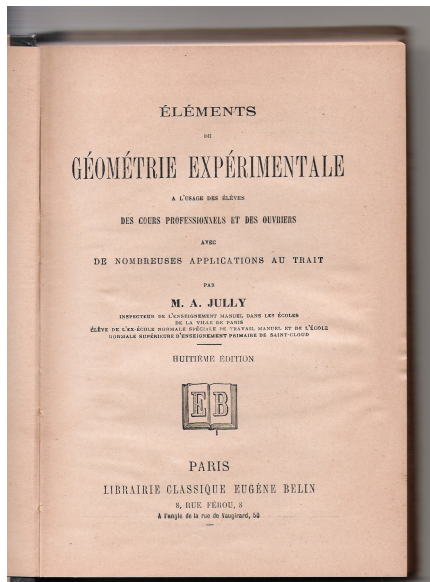
APPLICATIONS. — Ce raccord présente l'avantage de ne pas offrir de brusque changement de direction, la courbure variant graduel-

lement du sommet aux points de raccordement.

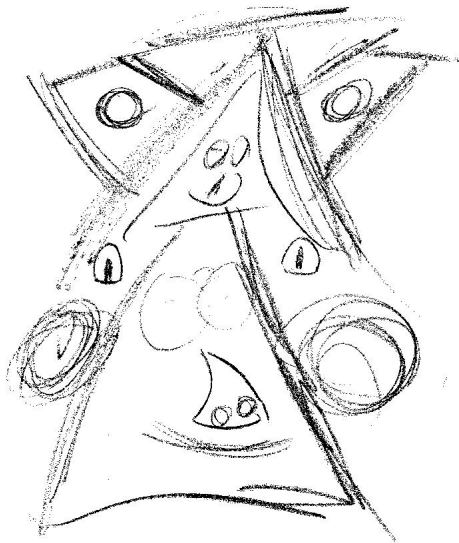
Les cintres constituent de véritables raccords opérés entre deux pieds-droits, au moyen d'une courbe. Quand les pieds-droits ont même hauteur, on emploie un demi-cercle, ou une moitié d'ellipse donnée par le grand axe si le cintre est surhaussé, et par le petit, s'il est surbaissé. Quand les pieds-droits sont inégaux et parallèles, on se sert pour l'arc rampant d'une demi-ellipse donnée par deux diamètres conjugués, mais si les pieds-droits ne sont ni égaux ni parallèles, le raccordement se fait suivant un arc de parabole tracé comme il vient d'être indiqué ci-dessus.



# Construction of curves from lines

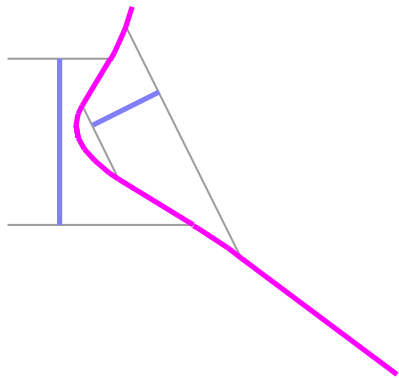


# Triangles and curves



[Florence, 1997]  
Triangular period

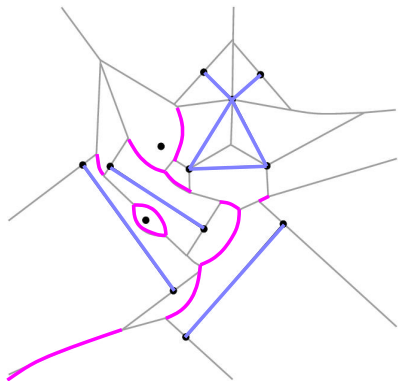
# Curves already appear for linear input



