



From triangles to curves

Monique Teillaud

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From triangles to curves

Monique Teillaud

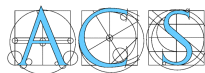
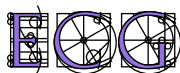


22nd European Workshop on Computational Geometry
March 2006 - $\Delta\epsilon\lambda\varphi\omicron\acute{\iota}$

Warning

- focus on practical methods
- non exhaustive, biased

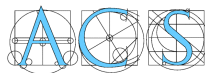
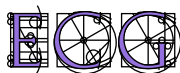
mostly (not only)



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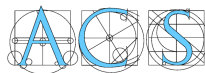
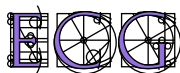


“Commercial”: ECG book coming out soon...

Warning

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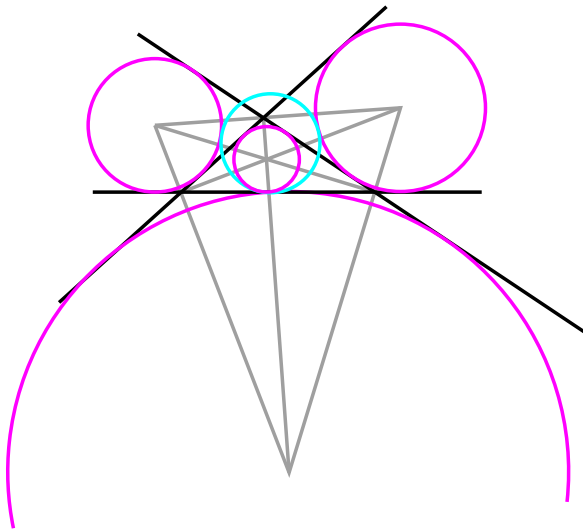
mostly (not only)



Advice to people having some knowledge of Computer Algebra:
you may leave the room

- non technical, superficial. . .

Circles are never far from triangles



Construction of curves from lines

Parabola: smooth connection between line segments

Du point Q on peut mener deux tangentes.

II. — Tracer, parallèlement à une droite donnée RR', une tangente à la parabole (fig. 107).

Abaisser du foyer F une perpendiculaire FM sur RR', puis élever une perpendiculaire QS au milieu de MF.

§ 93. **Raccord obtenu par un arc de parabole.**

— Décrire une parabole, et mener deux tangentes QC et QC' (fig. 108); tracer la corde des contacts CC', soit ED une tangente quelconque. Porter CD en QD₁, et joindre D₁E; constater que D₁E est parallèle à CC'.

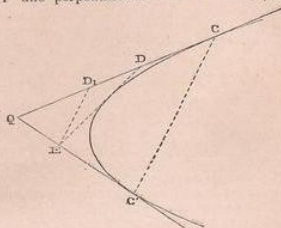


Fig. 108.

De cette constatation nous tirons la conclusion suivante :

$$\frac{QD_1}{QC} = \frac{QE}{QC'},$$

que nous pouvons énoncer ainsi :

Les points d'intersection D et E partagent les tangentes QC et QC' en segments proportionnels inversement placés par rapport au point Q.

Proposons-nous de raccorder par un arc de parabole les deux directrices concourantes QC et QC' (fig. 109) :

Partager les distances QC et QC' en un même nombre de parties égales, cinq par exemple, et numérotier les points de division de

Construction of curves from lines

Parabola: smooth connection between line segments

248

CHAPITRE II.

l'une à partir du sommet, et de l'autre à partir du raccord; joindre les points portant le même numéro.

Toutes les droites ainsi tracées seront tangentes à l'arc de parabole, et chaque contact se trouvera au milieu de la portion de

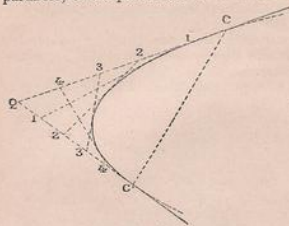


Fig. 109.

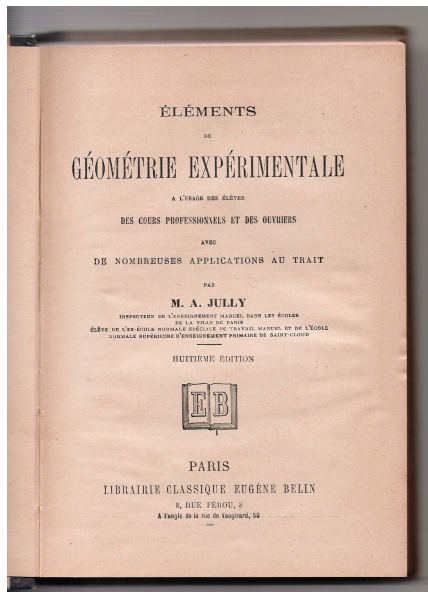
tangente comprise entre les tangentes voisines. On aura ainsi autant de points de la courbe qu'on voudra, et de plus, en chacun de ces points une tangente qui servira de limite, il sera donc très facile de tracer la courbe.

