

1 **DECOMPOSITION OF HIGH DIMENSIONAL AGGREGATIVE**
2 **STOCHASTIC CONTROL PROBLEMS ***

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5 **Abstract.** We consider the framework of high dimensional stochastic control problem, in which
6 the controls are aggregated in the cost function. As first contribution we introduce a modified
7 problem, whose optimal control is under some reasonable assumptions an ε -optimal solution of the
8 original problem. As second contribution, we present a decentralized algorithm whose convergence
9 to the solution of the modified problem is established. Finally, we study the application to a problem
10 of coordination of energy consumption and production of domestic appliances.

11 **Key words.** Stochastic optimization, Lagrangian decomposition, Uzawa’s algorithm, stochastic
12 gradient.

13 **AMS subject classifications.** 93E20,65K10, 90C25, 90C39, 90C15.

14 **1. Introduction.** The present article aims at solving a high dimensional
15 stochastic control problem (P_1) involving a large number n of agents indexed by
16 $i \in \{1, \dots, n\}$, of the form:

17 (1.1) $(P_1) \quad \begin{cases} \inf_{u \in \mathcal{U}} J(u) \\ J(u) := \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) + \frac{1}{n} \sum_{i=1}^n F_i(u^i, X^{i,u^i}) \right). \end{cases}$

18 The dynamics of the state of each agent X^{i,u^i} is driven by independent Brownian motions
19 W^i (no common noise) so that potential interactions between agents dynamics
20 is only due to the non anticipative controls u^i supposed to be progressively measur-
21 able w.r.t. to the Brownian noise $W = (W^i)_{i \in \{1, \dots, n\}}$. We emphasize, the specific
22 structure of that problem whose cost function is the sum of, on one side, additively
23 separable terms F_i between agents and a coupling term F_0 , function of the *aggregate*
24 strategies $\frac{1}{n} \sum_{i=1}^n u^i$.

25 **1.1. Motivations.** This work is motivated by its potential applications for large-
26 scale coordination of flexible appliances, to support power system operation in a con-
27 text of increasing penetration of renewables. One type of appliances that has been
28 consistently investigated in the last few years, for its intrinsic flexibility and potential

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for network support, includes thermostatically controlled loads (TCLs) such as refrigerators or air conditioners. Several papers have already investigated the potential of dynamic demand control and frequency response services of TCLs [21] and how the population recovers from significant perturbations [3]. The coordination of TCLs can be performed in a centralized way, like in [8]. However this approach raises challenging problems in terms of communication requirements and customer privacy. A common objective can be reached in a fully distributed approach, like in [25], where each TCL is able to calculate its own actions (ON/OFF switching) to pursue a common objective. This paper is related to the work of De Paola *et al.* [4], where each agent represents a flexible TCL device. In [4] a distributed solution is presented for the operation of a population of $n = 2 \times 10^7$ refrigerators providing frequency support and load shifting. They adopt a game-theory framework, modelling the TCLs as price-responsive rational agents that schedule their energy consumption and allocate their frequency response provision in order to minimize their operational costs. The potential practical application of our work also considers a large population of TCLs which, contrarily to [4], have stochastic dynamics. The proposed approach is able to minimize the overall system costs in a distributed way, with each TCL determining its optimal power consumption profile in response to price signals.

1.2. Related literature. The considered problem belongs to the class of stochastic control: looking for strategies minimizing the expectation of an objective function under specific constraints. One of the main approaches proposed in the literature to tackle this problem is, is to use random trees: this consists in replacing the almost sure constraints, induced by non-anticipativity, by a finite number of constraints to get a finite set of scenarios (see. [9] and [19]). Once the tree structure is built, the problem is solved by different decomposition methods such as scenario decomposition [18] or dynamic splitting [20]. The main objective of the scenario method is reducing the problem to an approximated deterministic one. The paper focuses on high dimensional noise problems with large number of time steps, this approach is not feasible. The idea of reducing a single high dimensional problem to a large number with low dimension has been widely studied in the deterministic case. In deterministic and stochastic problems a possibility is to use time decomposition thanks to the Dynamic Programming Principle [1] taking advantage of Markov property of the system. However, this method requires a specific time structure of the cost function and fails when applied to problems for which the state space dimension is greater than five. One can deal with the curse of dimensionality, under continuous linear-convex assumptions, by using the Stochastic Dual Dynamic Programming algorithm (SDDP) [15] to get upper and lower bounds of the value function, using polyhedral approximations. Though the almost-sure convergence of a broad class of SDDP algorithms has been proved [17], there is no guarantee on the speed of the convergence and there is no good stopping test. In [14], a stopping criteria based on a dual version of SDDP, which gives a deterministic upper-bound for the primal problem, is proposed. SDDP is well-adapted for medium sized population problems ($n \leq 30$), whereas it fails for problems with magnitude similar to one of the present paper ($n > 1000$). It is natural for this type of high dimensional problem to investigate decomposition techniques in the spirit of the Dual Approximation Dynamic Programming (DADP). DADP has been developed in PhD theses (see [7], [12]). This approach is characterized by a price decomposition of the problem, where the stochastic constraints are projected on subspaces such that the associated Lagrangian multiplier is adapted for dynamic programming. Then the optimal multiplier is estimated by implementing Uzawa's

78 algorithm. To this end in [12], the Uzawa’s algorithm, formulated in a Hilbert set-
 79 ting, is extended to a Banach space. DADP has been applied in different cases, such
 80 as storage management problem for electrical production in [7, chapter 4] and hydro
 81 valley management [2]. In the proposed paper, in the same vein as DADP we propose
 82 a price decomposition approach restricted to deterministic prices. This new approach
 83 takes advantage of the large population number in order to introduce an auxiliary
 84 problem where the coupling term is purely deterministic.

85 **1.3. Contributions.** We consider the following approximation of problem (P_1) :

$$86 \quad (1.2) \quad (P_2) \quad \begin{cases} \inf_{u \in \mathcal{U}} \tilde{J}(u) \\ \tilde{J}(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) + \frac{1}{n} \mathbb{E} \left(\sum_{i=1}^n F_i(u^i, X^{i,u^i}) \right). \end{cases}$$

87 As a first contribution, this paper shows that under some convexity and regularity
 88 assumptions on F_0 and $(F_i)_{i \in \{1, \dots, n\}}$, any solution of problem (P_2) is an ε_n -solution
 89 of (P_1) , with $\varepsilon_n \rightarrow 0$ when $n \rightarrow \infty$. In addition, an approach of price decomposition
 90 for (P_2) is easier than for (P_1) , since the Lagrange multiplier is deterministic for (P_2) ,
 91 whereas it is stochastic for (P_1) . Since computing the dual cost of (P_2) is expensive,
 92 we propose *Stochastic Uzawa* and *Sampled Stochastic Uzawa* algorithms relying on
 93 Robbins Monroe algorithm in the spirit of the stochastic gradient. Its convergence
 94 is established. We check the effectiveness of the *Stochastic Uzawa* algorithm on a
 95 linear quadratic Gaussian framework, and we apply the *Sampled Stochastic Uzawa*
 96 algorithm to a model of power system, inspired by the work of A. De Paola *et al.* [4].

97 **2. General framework.** Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability
 98 space on which $W = (W^i)_{i=1, \dots, n}$ is a n -dimensional Brownian motion, such that
 99 for any $t \in [0, T]$ and $i \in \{1, \dots, n\}$, W_t^i takes value in \mathbb{R} , and generates the filtration
 100 $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. \mathbb{P} stands for the Wiener measure associated with this filtration and
 101 \mathbb{F} for the augmented filtration by all \mathbb{P} -null sets.

102 The following notations are used:

$$\begin{aligned} \mathbb{X} &:= \{ \varphi : \Omega \rightarrow \mathcal{C}([0, T], \mathbb{R}) \mid \varphi(\cdot) \text{ is } \mathbb{F} \text{- adapted, } \|\varphi\|_{\infty, 2} := \mathbb{E} \left(\sup_{s \in [0, T]} |\varphi(s)|^2 \right)^{\frac{1}{2}} < \infty \}, \\ 103 \quad L^2(0, T) &:= \{ \varphi : [0, T] \rightarrow \mathbb{R} \mid \int_0^T |\varphi(t)|^2 dt < \infty \}, \\ \mathbb{U} &:= \{ \varphi : [0, T] \times \Omega \rightarrow \mathbb{R} \mid \varphi(\cdot) \text{ is } \mathbb{F} \text{- prog. measurable, } \mathbb{E} \int_0^T |\varphi(t)|^2 dt < \infty \}, \end{aligned}$$

104 and for any $i \in \{1, \dots, n\}$, the feasible set of controls is defined by:

$$105 \quad (2.1) \quad \mathcal{U}_i := \{ v \in \mathbb{U} \text{ and } v_t(\omega) \in [-M_i, M_i], \text{ for a.a. } (t, \omega) \in [0, T] \times \Omega \},$$

106 and we set $M := \max_{i \in \{1, \dots, n\}} M_i$. The set of admissible controls is $\mathcal{U} := \mathcal{U}_1 \times \dots \times \mathcal{U}_n$,

107 whose elements are denoted by $u := (u^1, \dots, u^n)$.

108 Each local agent $i = 1, \dots, n$ is supposed to control its state variable through the
 109 control process $u^i \in \mathcal{U}_i$ and suffers from independent uncertainties. More specifically,
 110 the state process of each agent, $X^{i, u^i} = (X_t^{i, u^i})_{t \in [0, T]}$, for $i = 1, \dots, n$ takes values in

111 \mathbb{R} and follows the dynamics

(2.2)

$$112 \quad \begin{cases} dX_t^{i,u^i} &= \mu_i(t, u_t^i, X_t^{i,u^i})dt + \sigma_i(t, X_t^{i,u^i})dW_t^i, \text{ for } t \in [0, T], \text{ for } i \in \{1, \dots, n\} \\ X_{0,u^i}^i &= x_0^i \in \mathbb{R}. \end{cases}$$

113 Without loss of generality, the initial states x_0^i are supposed to be deterministic.

114 The process X^i is \mathbb{F} -progressively measurable. For all i , \mathcal{F}^i stands for the natural
115 filtration of the Brownian motion W^i .

116 **2.1. On the well-posedness of (P_1) .** In this section, the assumptions needed
117 for (P_1) to be well posed are studied.

118 *Assumption 2.1.* For any $i \in \{1, \dots, n\}$, the functions μ_i and σ_i are continuous
119 w.r.t (u, x) uniformly in t . In addition there exists $K_i > 0$ such that, for any $t \in [0, T]$
120 and $\nu \in [-M, M]$:

(2.3)

$$121 \quad \begin{aligned} |\mu_i(t, \nu, x) - \mu_i(t, \nu, y)| + |\sigma_i(t, \nu, x) - \sigma_i(t, \nu, y)| &\leq K_i |x - y|, \\ |\mu_i(t, \nu, x)| + |\sigma_i(t, \nu, x)| &\leq K_i (1 + |x|), \end{aligned} \quad \text{for any } x, y \in \mathbb{R}.$$

122 **LEMMA 2.2.** *Let $i \in \{1, \dots, n\}$ and $v \in \mathcal{U}_i$ be a control process. If Assumption*
123 *2.1 holds, then there exists a unique process $X^{i,v} \in \mathbb{X}$ satisfying (2.2) (in the strong*
124 *sense) such that for any $p \in [1, \infty)$*

$$125 \quad (2.4) \quad \mathbb{E} \left(\sup_{0 \leq t \leq T} |X_t^{i,v}|^p \right) < C(p, T, x_0, K) < \infty .$$

126 *Proof.* The proof for the existence and uniqueness of a solution of (2.2) relies on
127 [13, Theorem 3.6, Chapter 2]. The inequality is a result of [13, Theorem 4.4, Chapter
128 2]. \square

129 Let $F_0 : L^2(0, T) \rightarrow \bar{\mathbb{R}}$ and $F_i : L^2(0, T) \times \mathcal{C}([0, T], \bar{\mathbb{R}}) \rightarrow \mathbb{R}$ be proper and lower
130 semi continuous functions, and there exists $\hat{u} \in \mathcal{U}$ such that:

$$131 \quad (2.5) \quad \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n \hat{u}^i \right) \right) < \infty.$$

132 Define $G_i : L^2(0, T) \rightarrow \bar{\mathbb{R}}$ by $G_i(u^i) = F_i(u^i, X^{i,u^i})$ for any $u^i \in \mathcal{U}_i$. Additional
133 assumptions are formulated below.