Bisecting curve

2D line segments  
arcs of parabolas

# Curves already appear for linear input

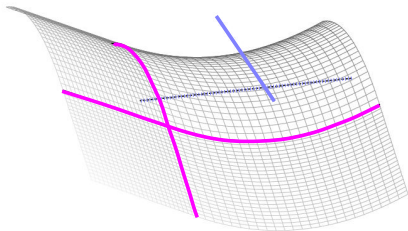


Voronoi diagram

2D line segments  
arcs of parabolas

© Karavelas - CGAL

# Curves already appear for linear input



Voronoi diagram

3D line segments

patches of quadric surfaces

More generally:

manipulations of algebraic curves and surfaces

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manipulations of algebraic curves and surfaces

Only considered here

Exact Geometric Computation

[Yap][...]

# Why we should not be afraid of Computer Algebra

- useful
- interesting
- not so hard to understand



# Why we should not be afraid of Computer Algebra

- useful
- interesting
- not so hard to understand
- people are nice

# Why we should not be afraid of Computer Algebra

trying to convince myself...

- useful
- interesting
- not so hard to understand (?)
- **some** people are nice

# One tool: Resultant

Resultant of a **system** of polynomial equations

= **necessary and sufficient condition**  
such that it **has a root**.

# One tool: Resultant

Resultant of a **system** of polynomial equations

= **necessary and sufficient condition**  
such that it **has a root**.

How to compute the resultant?

hard problem

# Sylvester resultant

Univariate case

$$\begin{cases} P &= a_0x^m + \cdots + a_m \\ Q &= b_0x^n + \cdots + b_n \end{cases}$$

$a_0 \neq 0, b_0 \neq 0, m > n,$   
coefficients in a field  $\mathbb{K}$  (algebraically closed).





# Demystifying Resultant - I

$$\begin{cases} ax + by - c = 0 \\ dx + ey - f = 0 \end{cases}$$

seen as:  $x$  unknown,  $y$  parameter



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$$\text{Sylvester Resultant} = \begin{vmatrix} a & d \\ by - c & ey - f \end{vmatrix}$$

# Demystifying Resultant - I

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seen as:  $x$  unknown,  $y$  parameter

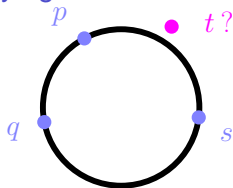
$$\begin{aligned} \text{Sylvester Resultant} &= \begin{vmatrix} a & d \\ by - c & ey - f \end{vmatrix} \\ &= a(ey - f) - d(by - c) \end{aligned}$$

Boils down to **eliminate**  $x$

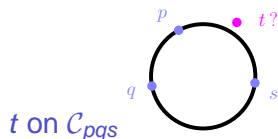
## Demystifying Resultant - II

$p, q, s$  three points in the plane,  
 $t$  a fourth point.

Is  $t$  lying on the circle  $C_{pqs}$ ?



## Demystifying Resultant - II

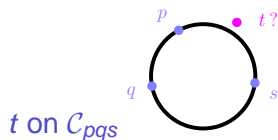


$C_{pqs}$  center  $(x_c, y_c)$  radius  $r$

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

$$\text{iff} \begin{cases} 2x_p x_c + 2y_p y_c + (r^2 - x_c^2 - y_c^2) - (x_p^2 + y_p^2) = 0 \\ 2x_q x_c + 2y_q y_c + (r^2 - x_c^2 - y_c^2) - (x_q^2 + y_q^2) = 0 \\ 2x_s x_c + 2y_s y_c + (r^2 - x_c^2 - y_c^2) - (x_s^2 + y_s^2) = 0 \\ 2x_t x_c + 2y_t y_c + (r^2 - x_c^2 - y_c^2) - (x_t^2 + y_t^2) = 0 \end{cases}$$

