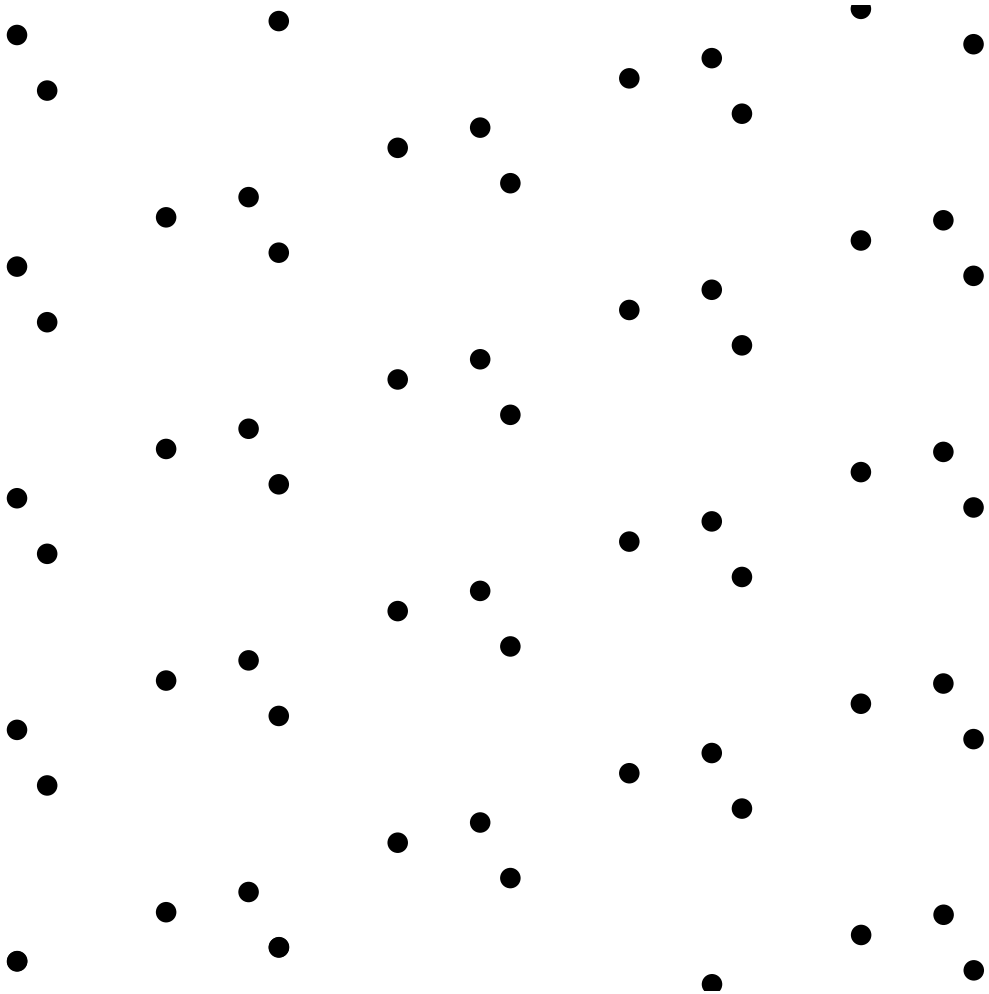
Generalizing CGAL Periodic Delaunay Triangulations

Georg Osang, Mael Rouxel-Labbé and Monique Teillaud

September 8th, 2020

Periodic Delaunay triangulations



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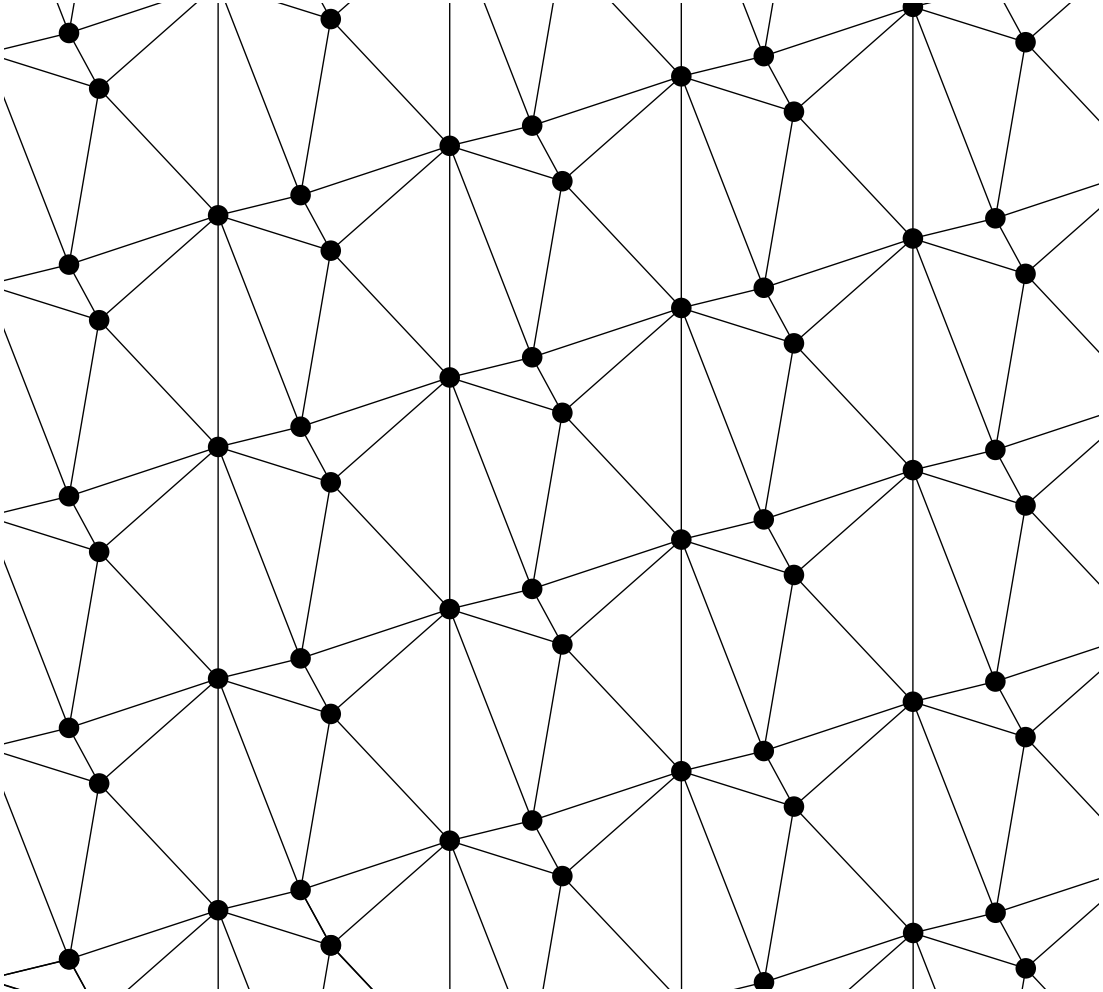

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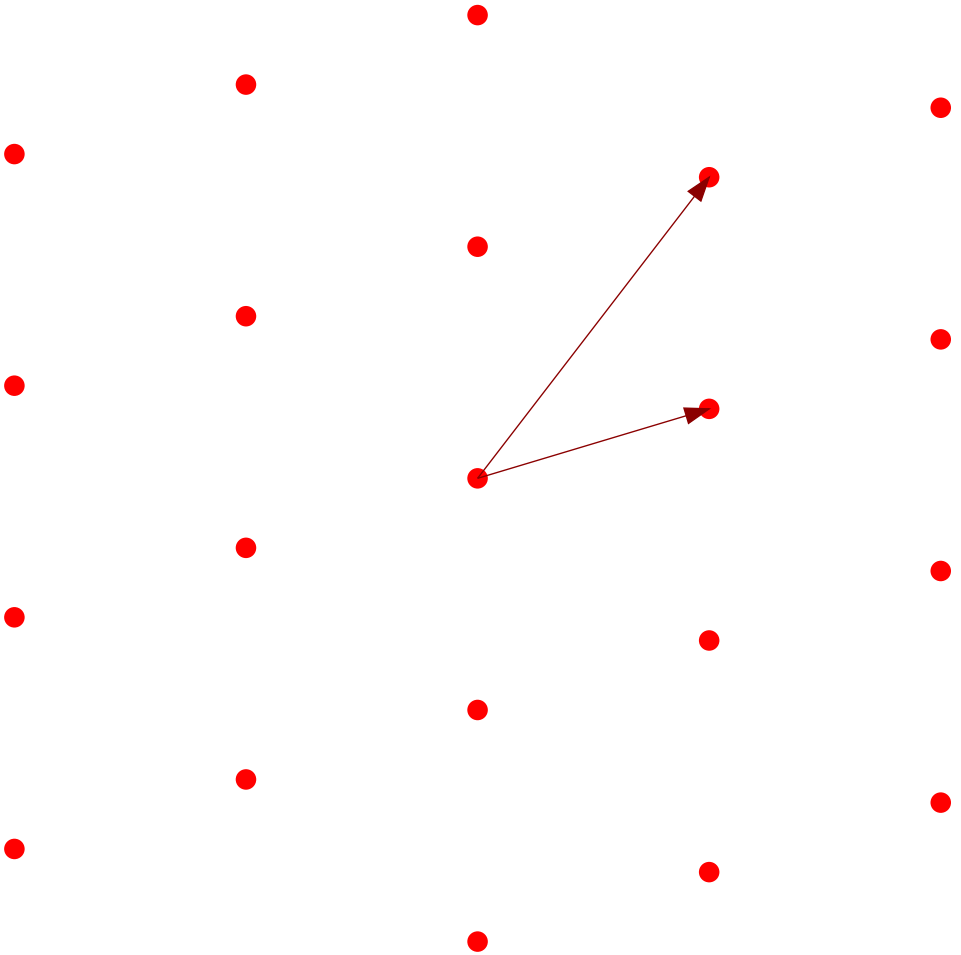

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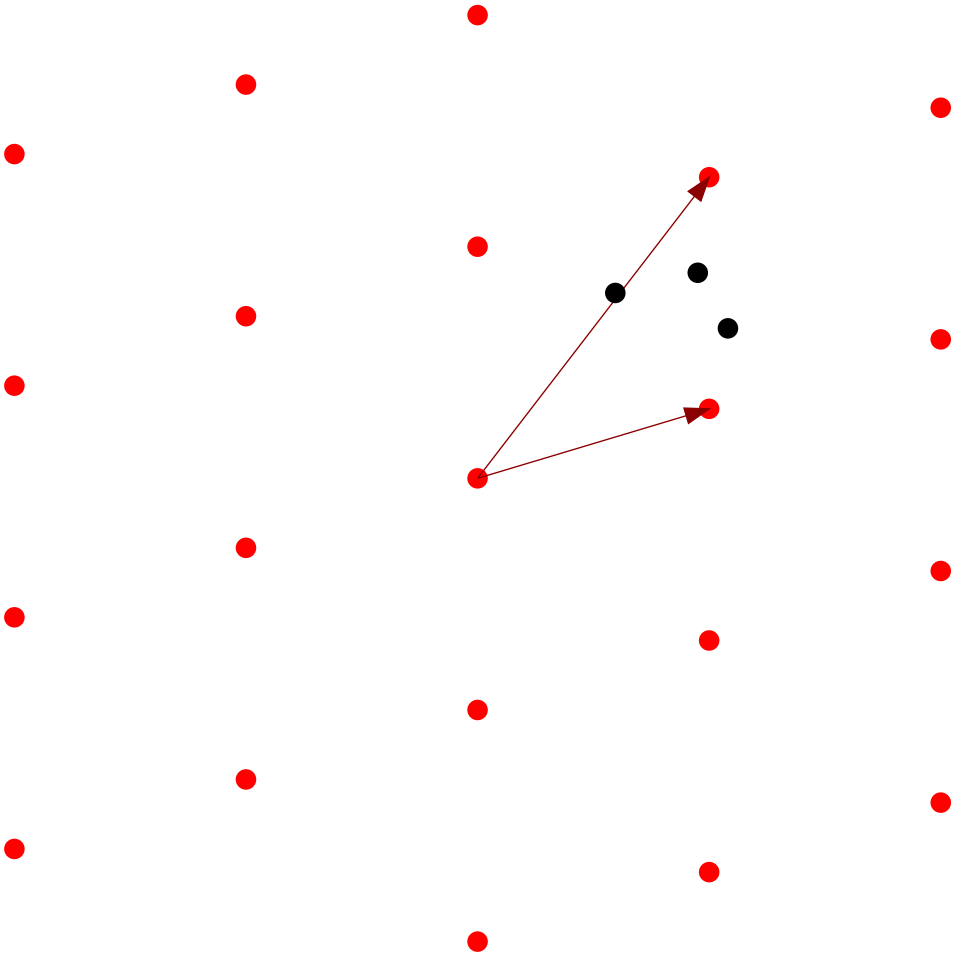
Periodic Delaunay triangulations



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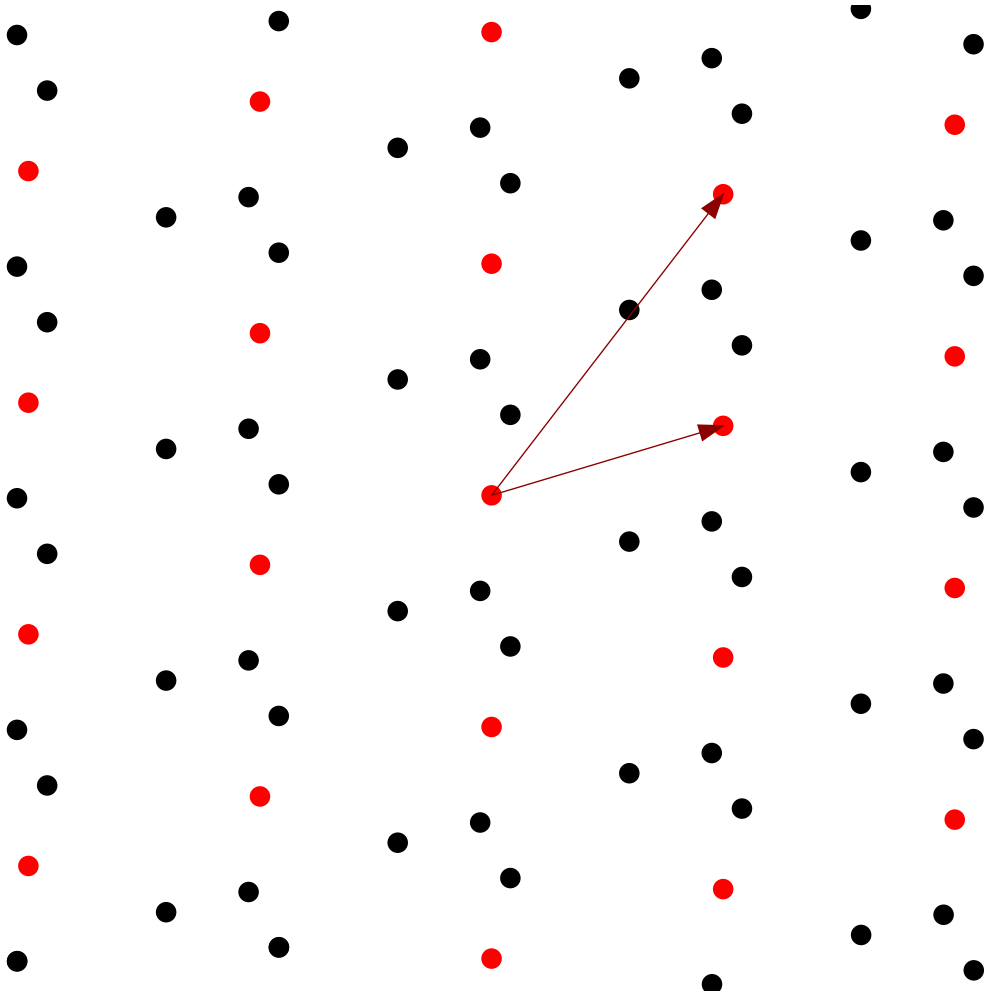

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$\rightarrow$ Forming periodic point set ΛX (in 2D or 3D)

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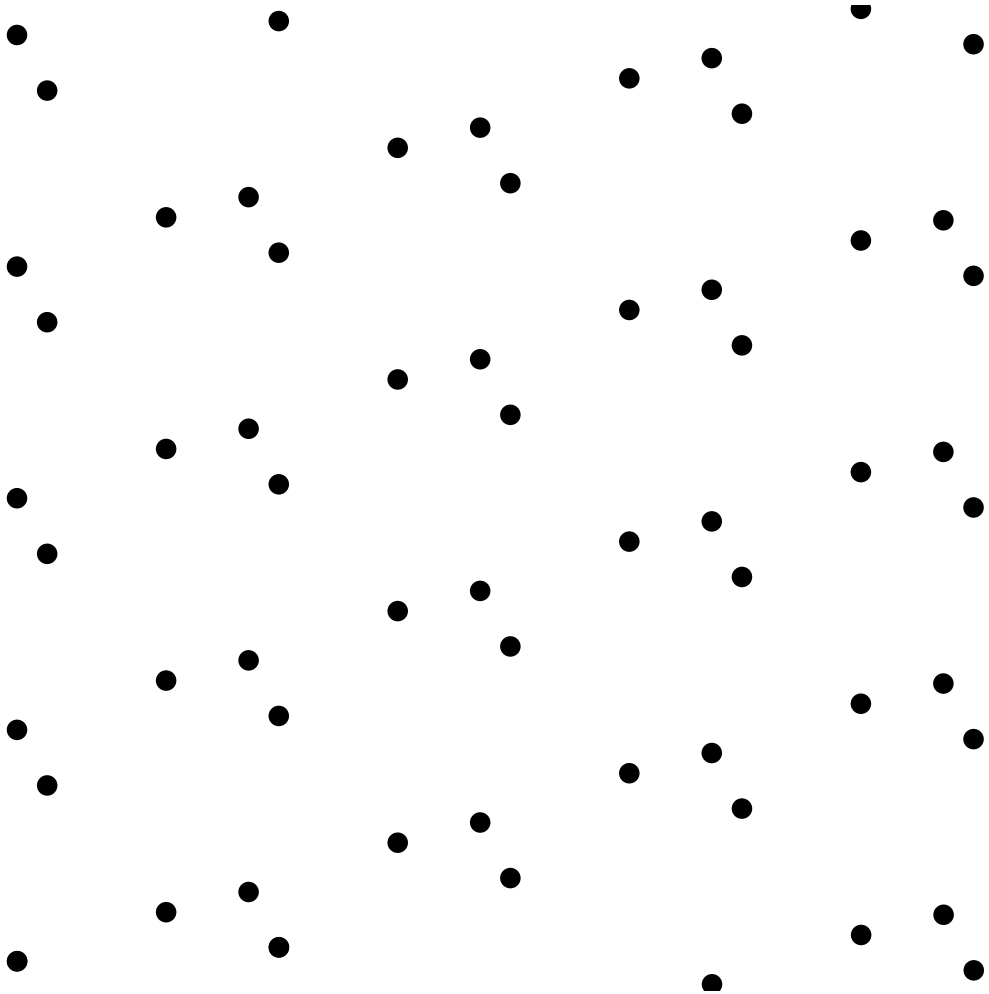

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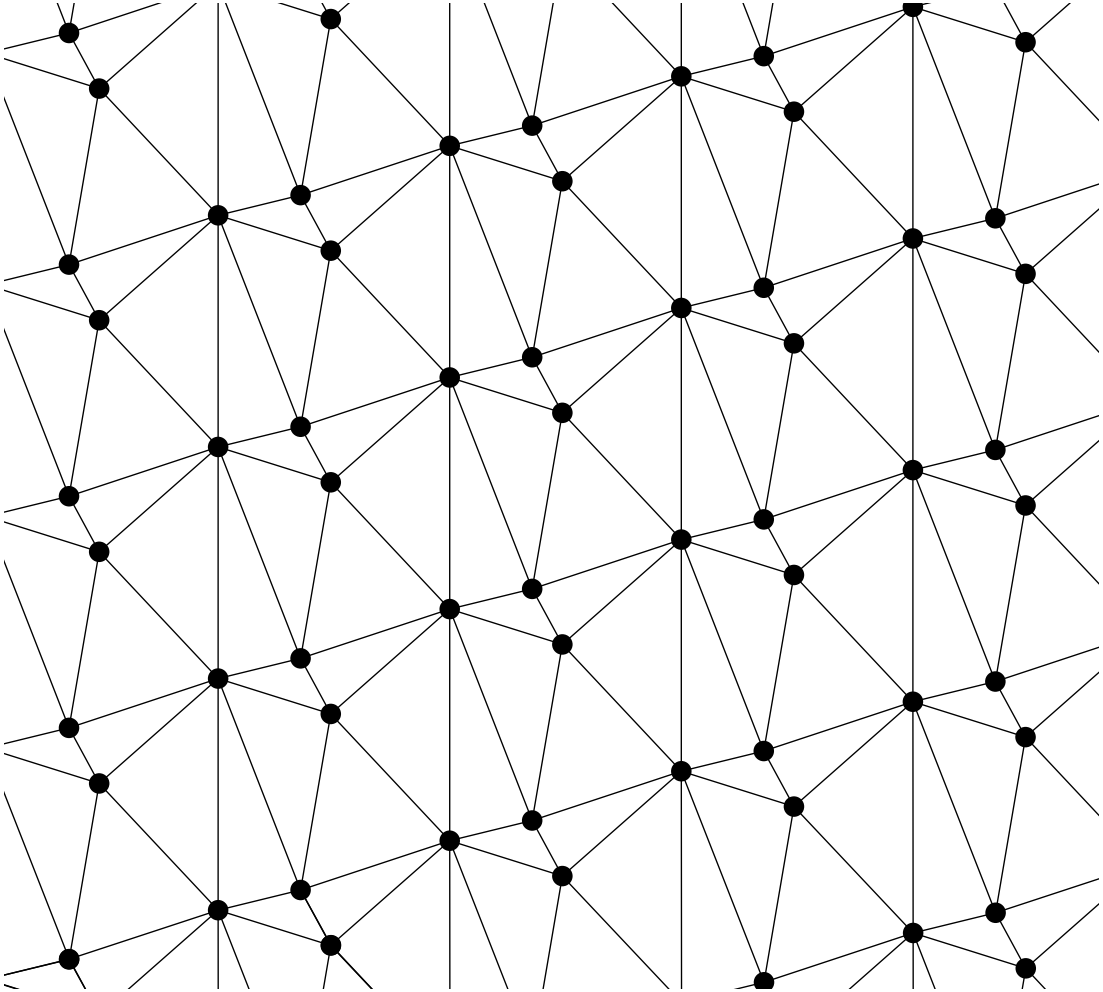

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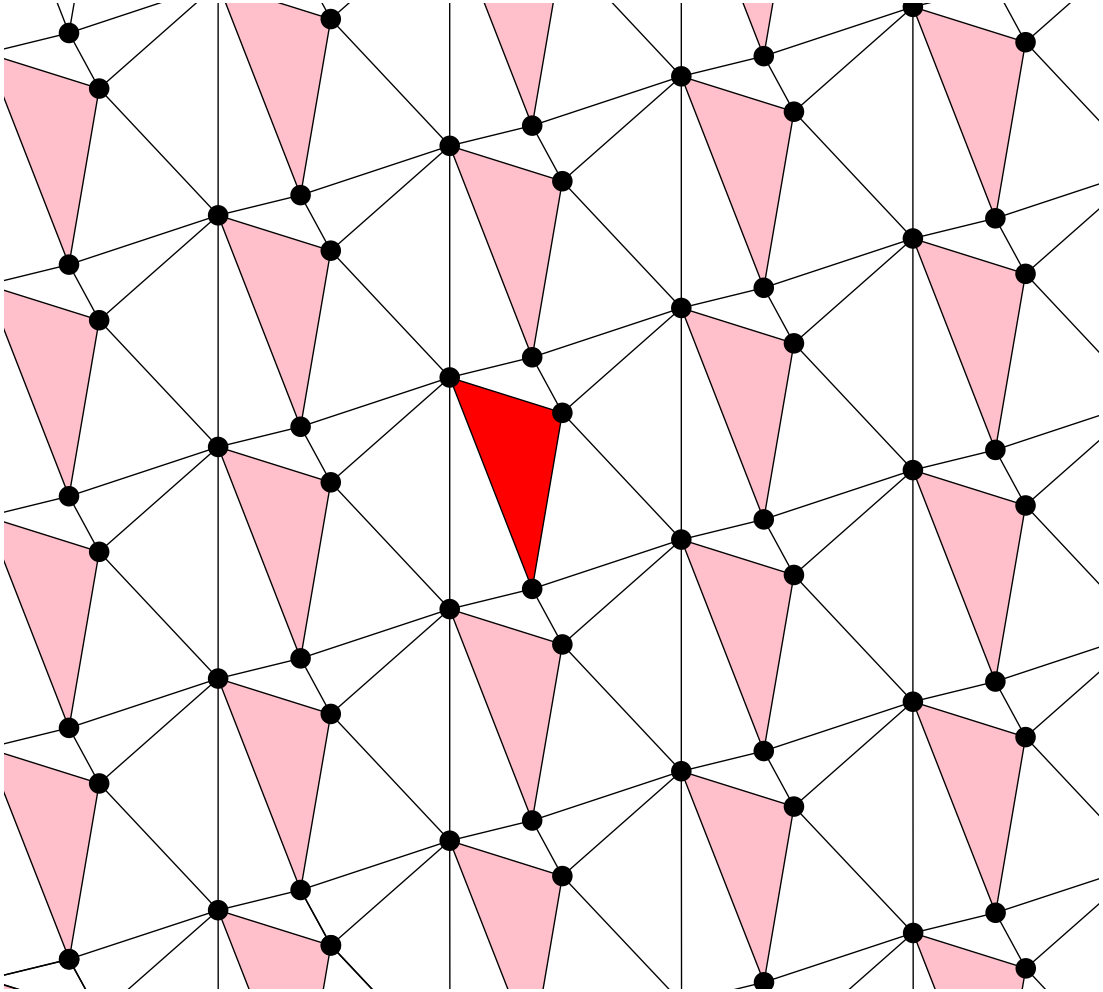