134 *Assumption 2.3.* For any $i \in \{1, \dots, n\}$:

- 135 (i) $u^i \mapsto \mathbb{E}(G_i(u^i))$ is strictly convex.
136 (ii) there exists a positive integer p such that F_i has p -polynomial growth, i.e
137 there exists $K > 0$ such that for any $x^i \in \mathcal{C}([0, T], \mathbb{R})$ and $u^i \in L^2(0, T)$:

$$138 \quad (2.6) \quad |F_i(u^i, x^i)| \leq K \left(1 + \sup_{0 \leq t \leq T} |x_t^i|^p \right).$$

139 Assumption 2.3.(1) holds in different cases, like in the example below.

140 *Example 2.4.* For any $i \in \{1, \dots, n\}$, there exists $g_i : L^2(0, T) \rightarrow \mathbb{R}$ and $h_i : L^2(0, T) \times \mathcal{C}[0, T] \rightarrow \mathbb{R}$
141 $\mathcal{C}[0, T] \rightarrow \mathbb{R}$ such that for any $(v, X) \in L^2(0, T) \times \mathcal{C}[0, T]$, $F_i(v, X) = g_i(v) + h_i(X)$

142 and there exists five $L^\infty([0, T])$ scalar functions $\alpha_i, \beta_i, \gamma_i, \xi_i$ and θ_i such that for any
 143 $(t, \nu, x) \in [0, T] \times [-M, M] \times \mathbb{R}$:

$$144 \quad (2.7) \quad \mu_i(t, \nu, x) = \alpha_i(t)\nu + \beta_i(t)x + \gamma_i(t) \quad \text{and} \quad \sigma_i(x, t) = \xi_i(t)x + \theta_i(t).$$

145 Then Assumption 2.3.(i) is satisfied if:

- 146 (i) g_i is strictly convex and h_i convex.
 - 147 (ii) for a.e. $t \in [0, T]$, $\alpha(t) \neq 0$, g_i is convex and h_i strictly convex.
- 148 Indeed, for any $i \in \{1, \dots, n\}$, $u, v \in \mathbb{U}$, $\delta \in [0, 1]$ and $t \in [0, 1]$, it holds from (2.7)
 149 :

$$150 \quad (2.8) \quad X_t^{i, \delta u + (1-\delta)v} = \delta X_t^{i, u} + (1-\delta)X_t^{i, v}.$$

151 If point (i) holds, then:

$$152 \quad (2.9) \quad h_i(X_t^{i, \delta u + (1-\delta)v}) \leq \delta h_i(X_t^{i, u}) + (1-\delta)h_i(X_t^{i, v}).$$

153 Assumption 2.3.(i) follows from (2.9) and strict convexity of g_i .

154 Similarly, if point (ii) holds, then the inequality in (2.9) is strict, and Assumption
 155 2.3.(i) follows using also the convexity of g_i .

156 *Remark 2.5.* If for any $i \in \{1, \dots, n\}$, Assumption 2.3.(i) holds, then G_i is w.l.s.c.
 157 on \mathcal{U}_i . Indeed, G_i being convex, finite valued and bounded on bounded subsets
 158 of $L^2(0, T)$ (from the polynomial growth of F_i and the inequality (2.4)), thus G_i is
 159 continuous on \mathcal{U}_i .

160 From now on, Assumptions 2.1 and 2.3 are in force in the sequel.

161 The following lemma ensures the well-posedness of (P_1) .

162 **LEMMA 2.6.** *Suppose that F_0 is convex. Then J reaches its minimum over \mathcal{U} at*
 163 *a unique point.*

164 *Proof.* Clearly the control $\hat{u} \in \mathcal{U}$ defined in (2.5) is feasible. The existence and
 165 uniqueness of a minimum is proved by considering a minimizing sequence $\{u_k\}$ of
 166 J over \mathcal{U} . The set \mathcal{U} being bounded and weakly closed, there exists a sub-sequence
 167 $\{u_{k_\ell}\}$ which weakly converges to a certain $u^* \in \mathcal{U}$. Using Assumptions 2.3.(i)(ii) and
 168 convexity of F_0 , it follows that $\liminf J(u_{k_\ell}) \geq J(u^*)$ and thus u^* is a solution of
 169 (P_1) . The uniqueness is due to the strict convexity of $u^i \mapsto \mathbb{E}(F_i(u^i, X^{i, u^i}))$. \square

170 *Remark 2.7.* This kind of stochastic optimization problem is illustrated in Section
 171 7 with a problem of coordination of a large population of domestic appliances, where
 172 a system operator has to meet the demand while producing at low cost. The state X_t^i
 173 can represent for instance the temperature or the battery level of the agent i at time t ,
 174 and u_t^i its proper power generation or consumption. F_0 can be assimilated to the cost
 175 function to satisfy the demand, and for any i , F_i to the cost function connected to the
 176 proper functioning of the TCLs (characterized by individual cost function, comfort
 177 constraints, etc...).

178 **3. Approximating the optimization problem.** In this section, the link be-
 179 tween problems (P_1) and (P_2) is analyzed.

180 *Assumption 3.1.* Problem (P_2) admits a unique solution.

181 Notice that by using the same techniques as for Lemma 2.6, one can prove that the
 182 above assumption is satisfied when F_0 is convex.

183 We have the following key result.

184 THEOREM 3.2. Under Assumption 3.1, \tilde{J} reaches its minimum over \mathcal{U} at a unique
 185 point, $\tilde{u} \in \mathcal{U}$, such that for any i , \tilde{u}^i is \mathcal{F}^i -adapted and thus for any $j \neq i$, \tilde{u}^i and \tilde{u}^j
 186 are mutually independent.

187 *Proof.* Fix $i \in \{1, \dots, n\}$, using the convexity and l.s.c properties of
 188 $u^i \mapsto \mathbb{E}(G_i(u^i))$ and Jensen's inequality, we get:

$$189 \quad (3.1) \quad \mathbb{E}(G_i(u^i)|W^i) \geq G_i(\mathbb{E}(u^i|W^i)).$$

190 On the other hand $(u^1, \dots, u^n) \mapsto F_0(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i))$ is invariant when taking the con-
 191 ditional expectation:

$$192 \quad (3.2) \quad F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i)\right) = F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(\mathbb{E}(u^i|W^i))\right).$$

193 From Assumption 3.2, we know that there exists a solution u^* to (P_2) .

194 We set $\tilde{u} := (\mathbb{E}(u^{*1}|W^1), \mathbb{E}(u^{*2}|W^2), \dots, \mathbb{E}(u^{*n}|W^n))$. For any i , $\tilde{u}^i :=$
 195 $\mathbb{E}(u^{*i}|W^i)$ is \mathcal{F}_i -adapted. Using the definition of u^* and (3.1), one can derive that:

$$196 \quad (3.3) \quad \inf_{u \in \mathcal{U}} \tilde{J}(u) = \tilde{J}(u^*) \geq \tilde{J}(\tilde{u}). \quad \square$$

197 Let $\hat{\mathcal{U}}$ be a subset of \mathcal{U} associated to decentralized controls, in the sense that:

$$198 \quad (3.4) \quad \hat{\mathcal{U}} := \{u \in \mathcal{U} \mid u^i \text{ is } \mathcal{F}^i\text{-adapted for all } i \in \{1, \dots, n\}\}$$

199 From Theorem 3.2, if Assumption 3.1 holds, then:

$$200 \quad (3.5) \quad \min_{u \in \hat{\mathcal{U}}} \tilde{J}(u) = \min_{u \in \mathcal{U}} \tilde{J}(u).$$

201

202 *Remark 3.3.* If Assumption 3.1 isn't satisfied, we can prove by same arguments
 203 that for any $\varepsilon > 0$ there exists an ε -optimal solution such that the individual controls
 204 are mutually independent.

205 LEMMA 3.4. If F_0 is Lipschitz with constant γ , then an optimal solution in $\hat{\mathcal{U}}$ of
 206 problem (P_2) is an ε -optimal solution in $\hat{\mathcal{U}}$ of problem (P_1) , with $\varepsilon = 2\gamma M \sqrt{T/n}$.

207 *Proof.* Indeed, there exists a number γ such that $\gamma > 0$ and for all $x, y \in H_1$ we
 208 have:

$$209 \quad (3.6) \quad |F_0(x) - F_0(y)| < \gamma \|x - y\|_{H_1}.$$

210 We set for any $u \in \mathcal{U}$:

$$211 \quad (3.7) \quad \hat{u}^i := u^i - \mathbb{E}(u^i).$$

212 Using the Jensen and Hölder inequalities, $(\mathbb{E}|Y|) \leq (\mathbb{E}|Y|^2)^{\frac{1}{2}}$, the fact that for any
 213 $j \neq i$, u_i and u_j are mutually independent, and that u^i is bounded by M , we have

214 $\forall u \in \hat{\mathcal{U}}:$
 (3.8)

$$\begin{aligned}
 |\mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) - F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) \right)| &\leq \mathbb{E} \left(\left| F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) - F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) \right| \right) \\
 &\leq \frac{\gamma}{n} \mathbb{E} \left(\left\| \sum_{i=1}^n \hat{u}^i \right\|_{L^2(0,T)} \right) \\
 &\leq \frac{\gamma}{n} \mathbb{E} \left(\left\| \sum_{i=1}^n \hat{u}^i \right\|_{L^2(0,T)}^2 \right)^{\frac{1}{2}} \\
 &= \frac{\gamma}{n} \left(\int_0^T \text{Var} \left(\sum_{i=1}^n u_t^i \right) dt \right)^{\frac{1}{2}} \\
 &\leq \frac{\gamma}{n^{\frac{1}{2}}} M \sqrt{T}.
 \end{aligned}$$

215
 216 Let \tilde{u}^* denote a minimizer of \tilde{J} on $\hat{\mathcal{U}}$, then using (3.8) for any $u \in \hat{\mathcal{U}}$ it holds:

$$217 \quad (3.9) \quad J(\tilde{u}^*) \leq \tilde{J}(\tilde{u}^*) + \frac{\gamma}{n^{\frac{1}{2}}} M \sqrt{T} \leq \tilde{J}(u) + \frac{\gamma}{n^{\frac{1}{2}}} M \sqrt{T} \leq J(u) + \frac{2\gamma}{n^{\frac{1}{2}}} M \sqrt{T}. \quad \square$$

218 *Assumption 3.5.* F_0 is Gâteaux differentiable with c -Lipschitz derivative.

219 **THEOREM 3.6.** *Suppose F_0 is convex, then the following ε -optimality results hold:*

- 220 (i) *For any $u \in \mathcal{U}$, $\tilde{J}(u) \leq J(u)$.*
 221 (ii) *Suppose Assumption 3.5 holds, then any optimal solution of problem (P_2) is*
 222 *an ε -optimal solution (where $\varepsilon = 2cTM^2/n$) of problem (P_1) .*

223 *Proof.* Proof of point (i).

224 By Jensen's inequality, we have that:

$$225 \quad (3.10) \quad F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) \leq \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) \right), \forall u \in \mathcal{U},$$

226 which gives the result.

227 Proof of point (ii).

228 Since F_0 is convex, differentiable, with a c -Lipschitz differential, one can derive for
 229 any $u \in \hat{\mathcal{U}}$ and a.s.:

$$\begin{aligned}
 &F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) - F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}[u^i] \right) \\
 &\leq \frac{1}{n} \langle \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right), \sum_{i=1}^n \hat{u}^i \rangle_{L^2(0,T)} \\
 230 \quad (3.11) \quad &= \frac{1}{n} \langle (\nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) - \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}[u^i] \right)), \sum_{i=1}^n \hat{u}^i \rangle_{L^2(0,T)} \\
 &\quad + \frac{1}{n} \langle \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right), \sum_{i=1}^n \hat{u}^i \rangle_{L^2(0,T)} \\
 &\leq \frac{c}{n^2} \left\| \sum_{i=1}^n \hat{u}^i \right\|_{L^2(0,T)}^2 + \frac{1}{n} \langle \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right), \sum_{i=1}^n \hat{u}^i \rangle_{L^2(0,T)},
 \end{aligned}$$

231 where \hat{u}^i is defined in (3.7). Taking the expectation of (3.11):

$$232 \quad (3.12) \quad \mathbb{E} \left(\left\langle \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}[u^i] \right), \sum_{i=1}^n \hat{u}^i \right\rangle_{H_1} \right) = 0,$$

233 and using the mutual independence of the controls and their boundedness we get as
234 in (3.8):

$$235 \quad (3.13) \quad \frac{c}{n^2} \mathbb{E}(\|\sum_{i=1}^n \hat{u}^i\|_{L^2(0,T)}^2) = \frac{c}{n^2} \int_0^T \sum_{i=1}^n \text{Var}(u_i^i) dt \leq \frac{c}{n} TM^2.$$

236 Let \tilde{u}^* denote a minimizer of \tilde{J} on $\tilde{\mathcal{U}}$, then using (3.5), (3.13) and (3.10), for any
237 $u' \in \mathcal{U}$ we have:

$$238 \quad (3.14) \quad J(\tilde{u}^*) \leq \tilde{J}(\tilde{u}^*) + \frac{c}{n} TM^2 \leq \tilde{J}(u') + \frac{c}{n} TM^2 \leq J(u') + \frac{2c}{n} TM^2.$$

239 Thus for $\varepsilon = 2cTM^2/n$, \tilde{u}^* constitutes an ε -optimal solution to the stochastic control
240 problem (P_1) . \square

241 **PROPOSITION 3.7.** *If F_0 is convex, then we have the following inequalities:*

$$242 \quad (3.15) \quad J(\tilde{u}) - \tilde{J}(\tilde{u}) \geq J(\tilde{u}) - J(u^*) \geq 0,$$

243 where \tilde{u} and u^* are respectively the optimal controls of problems (P_2) and (P_1) .