## Demystifying Resultant - II

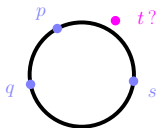


$$\text{iff } \begin{cases} 2x_p X + 2y_p Y + R - (x_p^2 + y_p^2)Z = 0 \\ 2x_q X + 2y_q Y + R - (x_q^2 + y_q^2)Z = 0 \\ 2x_s X + 2y_s Y + R - (x_s^2 + y_s^2)Z = 0 \\ 2x_t X + 2y_t Y + R - (x_t^2 + y_t^2)Z = 0 \end{cases}$$

has a non-trivial solution  $(X, Y, R, Z)$  and

$$\begin{aligned} X/Z &= x_c \\ Y/Z &= y_c \\ R/Z &= r^2 - x_c^2 - y_c^2. \end{aligned}$$

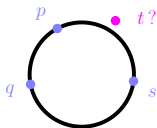
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$$\text{iff} \begin{vmatrix} x_p & y_p & 1 & x_p^2 + y_p^2 \\ x_q & y_q & 1 & x_q^2 + y_q^2 \\ x_s & y_s & 1 & x_s^2 + y_s^2 \\ x_t & y_t & 1 & x_t^2 + y_t^2 \end{vmatrix} = 0$$

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= resultant of the system

Allows to eliminate  $x_c, y_c, r^2$

# Resultant

- Resultant often used in simple cases without noticing



# Resultant

- Resultant often used in simple cases without noticing
- **Linear algebra** helps solve non-linear problems

# Digression on algebraic degree

One measure of **efficiency** and **precision** of a **predicate**:  
algebraic degree

# Digression on algebraic degree

If predicate = sign of a resultant

Resultant has **minimal degree**  $\implies$  optimal predicate?

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No:

- methods often return a multiple of the resultant  
     $\longrightarrow$  resultant hard to compute

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- the resultant may be factored
  - $\longrightarrow$  predicate can have a lower degree

# Digression on algebraic degree

If predicate = sign of a resultant

Resultant has **minimal degree**  $\implies$  optimal predicate?

No:

- methods often return a multiple of the resultant
  - $\longrightarrow$  resultant hard to compute
- the resultant may be factored
  - $\longrightarrow$  predicate can have a lower degree
- a factor may be  $P^2 + Q^2$ 
  - $\longrightarrow$  the degree does not mean so much

## Digression on algebraic degree

- filtering techniques used for efficiency
  - maybe not such an interesting measure ?

# Digression on algebraic degree

- Degree of a **predicate**  
→ not trivial
- Degree of an **algorithm**  
→ depends on the algebraic expressions of predicates
- Degree of a geometric **problem**  
→ ?

Digression  $\mapsto$  thread



## Another tool: Sturm sequences

$$\mathcal{P} = P_0, P_1, \dots, P_d \in \mathbb{R}[X]$$

$$\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$$

$\text{Var}(\mathcal{P}; \alpha)$  = number of sign variations in the sequence  
 $P_0(\alpha), P_1(\alpha), \dots, P_d(\alpha)$

$$\text{Var}(\mathcal{P}; \alpha, \beta) = \text{Var}(\mathcal{P}; \alpha) - \text{Var}(\mathcal{P}; \beta)$$

# Sturm sequences

$P, Q \in \mathbb{K}[X]$  signed remainder sequence of  $P$  and  $Q =$   
sequence  $\mathcal{S}(P, Q) : P_0, P_1, \dots, P_k$

$$P_0 = P$$

$$P_1 = Q$$

$$P_2 = -\text{Rem}(P_0, P_1)$$

$$\vdots$$

$$P_k = -\text{Rem}(P_{k-2}, P_{k-1})$$

$$P_{k+1} = -\text{Rem}(P_{k-1}, P_k) = 0$$

where

$\text{Rem}(A, B) =$  remainder of the Euclidean division of  $A$  by  $B$

# Sturm sequences

Sturm sequence of  $P$  =  
sequence  $\mathcal{S}(P, P')$  of signed remainders of  $P$  and  $P'$