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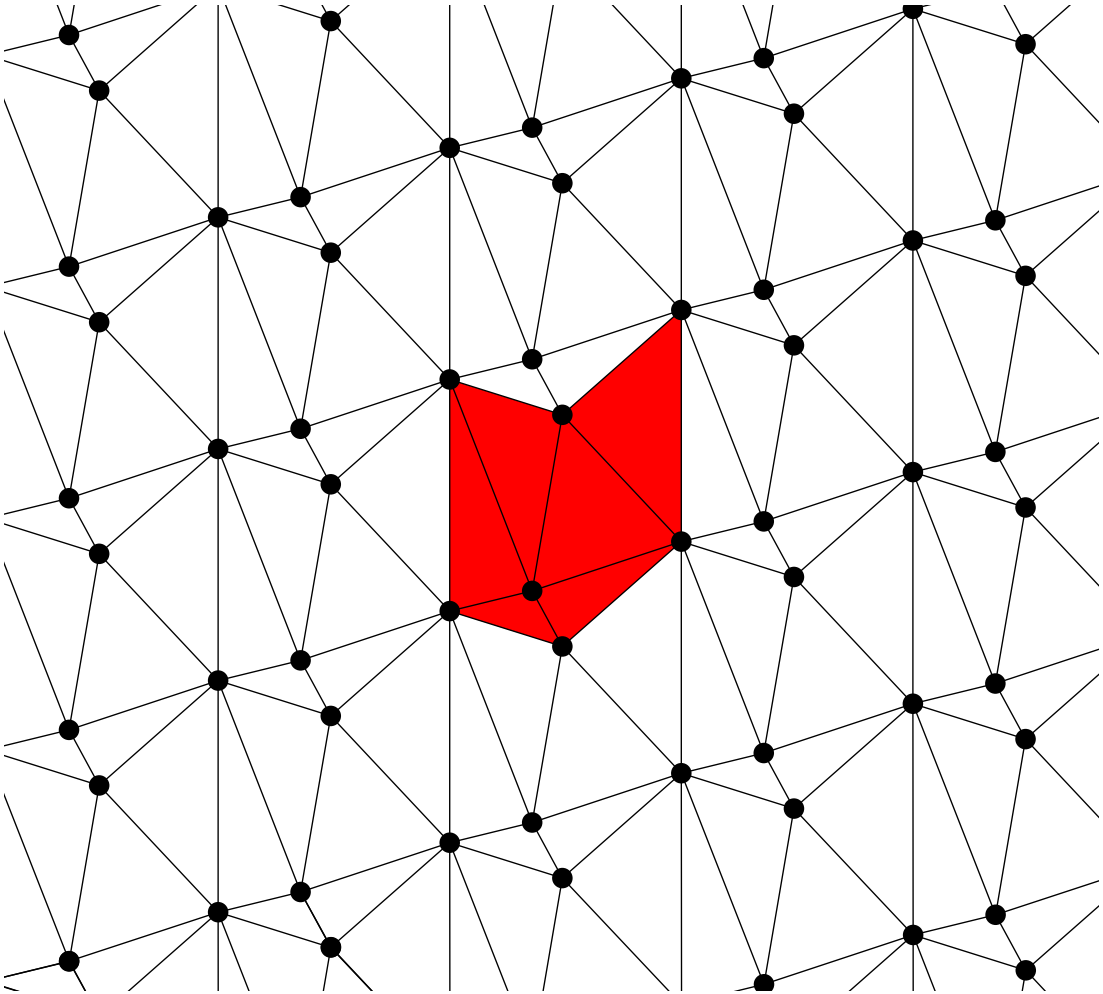

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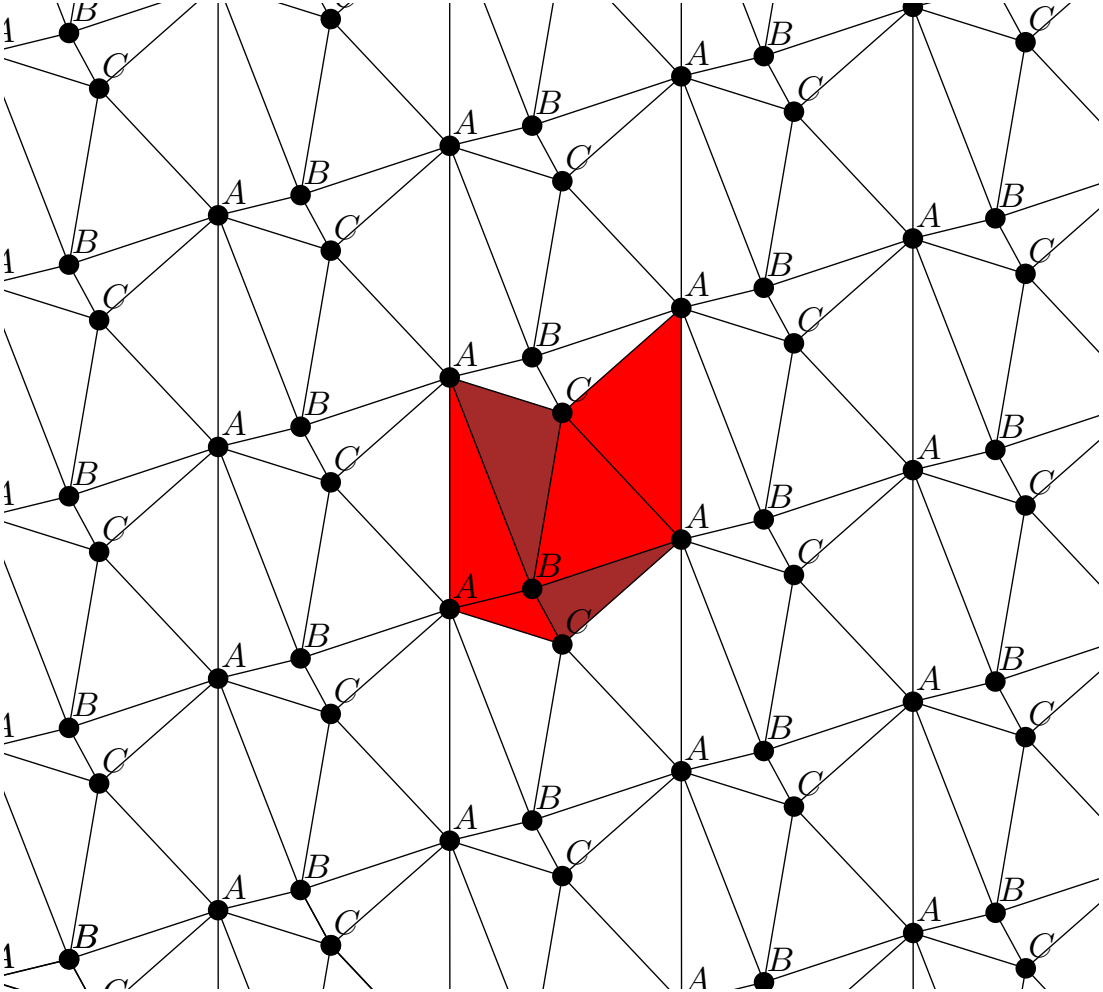

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Applications

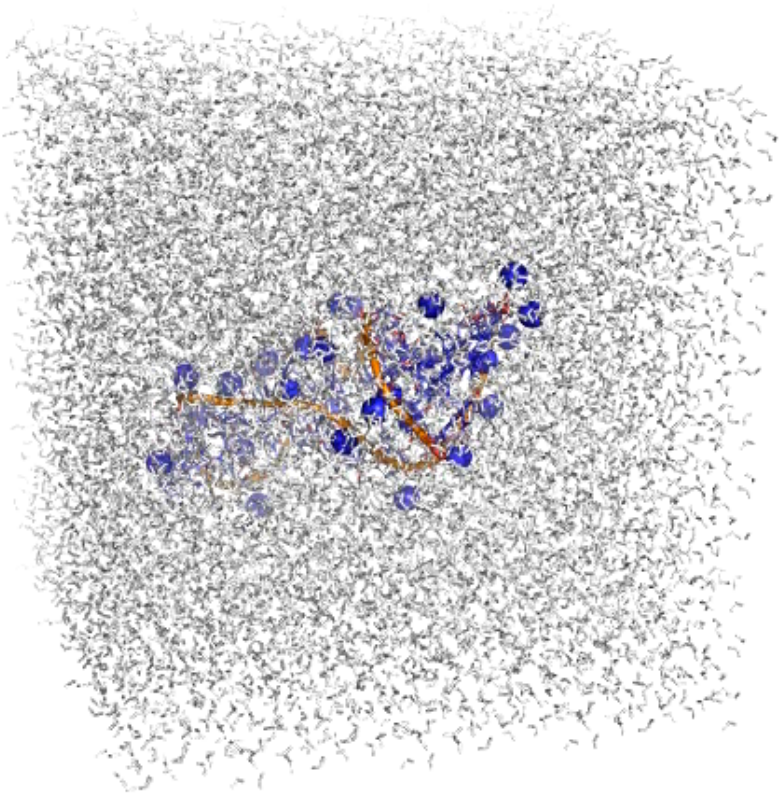
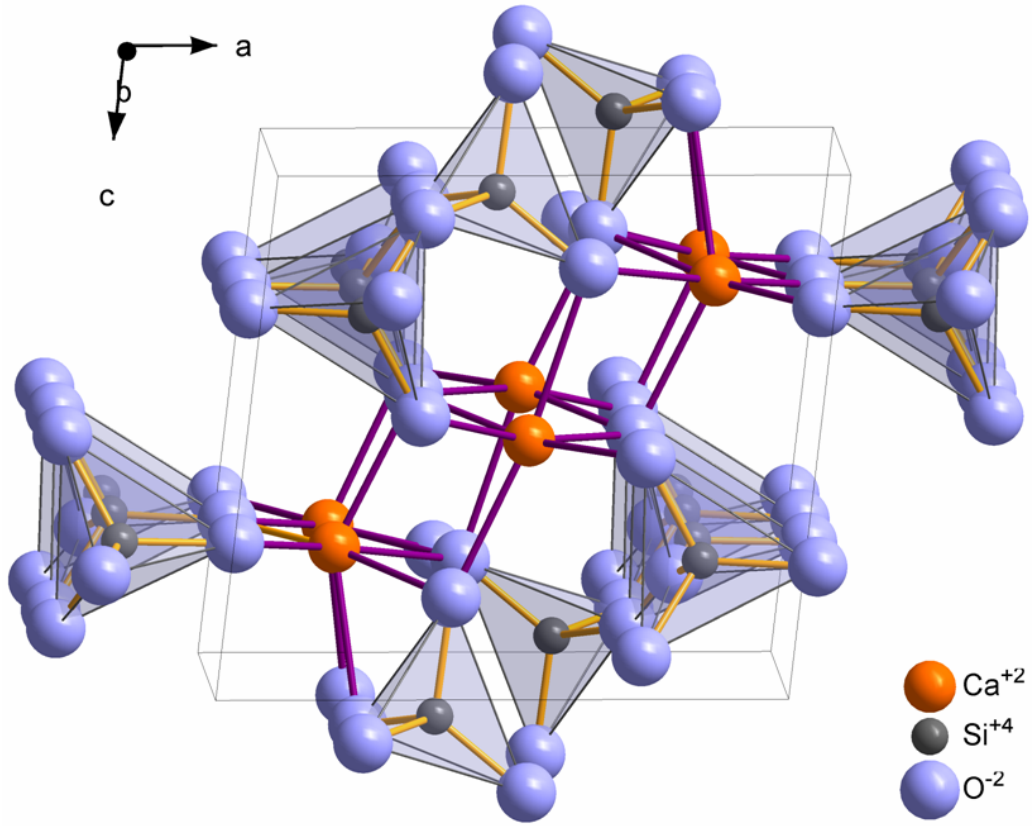


Image courtesy of Julie Bernauer



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$\Rightarrow$ Goal: provide similar user interface for generic case

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  - initially 3^d copies of everything
  - after certain criterion met, 1 copy of everything

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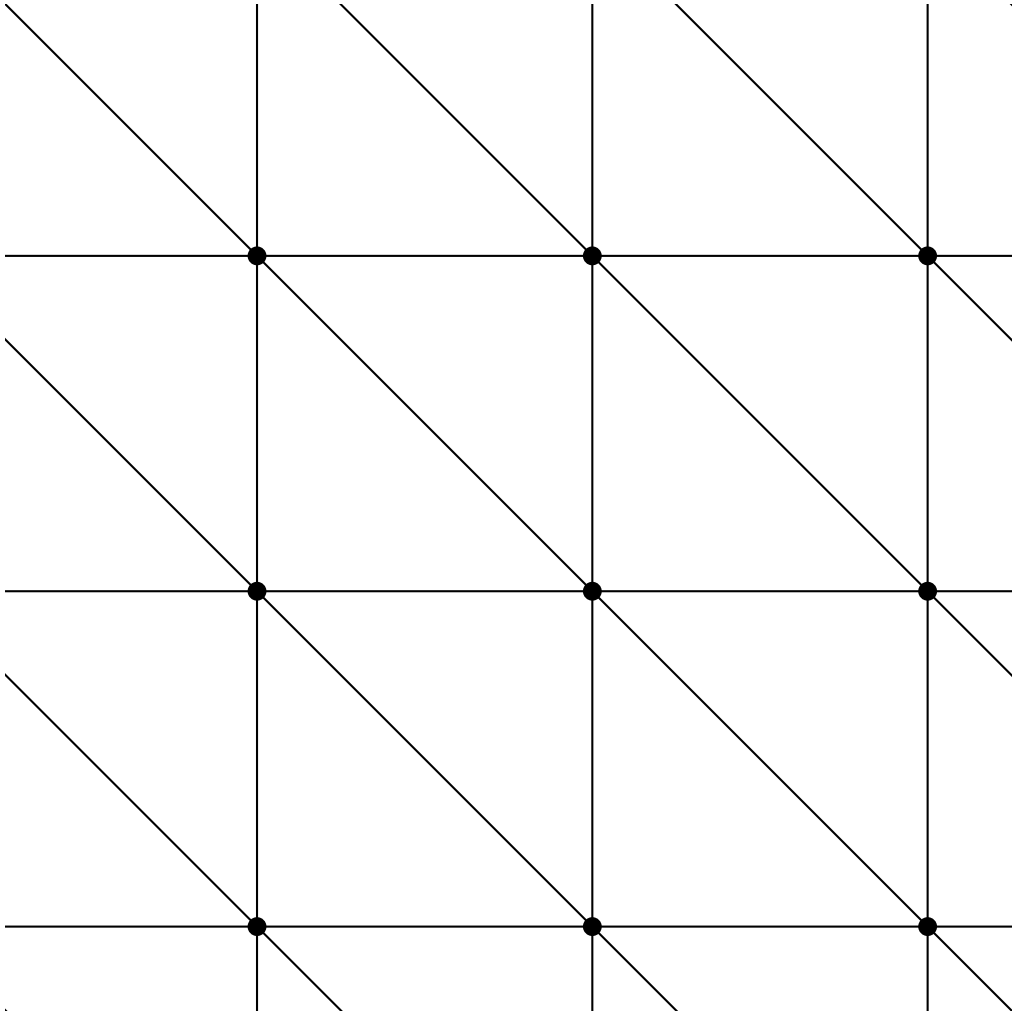

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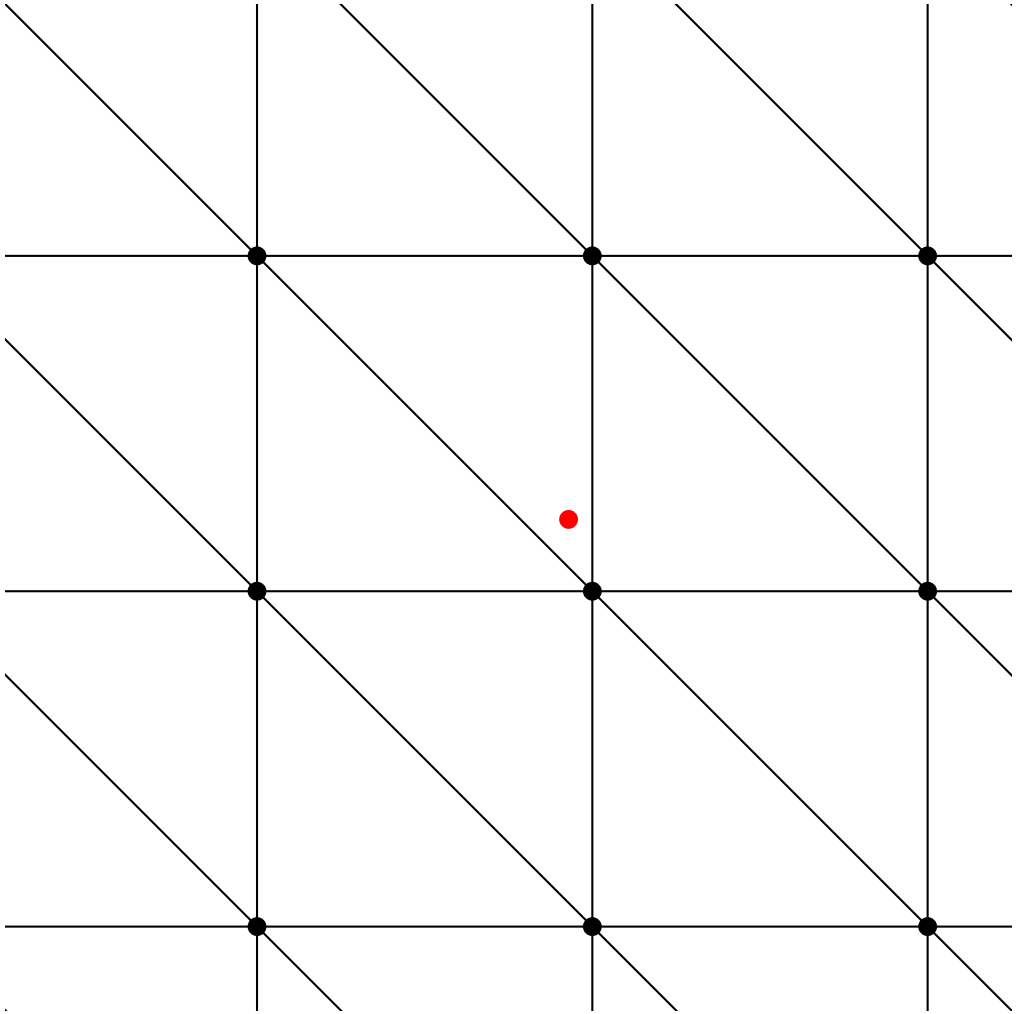

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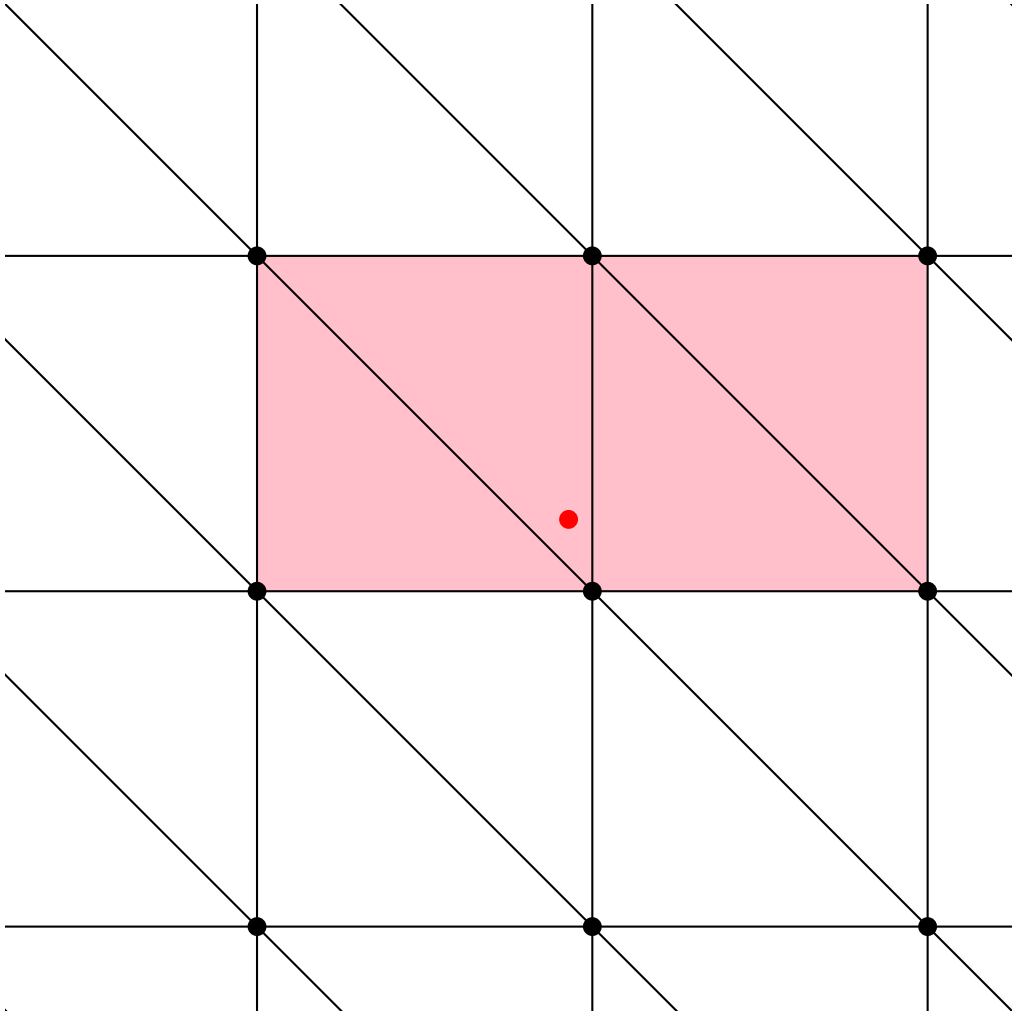

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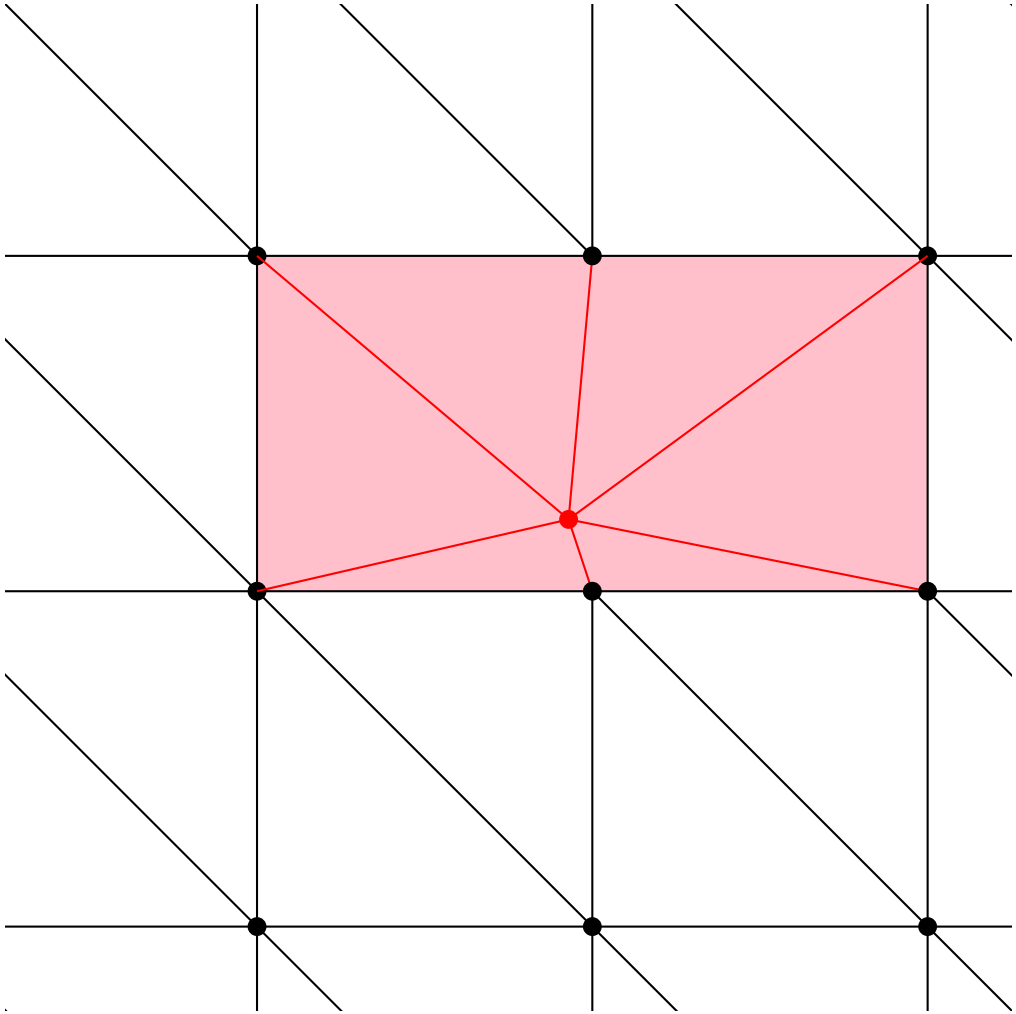