244 *Proof.* From Jensen inequality and by definition of \tilde{u} we have :

$$245 \quad (3.16) \quad J(u^*) \geq \tilde{J}(u^*) \geq \tilde{J}(\tilde{u}),$$

246 therefore from the two previous inequalities and adding $J(\tilde{u})$ we get (3.16). \square

247 *Remark 3.8.* An approximation scheme to compute \tilde{u} is provided in Section 5.
248 The practical interest of inequality (3.15) is that one can compute an upper bound
249 for the error $J(\tilde{u}) - J(u^*)$, that can be automatically derived from this approximation.

250 **4. Dualization and Decentralization of problem (P_2) .** From now on, in
251 addition to Assumptions 2.1 and 2.3, the assumption that F_0 is convex is in force in
252 the sequel. The problem (P_2) defined in (1.2) is dualized in order to decouple the
253 controls in this problem.

254

255 The optimization problem (P_2) is equivalent to:

$$256 \quad (4.1) \quad (P_3) \quad \begin{cases} \min_{u \in \mathcal{U}, v \in \mathcal{V}} \bar{J}(u, v), \\ \bar{J}(u, v) := F_0(v) + \frac{1}{n} \mathbb{E} \left(\sum_{i=1}^n F_i(u^i, X^{i, u^i}) \right), \\ \text{s.t. } g(u, v) = 0, \end{cases}$$

257 where $g(u, v) := \frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) - v$ and $\mathcal{V} := \{v \in L^2(0, T); |v(t)| \leq 2M \forall t \in [0, T]\}$.

258 The *Lagrangian* function associated with the constrained optimization problem (P_3)
259 is: $L : \mathcal{U} \times L^2(0, T) \times L^2(0, T) \rightarrow \mathbb{R}$ defined by:

$$260 \quad (4.2) \quad L(u, v, \lambda) := \bar{J}(u, v) + \langle \lambda, \frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) - v \rangle_{H_1}.$$

261 The dual problem (D) associated with (P_3) is:

$$262 \quad (4.3) \quad (D) \quad \max_{\lambda \in L^2(0, T)} \mathcal{W}(\lambda), \quad \text{where } \mathcal{W}(\lambda) := \min_{u \in \mathcal{U}, v \in \mathcal{V}} L(u, v, \lambda).$$

263 The problem is said to be qualified if it is still feasible after a small perturbation of
 264 the constraint, in the following sense:

$$265 \quad (4.4) \quad \text{There exists } \varepsilon > 0 \text{ such that } \mathcal{B}_{L^2(0,T)}(0, \varepsilon) \subset g(\mathcal{U}, \mathcal{V}),$$

266 where $\mathcal{B}_{L^2(0,T)}(0, \varepsilon)$ is the open ball of radius ε in $L^2(0, T)$ and g has been defined in
 267 (4.1).

268 LEMMA 4.1. *Problem (P_3) is qualified.*

269 *Proof.* Choose $\varepsilon := M$. Then

$$270 \quad (4.5) \quad \mathcal{B}_{L^2(0,T)}(0, \varepsilon) \subset \overline{\mathcal{B}_{L^2(0,T)}(0, 2M)} = g(0, \mathcal{V}) \subset g(\mathcal{U}, \mathcal{V}),$$

271 where $g(0, \mathcal{V})$ and $g(\mathcal{U}, \mathcal{V})$ are respectively the image by g of $\{0\} \times \mathcal{V}$ and $\mathcal{U} \times \mathcal{V}$. The
 272 conclusion follows. \square

273 By Assumption 2.3, Lemma 4.1 and the convexity of F_0 , the strong duality holds:

$$274 \quad (4.6) \quad \mathcal{W}(\lambda^*) = \tilde{J}(u^*),$$

275 where $\lambda^* \in \arg \max_{\lambda \in L^2(0,T)} \mathcal{W}(\lambda)$ and $u^* \in \arg \min_{u \in \mathcal{U}, v \in \mathcal{V}} L(\lambda^*, u, v)$.

276 Since the set of admissible controls $\mathcal{U} = \mathcal{U}_1 \times \dots \times \mathcal{U}_n$ is a Cartesian product
 277 and by strict convexity of $u^i \mapsto \mathbb{E}(F_i(u^i, X^{i,u^i}))$, each component u^{*i} can be uniquely
 278 determined by solving the following decentralized sub problem:

$$279 \quad (4.7) \quad u^{*i} = \arg \min_{u^i \in \mathcal{U}_i} \left\{ \mathbb{E} \left(F_i(u^i, X^{i,u^i}) + \langle \lambda^*, u^i \rangle_{L^2(0,T)} \right) \right\},$$

280 where $\hat{\mathcal{U}}_i := \{u \in \mathcal{U}_i \mid u^i \text{ is } \mathcal{F}^i\text{-adapted}\}$.

281 *Remark 4.2.* By using the same argument as in Theorem 3.2, one can prove that:

$$282 \quad (4.8) \quad \begin{aligned} & \min_{u^i \in \hat{\mathcal{U}}_i} \left\{ \mathbb{E} \left(F_i(u^i, X^{i,u^i}) + \langle \lambda^*, u^i \rangle_{L^2(0,T)} \right) \right\} \\ &= \min_{u^i \in \mathcal{U}_i} \left\{ \mathbb{E} \left(F_i(u^i, X^{i,u^i}) + \langle \lambda^*, u^i \rangle_{L^2(0,T)} \right) \right\}. \end{aligned}$$

283 5. Stochastic Uzawa and Sampled Stochastic Uzawa algorithms.

284 **5.1. Continuous time setting.** We recall that Assumptions 2.1 and 2.3 are in
 285 force, as well as convexity of F_0 .

286 This section aims at proposing an algorithm to find a solution of the dual problem
 287 (4.3).

288 For all $i \in \{1, \dots, n\}$, and $\lambda \in L^2(0, T)$, we define the optimal control $u^i(\lambda)$:

$$289 \quad (5.1) \quad u^i(\lambda) := \arg \min_{u^i \in \hat{\mathcal{U}}_i} \left\{ \mathbb{E} \left(F_i(u^i, X^{i,u^i}) + \langle \lambda, u^i \rangle_{L^2(0,T)} \right) \right\},$$

290 which is well defined since $u^i \mapsto \mathbb{E}(F_i(u^i, X^{i,u^i}))$ is strictly convex.

291 For any $\lambda \in L^2(0, T)$, the subset $V(\lambda)$ is defined by:

$$292 \quad (5.2) \quad V(\lambda) := \arg \min_{v \in \mathcal{V}} \{F_0(v) - \langle \lambda, v \rangle_{L^2(0,T)}\}.$$

293 Since F_0 is convex and \mathcal{V} is bounded, $V(\lambda)$ is a non empty subset of \mathcal{V} and is reduced
 294 to a singleton if F_0 is strictly convex.

295 For any $\lambda \in L^2(0, T)$, a function $v(\lambda)$, which is a selection of $V(\lambda)$, is associated.

296 Uzawa's algorithm seems particularly adapted for this problem. However at each
 297 dual iteration k and any $i \in \{1, \dots, n\}$, for the update of λ^{k+1} , one would have to
 298 compute the quantities $\mathbb{E}[u^i(\lambda^k)]$, which is hard in practice. Therefore two algorithms
 299 are proposed where at each iteration k , λ^{k+1} is updated thanks to a realization of
 300 $u^i(\lambda^k)$.

301 For any real valued function F defined on $L^2(0, T)$, F^* stands for its Fenchel
 302 conjugate.

303 LEMMA 5.1. *Assumption 3.5 holds iff F_0^* is proper and strongly convex.*

304 *Proof.* (i) Let Assumption 3.5 hold. Since F_0 is proper, convex and l.s.c., F_0^*
 305 is l.s.c. proper. From the Lipschitz property of the gradient of F_0 , it holds that
 306 $\text{dom}(F_0) = L^2(0, T)$.

307 Let $s, \tilde{s} \in \text{dom}(F_0^*)$ such that there exist $\lambda_s \in \partial F_0^*(s)$ and $\mu_{\tilde{s}} \in \partial F_0^*(\tilde{s})$. From
 308 the differentiability, l.s.c. and convexity of F_0 , it follows that:

$$309 \quad (5.3) \quad s = \nabla F_0(\lambda_s) \quad \text{and} \quad \tilde{s} = \nabla F_0(\mu_{\tilde{s}}).$$

310 By Assumption 3.5 and the extended Baillon-Haddad theorem [16, Theorem 3.1], ∇F_0
 311 is cocoervice. In other words:

$$312 \quad (5.4) \quad \begin{aligned} \langle s - \tilde{s}, \lambda_s - \mu_{\tilde{s}} \rangle_{L^2(0, T)} &= \langle \nabla F_0(\lambda_s) - \nabla F_0(\mu_{\tilde{s}}), \lambda_s - \mu_{\tilde{s}} \rangle_{L^2(0, T)} \\ &\geq \frac{1}{c} \|\nabla F_0(\lambda_s) - \nabla F_0(\mu_{\tilde{s}})\|_{L^2(0, T)}^2 \\ &= \frac{1}{c} \|s - \tilde{s}\|_{L^2(0, T)}^2. \end{aligned}$$

313 Therefore ∂F_0^* is strongly monotone, which implies the strong convexity of F_0^* .

314 (ii) Conversely, assume that F_0^* is proper and strongly convex. Then there exist
 315 $\alpha, \beta > 0$ such that for any $s \in \text{dom}(F_0^*)$:

$$316 \quad (5.5) \quad F_0^*(s) \geq \alpha \|s\|_{L^2(0, T)}^2 - \beta,$$

317 and F_0 being convex, l.s.c. and proper, for any $\lambda \in L^2(0, T)$ it holds:

$$318 \quad (5.6) \quad F_0(\lambda) \leq \sup_{s \in L^2(0, T)} \langle s, \lambda \rangle_{L^2(0, T)} - \alpha \|s\|_{L^2(0, T)}^2 + \beta = \|\lambda\|^2 / \alpha + \beta.$$

319 Thus F_0 is proper and uniformly upper bounded over bounded sets and therefore is
 320 locally Lipschitz. In addition, from the strong convexity of F_0^* and the convexity
 321 of F_0 , for any $\lambda \in L^2(0, T)$, $\partial F_0(\lambda)$ is a singleton. Thus F_0 is everywhere Gâteaux
 322 differentiable.