$Var(\mathcal{S}(P, P'); \alpha, \beta)$   
is the **number of roots** of  $P$  in the interval  $[\alpha, \beta]$

# Sturm sequences for dummies

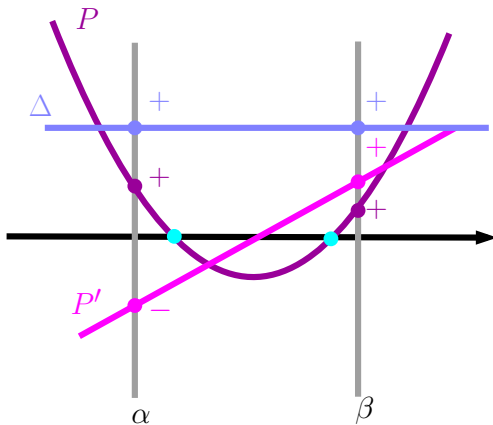
# Sturm sequences ~~for dummies~~ by a dummy

$$P = aX^2 + bX + c$$

$$P' = 2aX + b$$
$$P = P' \cdot \left(\frac{X}{2} + \frac{b}{4a}\right) - \left(\frac{b^2}{4a} - c\right)$$

# Sturm sequences ~~for dummies~~ by a dummy

$$P = aX^2 + bX + c$$



$$P' = 2aX + b$$

$$P = P' \cdot \left(\frac{X}{2} + \frac{b}{4a}\right) - \left(\frac{b^2}{4a} - c\right)$$

$$P_0 = P, P_1 = P', P_2 = \Delta$$

if  $\Delta > 0$

$$\alpha = -\infty, \beta = +\infty$$

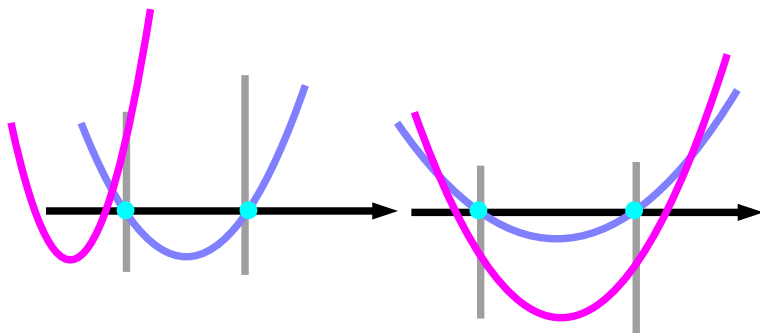
$$\text{Var}(P; \alpha) = 2$$

$$\text{Var}(P; \beta) = 0$$

2 roots

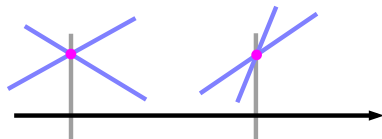
# Sturm sequences

Sequence  $S(P, P'Q)$  of signed remainders of  $P$  and  $P'Q$  counts the number of roots of  $P$  at which  $Q$  is positive



Sturm sequences allow to compare roots of  $P$  and  $Q$

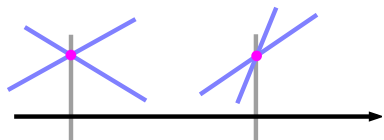
# Comparing intersection points



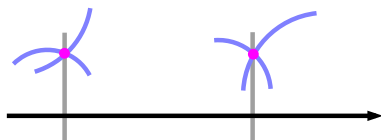
signs of  
polynomial expressions



# Comparing intersection points

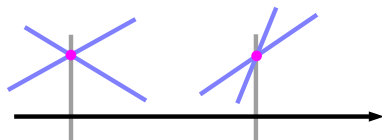


signs of  
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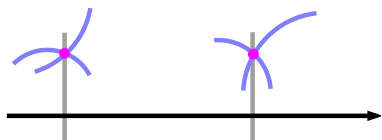