APPLICATIONS. — Ce raccord présente l'avantage de ne pas offrir de brusque changement de direction, la courbure variant graduel-

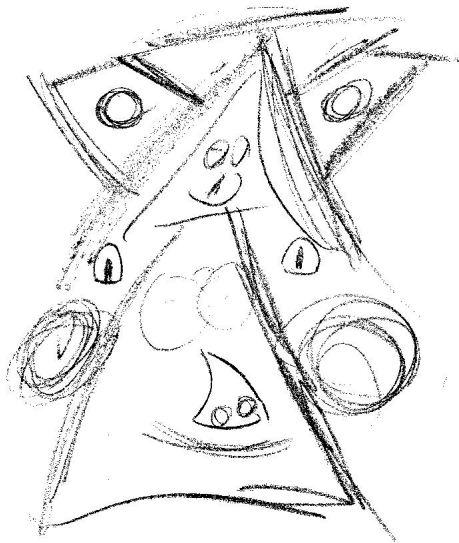
lement du sommet aux points de raccordement.

Les cintres constituent de véritables raccords opérés entre deux pieds-droits, au moyen d'une courbe. Quand les pieds-droits ont même hauteur, on emploie un demi-cercle, ou une moitié d'ellipse donnée par le grand axe si le cintre est surhaussé, et par le petit, s'il est surbaissé. Quand les pieds-droits sont inégaux et parallèles, on se sert pour l'arc rampant d'une demi-ellipse donnée par deux diamètres conjugués, mais si les pieds-droits ne sont ni égaux ni parallèles, le raccordement se fait suivant un arc de parabole tracé comme il vient d'être indiqué ci-dessus.

Construction of curves from lines

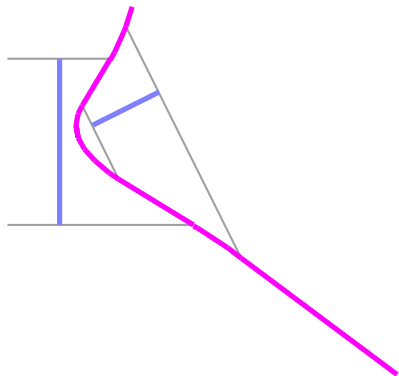


Triangles and curves



[Florence, 1997]
Triangular period

Curves already appear for linear input

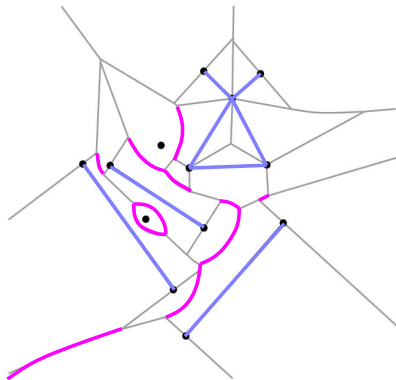


Bisecting curve

2D line segments

arcs of parabolas

Curves already appear for linear input

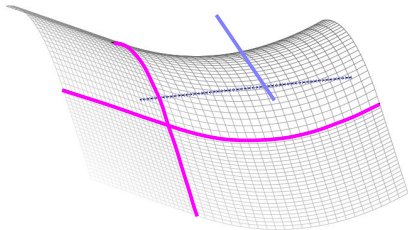


Voronoi diagram

2D line segments
arcs of parabolas

© Karavelas - CGAL

Curves already appear for linear input



Voronoi diagram

3D line segments
patches of quadric surfaces

More generally:

manipulations of algebraic curves and surfaces

More generally:

manipulations of algebraic curves and surfaces

Only considered here

Exact Geometric Computation

[Yap][...]

Why we should not be afraid of Computer Algebra

- useful
- interesting
- not so hard to understand

Why we should not be afraid of Computer Algebra

- useful
- interesting
- not so hard to understand
- people are nice

Why we should not be afraid of Computer Algebra

trying to convince myself...

- useful
- interesting
- not so hard to understand (?)
- some people are nice

One tool: Resultant

Resultant of a **system** of polynomial equations

= **necessary and sufficient condition**
such that it **has a root**.

One tool: Resultant

Resultant of a **system** of polynomial equations

= **necessary and sufficient condition**
such that it **has a root**.

How to compute the resultant?

hard problem

Sylvester resultant

Univariate case

$$\begin{cases} P &= a_0 x^m + \cdots + a_m \\ Q &= b_0 x^n + \cdots + b_n \end{cases}$$

$a_0 \neq 0, b_0 \neq 0, m > n,$

coefficients in a field \mathbb{K} (algebraically closed).