323 Let $\lambda, \mu \in L^2(0, T)$. Since F_0^* is strongly convex, the functions $F_0^*(s) - \langle \lambda, s \rangle_{L^2(0, T)}$
 324 (resp. $F_0^*(s) - \langle \mu, s \rangle_{L^2(0, T)}$) has a unique minimum point s_λ (resp. s_μ), characterized
 325 by:

$$326 \quad (5.7) \quad \lambda \in \partial F_0^*(s_\lambda) \quad \text{and} \quad \mu \in \partial F_0^*(s_\mu).$$

327 From the strong convexity of F_0^* , the strong monotonicity of ∂F_0^* holds:

$$328 \quad (5.8) \quad \langle \mu - \lambda, s_\mu - s_\lambda \rangle_{L^2(0, T)} \geq \frac{1}{c} \|s_\mu - s_\lambda\|_{L^2(0, T)}^2,$$

329 where $c > 0$ is a constant related to the strong convexity of F_0^* . Using that $s_\lambda =$
 330 $\nabla F_0(\lambda)$ and $s_\mu = \nabla F_0(\mu)$, it holds:

$$331 \quad (5.9) \quad \langle \mu - \lambda, \nabla F_0(\mu) - \nabla F_0(\lambda) \rangle_{L^2(0,T)} \geq \frac{1}{c} \|\nabla F_0(\mu) - \nabla F_0(\lambda)\|_{L^2(0,T)}^2,$$

332 meaning that ∇F_0 is cocoercive. Applying the Cauchy–Schwarz inequality to the left
 333 hand side of the previous inequality, the Lipschitz property of ∇F_0 follows. \square

334 **LEMMA 5.2.** *If Assumption 3.5 holds, then \mathcal{W} is strongly concave.*

335 *Proof.* For any $\lambda \in L^2(0, T)$, we have:

$$336 \quad (5.10) \quad \mathcal{W}(\lambda) = -(F_0^*(\lambda) + \frac{1}{n} \sum_{i=1}^n F_i^*(\lambda)),$$

337 where for any $i \in \{1, \dots, n\}$, F_i^* is convex and from Lemma 5.1 F_0^* is strongly convex. \square

338 We introduce the function $f : L^2(0, T) \rightarrow L^2(0, T)$ where for any $\lambda \in L^2(0, T)$:

$$339 \quad (5.11) \quad f(\lambda) := \frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i(\lambda)) - v(\lambda).$$

340 From the boundedness of \mathcal{U} and \mathcal{V} , it easily follows that there exists a finite positive
 341 real M_1 such that for any $\lambda \in L^2(0, T)$:

$$342 \quad (5.12) \quad \|f(\lambda)\|_{H_1}^2 \leq M_1.$$

343 For any $\lambda \in L^2(0, T)$, we denote by $\partial(-\mathcal{W}(\lambda))$ the subgradient of $-\mathcal{W}$ at λ . Therefore
 344 for any $\lambda \in L^2(0, T)$:

$$345 \quad (5.13) \quad \partial(-\mathcal{W}(\lambda)) \ni -f(\lambda).$$

346 The iterative algorithm, proposed as an approximation scheme for $\lambda^* \in \arg \max_{\lambda} \mathcal{W}(\lambda)$,
 347 is summarized in the *Stochastic Uzawa Algorithm 5.1*.

Algorithm 5.1 Stochastic Uzawa

- 1: Initialization $\lambda^0 \in L^2(0, T)$, set $\{\rho_k\}$ satisfying Assumption 5.4.
 - 2: $k \leftarrow 0$.
 - 3: **for** $k = 0, 1, \dots$ **do**
 - 4: $v^k \leftarrow v(\lambda^k)$ where $v(\lambda^k) \in V(\lambda^k)$, this set being defined in (5.2).
 - 5: $u^{i,k} \leftarrow u^i(\lambda^k)$ where $u^i(\lambda^k)$ is defined in (5.1) for any $i \in \{1, \dots, n\}$.
 - 6: Generate n independent realizations of Brownian motions $(W^{1,k+1}, \dots, W^{n,k+1})$, independent also with $\{W^{i,p} : 1 \leq i \leq n, p \leq k\}$.
 - 7: Compute the associated state realizations $(X^{1,u^1(\lambda^k)}, \dots, X^{n,u^n(\lambda^k)})$.
 - 8: $Y^{k+1} \leftarrow \frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(W^{i,k+1}) - v(\lambda^k)$.
 - 9: $\lambda^{k+1} \leftarrow \lambda^k + \rho_k Y^{k+1}$.
-

348 *Remark 5.3.* For the purpose of notation, $u^i(\lambda^k)(W^{i,k+1})$ in (8) corresponds to
 349 the realization of $u^i(\lambda^k)$ resulting from a realization of the Brownian $W^{i,k+1}$.

350 At any dual iteration k of Algorithm 5.1, Y^{k+1} is an estimator of
 351 $\mathbb{E}(\frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(W^{i,k+1}) - v(\lambda^k))$. Therefore an alternative approach proposed in the
 352 *Sampled Stochastic Uzawa* Algorithm 5.2 consists in performing less simulations at
 353 each iteration, by taking $m < n$, at the risk of performing more dual iterations, to
 estimate the quantity $\mathbb{E}(\frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(W^{i,k+1}) - v(\lambda^k))$.

Algorithm 5.2 Sampled Stochastic Uzawa

- 1: Initialization of m a positive integer and $\check{\lambda}^0 \in L^2(0, T)$, set $\{\rho_k\}$ satisfying Assumption 5.4.
 - 2: $k \leftarrow 0$.
 - 3: **for** $k = 0, 1, \dots$ **do**
 - 4: $v^k \leftarrow v(\check{\lambda}^k)$ where $v(\check{\lambda}^k) \in V(\check{\lambda}^k)$, this set being defined in (5.2).
 - 5: Generate m i.i.d discrete random variables I_1^k, \dots, I_m^k uniformly in $\{1, \dots, n\}$.
 - 6: $u^{I_j^k, k} \leftarrow u^{I_j^k}(\check{\lambda}^k)$ where $u^{I_j^k}(\check{\lambda}^k)$ is defined in (5.1) for any $j \in \{1, \dots, m\}$.
 - 7: Generate m independent realizations of Brownian motions $(W^{I_1^k, k+1}, \dots, W^{I_m^k, k+1})$, independent also with $\{W^{i,p} : 1 \leq i \leq m, p \leq k\}$.
 - 8: Compute the associated state realizations $(X^{I_1^k, u^{I_1^k}(\check{\lambda}^k)}, \dots, X^{I_m^k, u^{I_m^k}(\check{\lambda}^k)})$.
 - 9: $\check{Y}^{k+1} \leftarrow \frac{1}{m} \sum_{j=1}^m u^{I_j^k}(\check{\lambda}^k)(W^{I_j^k, k+1}) - v(\check{\lambda}^k)$
 - 10: $\check{\lambda}^{k+1} \leftarrow \check{\lambda}^k + \rho_k \check{Y}^{k+1}$.
-

354 The complexity of the *Stochastic Uzawa* Algorithm 5.2 is proportional to $m \times K$,
 355 where K is the total number of dual iterations and m the number of simulations
 356 performed at each iteration. The error $\mathbb{E}(\|\lambda^{k+1} - \lambda^*\|^2)$ for $\lambda^* \in S$ is the sum of
 357 the square of the bias (which only depends on K and not on m) and the variance
 358 (which both depends on K and m). Therefore this algorithm enables a bias variance
 359 trade-off for a given complexity. Similarly for a given error it enables to optimize the
 360 complexity of the algorithm.
 361

362 Some assumptions on the step size are introduced.

363 *Assumption 5.4.* The sequence $(\rho_k)_k$ is such that:

$$364 \quad (5.14) \quad \rho_k > 0, \quad \sum_{k=1}^{\infty} \rho_k = \infty, \quad \sum_{k=1}^{\infty} (\rho_k)^2 < \infty.$$

365 Note that a sequence of the form $\rho_k := \frac{a}{b+k}$, with $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+$, satisfies
 366 Assumption 5.4.

367 Let us denote $S := \arg \max_{\lambda \in L^2(0, T)} \mathcal{W}(\lambda)$, where S is nonempty because of the strong

368 convexity of \mathcal{W} .

369 The following result establishes the convergence of the *Stochastic Uzawa* Algo-
 370 rithm 5.1:

371 **THEOREM 5.5.** *Let Assumption 5.4 hold, then:*

372 (i) $\{\|\lambda^k - \lambda\|_{L^2(0, T)}^2\}$ converges a.s., for all $\lambda \in S$.

373 (ii) $\mathcal{W}(\lambda^k) \xrightarrow[k \rightarrow \infty]{} \max_{\lambda \in L^2(0, T)} \mathcal{W}(\lambda)$ a.s.

- 374 (iii) $\{\lambda^k\}$ weakly converges to some $\bar{\lambda} \in S$ in $L^2(0, T)$ a.s.
 375 (iv) If Assumption 3.5 holds, then a.s. $\{\lambda^k\}$ converges to $\bar{\lambda}$ in $L^2(0, T)$, with
 376 $S := \{\bar{\lambda}\}$.

377 Though the proof is similar to [6, Theorem 3.6], the current framework is different
 378 from the one of that reference, and for the convenience of the reader we provide the
 379 proof.

380 We first state two lemmas.

381 LEMMA 5.6 (Robbins-Siegmund). Let $\{\mathcal{G}_k\}$ be an increasing sequence of σ -
 382 algebra and d_k, a_k, b_k and c_k be nonnegative random variables adapted to \mathcal{G}_k . Assume
 383 that

$$384 \quad (5.15) \quad \mathbb{E}(d_{k+1}|\mathcal{G}_k) \leq d_k(1 + a_k) + b_k - c_k$$

385 and $\sum_{k=1}^{\infty} a_k < \infty$ a.s., $\sum_{k=1}^{\infty} b_k < \infty$ a.s. Then with probability one, $\{d_k\}$ is convergent

386 and it holds that $\sum_{k=1}^{\infty} c_k < \infty$.

387 *Proof.* See [5], Theorem 1.3.12. \square

388 LEMMA 5.7. Let $\{\alpha_k\}$ be a nonnegative deterministic sequence and $\{\beta_k\}$ a non-
 389 negative random sequence adapted to $\{\mathcal{G}_k\}$. Assume that $\sum_{k=1}^{\infty} \alpha_k = \infty$ a.s. and

390 $\mathbb{E}(\sum_{k=1}^{\infty} \alpha_k \beta_k) < \infty$ a.s. Moreover assume that $\beta_k - \mathbb{E}(\beta_{k+1}|\mathcal{G}_k) \leq c\alpha_k$ a.s. for all

391 k and some $c > 0$. Then $\beta_k \xrightarrow{a.s.} 0$.

392 *Proof.* See [6], Proposition 3.2. \square

393 *Proof of Theorem 5.5.* First consider point (i). Let $\lambda \in S$. For any k , \mathcal{G}_{k+1} is the
 394 filtration defined by:

$$395 \quad (5.16) \quad \mathcal{G}_{k+1} := \sigma(\{W^{i,p} : 1 \leq i \leq n, p \leq k+1\}).$$

396 Using the definition of $Y^{k+1} \in L^2(0, T)$ line 8 in the *Stochastic Uzawa Algorithm 5.1*,
 397 we have:

$$398 \quad (5.17) \quad \begin{aligned} \|\lambda^{k+1} - \lambda\|_{L^2(0,T)}^2 &= \|\lambda^k + \rho_k Y^{k+1} - \lambda\|_{L^2(0,T)}^2 \\ &= \|\lambda^k - \lambda\|_{L^2(0,T)}^2 + 2\rho_k \langle \lambda^k - \lambda, Y^{k+1} \rangle_{L^2(0,T)} \\ &\quad + (\rho_k)^2 \|Y^{k+1}\|_{L^2(0,T)}^2. \end{aligned}$$

399 Since Y^{k+1} is independent from \mathcal{G}^k , and using (5.12), it follows that:

$$400 \quad (5.18) \quad \mathbb{E}(\|Y^{k+1}\|_{L^2(0,T)}^2 | \mathcal{G}_k) = \mathbb{E} \left(\left\| \frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(W^{i,k+1}) - v(\lambda^k) \right\|_{L^2(0,T)}^2 \right) \leq M_1$$

401 Since λ^k is \mathcal{G}_k -measurable and that $\mathbb{E}[Y^{k+1}|\mathcal{G}_k] = f(\lambda^k)$, we have that:

$$\begin{aligned}
& \mathbb{E}[\|\lambda^{k+1} - \lambda\|_{L^2(0,T)}^2 | \mathcal{G}_k] \\
&= \|\lambda^k - \lambda\|_{L^2(0,T)}^2 + 2\rho_k \mathbb{E}(\langle \lambda^k - \lambda, Y^{k+1} \rangle | \mathcal{G}_k) + (\rho_k)^2 \mathbb{E}[\|Y^{k+1}\|_{L^2(0,T)}^2 | \mathcal{G}_k] \\
402 \quad (5.19) \quad & \leq \|\lambda^k - \lambda\|_{L^2(0,T)}^2 + 2\rho_k \langle \lambda^k - \lambda, f(\lambda^k) \rangle + (\rho_k)^2 M_1 \\
& \leq \|\lambda^k - \lambda\|_{L^2(0,T)}^2 + (\rho_k)^2 M_1 - 2\rho_k (\mathcal{W}(\lambda) - \mathcal{W}(\lambda^k)).
\end{aligned}$$

403 In the last inequality, we used the concavity of \mathcal{W} and (5.13). We set:

$$404 \quad (5.20) \quad a_k = 0, b_k = (\rho_k)^2 M_1, c_k = 2\rho_k (\mathcal{W}(\lambda) - \mathcal{W}(\lambda^k)),$$