comparison of  
algebraic numbers

# Comparing intersection points



signs of  
polynomial expressions

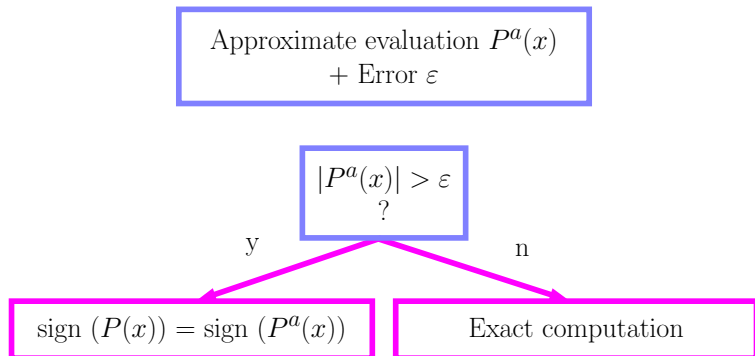


comparison of  
algebraic numbers

Sturm sequences  $\longrightarrow$   
signs of  
polynomial expressions

# Practical efficiency

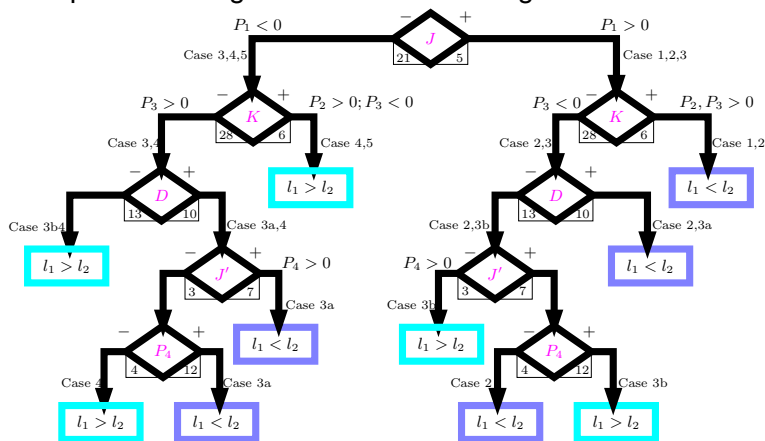
Arithmetic filters for sign computations:



Exact geometric computation  $\neq$  Exact arithmetics

# Practical efficiency

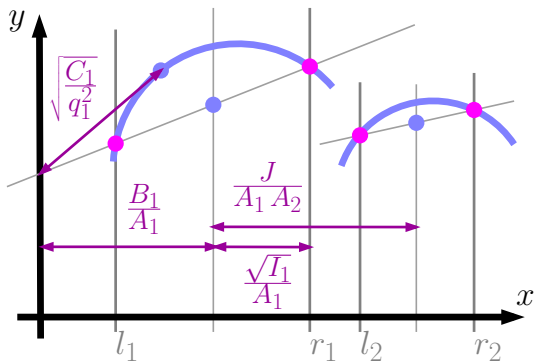
Comparison of algebraic numbers of degree 2:



polynomial expressions pre-computed  
static Sturm sequences

# Algebra is not just “computations”

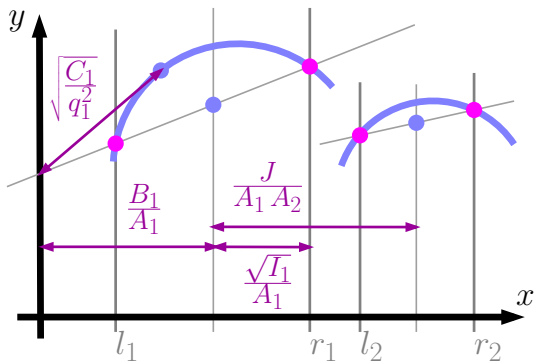
it has a meaning...!