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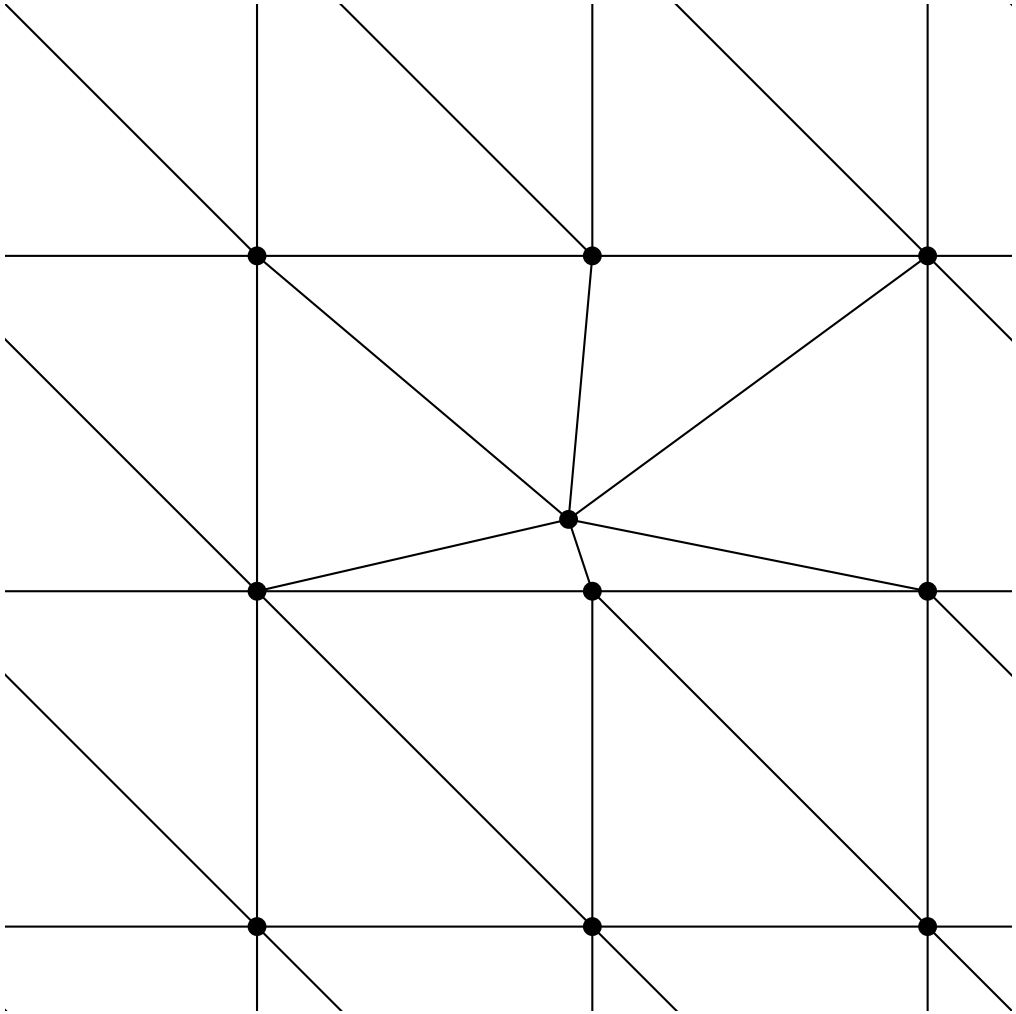

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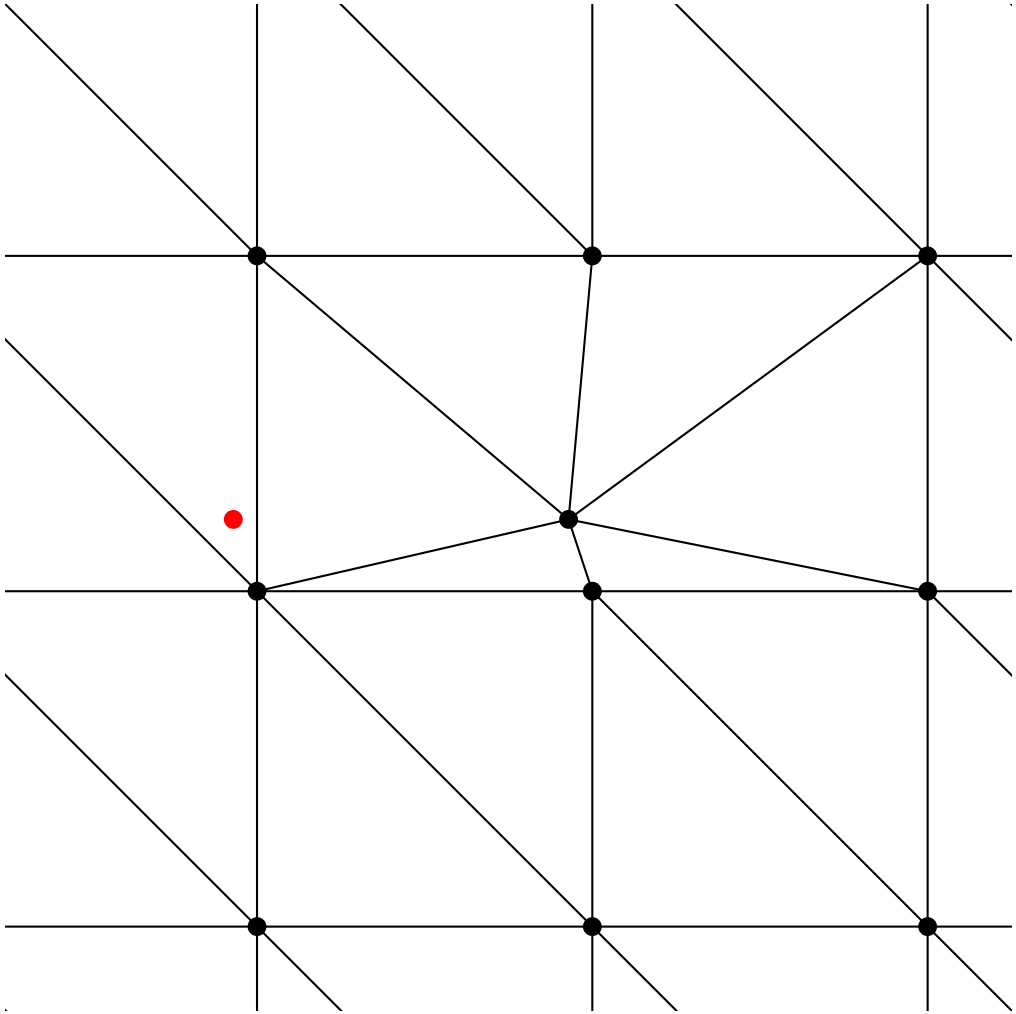

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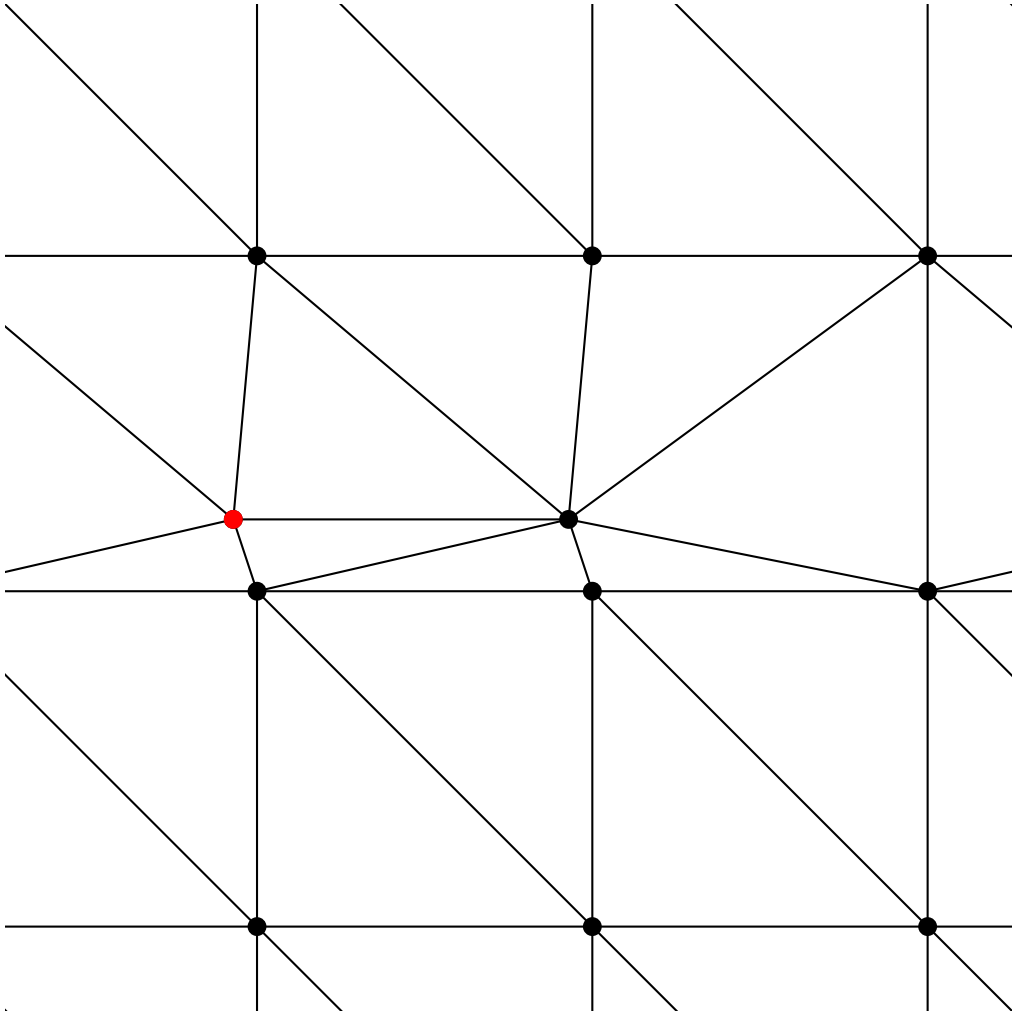

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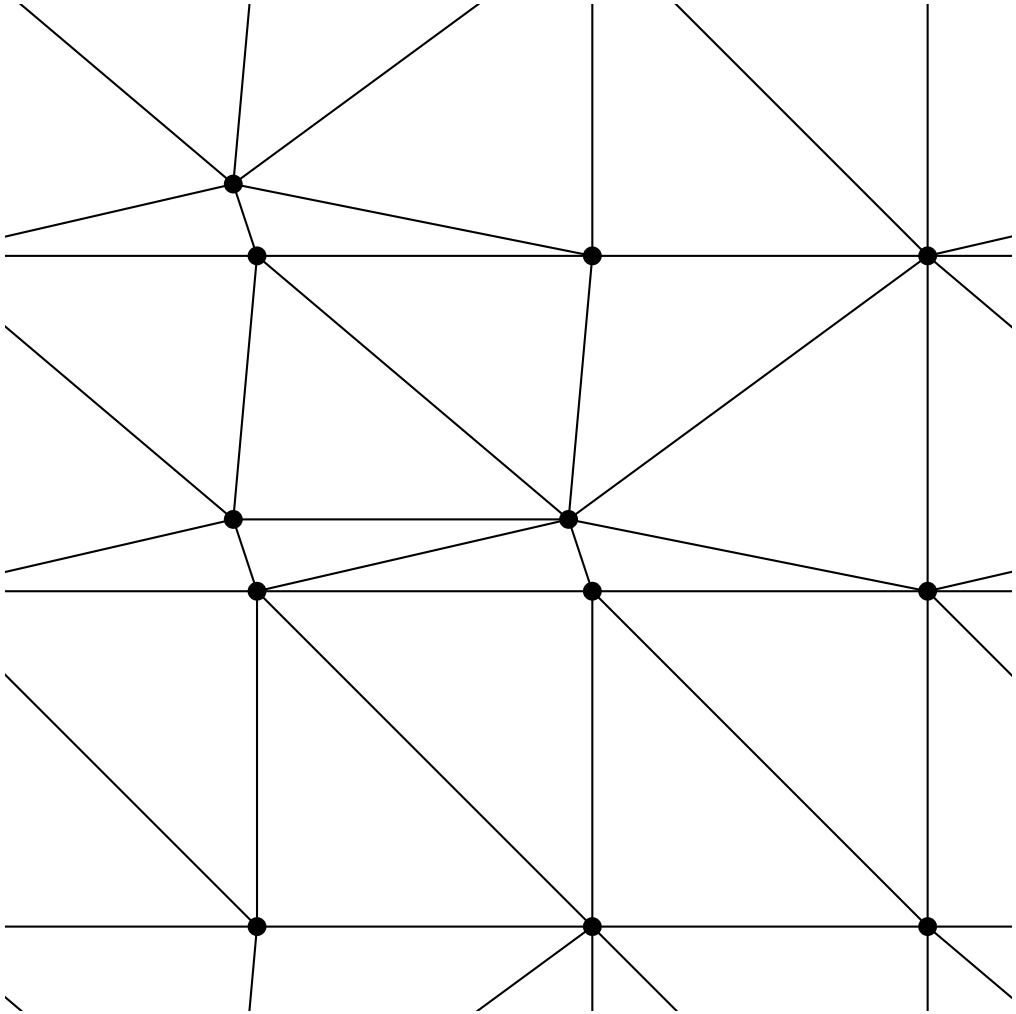

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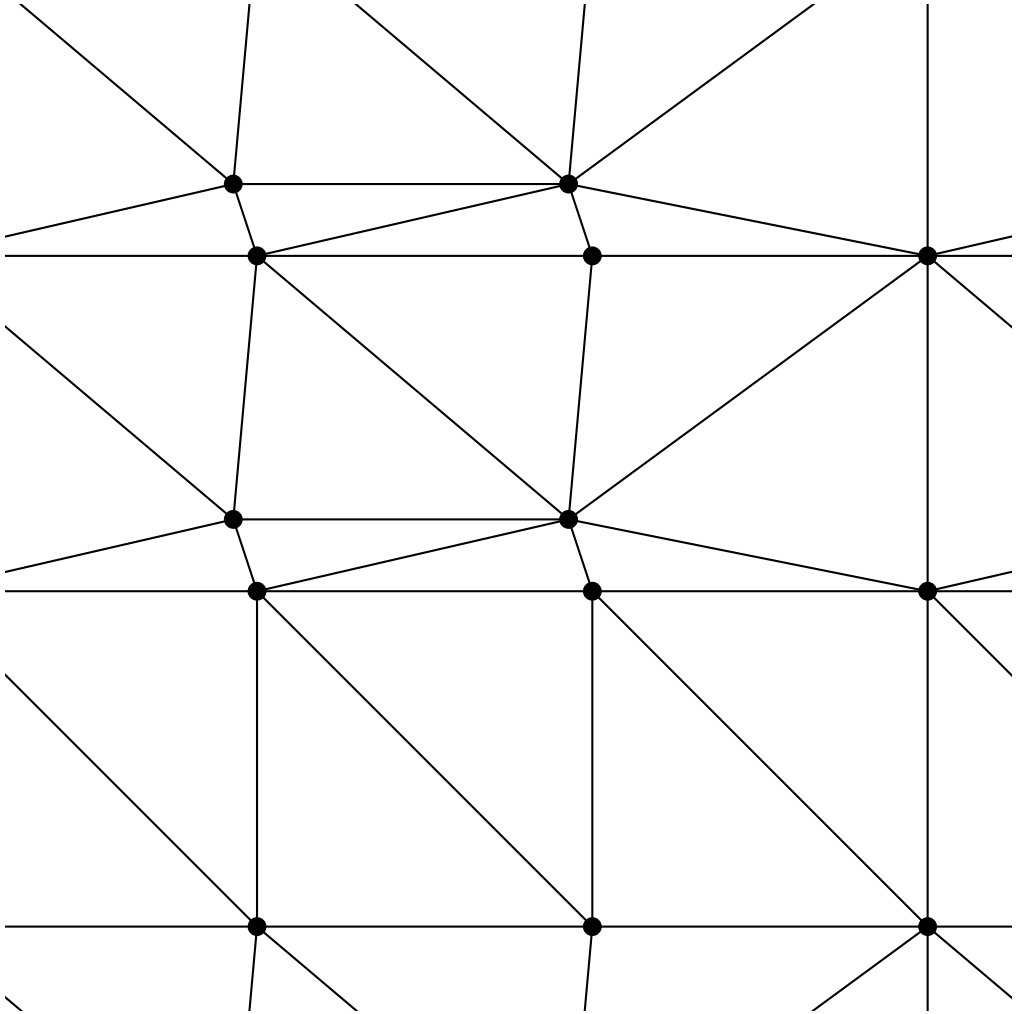

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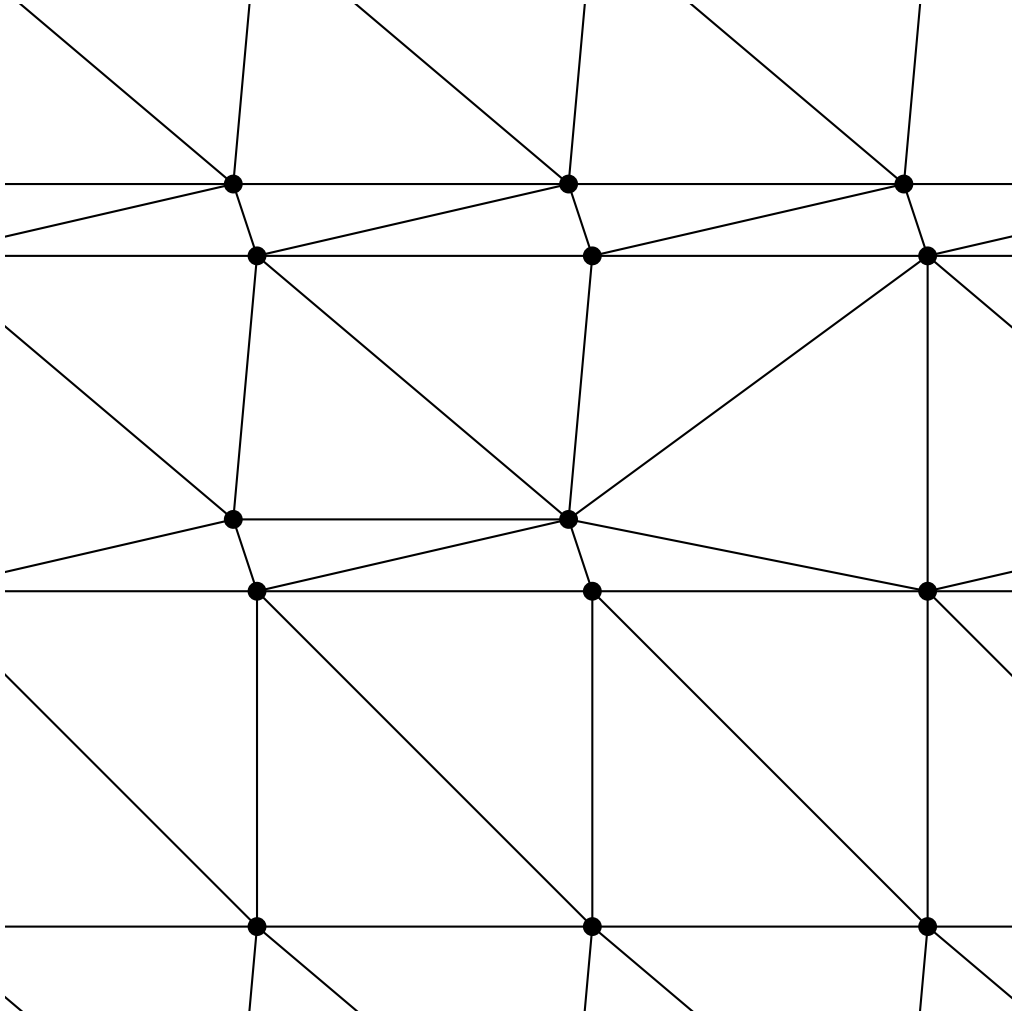

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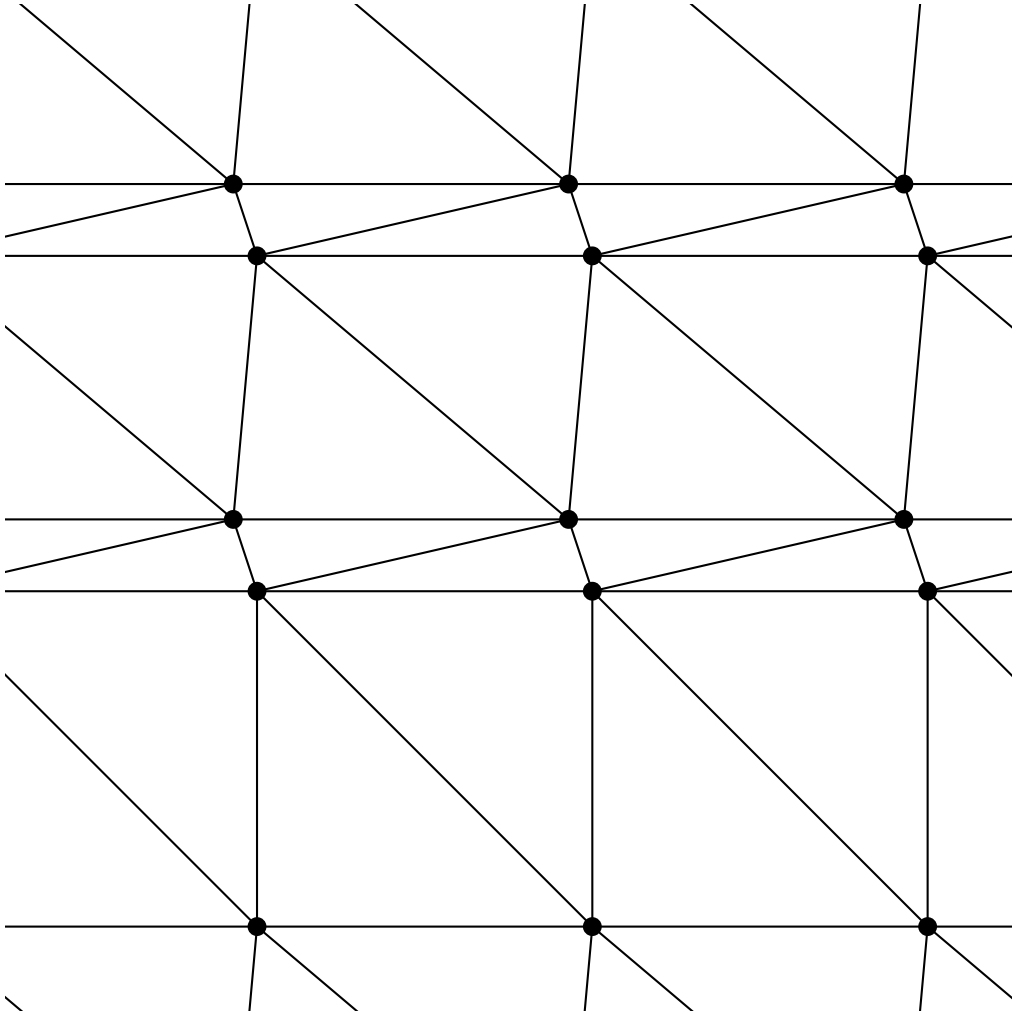

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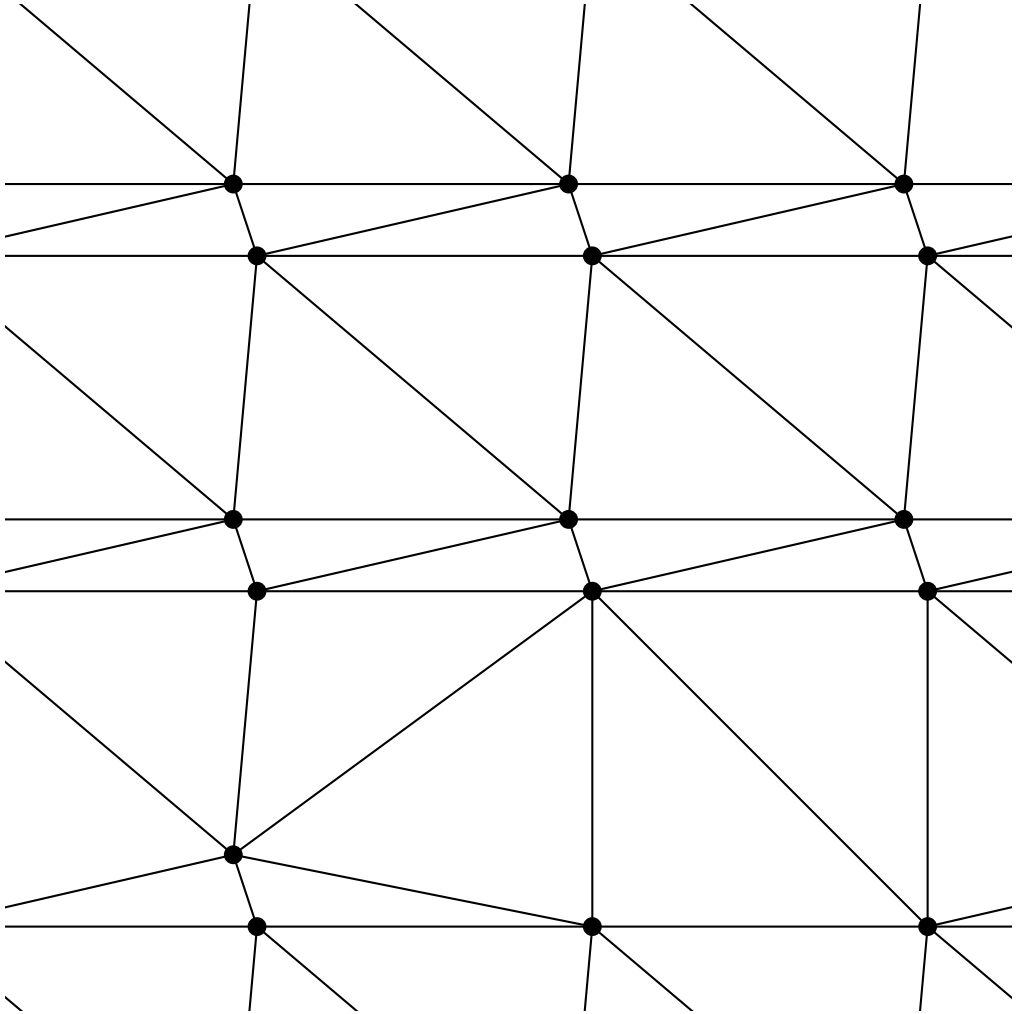