405 We have that $\sum_{k=1}^{\infty} a_k < \infty$ a.s. and $\sum_{k=1}^{\infty} b_k < \infty$ a.s. Clearly, a_k and b_k are nonnegative;
406 c_k is nonnegative since $\lambda \in S$. By Lemma 5.6, the sequence $\{\|\lambda^k - \lambda\|_{L^2(0,T)}^2\}$ converges
407 a.s. Now we show point (ii) thanks to Lemma 5.7.
408 By Lemma 5.6:

$$409 \quad (5.21) \quad \sum_{k=1}^{\infty} \rho_k (\mathcal{W}(\lambda) - \mathcal{W}(\lambda^k)) < \infty \text{ a.s.}$$

410 Taking the expected value in both side of (5.19), we get, using the deterministic
411 version of Lemma 5.6 that

$$412 \quad (5.22) \quad \mathbb{E} \left(\sum_{k=1}^{\infty} \rho_k (\mathcal{W}(\lambda) - \mathcal{W}(\lambda^k)) \right) < \infty.$$

413 By concavity of \mathcal{W} and the Cauchy-Schwarz inequality, we have:

$$414 \quad (5.23) \quad \mathcal{W}(\lambda^{k+1}) - \mathcal{W}(\lambda^k) \leq \langle f(\lambda^k), \lambda^{k+1} - \lambda^k \rangle \leq \rho_k \|f(\lambda^k)\| \|Y^{k+1}\|.$$

415 Let τ_M be the stopping time $\tau_M := \inf\{k : \|\lambda^k\| > M\}$ for $M \in \mathbb{N}$.

416 The sequence $\{\beta_k\}$ is defined by:

$$417 \quad (5.24) \quad \beta_k := \begin{cases} \mathcal{W}(\lambda) - \mathcal{W}(\lambda^k) & \text{if } \{\tau_M > k\}, \\ \mathcal{W}(\lambda) - \mathcal{W}(\lambda^{\tau_M}) & \text{otherwise, with } \beta_{\tau_M+k} = \beta_{\tau_M}, k \geq 1. \end{cases}$$

418 Notice that if $\|\lambda^k\| \leq M$, there exists by (5.18) and (5.12) $M' > 0$ such that

$$419 \quad (5.25) \quad \|f(\lambda^k)\| \mathbb{E}(\|Y^{k+1}\| | \mathcal{G}_k) \leq (M')^2.$$

420 Now

$$421 \quad (5.26) \quad \beta_k - \beta_{k+1} = \mathbb{1}_{\tau_M > k} (\mathcal{W}(\lambda^{k+1}) - \mathcal{W}(\lambda^k)),$$

422 and therefore by taking the conditional expectation on both sides, noticing that $\mathbb{1}_{\tau_M > k}$
 423 is \mathcal{G}_k -measurable, and considering (5.23) and (5.25), we get

$$424 \quad (5.27) \quad \beta_k - \mathbb{E}(\beta_{k+1} | \mathcal{G}_k) \leq \rho_k (M')^2.$$

425 By Lemma 5.7, on the set $B_M := \{\tau_M = \infty\}$, β_k converges to 0 and coincides with
 426 $\mathcal{W}(\lambda) - \mathcal{W}(\lambda^k)$. Since $\|\lambda^k - \lambda\|$ converges a.s., $\|\lambda^k\|$ is bounded in probability and
 427 therefore the probability of the set B_M can be made arbitrarily close to 1 by choosing
 428 M large. Since $\mathbb{P}(\cup_{M=1}^{\infty} B_M) = 1$, we may infer that $\mathcal{W}(\lambda) - \mathcal{W}(\lambda^k)$ converges to 0
 429 almost surely.

430
 431 For point (ii), since $\{\|\lambda^k - \lambda\|^2\}$ converges a.s. for all $\lambda \in S$, it is bounded
 432 in probability, so the sequence $\{\lambda^k\}$ generated by the algorithm has a.s. a weak
 433 accumulation point $\bar{\lambda}$ (the point $\bar{\lambda}$ is random in general). Let $\{\lambda^{k_m}\}$ such that $\lambda^{k_m} \rightharpoonup$
 434 $\bar{\lambda}$. Since \mathcal{W} is concave and upper semi-continuous, it is weakly upper semi-continuous,

$$435 \quad (5.28) \quad \mathcal{W}(\bar{\lambda}) \geq \lim_{m \rightarrow \infty} \mathcal{W}(\lambda^{k_m}) = \mathcal{W}(\bar{\lambda}).$$

436 In particular, $\bar{\lambda} \in S$. To show uniqueness, let $\lambda_1, \lambda_2 \in S$ be two distinct weak limits
 437 of $\{\lambda^k\}$, i.e. $\lambda^{k_m} \rightharpoonup \lambda_1$ and $\lambda^{k_l} \rightharpoonup \lambda_2$. Then

$$438 \quad (5.29) \quad \|\lambda^{k_m} - \lambda_2\|^2 = \|\lambda^{k_m} - \lambda_1\|^2 + \|\lambda_1 - \lambda_2\|^2 + 2\langle \lambda^{k_m} - \lambda_1, \lambda_1 - \lambda_2 \rangle,$$

439

$$440 \quad (5.30) \quad \|\lambda^{k_l} - \lambda_1\|^2 = \|\lambda^{k_l} - \lambda_2\|^2 + \|\lambda_2 - \lambda_1\|^2 + 2\langle \lambda^{k_l} - \lambda_2, \lambda_2 - \lambda_1 \rangle,$$

441 so by weak convergence of each subsequence, (5.29) and (5.30) are combined to obtain

$$442 \quad (5.31) \quad \lim_{m \rightarrow \infty} \|\lambda^{k_m} - \lambda_2\|^2 - \|\lambda^{k_m} - \lambda_1\|^2 = \|\lambda_2 - \lambda_1\|^2,$$

$$443 \quad (5.32) \quad \lim_{l \rightarrow \infty} \|\lambda^{k_l} - \lambda_1\|^2 - \|\lambda^{k_l} - \lambda_2\|^2 = \|\lambda_2 - \lambda_1\|^2.$$

444 By a.s. convergence of the sequence $\{\|\lambda^k - \lambda\|^2\}$ for all $\lambda \in S$, the limit of each
 445 subsequence is equal to the limit of the entire sequence with probability one, so
 446 $\lim_{m \rightarrow \infty} \|\lambda^{k_m} - \lambda_1\|^2 = \lim_{k \rightarrow \infty} \|\lambda^k - \lambda_1\|^2 =: l_1$ and similarly $\lim_{m \rightarrow \infty} \|\lambda^{k_m} - \lambda_2\|^2 =$
 447 $\lim_{k \rightarrow \infty} \|\lambda^k - \lambda_2\|^2 =: l_2$. Therefore (5.31) and (5.32) imply

$$448 \quad (5.33) \quad l_2 - l_1 = \|\lambda_1 - \lambda_2\|^2 = l_1 - l_2,$$

449 meaning $\|\lambda_1 - \lambda_2\|^2 = 0$ and thus the weak limits coincide. Therefore $\{\lambda^k\}$ is weakly
 450 convergent to a unique limit with probability one.

451 Finally, the last statement can now be proved. By strong convexity, $-\mathcal{W}$ has an
 452 unique minimum $\bar{\lambda}$, so $S = \{\bar{\lambda}\}$. By strong convexity, there exists a $\mu > 0$ such that

$$453 \quad (5.34) \quad \mathcal{W}(\bar{\lambda}) - \mathcal{W}(\lambda^k) \geq -\langle f(\bar{\lambda}), \lambda^k - \bar{\lambda} \rangle_{L^2(0,T)} + \frac{\mu}{2} \|\lambda^k - \bar{\lambda}\|_{L^2(0,T)}^2.$$

454 Since $-\langle f(\bar{\lambda}), \lambda^k - \bar{\lambda} \rangle_{L^2(0,T)} > 0$, by optimality of $\bar{\lambda}$, $\lim_{k \rightarrow \infty} \mathcal{W}(\bar{\lambda}) - \mathcal{W}(\lambda^k) = 0$ a.s.
 455 implies $\lim_{k \rightarrow \infty} \|\lambda^k - \bar{\lambda}\|_{L^2(0,T)} = 0$ a.s. \square

456 We recall the definition of $\bar{J}(u, v)$ in (4.1) and we define \bar{u} :

$$457 \quad (5.35) \quad \bar{u} := \arg \min_{u \in \mathcal{U}} \left\{ \mathbb{E} \left(\sum_{i=1}^n F_i(u^i, X^{i, u^i}) + \langle \bar{\lambda}, u^i \rangle_{L^2(0, T)} \right) \right\},$$

458 If F_0 is strictly convex, then we define:

$$459 \quad (5.36) \quad \bar{v} := \arg \min_{v \in \mathcal{V}} \{ F_0(v) + \langle \bar{\lambda}, v \rangle_{L^2(0, T)} \}.$$

460 If Assumption 3.5 holds and F_0 is strictly convex, $(\bar{u}, \bar{v}, \bar{\lambda})$ is a saddle point and \bar{u} is
461 the unique minimizer of \bar{J} in \mathcal{U} .

462 **THEOREM 5.8.** *Let the Assumptions 3.5 and 5.4 hold, then we have:*

463 (i) $\{u(\lambda^k)\}$ weakly converges a.s. to \bar{u} .

464 If F_0 is strictly convex, then:

465 (ii) $\bar{J}(u(\lambda^k)) \xrightarrow[k \rightarrow \infty]{} \bar{J}(\bar{u})$ a.s.

466 (iii) $\limsup_{k \rightarrow \infty} J(u(\lambda^k)) \leq \inf_{u \in \mathcal{U}} J(u) + 2\varepsilon$ a.s. where ε is defined in Theorem 3.6.(ii).

467 *Proof.* Proof of point (i). Since the sequence $\{(u(\lambda^k), v(\lambda^k))\}$ is bounded in $\mathbb{U} \times$
468 $L^2(0, T)$, there exists a weakly convergent sub-sequence $\{(u(\lambda^{\theta_k}), v(\lambda^{\theta_k}))\}$ such that:

$$469 \quad (5.37) \quad (u(\lambda^{\theta_k}), v(\lambda^{\theta_k})) \rightharpoonup (u^\theta, v^\theta) \in \mathcal{U} \times \mathcal{V}.$$

470 Using the definition of $\lambda \mapsto u(\lambda)$ in (5.1), it holds for any $k > 0$:

$$471 \quad (5.38) \quad \begin{aligned} & \mathbb{E} \left(F_i(\bar{u}^i, X^{i, \bar{u}^i}) + \langle \lambda^{\theta_k}, \bar{u}^i \rangle_{L^2(0, T)} \right) \\ & \geq \mathbb{E} \left(F_i(u^i(\lambda^{\theta_k}), X^{i, u^i(\lambda^{\theta_k})}) + \langle \lambda^{\theta_k}, u^i(\lambda^{\theta_k}) \rangle_{L^2(0, T)} \right). \end{aligned}$$

472 Using that $u^i \mapsto F_i(u^i(\lambda^k), X^{i, u^i(\lambda^k)})$ is w.l.s.c. (see Remark 2.5) and the a.s. con-
473 vergence of $\{\lambda^k\}$, resulting from Theorem 5.5.(iv), we have from (5.38) when $k \rightarrow \infty$
474 :

$$475 \quad (5.39) \quad \mathbb{E} \left(F_i(\bar{u}^i, X^{i, \bar{u}^i}) + \langle \bar{\lambda}, \bar{u}^i \rangle_{L^2(0, T)} \right) \geq \mathbb{E} \left(F_i(u^{i, \theta}, X^{i, u^{i, \theta}}) + \langle \bar{\lambda}, u^{i, \theta} \rangle_{L^2(0, T)} \right).$$

476 Since \bar{u} is uniquely defined (see (5.35)), it follows $u^\theta = \bar{u}$ and (5.39) is an equality.
477 Using that every weakly convergent sub sequence of $\{u(\lambda^k)\}$ has the same weak limit
478 \bar{u} , (i) is deduced.