Sylvester resultant

$$\begin{cases} P &= a_0 x^m + \cdots + a_m \\ Q &= b_0 x^n + \cdots + b_n \end{cases}$$

Sylvester resultant =

$$\begin{vmatrix} a_0 & & & & & & & & & & b_0 & & & & & & & & & \\ a_1 & a_0 & & & & & & & & & b_1 & b_0 & & & & & & & & \\ & a_1 & \ddots & & & & & & & & & b_1 & \ddots & & & & & & & \\ & & \ddots & \ddots & & & & & & & & & \ddots & & & & & & & b_0 \\ \vdots & & & \ddots & \ddots & & & & & & \vdots & & & \ddots & & & & & & b_1 \\ & \vdots & & & \ddots & & & & a_0 & & & \vdots & & & & & & & \\ & & \vdots & & & \ddots & & & a_1 & & & \vdots & & & & & & & \\ \vdots & & & \vdots & & & \ddots & & & \vdots & & \vdots & & & & & & & \\ & a_m & & & \vdots & & & \vdots & & \vdots & & \vdots & & & & & & & \\ & & a_m & & \vdots & & & \vdots & & \vdots & & \vdots & & & & & & & \\ & & & \ddots & & \vdots & & \vdots & & \vdots & & \vdots & & & & & & & \\ & & & & \ddots & & \vdots & & & \vdots & & \vdots & & & & & & & \\ & & & & & \ddots & & \vdots & & & b_n & & \vdots & & & & & & \\ & & & & & & \ddots & & & & & b_n & & \vdots & & & & & \\ & & & & & & & \ddots & & & & & & \ddots & & & & & \\ & & & & & & & & a_m & & & & & & & & & & b_n \end{vmatrix}$$

Sylvester resultant

$$\begin{cases} P &= a_0x^m + \cdots + a_m \\ Q &= b_0x^n + \cdots + b_n \end{cases}$$

Sylvester resultant =

$$= 0 \text{ iff}$$

P and Q have a
common root in \mathbb{K} .

Demystifying Resultant - I

$$\begin{cases} ax + by - c = 0 \\ dx + ey - f = 0 \end{cases}$$

seen as: x unknown, y parameter

Demystifying Resultant - I

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$$\text{Sylvester Resultant} = \begin{vmatrix} a & d \\ by - c & ey - f \end{vmatrix}$$

Demystifying Resultant - I

$$\begin{cases} ax + by - c = 0 \\ dx + ey - f = 0 \end{cases}$$

seen as: x unknown, y parameter

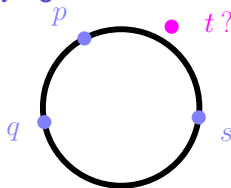
$$\begin{aligned} \text{Sylvester Resultant} &= \begin{vmatrix} a & d \\ by - c & ey - f \end{vmatrix} \\ &= a(ey - f) - d(by - c) \end{aligned}$$

Boils down to **eliminate** x

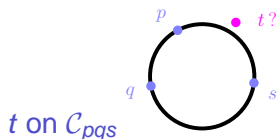
Demystifying Resultant - II

p, q, s three points in the plane,
 t a fourth point.

Is t lying on the circle C_{pqs} ?



Demystifying Resultant - II

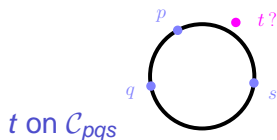


C_{pqs} center (x_c, y_c) radius r

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

$$\text{iff} \begin{cases} 2x_px_c + 2y_py_c + (r^2 - x_c^2 - y_c^2) - (x_p^2 + y_p^2) = 0 \\ 2x_qx_c + 2y_qy_c + (r^2 - x_c^2 - y_c^2) - (x_q^2 + y_q^2) = 0 \\ 2x_sx_c + 2y_sy_c + (r^2 - x_c^2 - y_c^2) - (x_s^2 + y_s^2) = 0 \\ 2x_tx_c + 2y_ty_c + (r^2 - x_c^2 - y_c^2) - (x_t^2 + y_t^2) = 0 \end{cases}$$

Demystifying Resultant - II

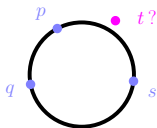


$$\text{iff} \begin{cases} 2x_p X + 2y_p Y + R - (x_p^2 + y_p^2)Z = 0 \\ 2x_q X + 2y_q Y + R - (x_q^2 + y_q^2)Z = 0 \\ 2x_s X + 2y_s Y + R - (x_s^2 + y_s^2)Z = 0 \\ 2x_t X + 2y_t Y + R - (x_t^2 + y_t^2)Z = 0 \end{cases}$$

has a non-trivial solution (X, Y, R, Z) and

$$\begin{aligned} X/Z &= x_c \\ Y/Z &= y_c \\ R/Z &= r^2 - x_c^2 - y_c^2. \end{aligned}$$