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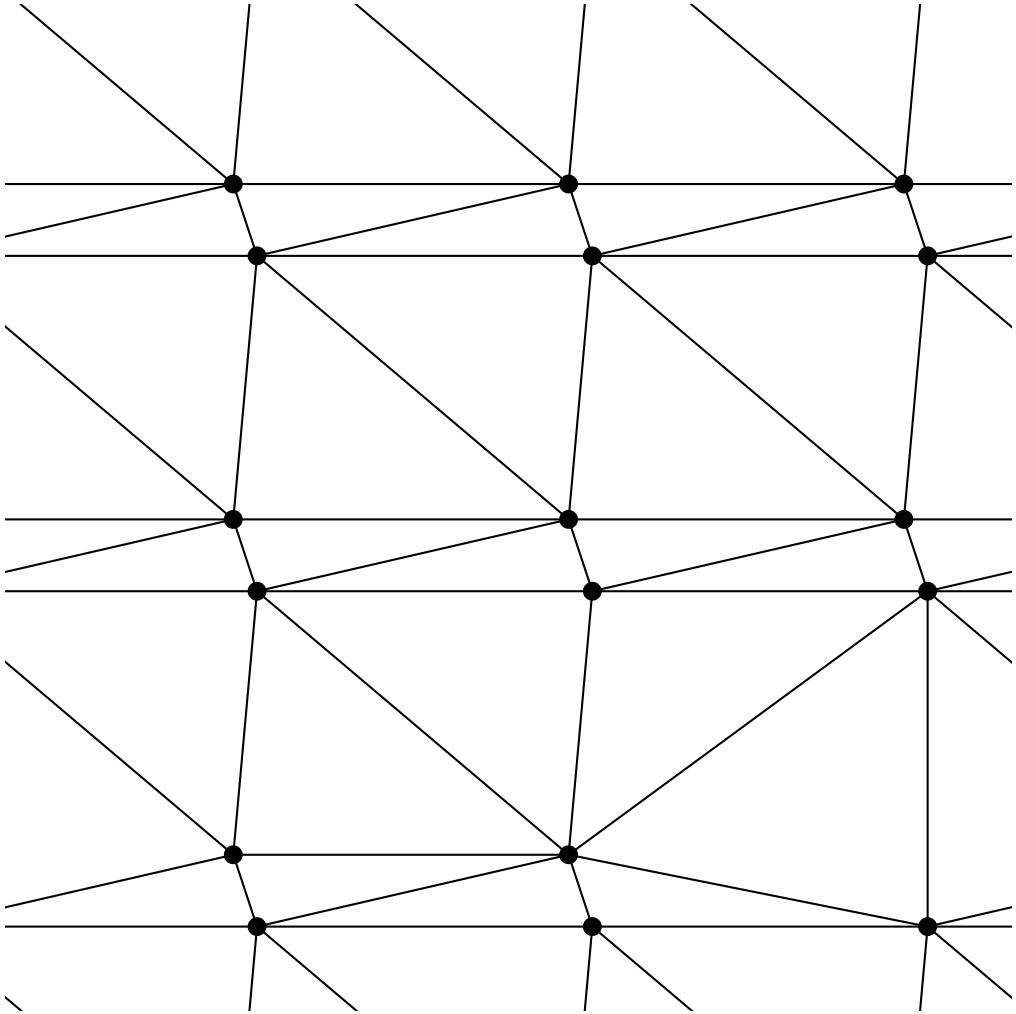

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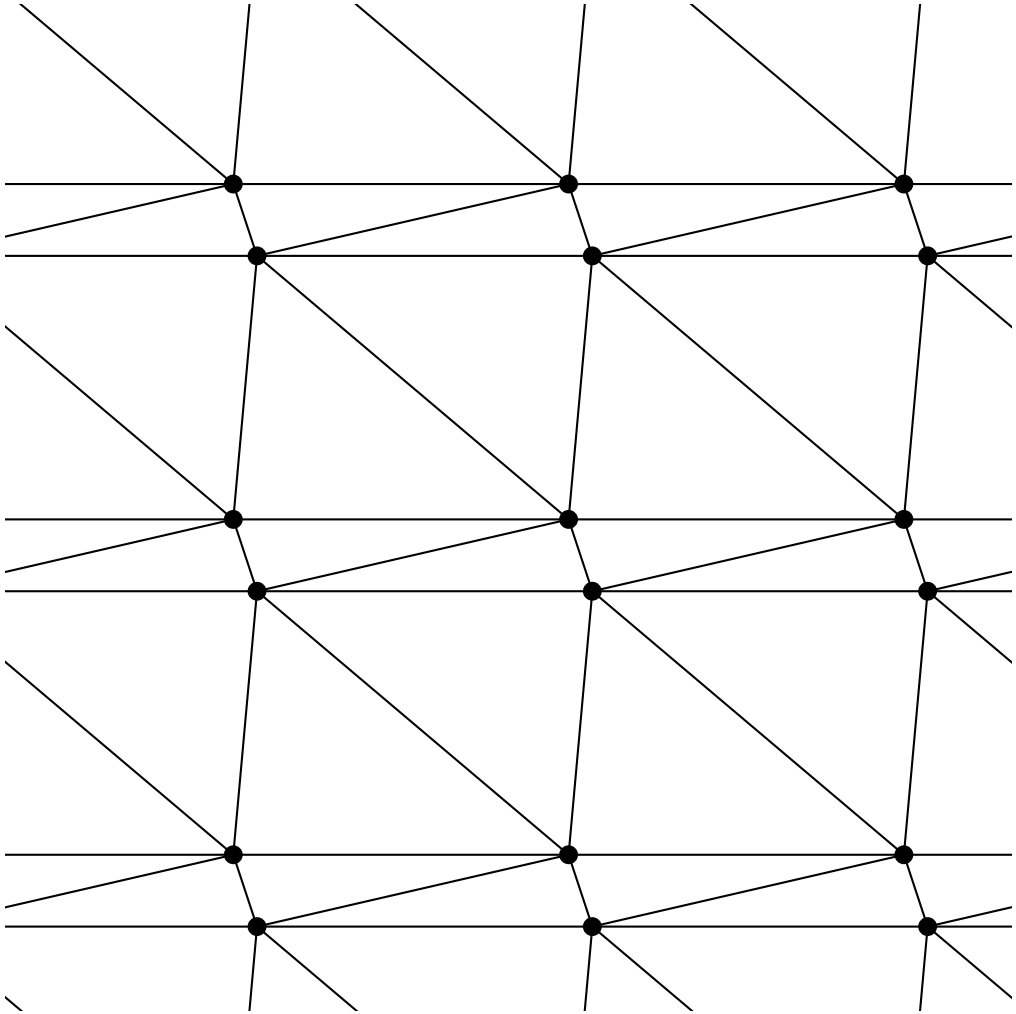

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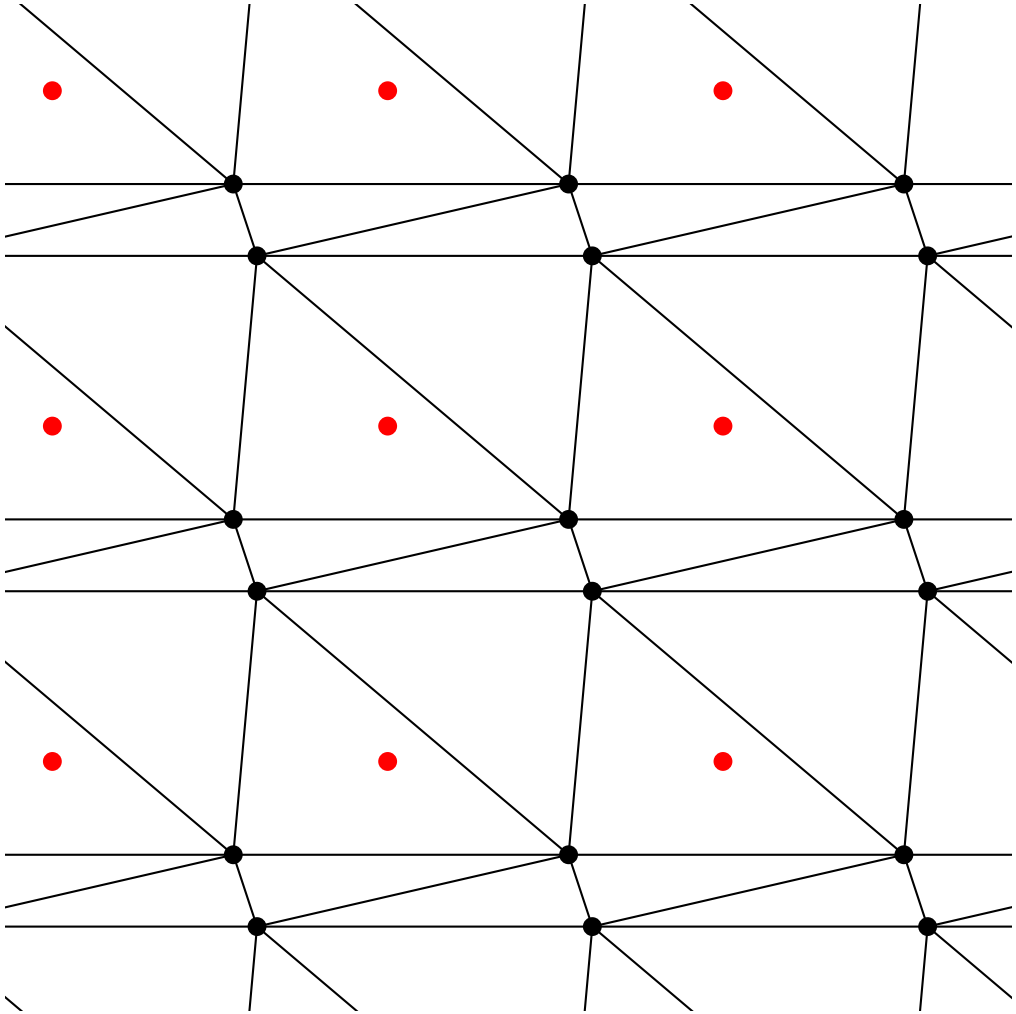

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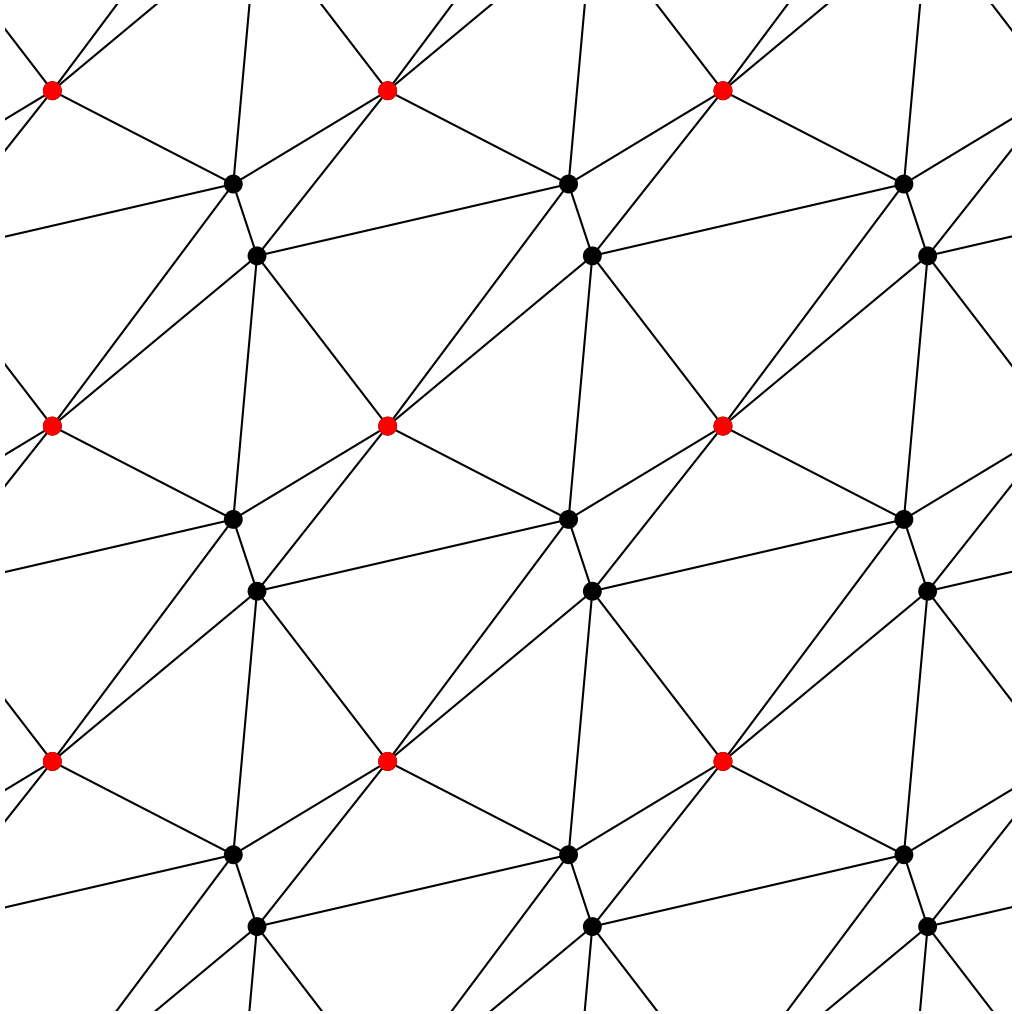

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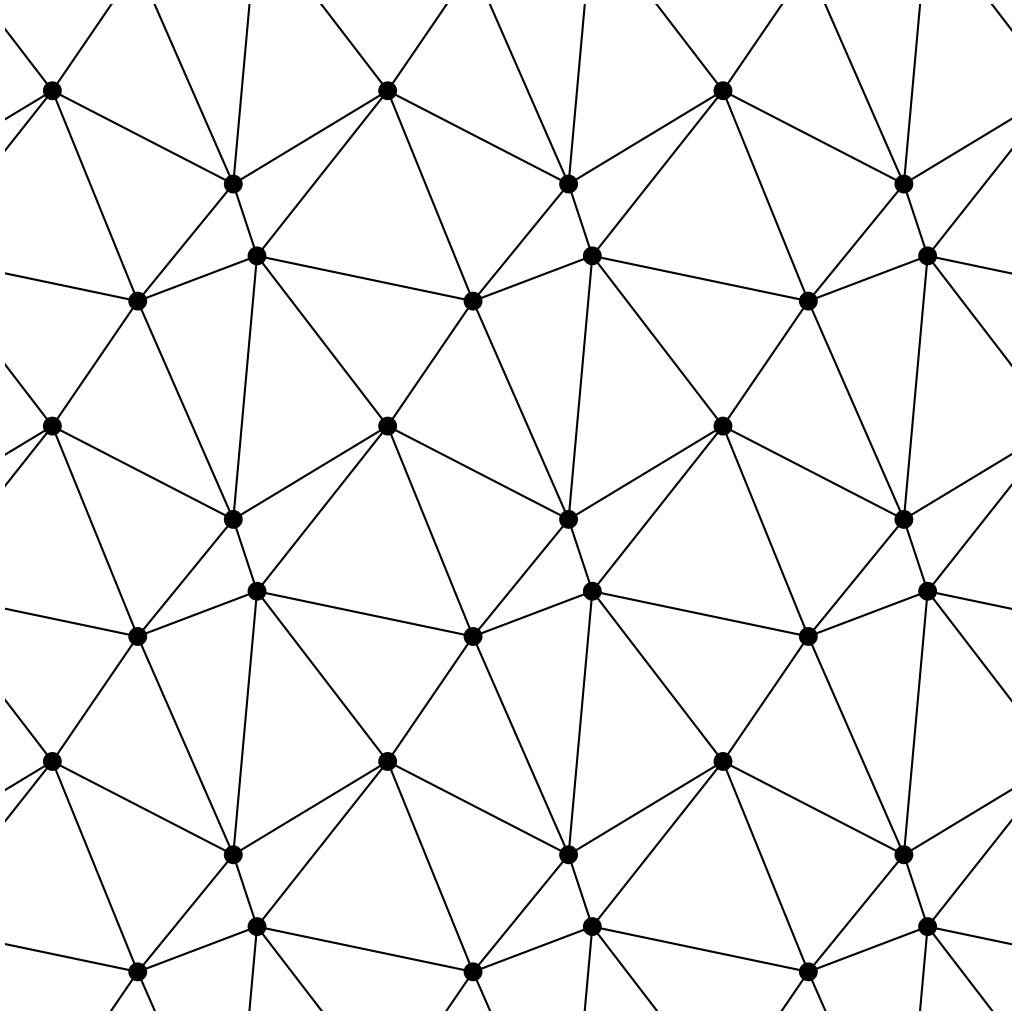

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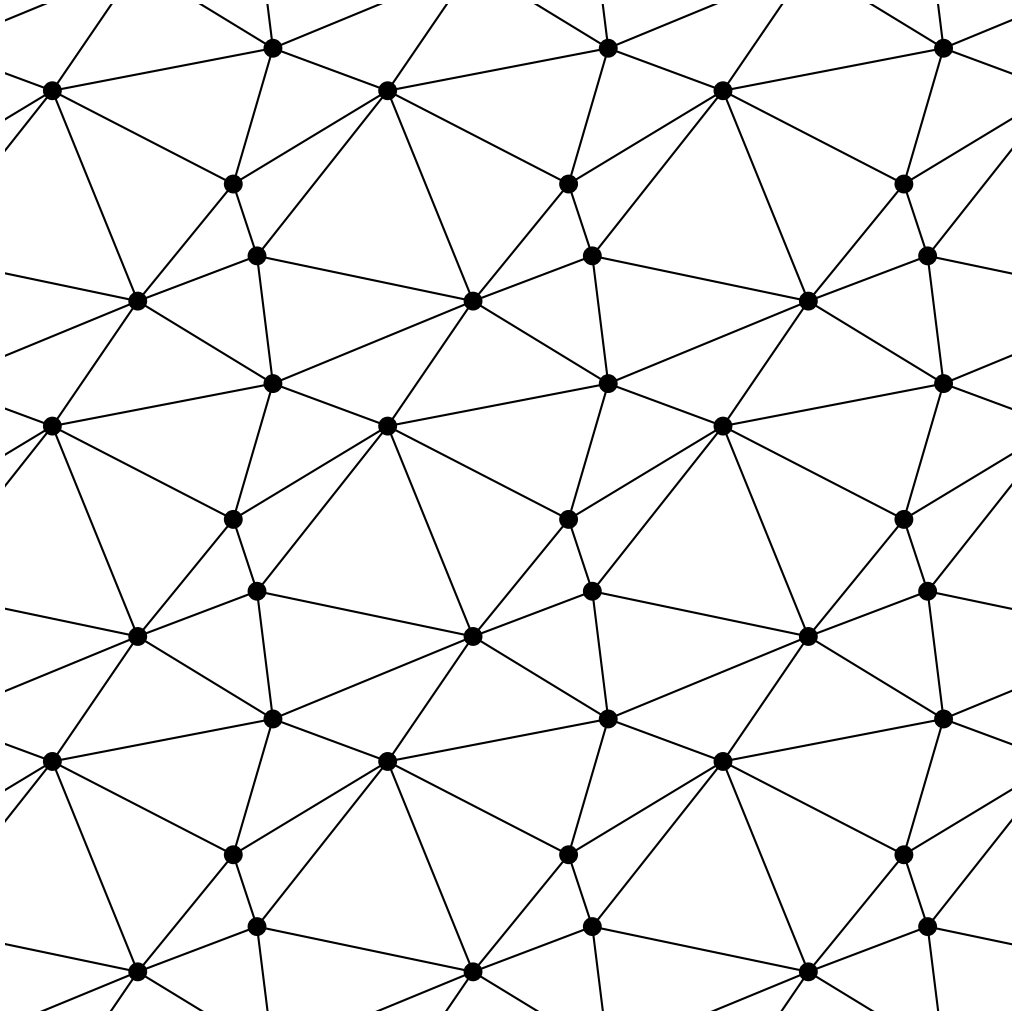

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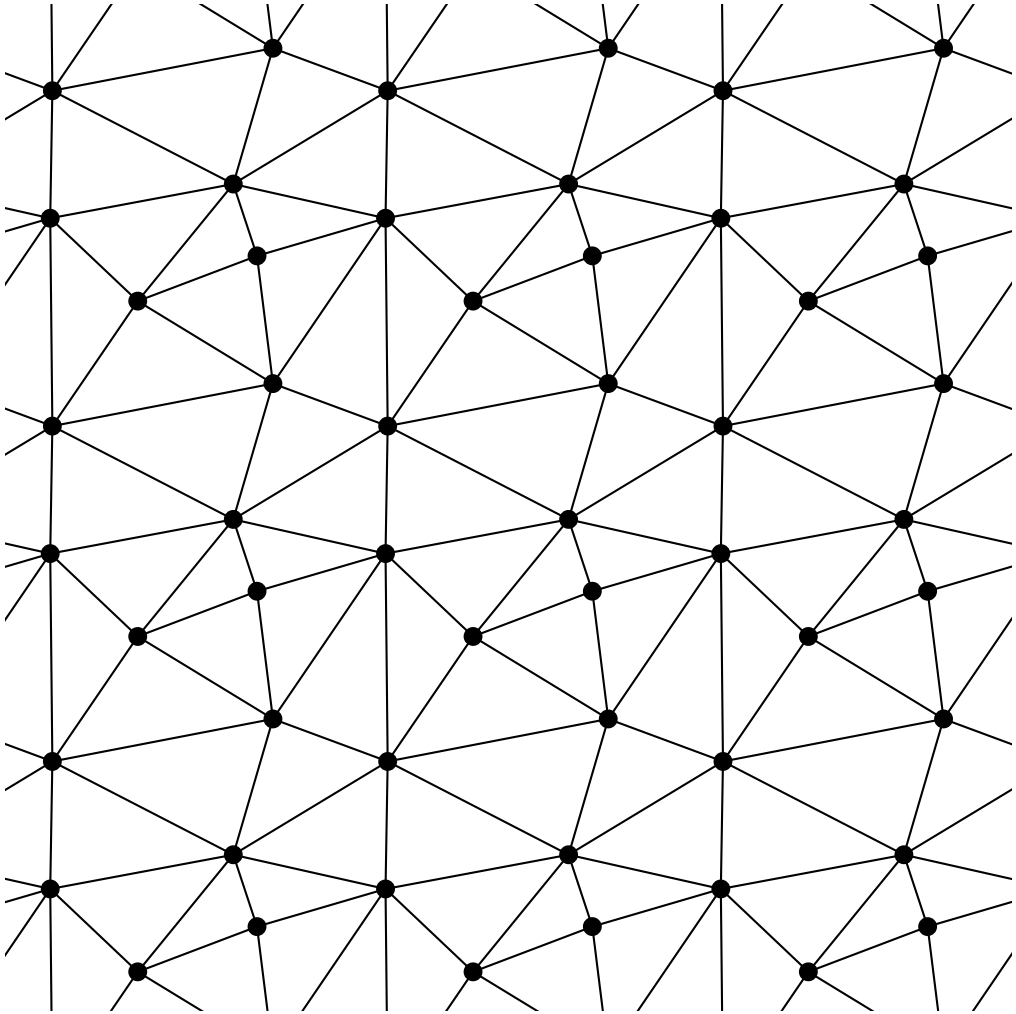

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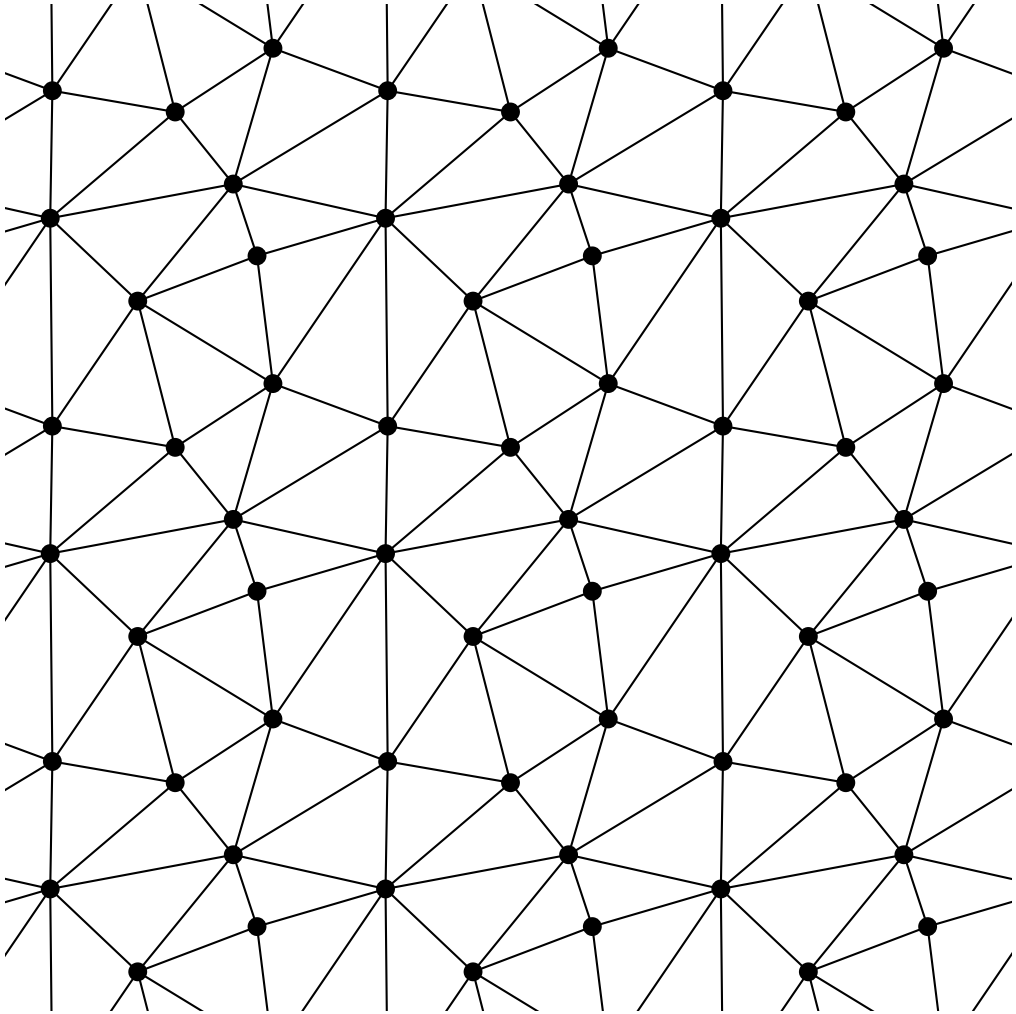