$K = 0 \iff l_1, r_1, l_2, r_2$  harmonic division

# Algebra is not just “computations”

it has a meaning...!

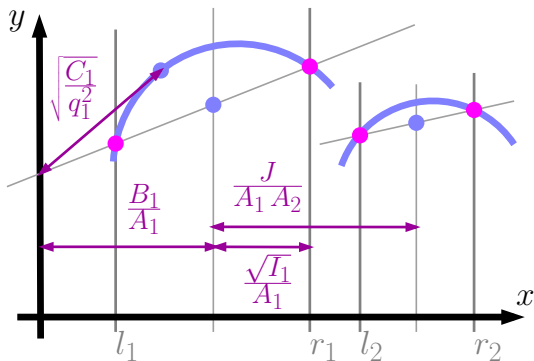


$K = 0 \iff l_1, r_1, l_2, r_2$  harmonic division

- Geometric interpretation in more complicated cases...?

# Algebra is not just “computations”

it has a meaning...!



$K = 0 \iff l_1, r_1, l_2, r_2$  harmonic division

- Geometric interpretation in more complicated cases...?
- Optimal degree...?



*Computational Geometry Algorithms Library*

Open Source Project

[www.cgal.org](http://www.cgal.org)





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[www.cgal.org](http://www.cgal.org)

Release 3.2 soon



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Release 3.2 soon

**Exclusive** news: Out before Microsoft new OS!



*Computational Geometry Algorithms Library*

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Release 3.2 soon

- **new:** 2D Circular Kernel  
manipulations of circular arcs



*Computational Geometry Algorithms Library*

Open Source Project

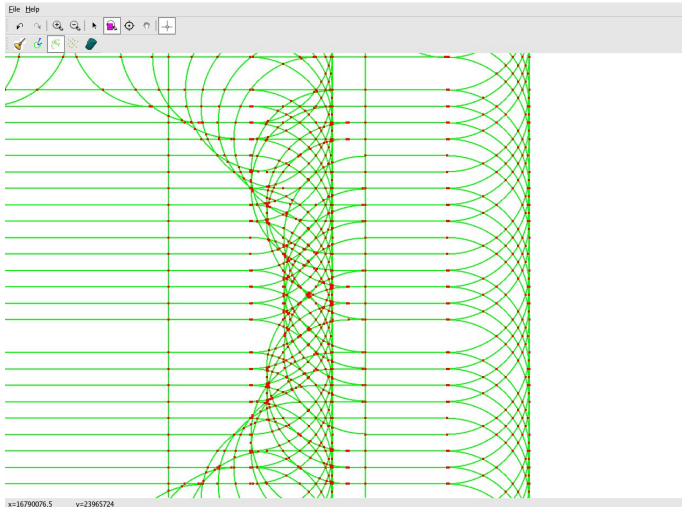
[www.cgal.org](http://www.cgal.org)

Release 3.2 soon

- **new:** 2D Circular Kernel  
manipulations of circular arcs
- Arrangement package **redesigned**
- ...



# VLSI - CAD



# Intersection of two quadrics $Q_S$ and $Q_T$

## Levin's pencil method

- find a “good” quadric in the pencil  $Q_{R(\lambda)=\lambda S-T}$   
 $\lambda$  root of degree 3 pol.
- **Diagonalize**  $R(\lambda)$ .  
Eigenvalues = roots of degree 2 pol.  $\in \mathbb{Q}(\lambda)$ .  
**Normalize** eigenvectors.
- Plug the parameterization of  $Q_{R(\lambda)}$  in  $Q_T$ .  
Degree 2 in one of the parameters. **Solve**