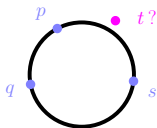
Demystifying Resultant - II



$$\text{iff} \begin{cases} 2x_p x_c + 2y_p y_c + (r^2 - x_c^2 - y_c^2) - (x_p^2 + y_p^2) = 0 \\ 2x_q x_c + 2y_q y_c + (r^2 - x_c^2 - y_c^2) - (x_q^2 + y_q^2) = 0 \\ 2x_s x_c + 2y_s y_c + (r^2 - x_c^2 - y_c^2) - (x_s^2 + y_s^2) = 0 \\ 2x_t x_c + 2y_t y_c + (r^2 - x_c^2 - y_c^2) - (x_t^2 + y_t^2) = 0 \end{cases}$$

$$\text{iff} \begin{vmatrix} x_p & y_p & 1 & x_p^2 + y_p^2 \\ x_q & y_q & 1 & x_q^2 + y_q^2 \\ x_s & y_s & 1 & x_s^2 + y_s^2 \\ x_t & y_t & 1 & x_t^2 + y_t^2 \end{vmatrix} = 0$$

Demystifying Resultant - II



$$\text{iff} \begin{cases} 2x_p x_c + 2y_p y_c + (r^2 - x_c^2 - y_c^2) - (x_p^2 + y_p^2) = 0 \\ 2x_q x_c + 2y_q y_c + (r^2 - x_c^2 - y_c^2) - (x_q^2 + y_q^2) = 0 \\ 2x_s x_c + 2y_s y_c + (r^2 - x_c^2 - y_c^2) - (x_s^2 + y_s^2) = 0 \\ 2x_t x_c + 2y_t y_c + (r^2 - x_c^2 - y_c^2) - (x_t^2 + y_t^2) = 0 \end{cases}$$

$$\text{iff} \begin{vmatrix} x_p & y_p & 1 & x_p^2 + y_p^2 \\ x_q & y_q & 1 & x_q^2 + y_q^2 \\ x_s & y_s & 1 & x_s^2 + y_s^2 \\ x_t & y_t & 1 & x_t^2 + y_t^2 \end{vmatrix} = 0$$

= **resultant** of the system

Allows to **eliminate** x_c, y_c, r^2

Resultant

- Resultant often used in simple cases without noticing

Resultant

- Resultant often used in simple cases without noticing
- Linear algebra helps solve non-linear problems

Digression on algebraic degree

One measure of efficiency and precision of a predicate:
algebraic degree

Digression on algebraic degree

If predicate = sign of a resultant

Resultant has **minimal degree** \implies optimal predicate?

Digression on algebraic degree

If predicate = sign of a resultant

Resultant has **minimal degree** \implies optimal predicate?

No:

- methods often return a multiple of the resultant
 \longrightarrow resultant hard to compute

Digression on algebraic degree

If predicate = sign of a resultant

Resultant has **minimal degree** \implies optimal predicate?

No:

- methods often return a multiple of the resultant
 - \longrightarrow resultant hard to compute
- the resultant may be factored
 - \longrightarrow predicate can have a lower degree

Digression on algebraic degree

If predicate = sign of a resultant

Resultant has **minimal degree** \implies optimal predicate?

No:

- methods often return a multiple of the resultant
 - \longrightarrow resultant hard to compute
- the resultant may be factored
 - \longrightarrow predicate can have a lower degree
- a factor may be $P^2 + Q^2$
 - \longrightarrow the degree does not mean so much

Digression on algebraic degree

- filtering techniques used for efficiency
→ maybe not such an interesting measure ?

Digression on algebraic degree

- Degree of a **predicate**
→ not trivial
- Degree of an **algorithm**
→ depends on the algebraic expressions of predicates
- Degree of a geometric **problem**
→ ?

Digression \mapsto thread

Another tool: Sturm sequences

$$\mathcal{P} = P_0, P_1, \dots, P_d \in \mathbb{R}[X]$$

$$\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$$

$Var(\mathcal{P}; \alpha)$ = number of sign variations in the sequence
 $P_0(\alpha), P_1(\alpha), \dots, P_d(\alpha)$

$$Var(\mathcal{P}; \alpha, \beta) = Var(\mathcal{P}; \alpha) - Var(\mathcal{P}; \beta)$$

Sturm sequences

$P, Q \in \mathbb{K}[X]$ signed remainder sequence of P and $Q =$
sequence $\mathcal{S}(P, Q) : P_0, P_1, \dots, P_k$

$$P_0 = P$$

$$P_1 = Q$$

$$P_2 = -\text{Rem}(P_0, P_1)$$

$$\vdots$$

$$P_k = -\text{Rem}(P_{k-2}, P_{k-1})$$

$$P_{k+1} = -\text{Rem}(P_{k-1}, P_k) = 0$$

where

$\text{Rem}(A, B) = \text{remainder}$ of the Euclidean division of A by B

Sturm sequences

Sturm sequence of P =
sequence $\mathcal{S}(P, P')$ of signed reminders of P and P'