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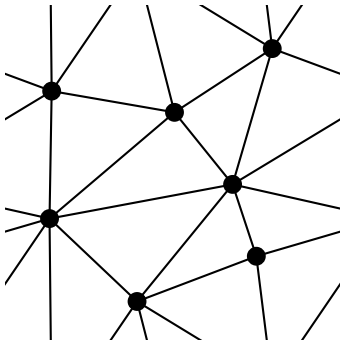

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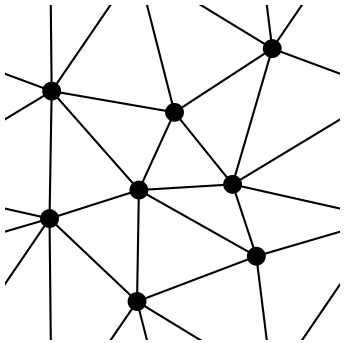

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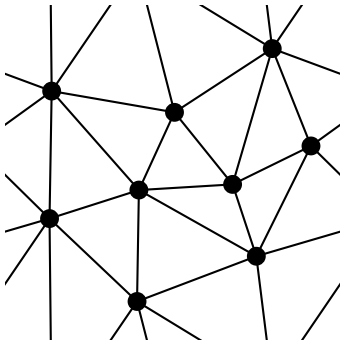

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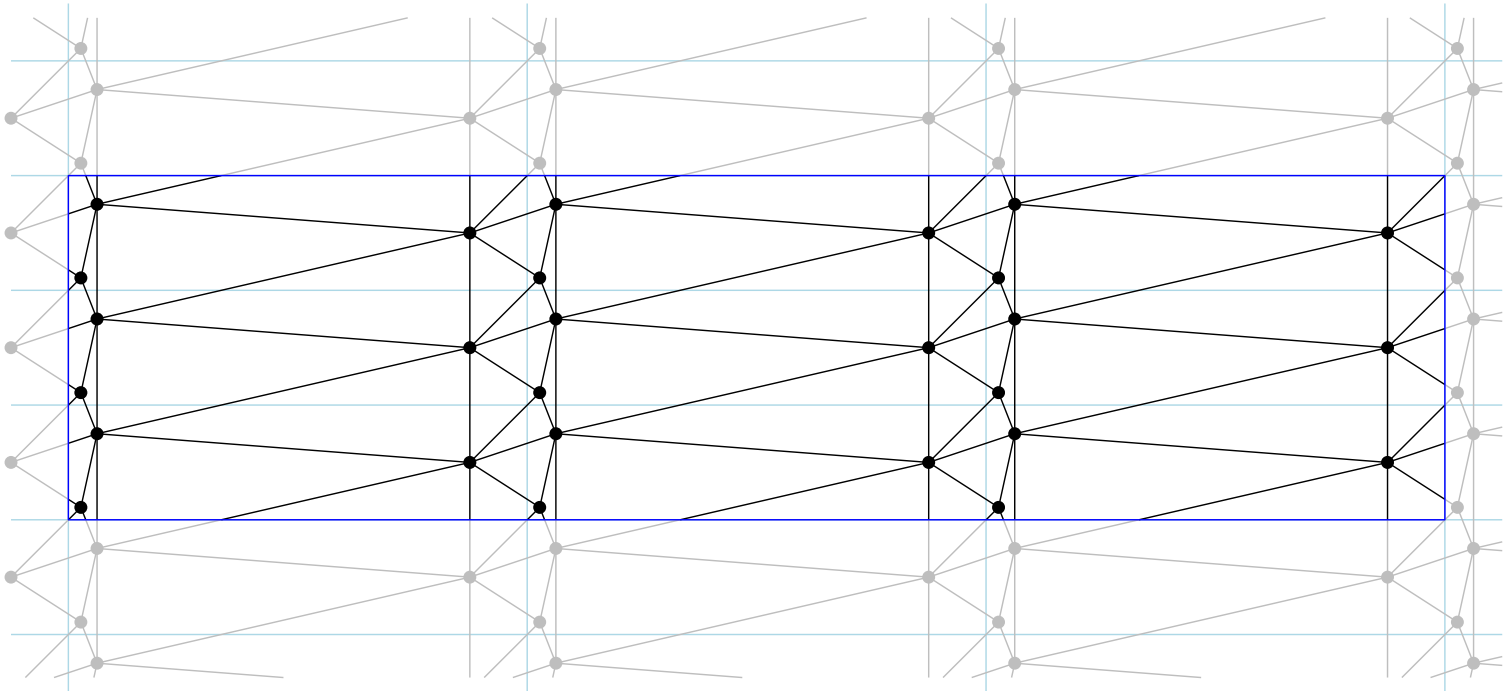
  - initial 3^d copies not sufficient



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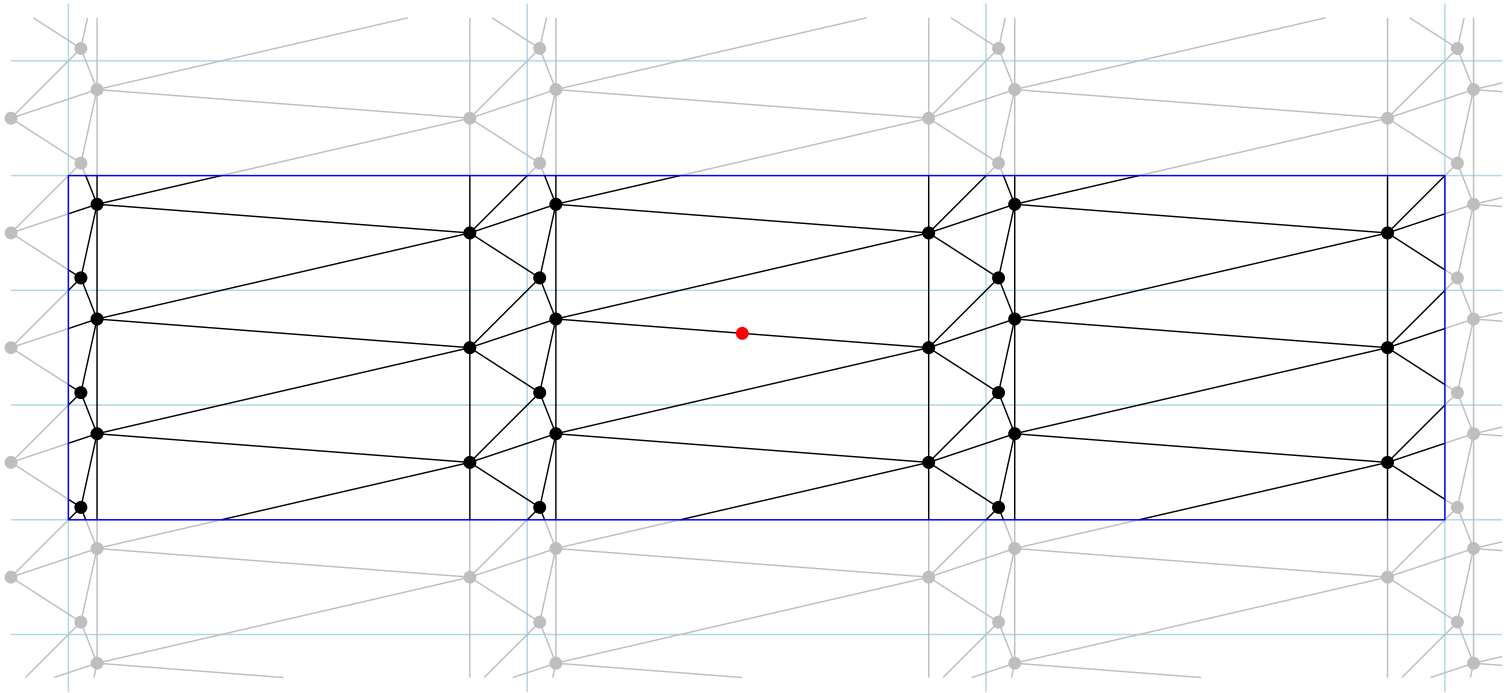

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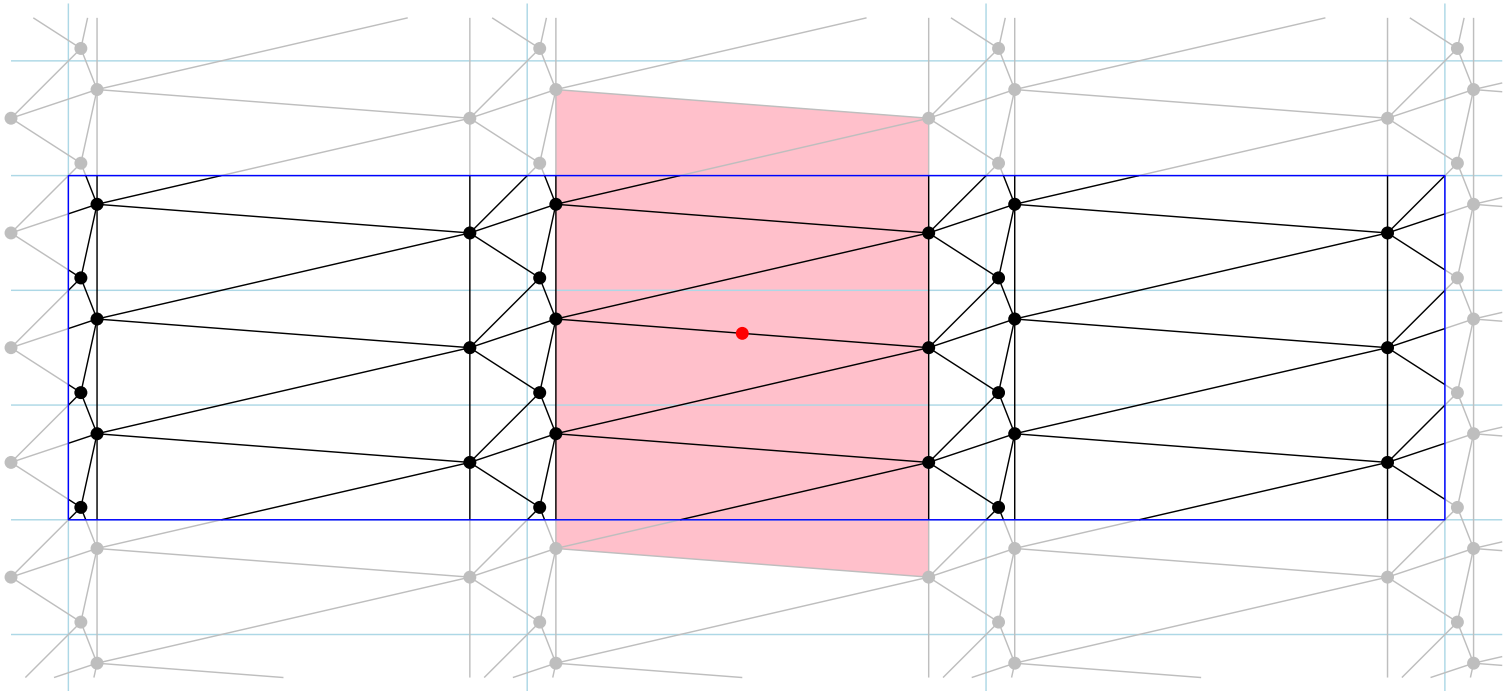

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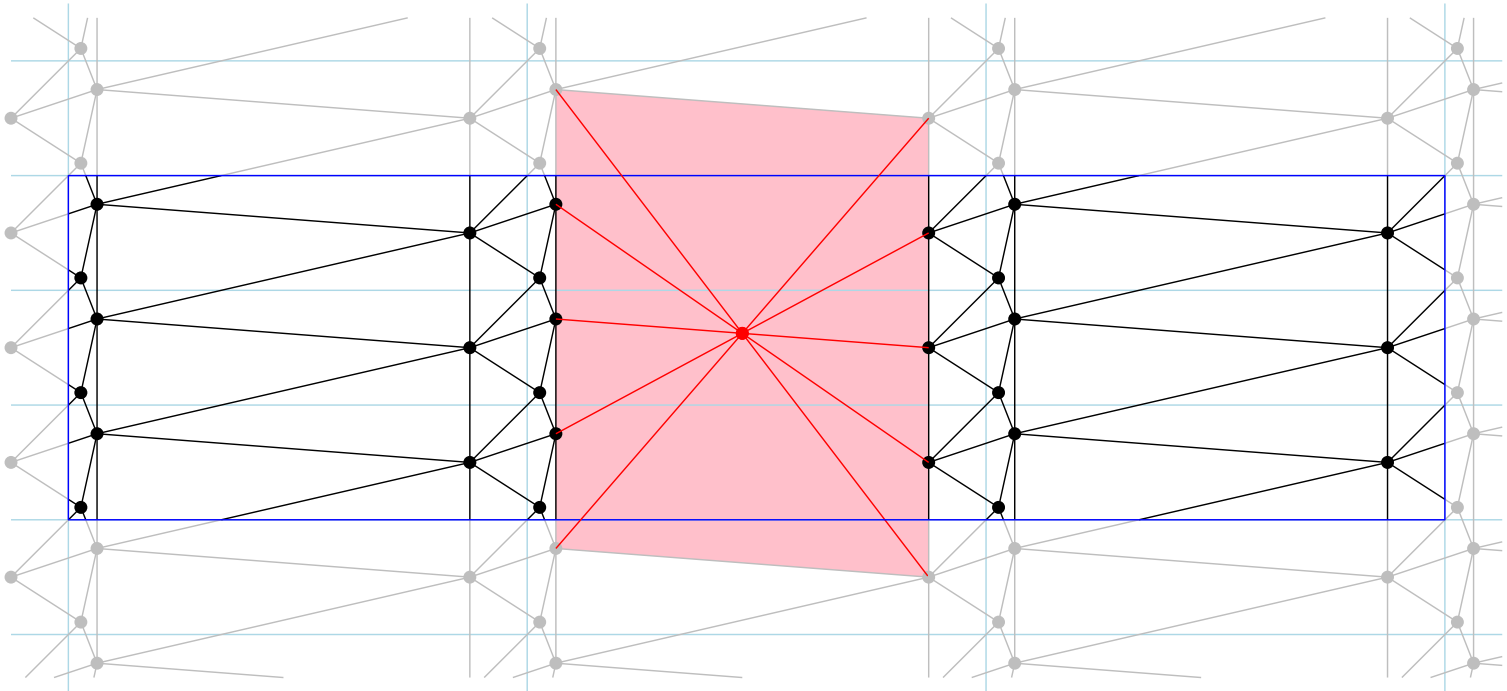
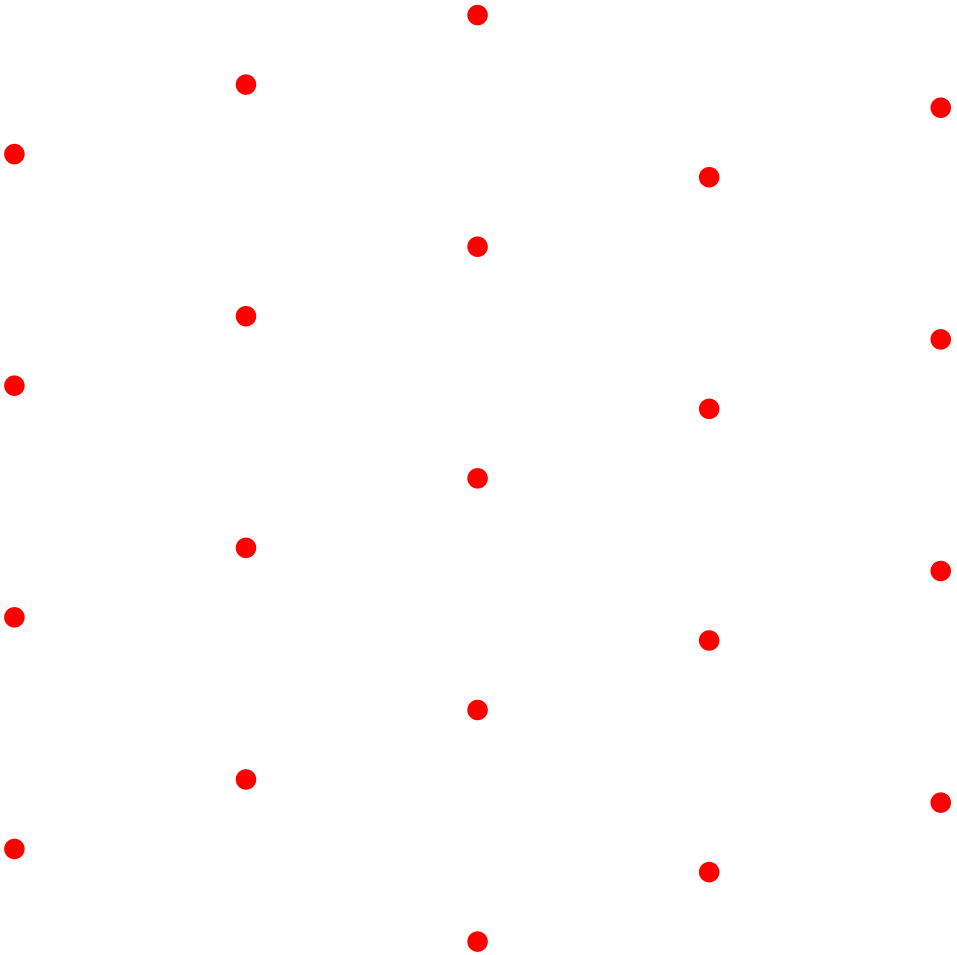


Algorithm by Dolbilin & Huson, '97



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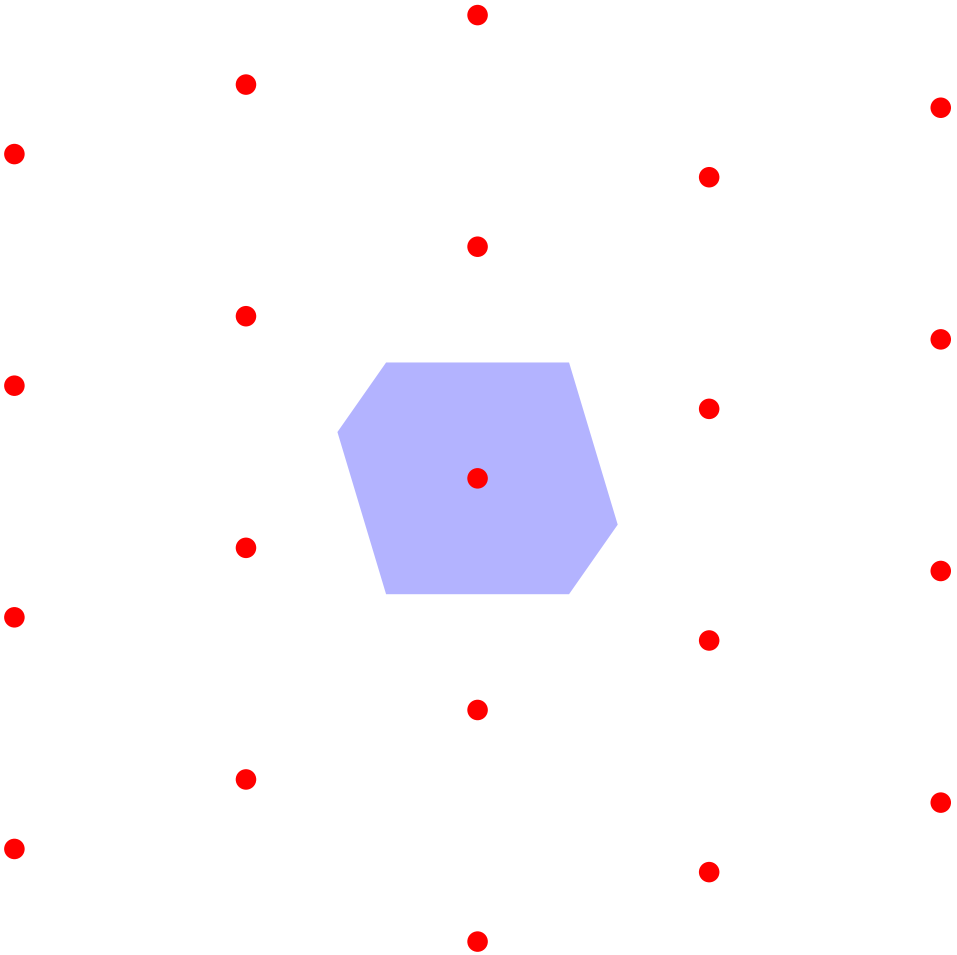

- $\mathrm{dom}(0, \Lambda)$: Voronoi domain of origin

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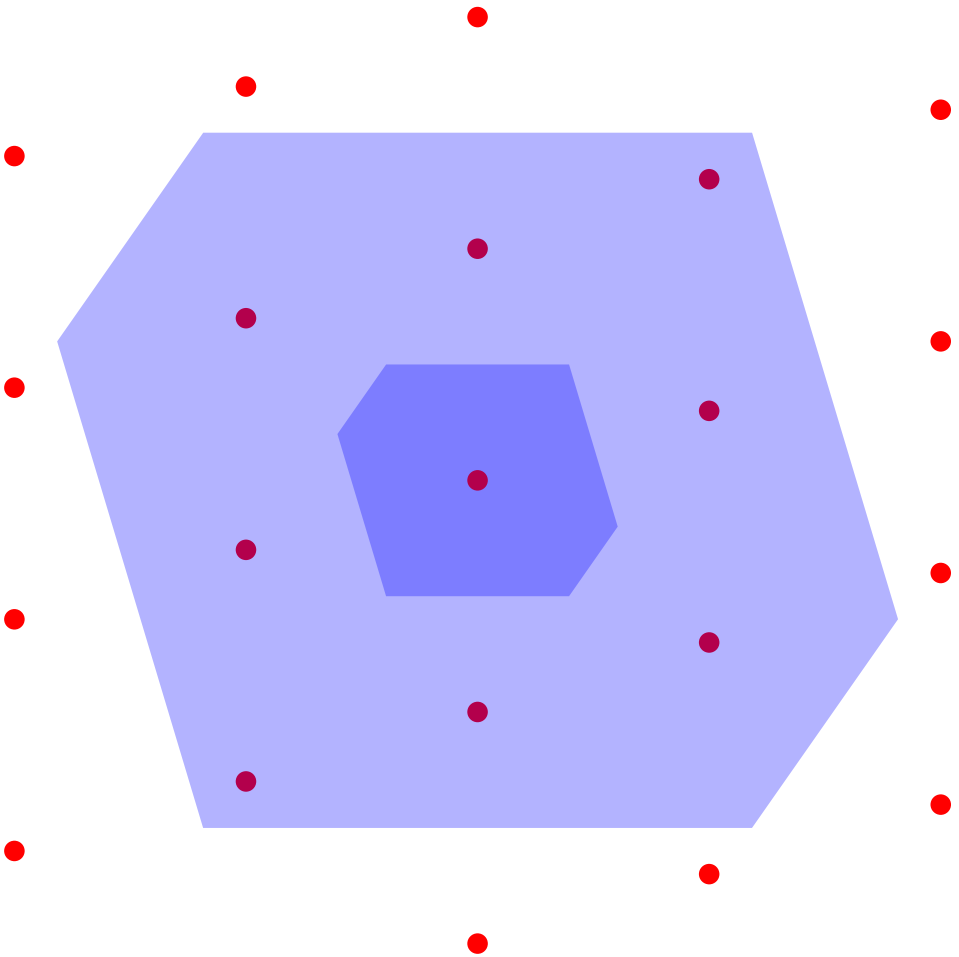

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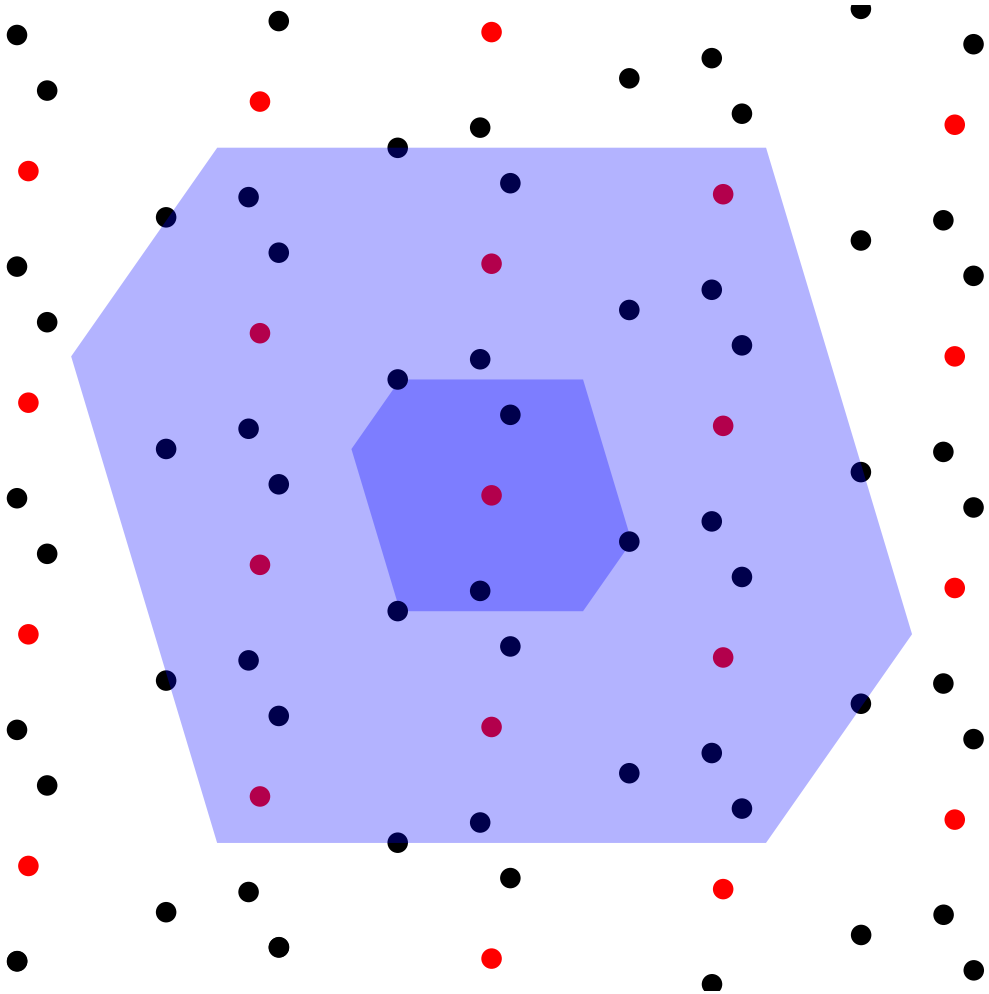