“good” = simple ruled

$$\left| \begin{array}{ccc|c} x & x & x & \\ x & x & x & \vdots \\ x & x & x & \\ \dots & & & \cdot \end{array} \right|$$

principal subdeterminant = 0

# Intersection of two quadrics $Q_S$ and $Q_T$

## Levin's pencil method

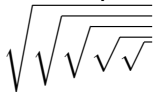
- find a “good” quadric in the pencil  $Q_{R(\lambda)=\lambda S-T}$   
 $\lambda$  root of degree 3 pol.
- Diagonalize  $R(\lambda)$ .  
Eigenvalues = roots of degree 2 pol.  $\in \mathbb{Q}(\lambda)$ .  
Normalize eigenvectors.
- Plug the parameterization of  $Q_{R(\lambda)}$  in  $Q_T$ .  
Degree 2 in one of the parameters. Solve

## Improvement

- work in  $\mathbb{P}^3$
- Relax the constraint on  $Q_{R(\lambda)}$   
Rational, ruled.
- Apply Gauss reduction of the quadratic form:  
 $P^T R P$  diagonal.  
Rational transformation.
- Plug the parameterization in  $Q_T$ .  
Degree 2 in one of the parameters. Solve

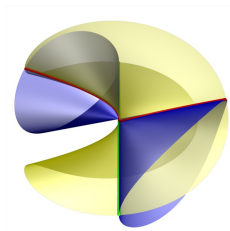
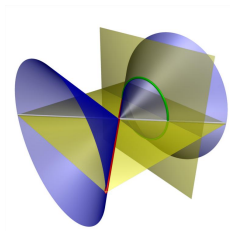
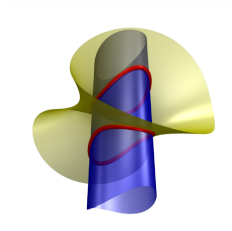
# Intersection of quadrics

Levin's pencil method



- New parameterization
- rational when it exists,
  - involves  $\sqrt{\text{pol.}}$  otherwise.
  - quasi-optimal in  $\sqrt{\cdot}$ .

Implemented

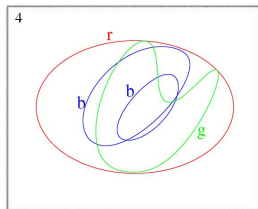
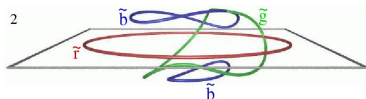
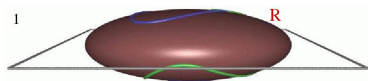


© Dupont et al



# Arrangement of quadrics

Projection approach



© Wolpert

Planar arrangement of curves of degree 4

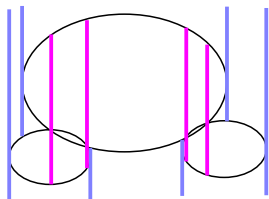
a curve can have 6 singular points

Sort out (upper, lower) → arrangement on each quadric

Surfacic approach

# Arrangement of quadrics

Sweeping approach



**Volumic** approach:  
vertical decomposition

Sweeping plane:  
Trapezoidal map of evolving conics

# Arrangement of quadrics

Sweeping approach

Events:

- new quadric
- features in the map intersect

# Arrangement of quadrics

Sweeping approach

Events:

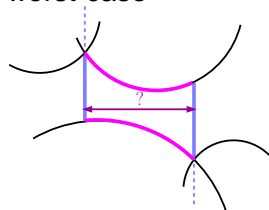
- new quadric
- features in the map intersect

$x$  solution of

$$\exists y, z_1, z_2 \text{ s.t. } \begin{cases} Q_i(x, y, z_1) = 0 \\ Q_j(x, y, z_1) = 0 \end{cases} \text{ and } \begin{cases} Q_k(x, y, z_2) = 0 \\ Q_l(x, y, z_2) = 0 \end{cases}$$

$x$  in an extension field of **degree 16**

worst-case



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Sweeping approach

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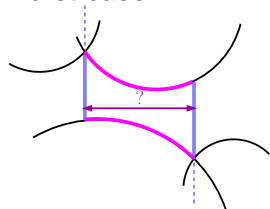
$x$  in an extension field of **degree 16**