479 *Proof of point (ii).*

480 From point (i) and (5.39), it follows for any $i \in \{1, \dots, n\}$:

$$481 \quad (5.40) \quad \lim_{k \rightarrow \infty} \mathbb{E} \left(F_i(u^i(\lambda^k), X^{i, u^i(\lambda^k)}) \right) = \mathbb{E} \left(F_i(\bar{u}^i, X^{i, \bar{u}^i}) \right).$$

482 Using 5.37, the w.l.s.c of F_0 , equation (5.36), and applying the same previous argument
483 to $\{v(\lambda^{\theta_k})\}$, it holds that:

$$484 \quad (5.41) \quad \lim_{k \rightarrow \infty} F_0(v(\lambda^k)) - \langle \lambda^k, v(\lambda^k) \rangle_{L^2(0, T)} = F_0(\bar{v}) - \langle \bar{\lambda}, \bar{v} \rangle_{L^2(0, T)},$$

485 and

$$486 \quad (5.42) \quad v(\lambda^k) \rightharpoonup \bar{v}.$$

487 From the two previous equalities and the a.s. convergence of $\{\lambda^k\}$, it follows:

$$488 \quad (5.43) \quad \lim_{k \rightarrow \infty} F_0(v(\lambda^k)) = F_0(\bar{v}).$$

489 Using that $(\bar{u}, \bar{v}, \bar{\lambda})$ is a saddle point, it follows:

$$490 \quad (5.44) \quad \frac{1}{n} \sum_{i=1}^n \mathbb{E}(\bar{u}^i) = \bar{v}.$$

491 From (5.43) and (5.44), it holds:

$$492 \quad (5.45) \quad \lim_{k \rightarrow \infty} F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i(\lambda^k)) \right) = F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(\bar{u}^i) \right).$$

493 Then adding (5.40) and (5.45): $\lim_{k \rightarrow \infty} \tilde{J}(u(\lambda^k)) = \tilde{J}(\bar{u})$.

494 Proof of point (iii). From point (ii), inequality (3.14) and Theorem 3.6.(ii), it
495 holds:

$$496 \quad (5.46) \quad \limsup_{k \rightarrow \infty} J(u(\lambda^k)) \leq \limsup_{k \rightarrow \infty} \tilde{J}(u(\lambda^k)) + \varepsilon = \inf_{u \in \mathcal{U}} \tilde{J}(u) + \varepsilon \leq \inf_{u \in \mathcal{U}} J(u) + 2\varepsilon, \quad \square$$

497 where $\varepsilon = 2cTM^2/n$. The conclusion follows.

498 *Assumption 5.9.* (i) F_0 is strongly convex.

499 (ii) For any $i \in \{1, \dots, n\}$, the function $u^i \mapsto \mathbb{E}(G_i(u^i))$ is strongly convex.

500 LEMMA 5.10. *Let Assumption 5.9.(i) hold, then the function $\lambda \mapsto v(\lambda)$ is Lips-*
501 *chitz on $L^2(0, T)$.*

502 *Proof.* From the definition of v in (5.2), we have for any $\lambda \in L^2(0, T)$:

$$503 \quad (5.47) \quad \lambda \in \partial F_0(v(\lambda)).$$

504 Thus for any $\lambda, \mu \in L^2(0, T)$, we have from the strong convexity of F_0 :

$$505 \quad (5.48) \quad \begin{cases} F_0(v(\mu)) & \geq F_0(v(\lambda)) + \langle \lambda, v(\mu) - v(\lambda) \rangle_{L^2(0, T)} + \alpha \|v(\mu) - v(\lambda)\|_{L^2(0, T)}^2 \\ F_0(v(\lambda)) & \geq F_0(v(\mu)) + \langle \mu, v(\lambda) - v(\mu) \rangle_{L^2(0, T)} + \alpha \|v(\lambda) - v(\mu)\|_{L^2(0, T)}^2. \end{cases}$$

506 Adding the two previous inequalities, after simplifications, we get:

$$507 \quad (5.49) \quad \langle \lambda - \mu, v(\lambda) - v(\mu) \rangle_{L^2(0, T)} \geq 2\alpha \|v(\lambda) - v(\mu)\|_{L^2(0, T)}^2.$$

508 Applying Cauchy-Schwarz inequality and simplifying by $\|v(\lambda) - v(\mu)\|_{L^2(0, T)}$, we get
509 the desired Lipschitz inequality. \square

510 LEMMA 5.11. *Let Assumption 5.9.(ii) hold, thus the function $\lambda \mapsto u(\lambda)$ is Lips-*
511 *chitz on $L^2(0, T)$.*

512 *Proof.* The proof is similar to the proof of Lemma 5.10. \square

513 THEOREM 5.12. *Let the Assumption 3.5, 5.4, and 5.9 hold, then: $u(\lambda^k) \xrightarrow[k \rightarrow \infty]{} u(\bar{\lambda})$ a.s.*

515 *Proof.* The convergence follows from the Lipschitz property of $\lambda \mapsto u(\lambda)$ (as a
516 result of assumption 5.9) associated with the a.s. convergence of $\{\lambda^k\}$. \square

517 *Remark 5.13.* Note that Theorems 5.5, 5.8 and 5.12 still hold when replacing λ^k
 518 by $\check{\lambda}^k$ and Y^k by \check{Y}^k (defined resp. line 9 and 10 in the *Sampled Stochastic Uzawa*
 519 *Algorithm 5.2*). This can be proved by same argument, using that \check{Y}^k is bounded a.s.
 520 and $\mathbb{E}(\check{Y}^k | \check{\mathcal{G}}_k) = f(\check{\lambda}^k)$ for any k , where:

$$521 \quad (5.50) \quad \check{\mathcal{G}}_k = \sigma \left(\{W^{\ell,p}\} : 1 \leq \ell \leq m, p \leq k \right) \vee \sigma \left(\{I_\ell^p\} : 1 \leq \ell \leq m, p \leq k \right),$$

522 with $W^{\ell,p}$ and I_ℓ^k defined respectively at lines 7 and 5 of the *Sampled Stochastic*
 523 *Uzawa Algorithm 5.2*.

524 *Remark 5.14.* From a practical point of view, this algorithm can be implemented
 525 in a decentralized way, where the system operator sends the signal λ , which can
 526 be assimilated to a price, to the domestic appliances, which compute their optimal
 527 solution $u(\lambda)$, depending on their local parameters.

528 In (2.2), the states and controls of the agents are described in a continuous time
 529 setting with finite horizon. However all the previous results are easy to extend if
 530 we consider a discrete time setting with finite horizon, the proofs using the same
 531 arguments as in continuous time setting.

532 **5.2. Extension to the discrete time setting.** The results of the previous
 533 sections are extended to the discrete time setting in this subsection.

534 The following notations are used:

- 535 • Let $n \in \mathbb{N}^*$ be the number of agents and $T \in \mathbb{N}^*$ the finite time horizon.
- 536 • For any matrix M , M^\top denotes its transpose
- 537 • For any $i \in \{1, \dots, n\}$, $X^{i,u^i} := (x_0^i, \dots, x_T^i) \in \mathbb{R}^T$ is the state trajectory
 538 of agent i controlled by $u^i := (u_0^i, \dots, u_{T-1}^i) \in \mathbb{R}^T$. Similarly, for any $t \in$
 539 $\{0, \dots, T\}$ $X_t^u := (x_t^1, \dots, x_t^n) \in \mathbb{R}^n$ is the state vector of all the agents
 540 controlled by $u_j := (u_t^1, \dots, u_t^n) \in \mathbb{R}^n$. We have the following dynamics:

$$541 \quad (5.51) \quad \begin{cases} X_{t+1}^u &= AX_t^u + Bu_t + CW_{t+1}, \quad \text{for } t \in \{0, \dots, T-1\}, \\ X_0^u &= x_0 \in \mathbb{R}^n, \end{cases}$$

542 where A and B are diagonal matrices, C is a positive diagonal matrix of size
 543 n . The global noise process is a sequence of independent random variables
 544 (W_1, \dots, W_T) , where for any $t \in \{1, \dots, T\}$, W_t is a vector of centered, re-
 545 duced and independent Gaussian variables, defined on the probability space
 546 $(\Omega, \mathcal{F}, \mathbb{P})$:

$$547 \quad (5.52) \quad W_t := (W_t^1, \dots, W_t^n).$$

548 For any $i \in \{1, \dots, n\}$ and $t \in \{1, \dots, T\}$ we define $\mathcal{F}_t^i := \sigma(W_1^i, \dots, W_t^i)$.

- 549 • For any $i \in \{1, \dots, n\}$, we define $\mathcal{U}^i := \prod_{t=0}^{T-1} U_t^i$ the control space of agent i

550 where

$$551 \quad (5.53) \quad U_t^i := \{\alpha : \Omega \mapsto \mathbb{R} \mid \alpha \text{ is } \mathcal{F}_t^i \text{-measurable and } \alpha(\omega) \in [-M, M] \text{ } \mathbb{P}\text{-a.s.}\},$$

552 where $M > 0$. We finally set $\mathcal{U} := \prod_{i=1}^n \mathcal{U}^i$.

553 Now for any $n \in \mathbb{T}^*$ the optimization problems (P_1^d) and (P_2^d) can be clearly
 554 defined:

$$555 \quad (5.54) \quad (P_1^d) \left\{ \begin{array}{l} \inf_{u \in \mathcal{U}} J^d(u) \\ J^d(u) := \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) + \frac{1}{n} \sum_{i=1}^n F_i(u^i, X^{i,u^i}) \right), \end{array} \right.$$

556 and

$$557 \quad (5.55) \quad (P_2^d) \left\{ \begin{array}{l} \inf_{u \in \mathcal{U}} \tilde{J}^d(u) \\ \tilde{J}^d(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) + \frac{1}{n} \mathbb{E} \left(\sum_{i=1}^n F_i(u^i, X^{i,u^i}) \right), \end{array} \right.$$

558 where $F_0 : \mathbb{R}^T \rightarrow \bar{\mathbb{R}}$ and $F_i : \mathbb{R}^T \times \mathbb{R}^T \rightarrow \mathbb{R}$ are proper and lower semi continuous,
 559 and there exists $\hat{u} \in \mathcal{U}$ such that:

$$560 \quad (5.56) \quad \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n \hat{u}^i \right) \right) < \infty.$$

561 In addition we suppose that F_0 is convex and differentiable with c -Lipschitz derivative
 562 and for any $i \in \{1, \dots, n\}$, $u^i \mapsto \mathbb{E}(F_i(u^i, X^{i,u^i}))$ is strictly convex.

563 **COROLLARY 5.15.** (i) Problems (P_1^d) and (P_2^d) admit both a unique solution.
 564 (ii) Any optimal solution of problem (P_2^d) is an ε -optimal solution, where $\varepsilon =$
 565 $2cNM^2/n$, of problem (P_1^d) .

566 *Proof.* The proof of point (i) is the same as for the Lemma 2.6. Similarly, point
 567 (ii) is obtained by using the same proof of Theorem 3.6.(ii). \square

568 By adapting the *Stochastic Uzawa* (Algo 5.1) and the *Sampled Stochastic Uzawa*
 569 (Algo 5.2) to this discrete time setting, one can obtain similar results to Theorems
 570 5.5, 5.8 and 5.12.

571 **6. A numerical example: the LQG (Linear Quadratic Gaussian) prob-**
 572 **lem.** This sections aims at illustrating numerically the convergence of the *Stochastic*
 573 *Uzawa* (Algo 5.1) on a simple example. The algorithm speed of convergence is stud-
 574 ied, depending on the number of dual iterations and of agents. A linear quadratic
 575 formulation is considered, with n agents in a discrete setting problem (P_2^{LQG}) . We
 576 use the notations of Section 5.2.

577 This framework constitutes a simple test case, since the (deterministic) Uzawa's
 578 algorithm can be performed, and one can compare the resulting multiplier estimate
 579 with the one provided by the *Stochastic Uzawa* algorithm. Besides all the assump-
 580 tions required for the convergence of the *Stochastic Uzawa* (Algo 5.1) are satisfied
 581 for problem (P_2^{LQG}) . In addition the local problems (line 5 of this algorithm) can be
 582 resolved analytically.