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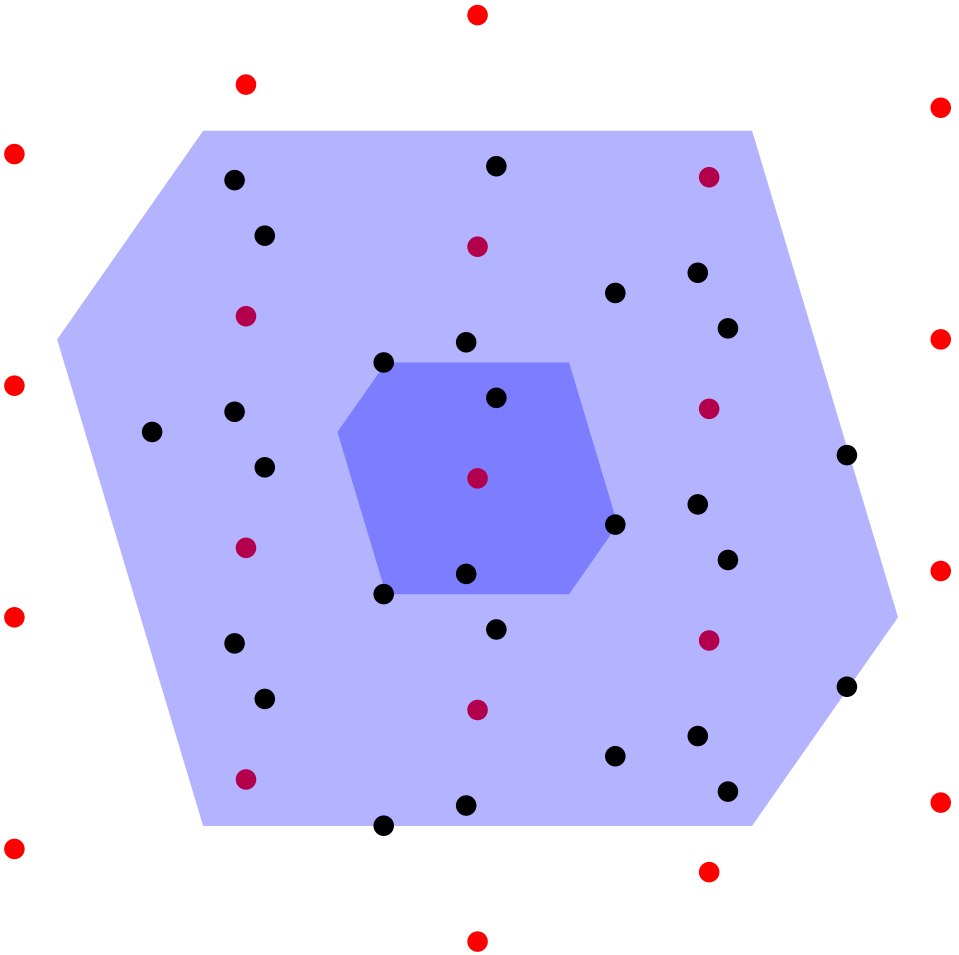

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- triangulate $\Lambda X \cap \mathrm{dom}(0, 3\Lambda)$

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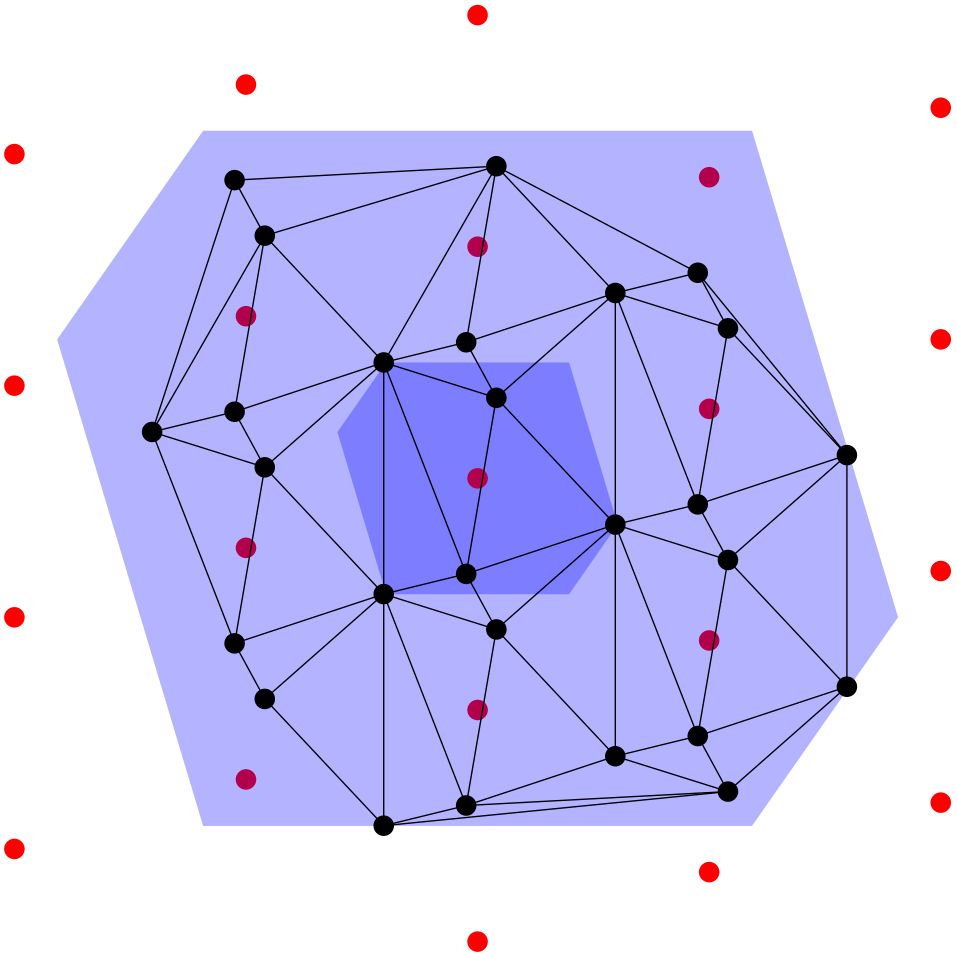

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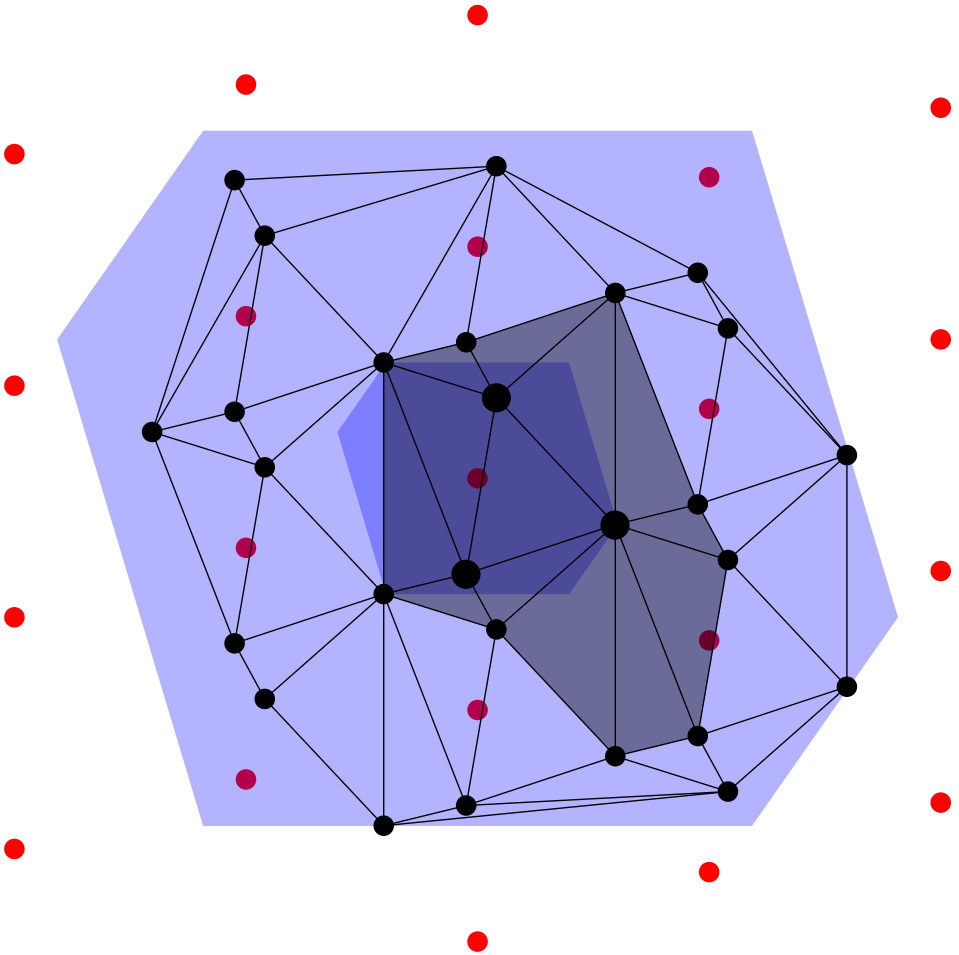

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- triangulate $\Lambda X \cap \mathrm{dom}(0, 3\Lambda)$
- Cells with a vertex in $\mathrm{dom}(0, \Lambda)$ are "good", i.e. part of periodic triangulation

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  - i.e., incrementally insert 3^d copies of each point
  - provide interface for access to (implicit) periodic triangulation

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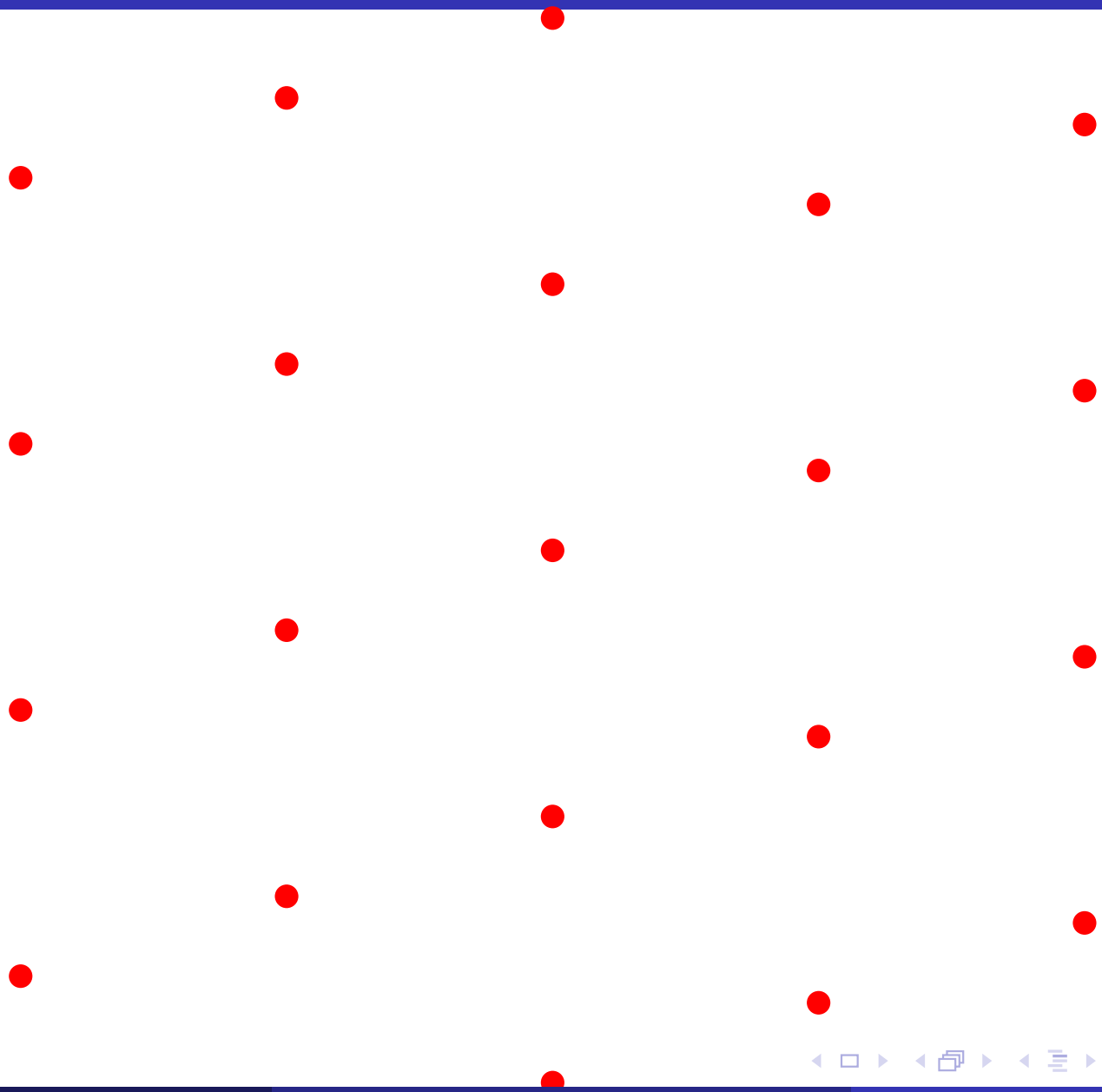

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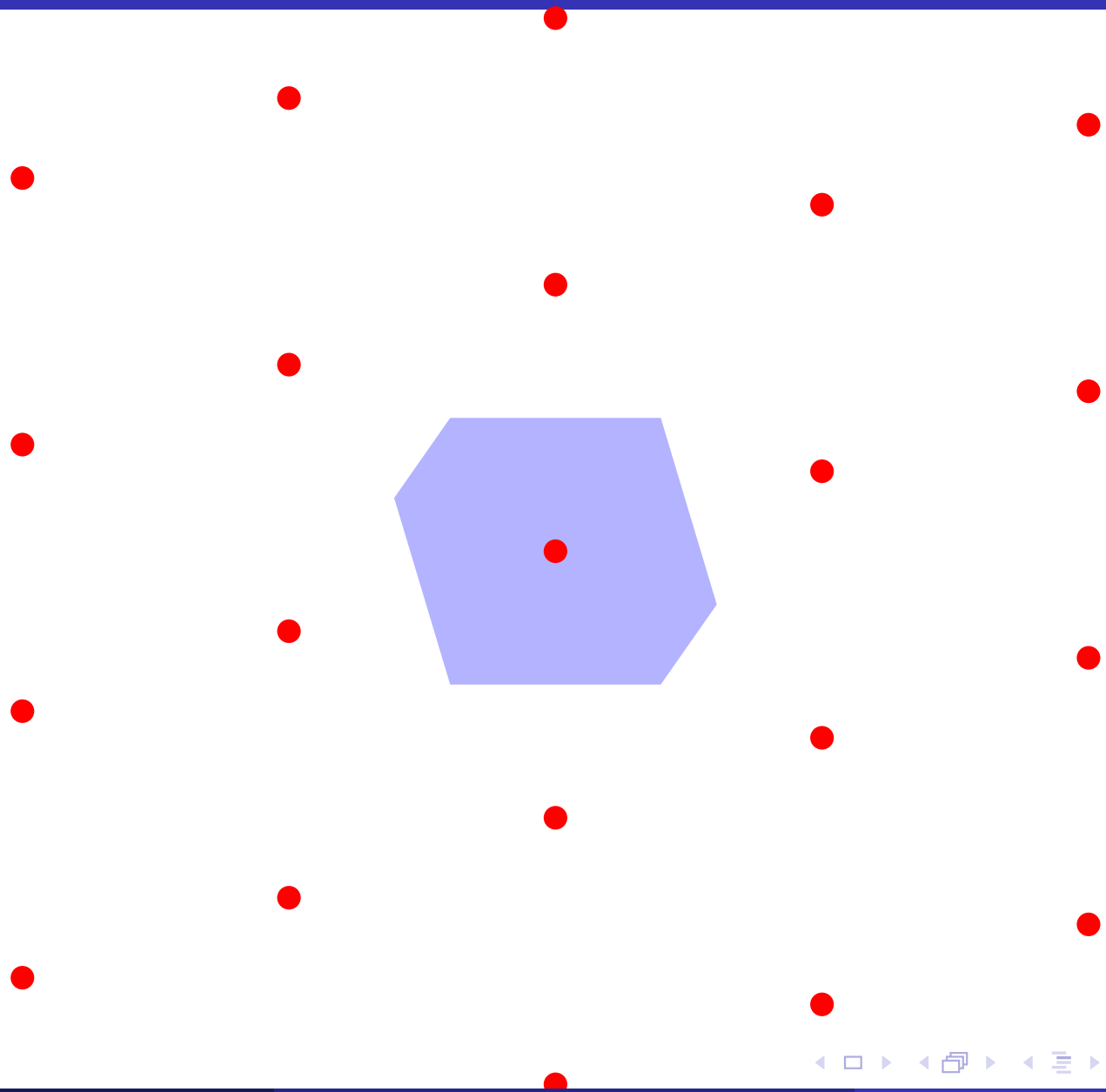

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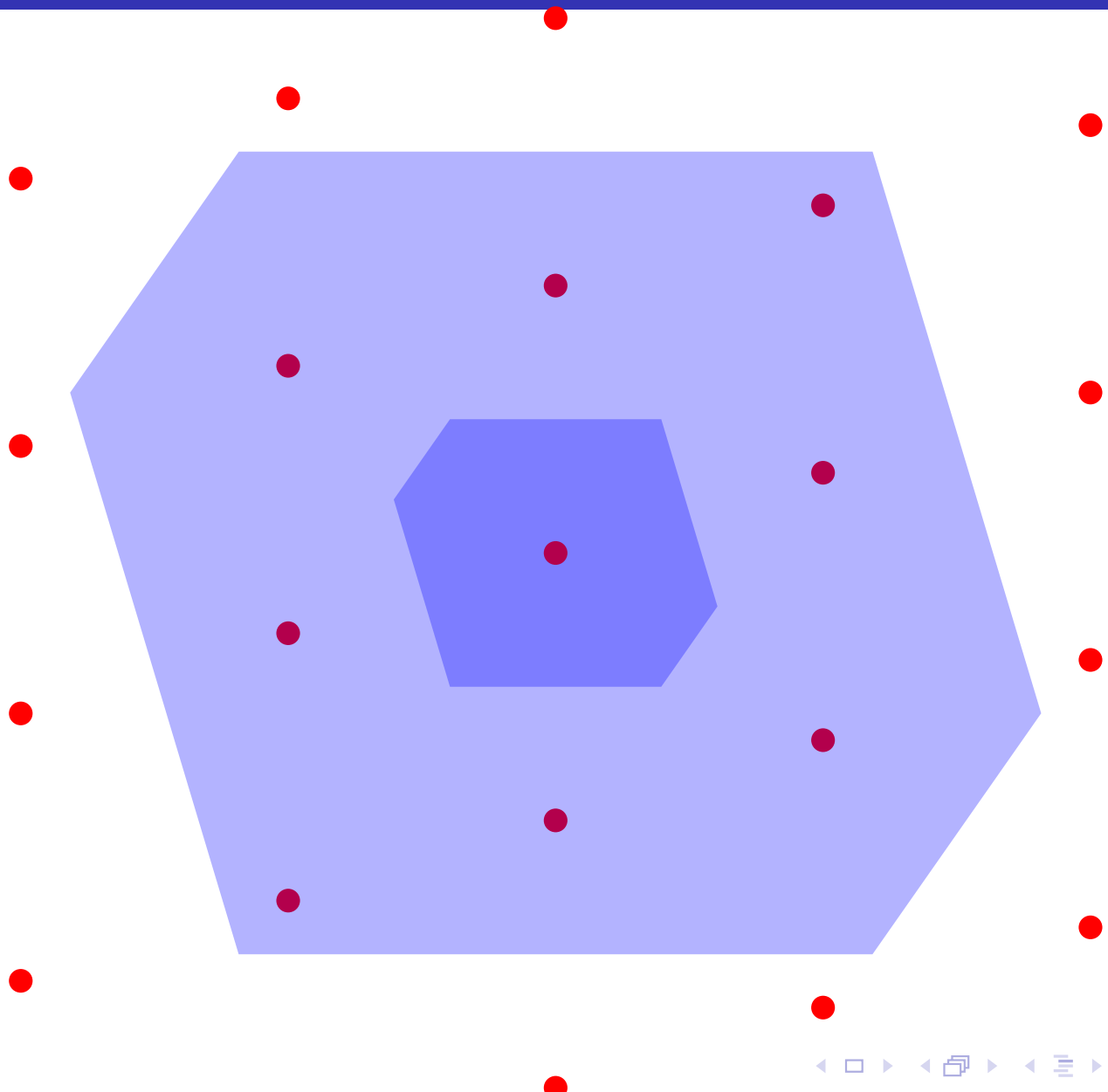

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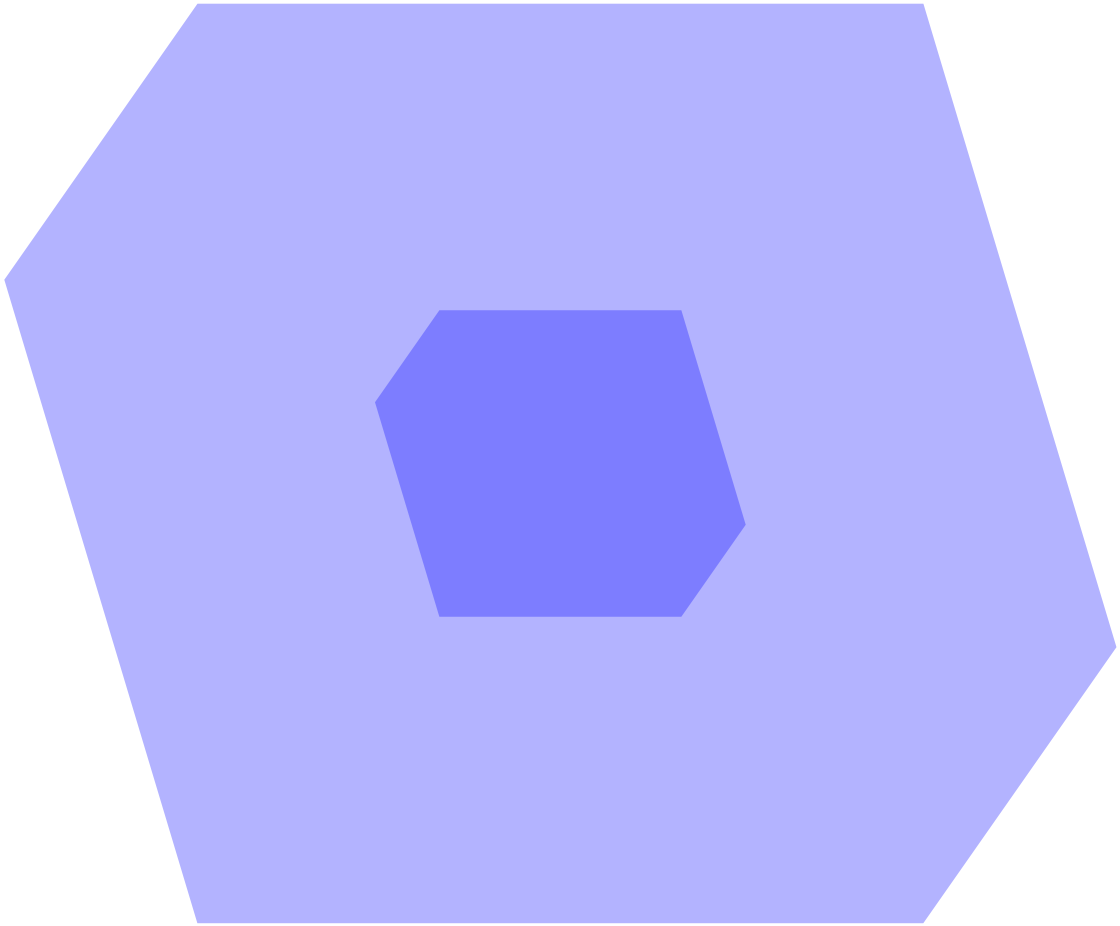