$Var(\mathcal{S}(P, P'); \alpha, \beta)$
is the **number of roots** of P in the interval $[\alpha, \beta]$

Sturm sequences for dummies

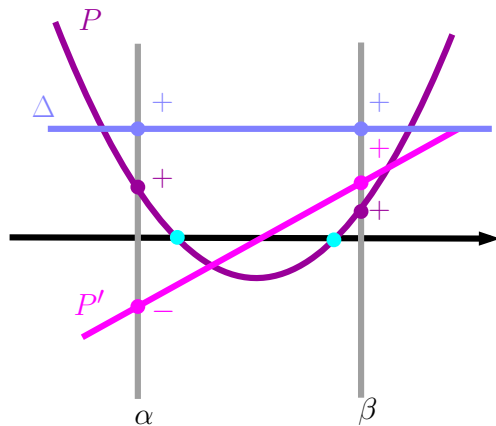
Sturm sequences ~~for dummies~~ by a dummy

$$P = aX^2 + bX + c$$

$$P' = 2aX + b$$
$$P = P' \cdot \left(\frac{X}{2} + \frac{b}{4a}\right) - \left(\frac{b^2}{4a} - c\right)$$

Sturm sequences ~~for dummies~~ by a dummy

$$P = aX^2 + bX + c$$



$$P' = 2aX + b$$
$$P = P' \cdot \left(\frac{X}{2} + \frac{b}{4a}\right) - \left(\frac{b^2}{4a} - c\right)$$

$$P_0 = P, P_1 = P', P_2 = \Delta$$

if $\Delta > 0$

$$\alpha = -\infty, \beta = +\infty$$

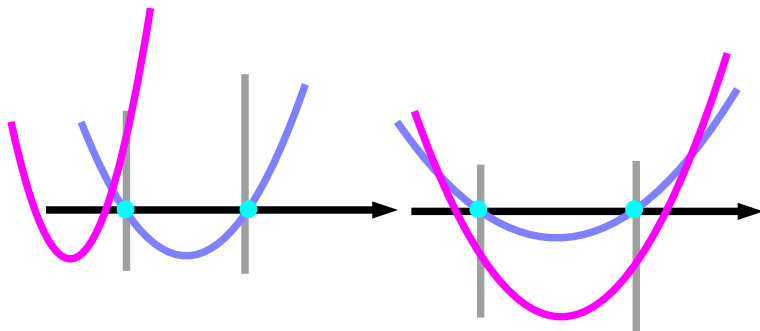
$$\text{Var}(\mathcal{P}; \alpha) = 2$$

$$\text{Var}(\mathcal{P}; \beta) = 0$$

2 roots

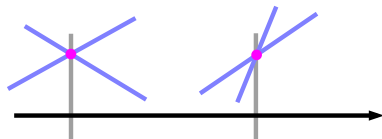
Sturm sequences

Sequence $S(P, P'Q)$ of signed remainders of P and $P'Q$ counts the number of roots of P at which Q is positive



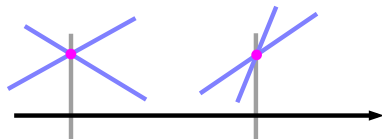
Sturm sequences allow to compare roots of P and Q