583 Problem (P_2^{LQG}) is similar to (P_2^d) defined in (5.55), but in this specific case, the
 584 function F_0 is a quadratic function of the aggregate strategies of the agents

$$585 \quad (6.1) \quad F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) := \frac{\nu}{2} \sum_{t=0}^T \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u_t^i) - r_t \right)^2,$$

586 where $\nu > 0$, $\{r_t\}$ is a deterministic target sequence. Similarly, the cost functions F_i
587 of the agents is expressed in a quadratic form of its state X^{i,u^i} and control u^i .

$$588 \quad (6.2) \quad F_i(u^i, X^{i,u^i}) := \frac{1}{2} \left(\sum_{t=0}^T d_i (X_t^{i,u^i})^2 + q_i (u_t^i)^2 \right) + \frac{d_i^f}{2} (X_T^{i,u^i})^2,$$

589 where for any $i \in \{1, \dots, n\}$, $q_i > 0$ and $d_i > 0$. Defining the matrices $D =$
590 $\text{diag}(d_1, \dots, d_n)$, $Q = \text{diag}(q_1, \dots, q_n)$ and $D^f = \text{diag}(d_1^f, \dots, d_n^f)$, we get:

$$591 \quad (6.3) \quad \sum_{i=1}^n F_i(u^i, X^{i,u^i}) = \frac{1}{2} \left(\sum_{t=0}^T X_t^{u^\top} D X_t^u + u_t^\top Q u_t \right) + \frac{1}{2} X_T^{u^\top} D^f X_T^u.$$

592 Now the optimization problem (P_2^{LQG}) is clearly defined.

593 To find the optimal multiplier and control of (P_2^{LQG}) , the *Stochastic Uzawa* Al-
594 gorithm 5.1 is applied where in this specific case the lines 4 and 6 take respectively
595 the following form at any dual iteration k :

$$596 \quad (6.4) \quad u^i(\lambda^k) := \arg \min_{u^i \in \tilde{U}^i} \left\{ \mathbb{E} \left(\frac{1}{2} \left(\sum_{t=0}^T d_i (X_t^{i,u^i})^2 + q_i (u_t^i)^2 + \lambda_t^k u_t^i \right) + \frac{d_i^f}{2} (X_T^{i,u^i})^2 \right) \right\},$$

597

$$598 \quad (6.5) \quad v(\lambda^k) := \arg \min_{v \in \mathbb{R}^T} \left\{ \left(\sum_{t=0}^T \nu (v_t - r_t)^2 - \lambda_t^k v_t \right) \right\}.$$

599 The optimization problem (6.4) solved by each local agent is also in the LQG frame-
600 work. One can solve these problems using the results of [23]. The resolution via
601 Riccati equations of (6.4) shows that $u^i(\lambda^k)$ is a linear function of the state X^{i,u^i}
602 and of the price λ^k . Therefore, in this specific example, for any t one can explicitly
603 compute $\mathbb{E}(u_t^i(\lambda^k) | \mathcal{G}_k)$, where \mathcal{G}_k is defined in (5.16). It allows us to implement the
604 (deterministic) Uzawa's algorithm as a reference to evaluate the performances of the
605 *Stochastic Uzawa* algorithm.

606 Different population sizes n are considered, with n ranging between 1 and 10^4 .
607 Similarly the algorithm is stopped for different numbers of dual iteration k , ranging
608 between 1 and 10^4 . In order to evaluate the bias and variance of the *Stochastic Uzawa*
609 algorithm, we have performed $J = 1000$ runs of the *Stochastic Uzawa* algorithm.

610 For any n , given the strong convexity of the dual function associated with (P_2^{LQG}) ,
611 there exists a unique optimal multiplier $\bar{\lambda}^n$. For any n , $\lambda^{k,n,j}$ denotes the dual price
612 computed during the j^{th} simulations ($j = 1, \dots, J$) of the *Stochastic Uzawa* algorithm,
613 after k dual iterations.

614 For any n , the deterministic multiplier $\bar{\lambda}^n$ is obtained by applying Uzawa's al-
615 gorithm, after 10^4 dual iterations. To this end, we applied the *Stochastic Uzawa*
616 Algorithm 5.1 where we ignored the line 8 and we replaced the update of λ^k line 9
617 by:

$$618 \quad (6.6) \quad \bar{\lambda}^{k+1} \leftarrow \bar{\lambda}^k + \rho_k \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i(\bar{\lambda}^k)) - v(\bar{\lambda}^k) \right).$$

619 At each dual iteration k , the computation of $\mathbb{E}(u^i(\lambda^k))$ is easy in this specific case,
620 $u^i(\lambda^k)$ being a linear function of X^{i,u^i} and λ^k as explained in the previous subsection.

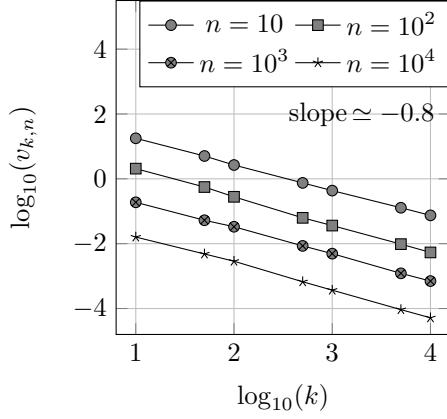


FIGURE 6.0.1. Variation of $v_{k,n}$ depending on the number of iterations k , for different number of agents n

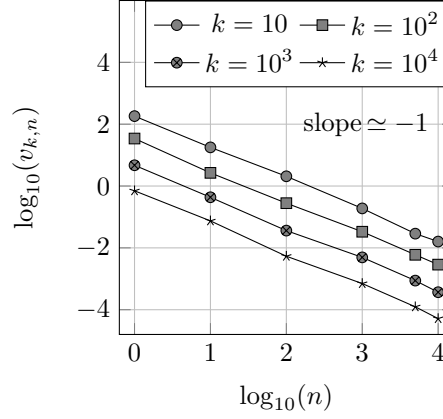


FIGURE 6.0.2. Variation of $v_{k,n}$ depending on the number of agents n , for different number of iterations k

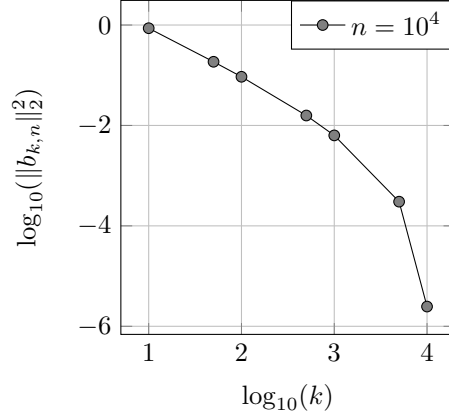


FIGURE 6.0.3. Variation of $\|b_{k,n}\|_2^2$ depending on the number of iterations k , for a fixed number of agents $n = 10^4$

621 The following results compare the multipliers $\lambda^{k,n,j}$ and $\bar{\lambda}^n$, obtained respectively
 622 by applying the *Stochastic Uzawa* and *Uzawa* algorithms.

623 For any k and n , $b_{k,n}$, $v_{k,n}$ and $\ell_{k,n}$ denotes respectively an estimation of the
 624 bias, the variance and the L2 norm of the error, via Monte Carlo method with J
 625 simulations. Thus we have for any k and n :

$$\begin{aligned}
 b_{k,n} &= \frac{1}{J} \sum_{j=1}^J \lambda^{k,n,j} - \bar{\lambda}^n, \\
 626 \quad (6.7) \quad v_{k,n} &= \frac{1}{J} \sum_{j=1}^J \|\lambda^{k,n,j} - \bar{\lambda}^n - b_{k,n}\|_2^2, \\
 \ell_{k,n} &= v_{k,n} + \|b_{k,n}\|_2^2.
 \end{aligned}$$

627 On Figure 6.0.1, we observe a behavior in $1/k^\alpha$ (with $\alpha \simeq 0.8$) of the variance

628 $v_{k,n}$ w.r.t. the number of iterations k . This rate of convergence is consistent with [5,
629 Theorem 2.2.12, Chapter 2] for Robbins Monro algorithm where the convergence is
630 proved to be of order at most in $1/k$.

631 On Figure 6.0.2 we observe a behavior in $1/n^\beta$ (with $\beta \simeq 1$) of the variance $v_{k,n}$
632 w.r.t. the number of agents n . This is expected, see [5, Theorem 2.2.12, Chapter 2]
633 and observing that the variance of Y^{k+1} is of order $1/n$ for any iteration k .

634 On Figure 6.0.3 we observe a faster behavior than $1/k$ of the bias $\|b_{k,n}\|^2$ w.r.t. the
635 number of iterations k . Thus for a large number of iterations ($k > 0$), the dominant
636 term impacting the error $l_{k,n}$ is the variance $v_{k,n}$.

637 **7. Price-based coordination of a large population of thermostatically**
638 **controlled loads.** The goal of this section is to demonstrate the applicability of the
639 presented approach for the coordination of thermostatic loads in a smart grid context.
640 The problem analyses the daily operation of a power system with a large penetration
641 of price-responsive demand, adopting a modelling framework similar to [4]. Two dis-
642 tinct elements are considered: i) a system operator, that must schedule a portfolio
643 of generation assets in order to satisfy the energy demand at a minimum cost, and
644 ii) a population of price-responsive loads (TCLs) that individually determine their
645 ON/OFF power consumption profile in response to energy prices with the objective
646 of minimizing their operating cost while fulfilling users' requirements. Note that the
647 operations of the two elements are interconnected, since the aggregate power consump-
648 tion of the TCLs will modify the demand profile that needs to be accommodated by
649 the system operator.

650 **7.1. Formulation of the problem.** In the considered problem, the function F_0
651 represents the minimized power production cost and corresponds to the resolution of
652 an Unit Commitment (UC) problem. The UC determines generation scheduling deci-
653 sions (in terms of energy production and frequency response (FR) provision) in order
654 to minimize the short term operating cost of the system while matching generation
655 and demand. The latter is the sum of an inflexible deterministic component (denoted
656 for any instant $t \in [0, T]$ by $\bar{D}(t)$) and of a stochastic part, which corresponds to the
657 total TCL demand profile $nU_{TCL}(t)$.

658 For simplicity, a Quadratic Programming (QP) formulation in a discrete time
659 setting is adopted for the UC problem. The central planner disposes of Z genera-
660 tion technologies (gas, nuclear, wind) and schedules their production and allocated
661 response by slot of 30 min every day. For any $j \in \{1, \dots, Z\}$ and $\ell \in \{1, \dots, 48\}$,
662 $H_j(t_\ell)$, $G_j(t_\ell)$ and $R_j(t_\ell)$ are respectively the commitment, the power production
663 and response [MWh] from unit j during the time interval $[t_\ell, t_{\ell+1}]$. The associated
664 vectors are denoted by $H(t_\ell) = [H_1(t_\ell), \dots, H_Z(t_\ell)]$, $G(t_\ell) = [G_1(t_\ell), \dots, G_Z(t_\ell)]$ and
665 $R(t_\ell) = [R_1(t_\ell), \dots, R_Z(t_\ell)]$.

666 The cost sustained at time t_ℓ by unit j is linear with respect to the commit-
667 ment $H_j(t_\ell)$ and quadratic with respect to generation $G_j(t_\ell)$ and can be expressed
668 as $c_{1,j}H_j(t_\ell)G_j^{Max}(t_\ell) + c_{2,j}G_j(t_\ell) + c_{3,j}G_j(t_\ell)^2$, with G_j^{Max} as the limit of produc-
669 tion allocated by each generation technology, $c_{1,j}$ [€/MWh] as no-load cost and $c_{2,j}$
670 [€/MWh] and $c_{3,j}$ [€/MW²h] as production cost of the generation technology j . The
671 optimization of F_0 must satisfy the following constraints:

$$672 \quad (7.1) \quad \sum_{j=1}^Z G_j(t_\ell) - \int_{t_\ell}^{t_{\ell+1}} (\bar{D}(t) + nU_{TCL}(t))dt = 0, \quad \forall \ell \in \{1, \dots, 48\},$$

673 (7.2) $0 \leq H_j(t_\ell) \leq 1, \quad \forall j \in \{1, \dots, Z\} \text{ and } \ell \in \{1, \dots, 48\},$

674 (7.3) $R_j(t_\ell) - r_j H_j(t_\ell) G_j^{max}(t_\ell) \leq 0, \quad \forall j \in \{1, \dots, Z\} \text{ and } \ell \in \{1, \dots, 48\},$

(7.4)
675 $R_j(t_\ell) - s_j (H_j(t_\ell) G_j^{max}(t_\ell) - G_j(t_\ell)) \leq 0, \quad \forall j \in \{1, \dots, Z\} \text{ and } \ell \in \{1, \dots, 48\},$

(7.5)
676 $\Delta G_L - \Lambda (\bar{D}(t_\ell) + n(\bar{U}_{TCL}(t_\ell) - \bar{R}_{TCL}(t_\ell))) \Delta f_{qss}^{max} - \hat{R}(t_\ell) \leq 0, \quad \forall \ell \in \{1, \dots, 48\}$

677 (7.6) $2\Delta G_L t_{ref} t_d - t_{ref}^2 \hat{R}(t_\ell) - 4\Delta f_{ref} t_d \hat{H}(t_\ell) \leq 0, \quad \forall \ell \in \{1, \dots, 48\}$

678 (7.7) $\bar{q}(t) - \hat{H}(t_\ell) \hat{R}(t_\ell) \leq 0 \quad \forall \ell \in \{1, \dots, 48\}$

679 (7.8) $\mu r_j H_j(t_\ell) G_j^{max}(t_\ell) - G_j(t_\ell) \leq 0, \quad \forall j \in \{1, \dots, Z\} \text{ and } \ell \in \{1, \dots, 48\}$

680 where (7.1) equals production and aggregated demand (i.e. the system inelastic
681 demand \bar{D} and the TCL flexible demand nU_{TCL}). The quantities \hat{R} and \hat{H} denote
682 the total reserve and inertia of the system, respectively, and are defined for any
683 $\ell \in \{1, \dots, 48\}$ as:

684 (7.9) $\hat{R}(t_\ell) = \sum_{j=1}^Z R_j(t_\ell) + nR_{TCL}(t_\ell) \quad \text{and} \quad \hat{H}(t_\ell) = \sum_{j=1}^Z \frac{h_j H_j(t_\ell) G_j^{max} - h_L \Delta G_L}{f_0}$

685 Assuming that for any generic generation technology j , the size of single plants
686 included in j is quite smaller than the aggregate installed capacity of j , inequality
687 (7.2) sets that commitment decisions can be extended to the fleet and expressed by
688 continuous variables $H_j(t_\ell) \in [0, 1]$.