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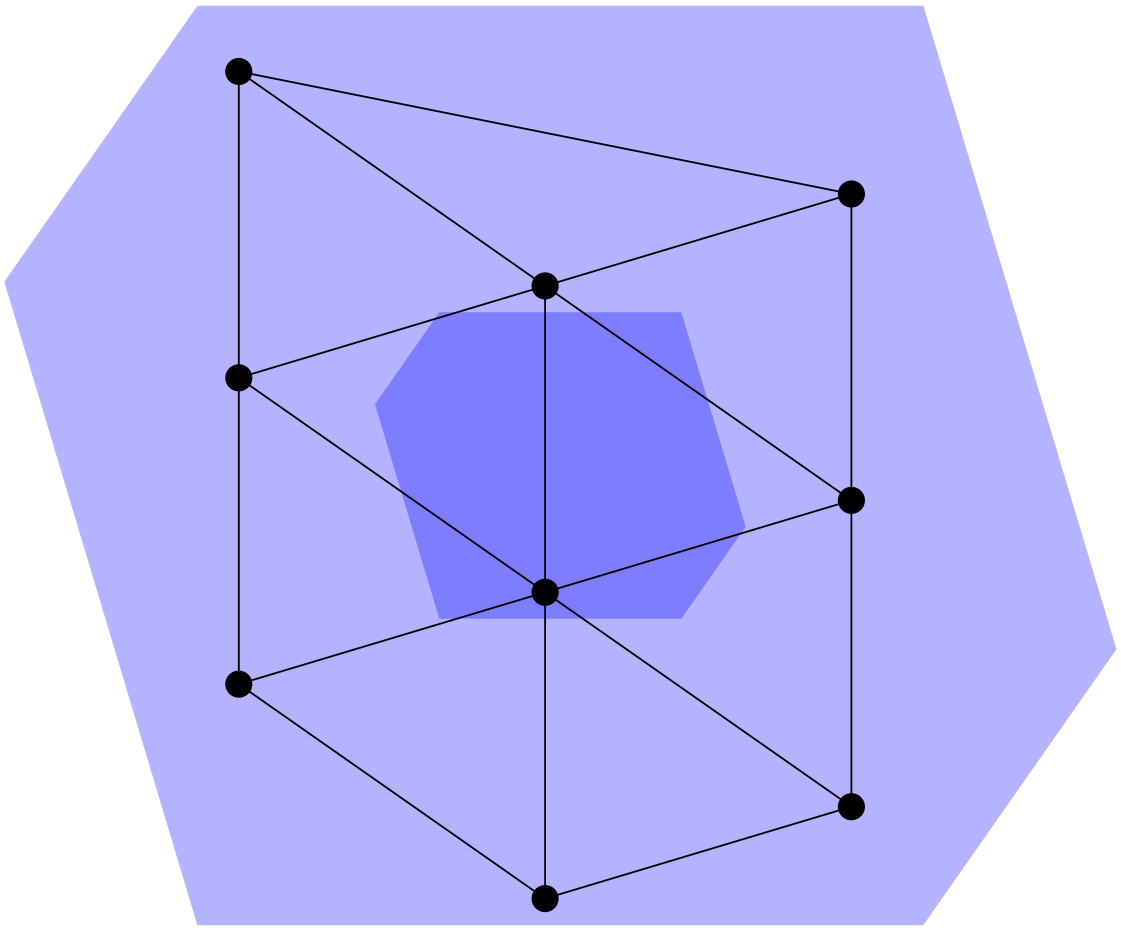

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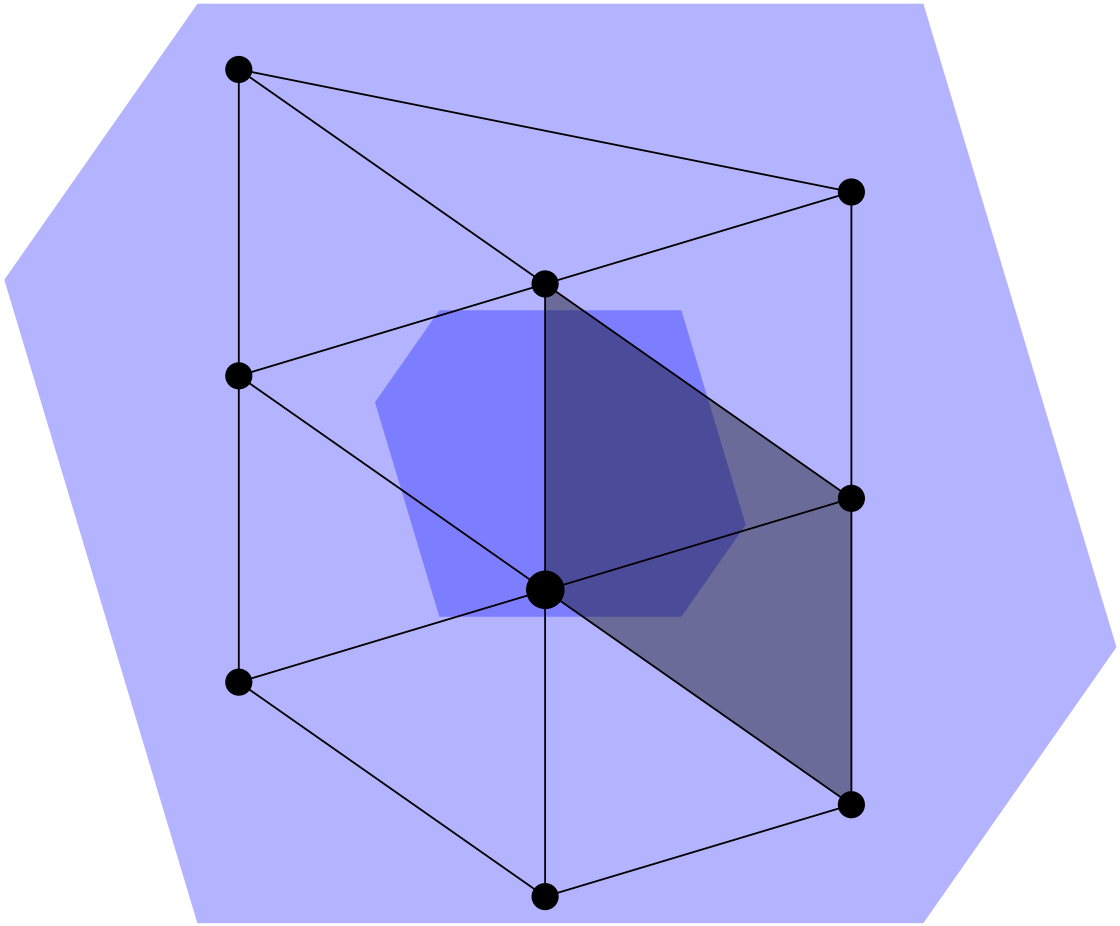

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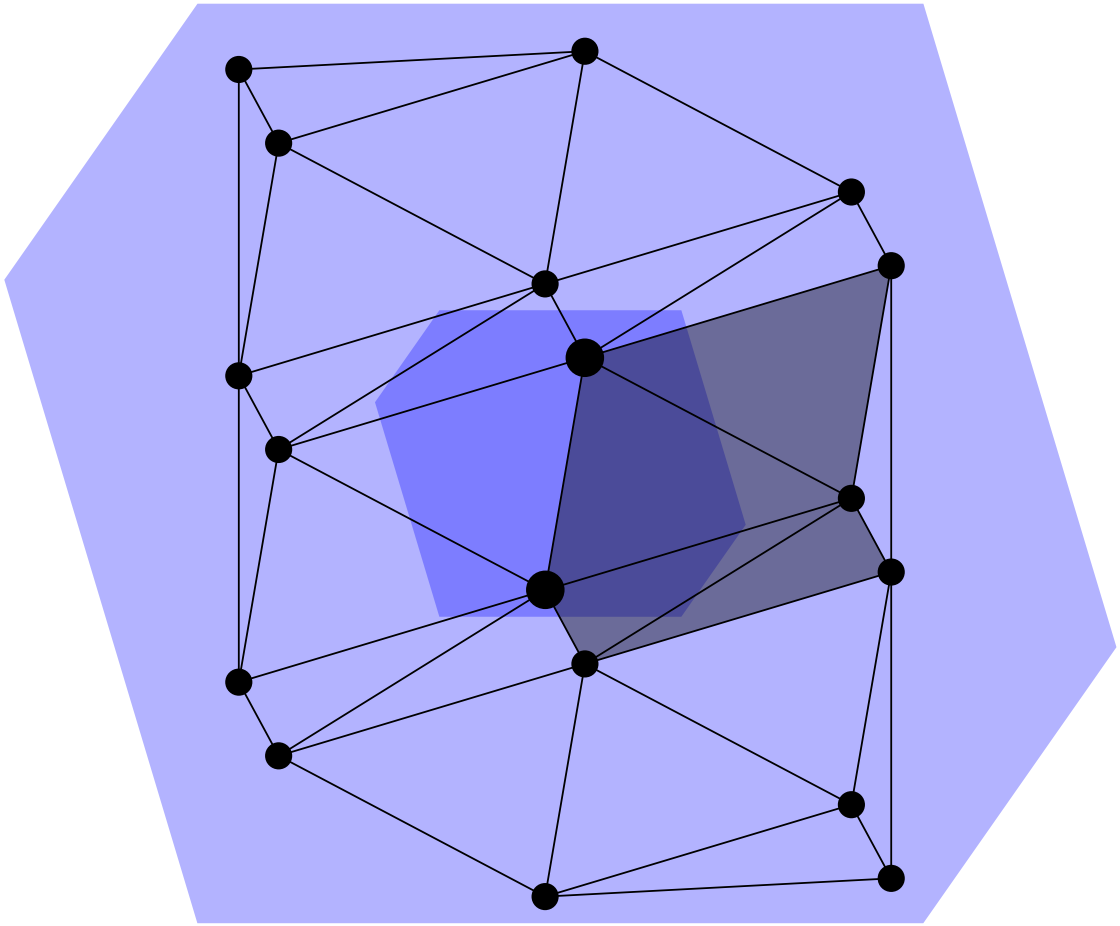

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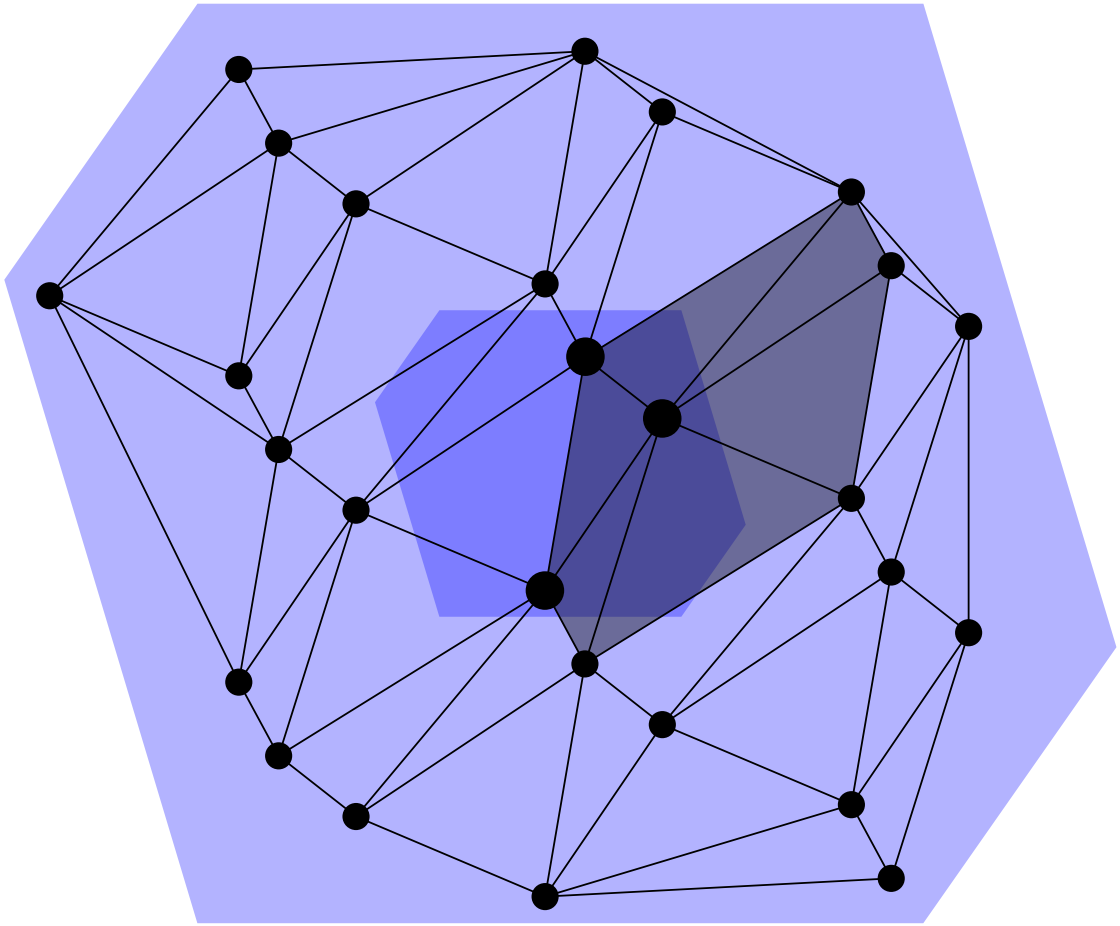

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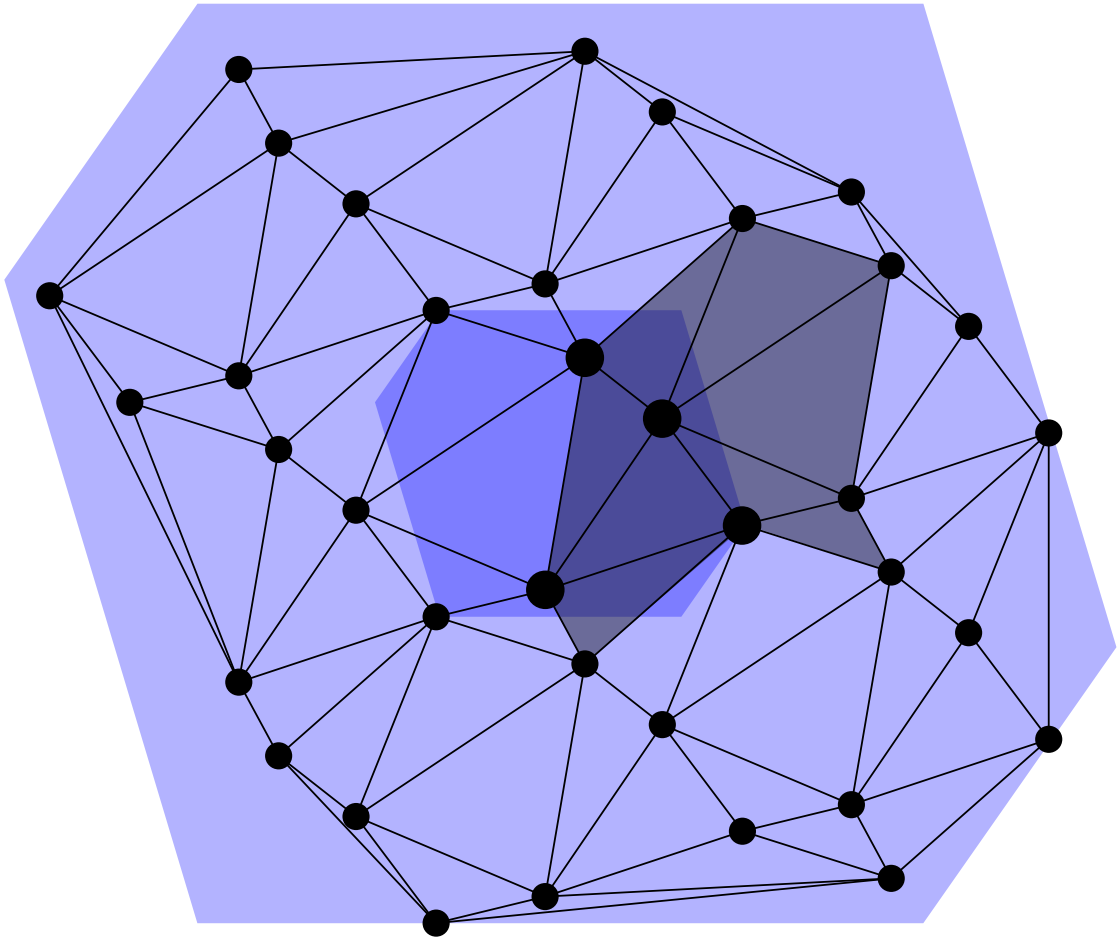

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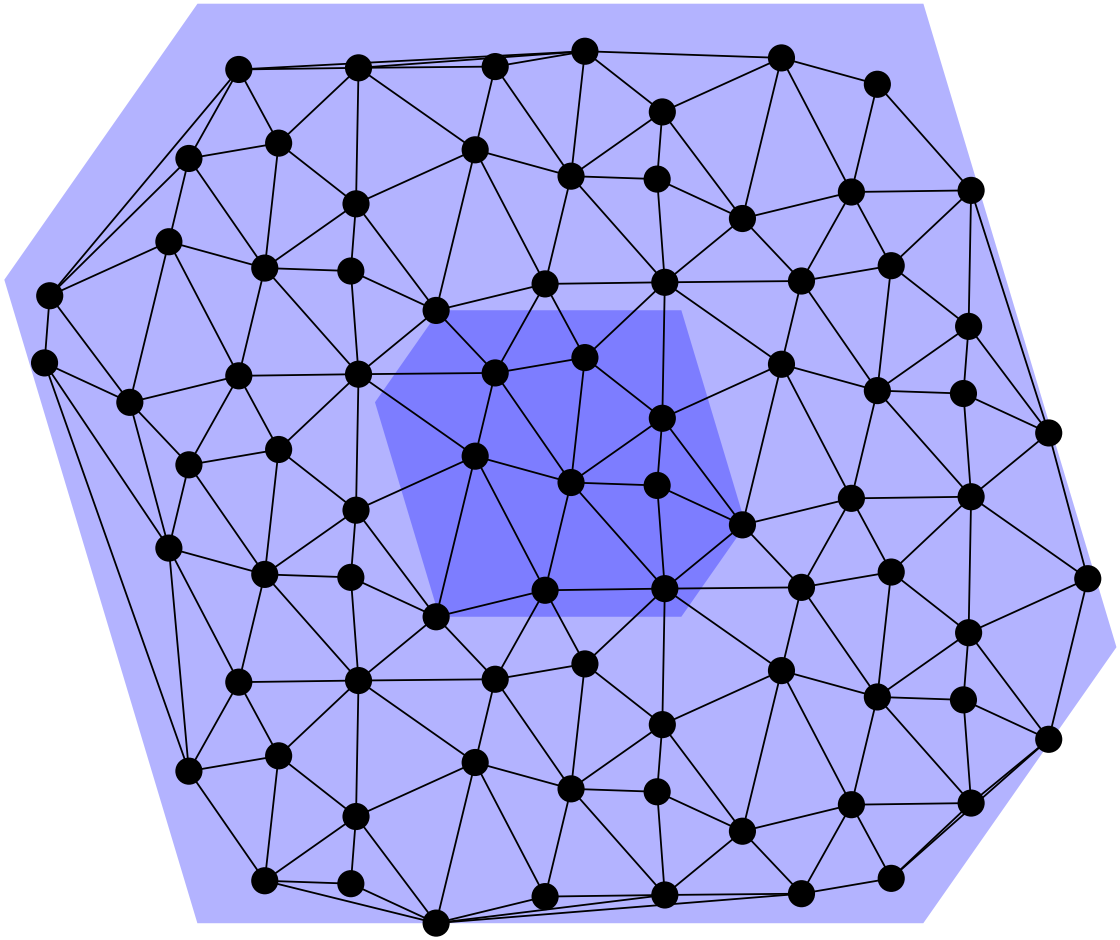

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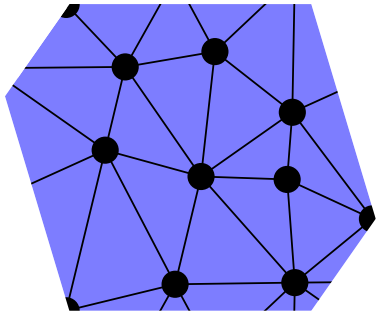

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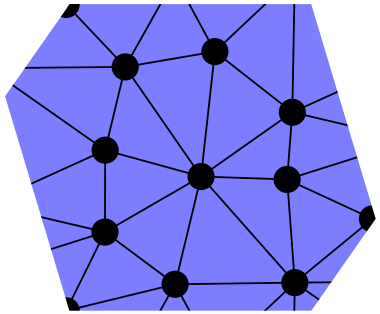

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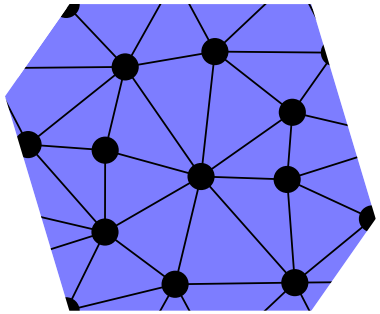

Voronoi domain

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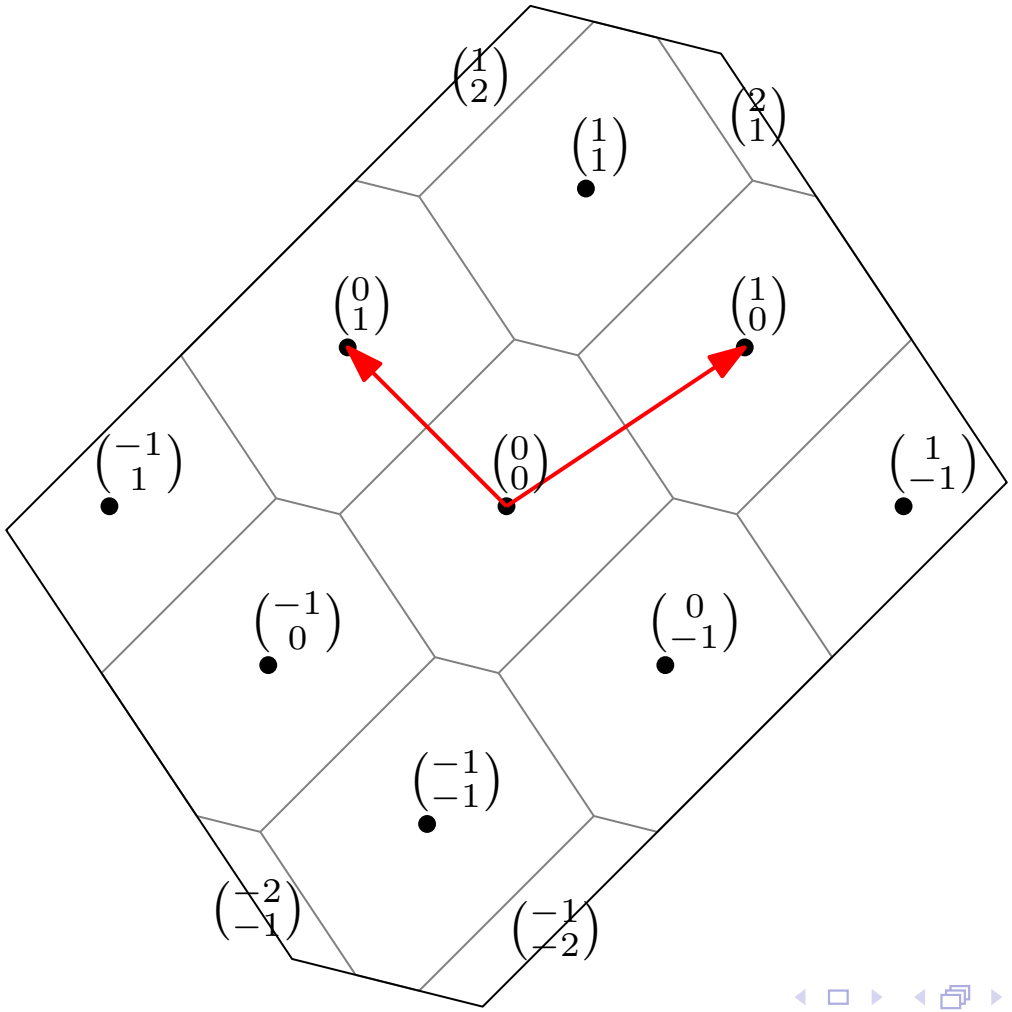
Interface in phase 1

- Output: associate *offsets* to vertices of a cell



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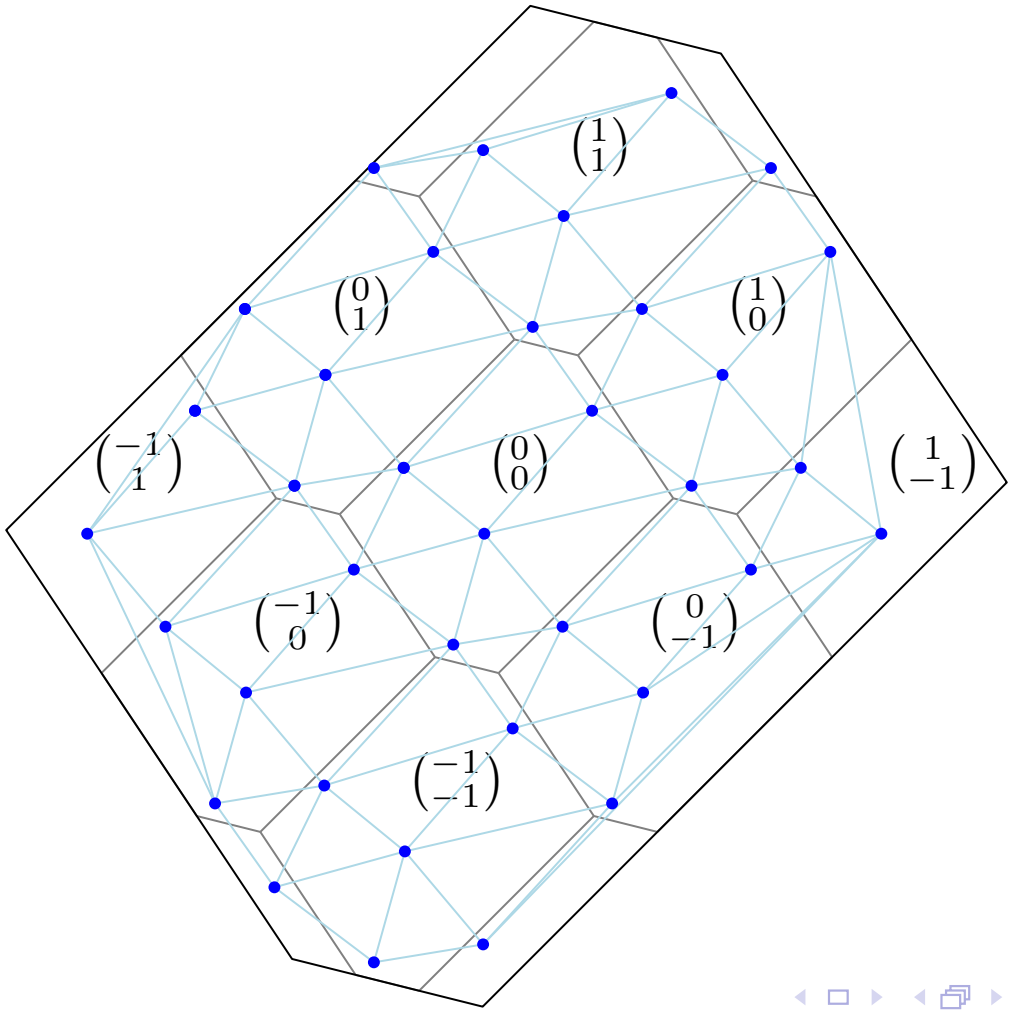