Comparing intersection points

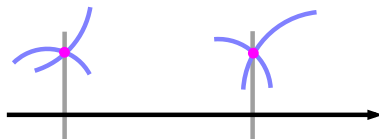


signs of
polynomial expressions

Comparing intersection points

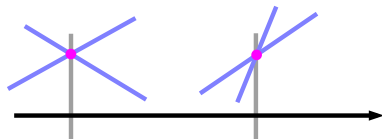


signs of
polynomial expressions

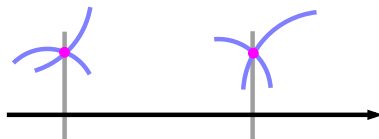


comparison of
algebraic numbers

Comparing intersection points



signs of
polynomial expressions

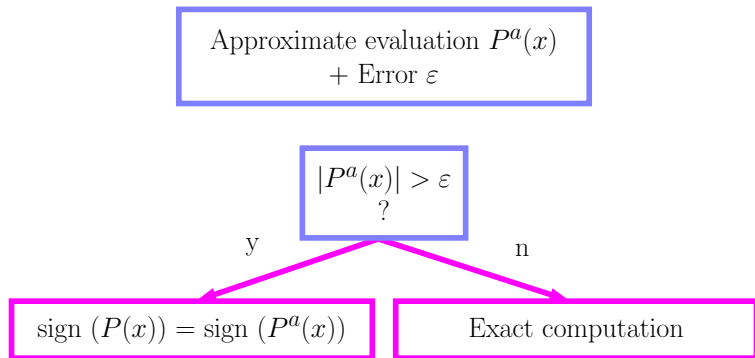


comparison of
algebraic numbers

Sturm sequences \longrightarrow
signs of
polynomial expressions

Practical efficiency

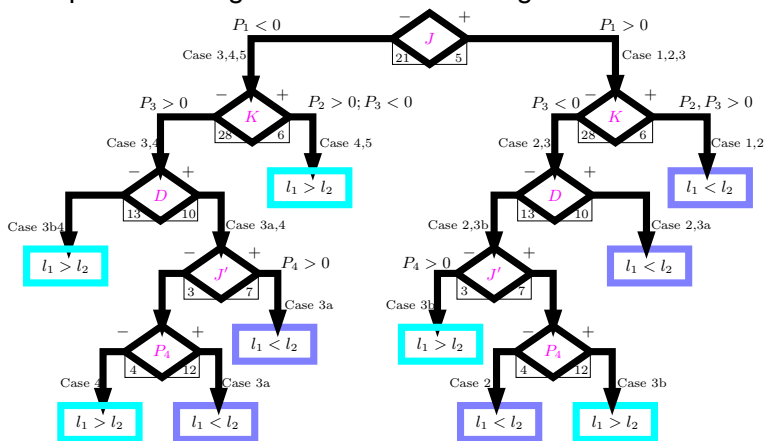
Arithmetic filters for sign computations:



Exact geometric computation \neq Exact arithmetics

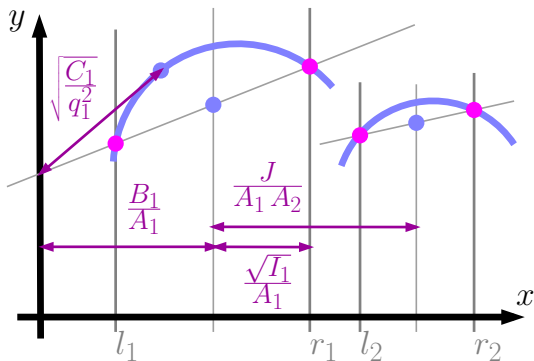
Practical efficiency

Comparison of algebraic numbers of degree 2:



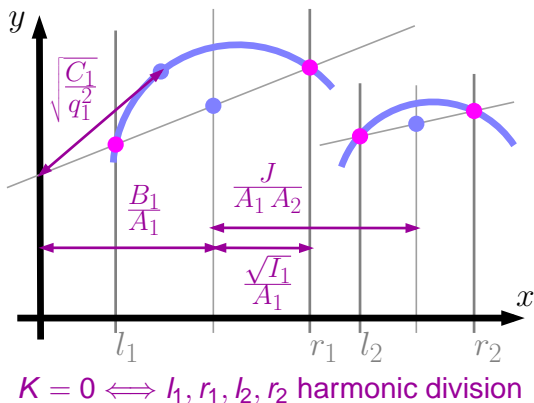
polynomial expressions **pre-computed**
static Sturm sequences

Algebra is not just “computations” it has a meaning...!



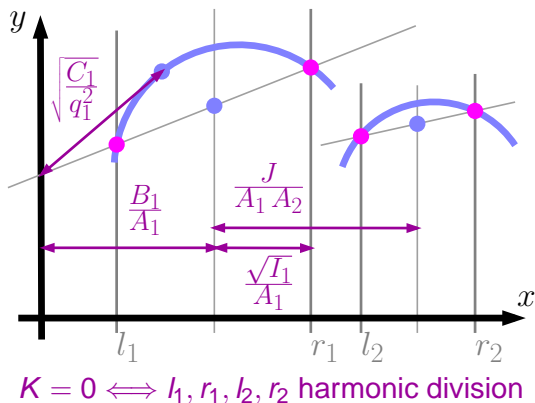
$K = 0 \iff l_1, r_1, l_2, r_2$ harmonic division

Algebra is not just “computations” it has a meaning...!



- Geometric interpretation in more complicated cases...?

Algebra is not just “computations” it has a meaning...!



- Geometric interpretation in more complicated cases...?
- Optimal degree...?



Computational Geometry Algorithms Library

Open Source Project

www.cgal.org



Computational Geometry Algorithms Library

Open Source Project

www.cgal.org

Release 3.2 soon



Computational Geometry Algorithms Library

Open Source Project

www.cgal.org

Release 3.2 soon

Exclusive news: Out before Microsoft new OS!