Comparison of events:

difference of events in an extension field of **degree 256...**

- Optimal degree...?

worst-case



# Apollonius diagram

Additively weighted Voronoi diagram

Weighted points  $\sigma_i = (p_i, r_i)$ ,  $p_i \in \mathbb{R}^2$ ,  $r_i \in \mathbb{R}$

$$\delta_i(\mathbf{x}) = \|\mathbf{x} - p_i\| - r_i$$

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Apollonius diagram =

lower envelope of the half-cones.

Bisector of  $\sigma_i$  and  $\sigma_j$  =

projection of a plane conic section  $C_i \cap C_j$ .



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$X_i$  projection of  $x$  onto  $C_i$

$$x \in A(\sigma_i) \quad \text{iff} \quad \|\mathbf{x} - p_i\| - r_i < \|\mathbf{x} - p_j\| - r_j \quad (\forall j)$$
$$\quad \text{iff} \quad \text{pow}(X_i, \Sigma_i) < \text{pow}(X_i, \Sigma_j)$$

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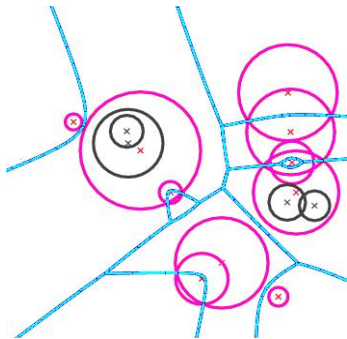
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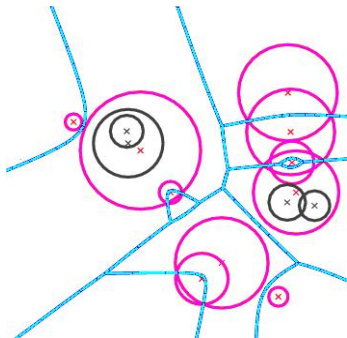
$A(\sigma_i)$  = projection of the intersection of  
the half-cone  $C_i$  with the power region of  $\Sigma_i$

Same in  $\mathbb{R}^d$



Tricky predicates  
Degree 16

© Karavelas



Tricky predicates  
Degree 16

©Karavelas

- Implementation degree 20:  
degree 16 requires  $\sim 100$  times as many arithmetic operations...
- Optimal degree...?

# Challenges

- **theoretical**: questions on degree...

# Challenges

- **theoretical**: questions on degree...
- Robust (Exact?) computation on higher degree curves and surfaces

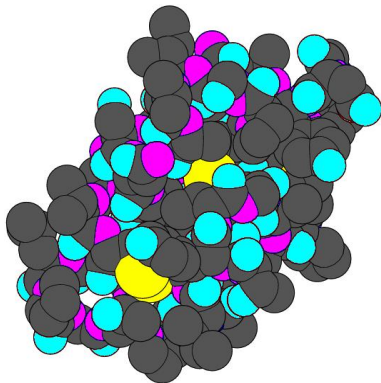
# Challenges

- **theoretical**: questions on degree...
- Robust (Exact?) computation on higher degree curves and surfaces
- Improvement of **practical efficiency** for low degree curves  
CAD-VLSI (circular arcs):
  - ~ 10 times slower than industrial non-robust code
  - good start!**



# Challenges

- Applications to **Structural biology**  
Manipulations of a large number of spheres  
(low degree surfaces. . . )



# ΕΥΧΑΡΙΣΤΩ

Material taken from:

- Greece
  - National University of Athens
  - University of Crete
- Germany
  - Max-Planck Institut für Informatik
  - Universität des Saarlandes
- Israel
  - Tel-Aviv University
- France
  - Loria
  - INRIA Sophia Antipolis

# ΕΥΧΑΡΙΣΤΩ