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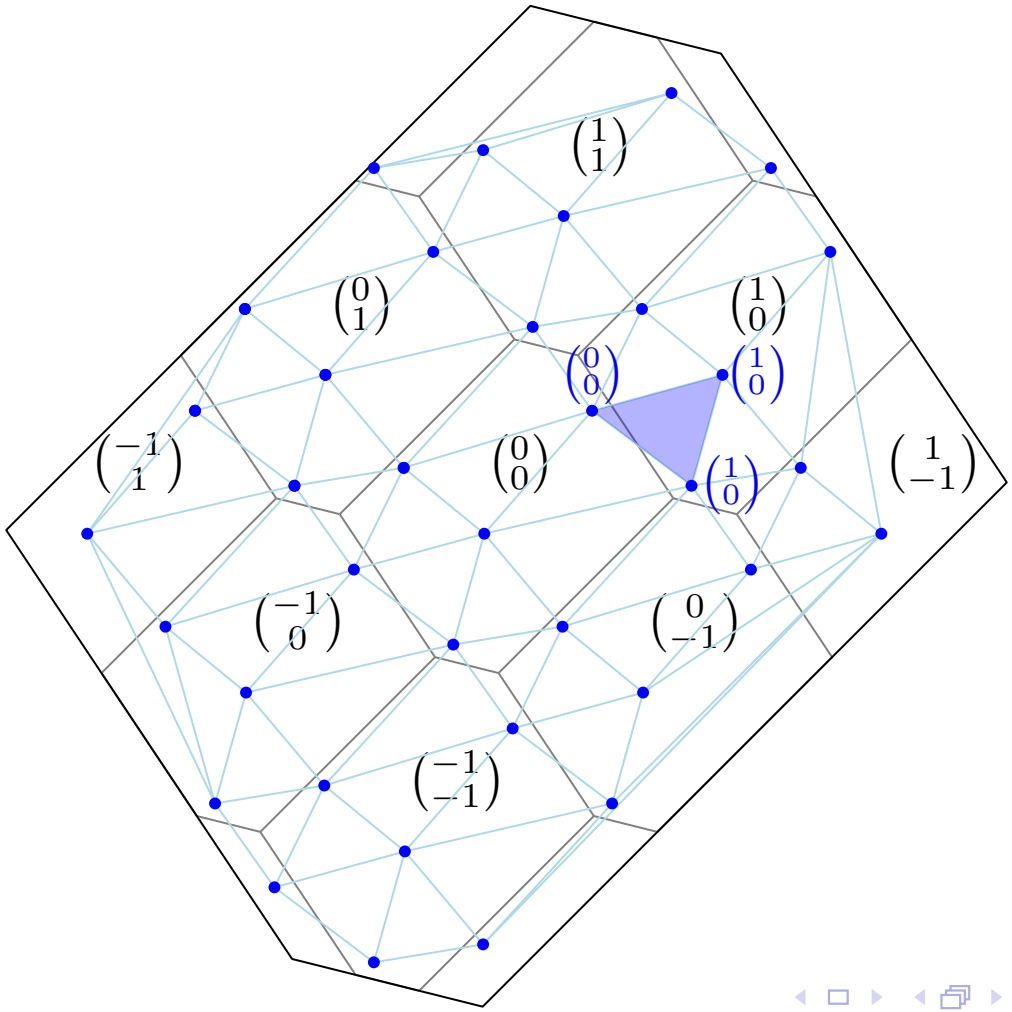

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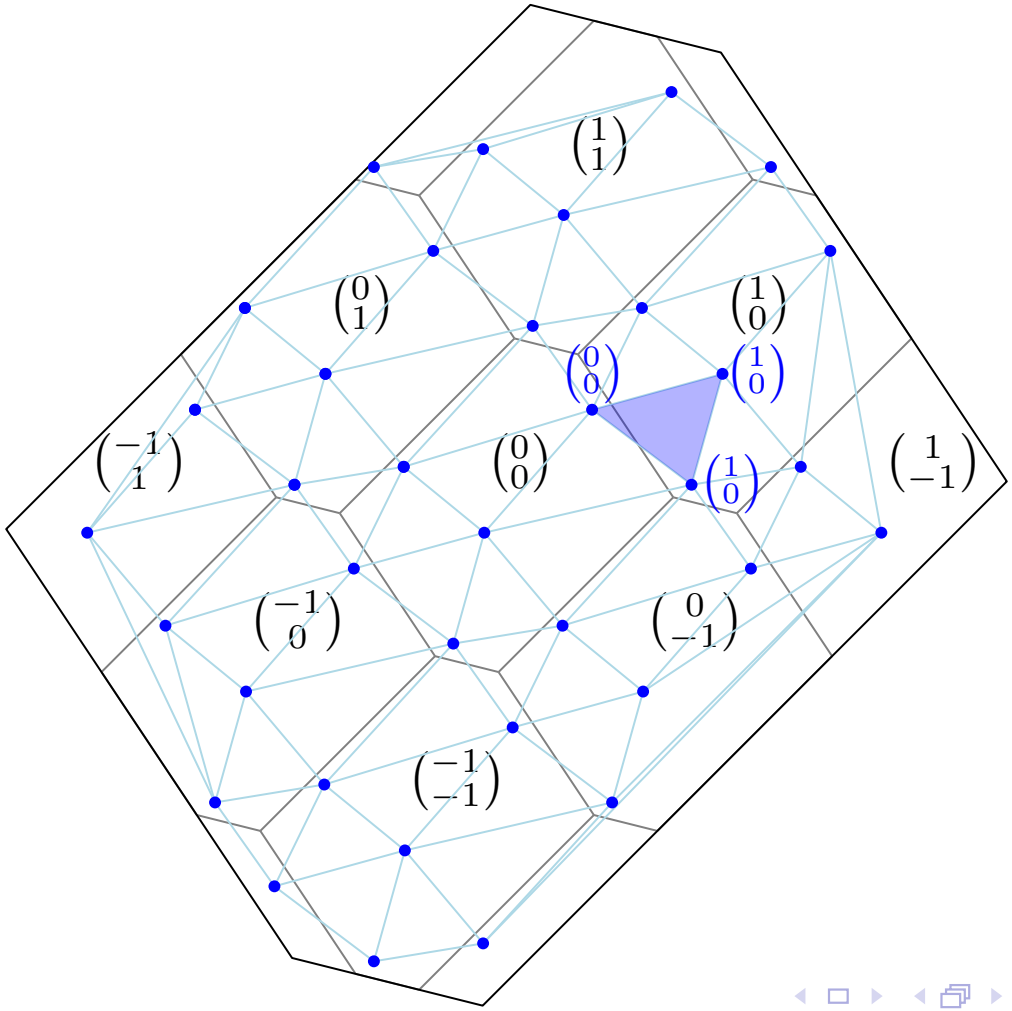

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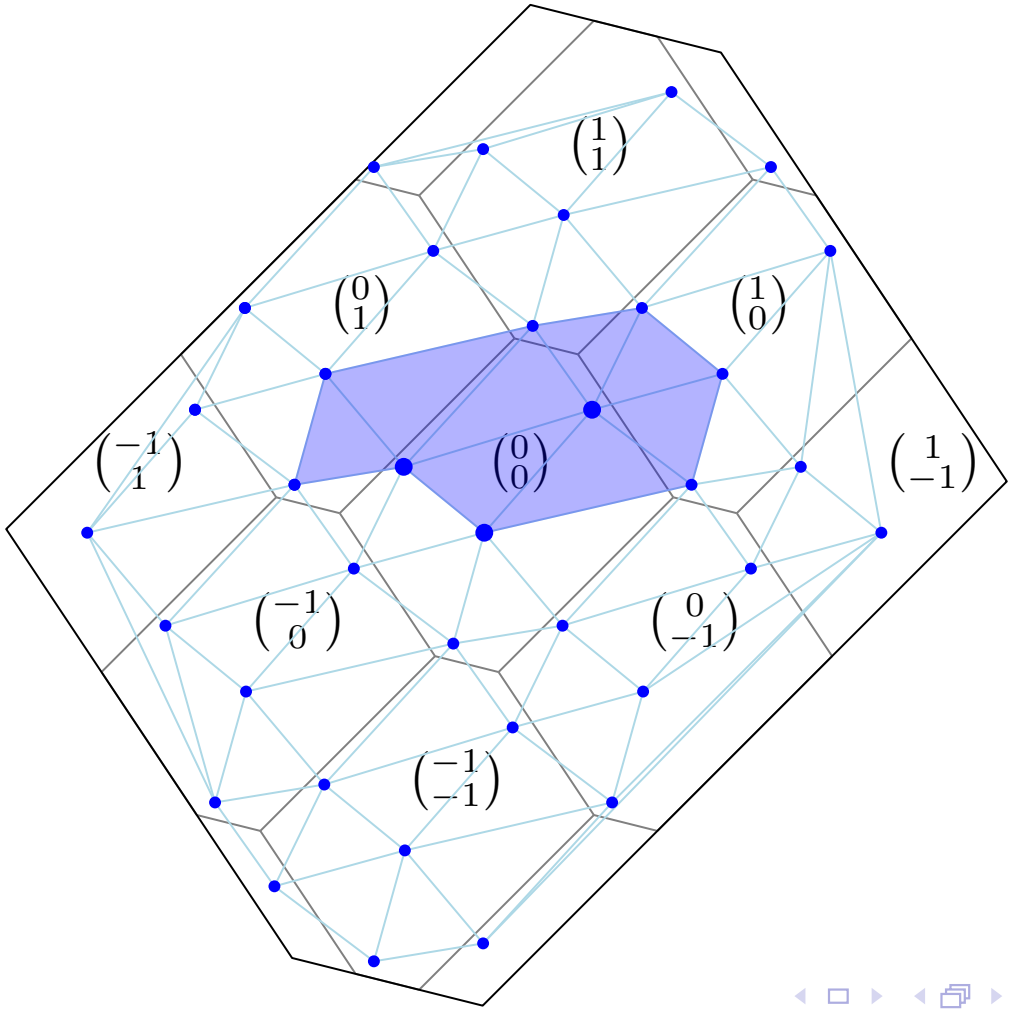

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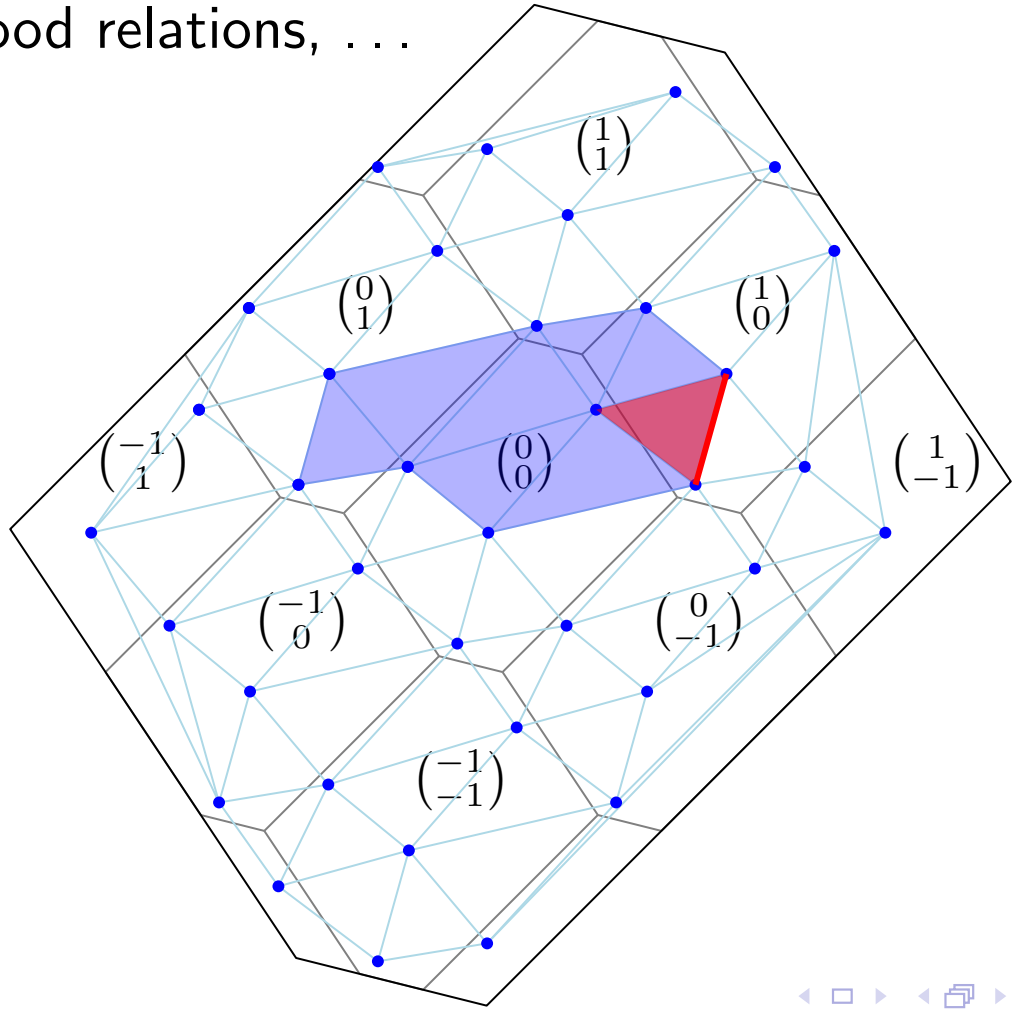

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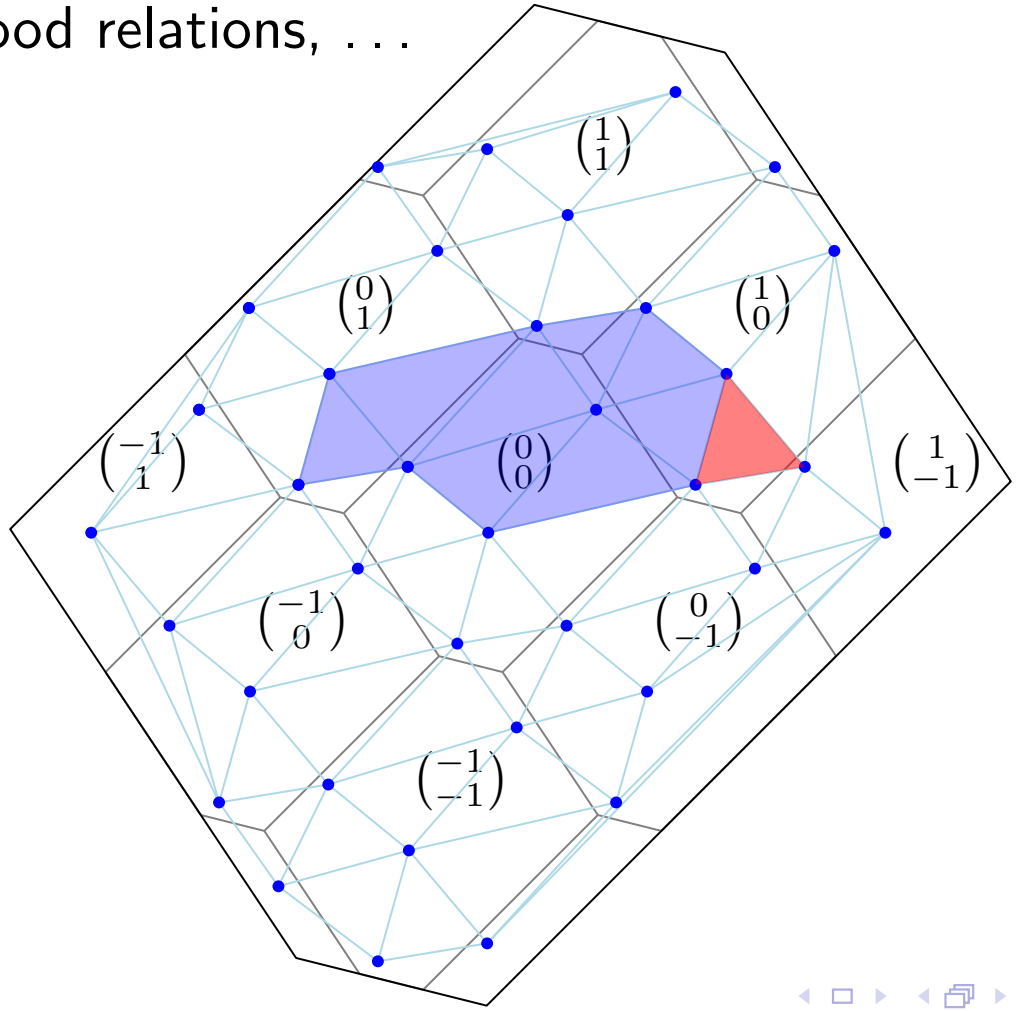

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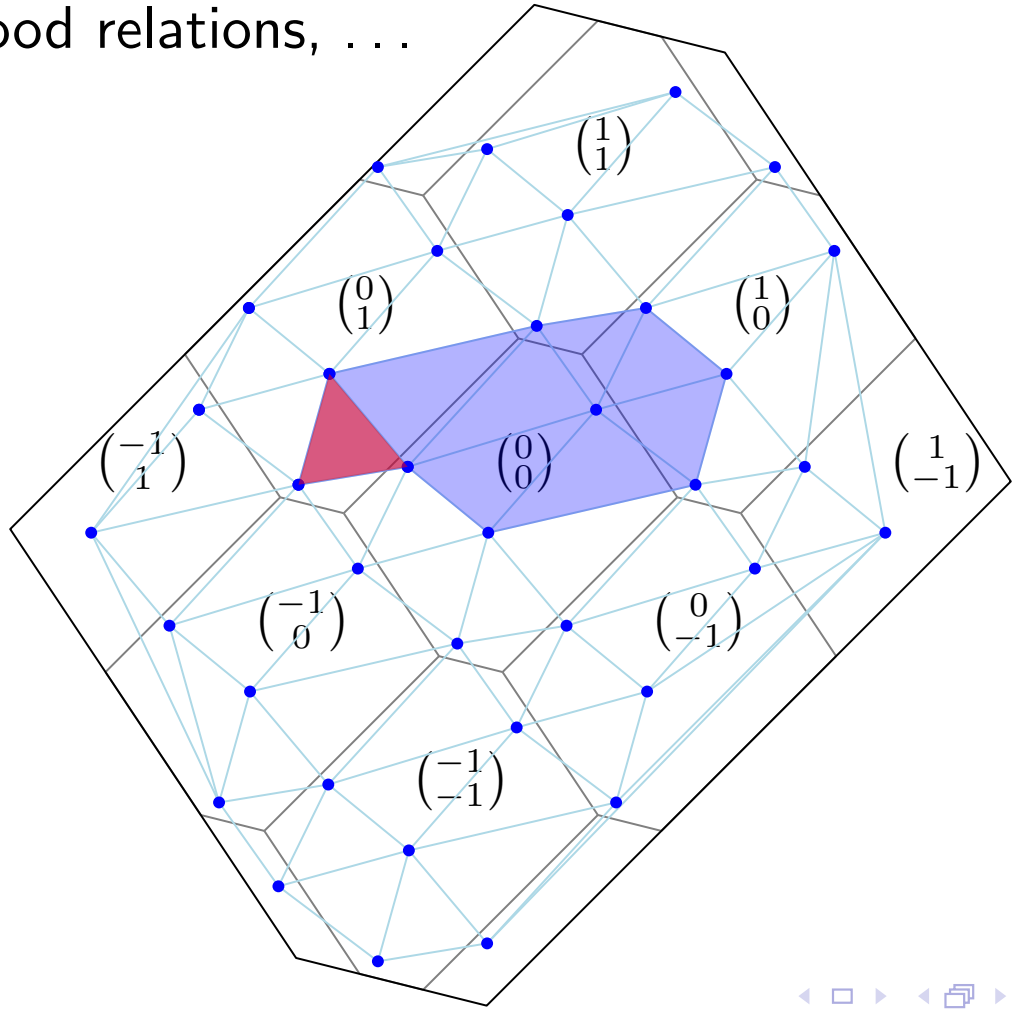

Converting to phase 2

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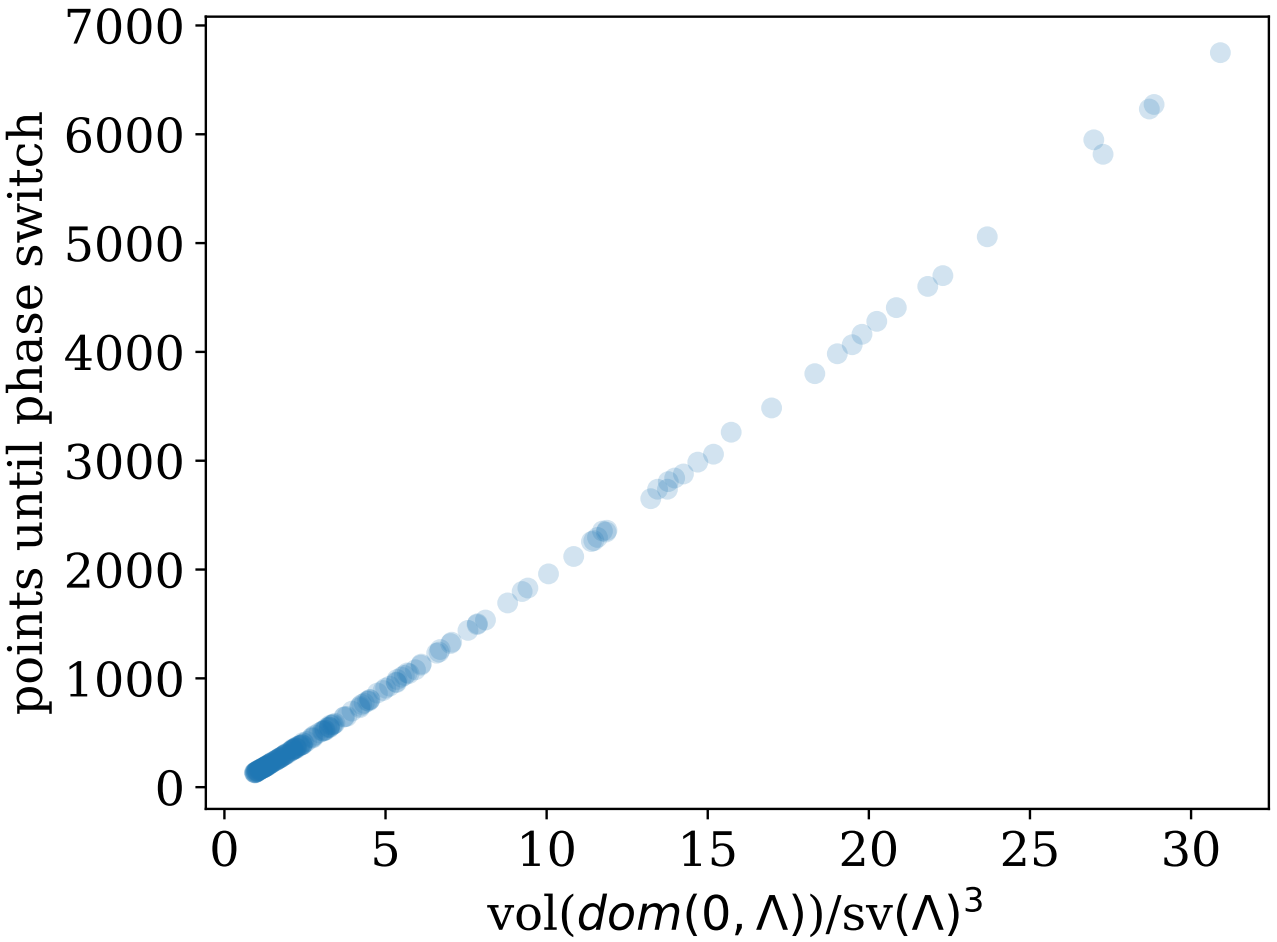
Number of points until transition:

- depends on shape of Voronoi domain
- in practice: earlier transition if points evenly distributed

Points until switch



Number of random points inserted until switching criterion is fulfilled, for various 3D lattices. Each point in the plot represents a lattice.

Running times (cubic case)

Implementation	CGAL Euclidean	CGAL Cubic	Lattice (new)
10^0	0.0000	0.0001	0.0002
10^1	0.0000	0.0160	0.0033
10^2	0.0004	0.1848	0.0461
10^3	0.0049	0.5957	0.0858
10^4	0.0487	0.9591	0.3832
10^5	0.5679	4.8119	4.5153
10^6	6.5974	93.9327	95.1447
10^7	59.5152	2314.3618	2317.7867

Running time (in seconds) of various algorithms on random point sets of various sizes with cubic periodicity.

Running times (other lattices)

Lattice	Cubic	FCC	Λ_1	Λ_2
$\frac{\mathrm{vol}(\mathrm{dom}(0,\Lambda))}{\mathrm{sv}(\Lambda)^3}$	1.0	0.71	12.5	346.41
n_{switch}	141	94	2519	89950
10^0	0.0002	0.0001	0.0003	0.0004
10^1	0.0033	0.0026	0.0044	0.0035
10^2	0.0461	0.0287	0.0460	0.0380
10^3	0.0858	0.0446	0.9812	0.6372
10^4	0.3832	0.1642	4.6602	16.5759
10^5	4.5153	2.7868	10.8139	362.7956
10^6	95.1447	51.5945	58.1568	517.9715
10^7	2317.7867	1215.2648	1799.4515	2983.0943

Running times (in seconds) of our algorithm for random point sets of various sizes and various lattices.

n_{switch}: average number of points until switch to phase 2

Summary

- periodic 2D and 3D Delaunay triangulations for arbitrary lattices within branch of CGAL codebase, aim to integrate into CGAL
- performance comparable to CGAL implementation for cubic lattice
- internally operates in 2 phases
  - Phase 1: Euclidean triangulation with 3^d copies of each point
  - Phase 2: Periodic triangulation with 1 copy of each point
  - Transition once sufficiently many points inserted
- provides uniform interface to periodic triangulation (e.g. iterators, adjacency relations) that hides the internal phase