Computational Geometry Algorithms Library

Open Source Project

www.cgal.org

Release 3.2 soon

- new: 2D Circular Kernel
manipulations of circular arcs



Computational Geometry Algorithms Library

Open Source Project

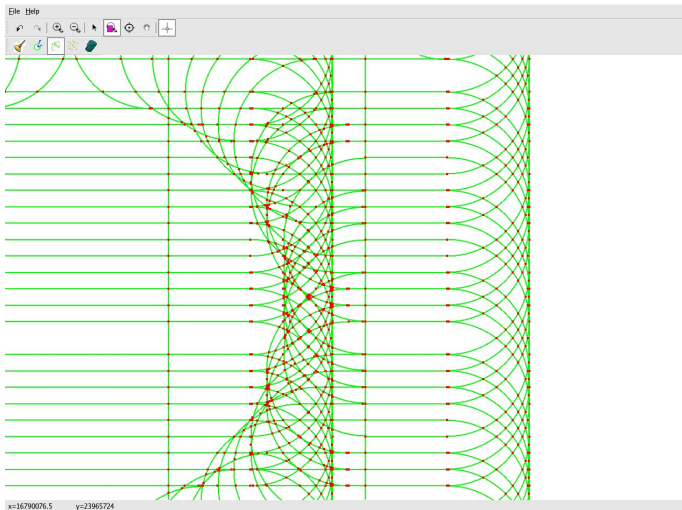
www.cgal.org

Release 3.2 soon

- new: 2D Circular Kernel
manipulations of circular arcs
- Arrangement package redesigned
- ...



VLSI - CAD



Intersection of two quadrics Q_S and Q_T

Levin's pencil method

- find a “good” quadric in the pencil $Q_{R(\lambda)=\lambda S-T}$
 λ root of degree 3 pol.
- Diagonalize $R(\lambda)$.
Eigenvalues = roots of degree 2 pol. $\in \mathbb{Q}(\lambda)$.
Normalize eigenvectors.
- Plug the parameterization of $Q_{R(\lambda)}$ in Q_T .
Degree 2 in one of the parameters. Solve

“good” = simple ruled

$$\left| \begin{array}{ccc|c} x & x & x & \\ x & x & x & \vdots \\ x & x & x & \\ \hline & \dots & & . \end{array} \right|$$

principal subdeterminant = 0

Intersection of two quadrics Q_S and Q_T

Levin's pencil method

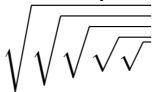
- find a “good” quadric in the pencil $Q_{R(\lambda)=\lambda S-T}$
 λ root of degree 3 pol.
- Diagonalize $R(\lambda)$.
Eigenvalues = roots of degree 2 pol. $\in \mathbb{Q}(\lambda)$.
Normalize eigenvectors.
- Plug the parameterization of $Q_{R(\lambda)}$ in Q_T .
Degree 2 in one of the parameters. Solve

Improvement

- work in \mathbb{P}^3
- Relax the constraint on $Q_{R(\lambda)}$
Rational, ruled.
- Apply Gauss reduction of the quadratic form:
 $P^T R P$ diagonal.
Rational transformation.
- Plug the parameterization in Q_T .
Degree 2 in one of the parameters. Solve

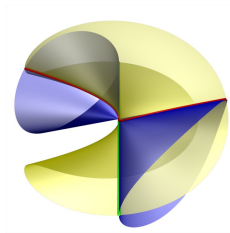
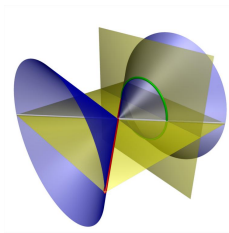
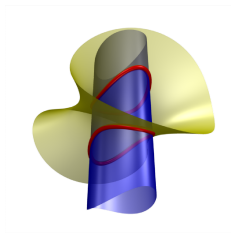
Intersection of quadrics

Levin's pencil method



- New parameterization
- rational when it exists, involves $\sqrt{\text{pol.}}$ otherwise.
 - quasi-optimal in $\sqrt{}$.

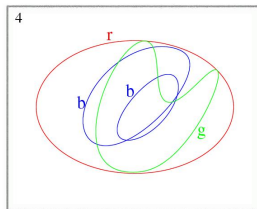
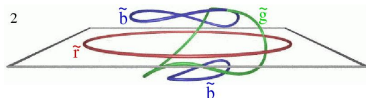
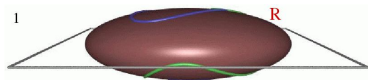
Implemented



©Dupont et al

Arrangement of quadrics

Projection approach



© Wolpert

Planar arrangement of curves of degree 4

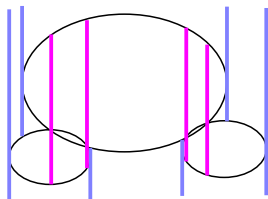
a curve can have 6 singular points

Sort out (upper, lower) \rightarrow arrangement on each quadric

Surfacic approach

Arrangement of quadrics

Sweeping approach



Sweeping plane:
Trapezoidal map of evolving conics

Volumic approach:
vertical decomposition

Arrangement of quadrics

Sweeping approach

Events:

- new quadric
- features in the map intersect

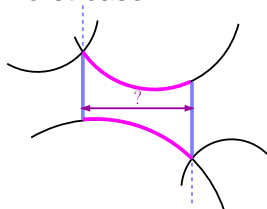
Arrangement of quadrics

Sweeping approach

Events:

- new quadric
- features in the map intersect

worst-case



\mathbf{x} solution of

$$\exists y, z_1, z_2 \text{ s.t. } \begin{cases} Q_i(\mathbf{x}, y, z_1) = 0 \\ Q_j(\mathbf{x}, y, z_1) = 0 \end{cases} \text{ and } \begin{cases} Q_k(\mathbf{x}, y, z_2) = 0 \\ Q_l(\mathbf{x}, y, z_2) = 0 \end{cases}$$

\mathbf{x} in an extension field of degree 16

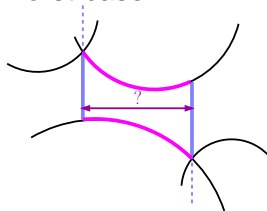
Arrangement of quadrics

Sweeping approach

Events:

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\mathbf{x} in an extension field of degree 16

Comparison of events:

difference of events in an extension field of degree 256...

- Optimal degree...?

Apollonius diagram

Additively weighted Voronoi diagram

Weighted points $\sigma_i = (p_i, r_i)$, $p_i \in \mathbb{R}^2$, $r_i \in \mathbb{R}$

$$\delta_i(\mathbf{x}) = \|\mathbf{x} - p_i\| - r_i$$

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Apollonius diagram =
lower envelope of the half-cones.

Bisector of σ_i and σ_j =
projection of a plane conic section $C_i \cap C_j$.

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X_i projection of \mathbf{x} onto C_i

$$\begin{aligned} \mathbf{x} \in A(\sigma_i) \quad & \text{iff} \quad \|\mathbf{x} - p_i\| - r_i < \|\mathbf{x} - p_j\| - r_j \quad (\forall j) \\ & \text{iff} \quad \text{pow}(X_i, \Sigma_i) < \text{pow}(X_i, \Sigma_j) \end{aligned}$$

Apollonius diagram

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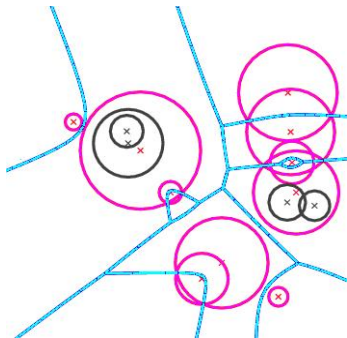
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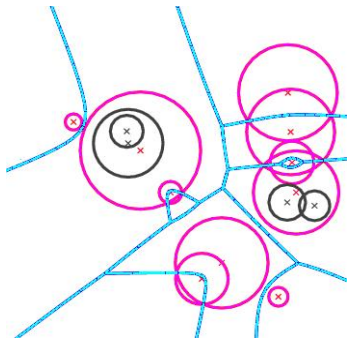
$A(\sigma_i)$ = projection of the intersection of
the half-cone C_i with the power region of Σ_i

Same in \mathbb{R}^d



Tricky predicates
Degree 16

© Karavelas



Tricky predicates
Degree 16

©Karavelas

- Implementation degree 20:
degree 16 requires ~ 100 times as many arithmetic operations...
- Optimal degree...?

Challenges

- **theoretical**: questions on degree...

Challenges

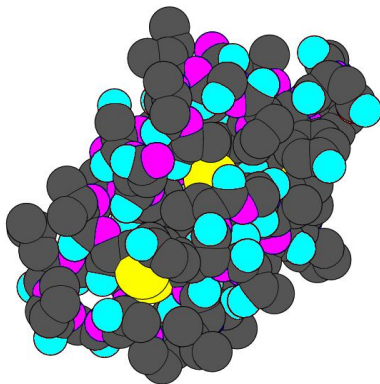
- **theoretical**: questions on degree...
- Robust (Exact?) computation on higher degree curves and surfaces

Challenges

- **theoretical**: questions on degree...
- Robust (Exact?) computation on higher degree curves and surfaces
- Improvement of **practical efficiency** for low degree curves
CAD-VLSI (circular arcs):
 - ~ 10 times slower than industrial non-robust code
 - good start!**

Challenges

- Applications to **Structural biology**
Manipulations of a large number of spheres
(low degree surfaces. . .)



ευχαριστώ

Material taken from:

- Greece
 - National University of Athens
 - University of Crete
- Germany
 - Max-Planck Institut für Informatik
 - Universität des Saarlandes
- Israel
 - Tel-Aviv University
- France
 - Loria
 - INRIA Sophia Antipolis

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