689 The amount of response allocated by each generation technology is limited by
690 the headroom $r_j H_j(t_\ell) G_j^{max}(t_\ell)$ in (7.3) and the slope s_j linking the FR with the
691 dispatch level (7.4). Constraints (7.5) to (7.8) deal with frequency response provision
692 and R_{TCL} (the mean of FR allocated by TCLs). They guaranty secure frequency
693 deviations following sudden generation loss ΔG_L . Inequality (7.5) allocates enough
694 FR (with delivery time t_d) such that the quasi-steady-state frequency remains above
695 Δf_{qss}^{max} , with Λ accounting for the damping effect introduced by the loads [11]. Fi-
696 nally (7.7) constraints the maximum tolerable frequency deviation Δf_{nad} , following
697 the formulation and methodology presented in [22] and [24]. The rate of change of
698 frequency is taken into account in (7.6) where at t_{ref} the frequency deviation remains
699 above Δf_{ref} . Constraint (7.8) prevents trivial unrealistic solutions that may arise
700 in the proposed formulation, such as high values of committed generation $H_j(t_\ell)$ in

701 correspondence with low (even zero) generation dispatch $G_j(t_\ell)$. The reader can refer
702 to [4] for more details on the UC problem.

703 The solution of the UC problem, corresponding to the function F_0 , can be de-
704 scribed by the following optimization problem:

(7.10)

$$705 \quad F_0(U_{TCL}, R_{TCL}) := \min_{H,G,R} \sum_{\ell=1}^{48} \sum_{j=1}^Z c_{1,j} H_j(t_\ell) G_j^{max}(t_\ell) + c_{2,j} G_j(t_\ell) + c_{3,j} G_j(t_\ell)^2,$$

706 subject to equations (7.1)-(7.8).

707 Note that the formulation of the present problem does not fulfill all the assumption
708 presented in Sections 2 and 5. In particular, the function F_0 is not strictly convex, as
709 instead supposed in Theorem 5.8.(ii).(iii). Nevertheless, the numerical simulations of
710 Section 7.2 shows that the proposed approach is still able to achieve convergence.

711 Regarding the modelling of the individual price-responsive TCLs, each TCL
712 $i \in \{1, \dots, n\}$ is characterized at any time $t \in [0, T]$ by its temperature X_t^{i,u^i} [$^\circ C$]
713 controlled by its power consumption u_t^i [W]. The thermal dynamic X_t^{i,u^i} of a single
714 TCL i is given by:

$$715 \quad (7.11) \quad \begin{cases} dX_t^{i,u^i} &= -\frac{1}{\gamma_i} (X_t^{i,u^i} - X_{OFF}^i + \zeta_i u_t^i) dt + \sigma_i dW_t^i, \quad \text{for } t \in [0, T], \\ X_{0,u^i}^i &= x_0^i \in \mathbb{R}, \end{cases}$$

716 where:

- 717 • γ_i is its thermal time constant [s].
- 718 • X_{OFF}^i is the ambient temperature [$^\circ C$].
- 719 • ζ_i is the heat exchange parameter [$^\circ C/W$].
- 720 • σ_i is a positive constant [$(^\circ C)s^{\frac{1}{2}}$],
- 721 • W^i is a Brownian Motion [$s^{\frac{1}{2}}$], independent from W^j for any $j \neq i$.

722 For any $i \in \{1, \dots, n\}$, the set of control \mathcal{U}_i is defined by:

$$723 \quad (7.12) \quad \mathcal{U}_i := \{\nu \in H_i \text{ and } \nu_t(\omega) \in \{0, P_{ON,i}\} \text{ for a.a. } (t, \omega) \in [0, T] \times \Omega^i\}.$$

724 The TCLs dynamics in (7.11) have been derived according to [10], with the addition
725 of the stochastic term $\sigma_i dW_t^i$ to account for the influence of the environment (open-
726 ing/closing of the fridge, environment temperature etc) on the evolution of the TCL
727 temperature.

728 By combining the objective functions of the systems, the system operator has to
729 solve the following optimization problem:

(7.13)

$$730 \quad (P_1^{TCL}) \quad \begin{cases} \inf_{u \in \mathcal{U}} J(u) \\ J(u) := \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i, \frac{1}{n} \sum_{i=1}^n r_i(u^i, X^{i,u^i}) \right) \right) \\ \quad + \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n \int_0^T f_i(u_s^i, X_s^{i,u^i}) ds + \gamma_i (X_T^{i,u^i} - \bar{X}^i)^2 \right), \end{cases}$$

731 where, for any $i \in \{1, \dots, n\}$ and any $s \in [0, T]$:

- 732 • $r_i(u^i, X^{i,u^i})(s)$ is the maximum amount of FR allocated by TCL i at time s :

$$733 \quad (7.14) \quad r_i(u^i, X^{i,u^i})(s) := u_s^i \frac{X_s^{i,u^i} - X_{min}^i}{X_{max}^i - X_{min}^i}.$$

734 • $f_i(u_s^i, X_s^{i,u^i})$ is the individual discomfort term of the TCL i at time s :
 (7.15)

735
$$f_i(u_s^i, X_s^{i,u^i}) := \alpha_i (X_s^{i,u^i} - \bar{X}^i)^2 + \beta_i ((X_{\min}^i - X_s^{i,u^i})_+^2 + (X_s^{i,u^i} - X_{\max}^i)_+^2),$$

736 where:

737 – $\alpha_i (X_s^{i,u^i} - \bar{X}^i)^2$ is a discomfort term penalizing temperature deviation
 738 from some comfort target \bar{X} [$^{\circ}\text{C}$], with α_i a discomfort term parameter
 739 [$\text{£}/h(^{\circ}\text{C})^2$].

740 – $\beta_i ((X_s^{i,u^i} - X_{\min}^i)_+^2 + (X_{\max}^i - X_s^{i,u^i})_+^2)$ is a penalization term to keep
 741 the temperature in the interval $[X_{\min}^i, X_{\max}^i]$, with β_i a target term pa-
 742 rameter [$\text{£}/s(^{\circ}\text{C})^2$] and for any $x \in \mathbb{R}$, $(a)_+ = \max(0, a)$.

743 • $\gamma_i (X_T^{i,u^i} - \bar{X}^i)^2$ is a terminal cost imposing periodic constraints, with γ a
 744 target term parameter [$\text{£}/s(^{\circ}\text{C})^2$].

745 Note that the control set \mathcal{U} is not convex. We can mention a possible relaxation
 746 of the problem by taking the control in the interval $[0, P_{ON,i}]$.

747 The modified problem (P_2^{TCL}) is studied to solve (P_1^{TCL}) .
 (7.16)

748
$$(P_2^{TCL}) \left\{ \begin{array}{l} \inf_{u \in \mathcal{U}} \tilde{J}(u) \\ \tilde{J}(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i), \frac{1}{n} \sum_{i=1}^n \mathbb{E}(r_i(u^i, X^{i,u^i})) \right) \\ + \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n \int_0^T f_i(u_s^i, X_s^{i,u^i}) ds + \gamma_i (X_T^{i,u^i} - \bar{X}^i)^2 \right). \end{array} \right.$$

749 **7.2. Decentralized implementation.** The *Sampled Stochastic Uzawa* Algo-
 750 rithm 5.2 is applied to solve (P_2^{TCL}) , with $m = 317$ simulations per iteration. At each
 751 iteration k , the lines 4 and 6 correspond respectively to the solution of a deterministic
 752 UC problem and of an Hamilton Jacobi Bellman (HJB) equation. The time steps
 753 $\Delta t = 7.6$ s and temperature steps $\Delta T = 0.15^{\circ}\text{C}$ are chosen for the discretization of
 754 the HJB equation. Let us note that at line 6, each TCL solves its own local problem
 755 on the basis of the received price signal $\lambda^k = (p^k, \rho^k)$:

756 (7.17)
$$\inf_{u^i \in \mathcal{U}_i} \int_0^T f_i(u_s^i, X_s^{i,u^i}) + u_s^i p_s^k - r_i(u^i, X^{i,u^i})(s) \rho_s^k ds,$$

757 where $f_i(u_s^i, X_s^{i,u^i})$ is a discomfort term defined in (7.15), $u_s^i p_s^k$ can be interpreted
 758 as consumption cost and $r_i(u^i, X^{i,u^i})(s) \rho_s^k$ as fee awarded for FR provision. This
 759 implementation has a practical sense: each TCL uses local information and a price
 760 that is communicated to them to schedule its power consumption on the time interval
 761 $[0, T]$. It follows that, with the proposed approach, it is possible to optimize the overall
 762 system costs in (P_1^{TCL}) in a distributed manner, with each TCL acting independently
 763 and pursuing the minimization of its own costs.

764 **7.3. Results.** The generation technologies available in the system are nuclear,
 765 combined cycle gas turbines (CCGT), open cycle gas turbines (OCGT) and wind.
 766 The characteristics and parameters of the UC in this simulation are the same as in
 767 [4].

768 It is assumed that a population of $n = 2 \times 10^7$ fridges with built-in freeze compart-
 769 ment operates in the system according to the proposed price-based control scheme.

770 For any agent i we set the consumption parameter $P_{ON,i} = 180W$. The values of the
 771 TCL dynamic parameters γ_i and X_{OFF}^i of (7.11) are equal to the ones taken in [4].
 772 Note that it is possible to take a population of heterogeneous TCLs with different
 773 parameter values. The initial temperature are picked randomly uniformly between
 774 -21°C and -14°C . For any agent i , the parameters of the individual cost function f_i ,
 775 defined in (7.15), are: $\alpha_i = 0.2 \times 10^{-4} \text{ £/s}(\text{°C})^2$, $\beta_i = 50 \text{ £/s}(\text{°C})^2$, $\bar{X}^i = -17.5^\circ\text{C}$ and
 776 $X_{max} = -14^\circ\text{C}$, $X_{min} = -21^\circ\text{C}$. The parameter β_i is taken intentionally very large
 777 to make the temperature stay in the interval $[X_{max}^i, X_{min}^i]$. Note that the individual
 778 problems solved by the TCLs are distinct than the ones in [4] (different terms and
 779 parameters).

780 Simulations are performed for different values of volatility $\sigma_i := 0, 1, 2$ (all the
 781 TCLs have the same volatility in the simulations), where σ_i is defined in (7.11).
 782 The *Sampled Stochastic Uzawa* Algorithm is stopped after 75 iterations or when the
 783 relative variation $2\|\lambda^{k+1} - \lambda^k\|_2^2 / \|\lambda^{k+1} + \lambda^k\|_2^2$ between two successive prices λ^k and
 784 λ^{k+1} is less than 10^{-4} .

785 The resulting profile of total power consumption nU_{TCL} and total allocated re-
 786 sponse nR_{TCL} by the TCLs population, at different values of σ , are reported in Figure
 787 7.3.1. The electricity prices p and response availability prices ρ are shown in Figure
 788 7.3.2. As observed in [4], the total consumption nU_{TCL} is higher when the price p
 789 is lower and inversely the total allocated response nR_{TCL} is higher when the price
 790 signal ρ is also higher. This can be observed during the first hours of the day, be-
 791 tween $t = 1\text{h}$ and $t = 5\text{h}$. The power U_{TCL} then oscillates during the day in order to
 792 maintain feasible levels of the internal temperature of the TCLs. Though the prices
 793 seem not to be sensitive to the values taken by σ , the average consumption U_{TCL}
 794 and response R_{TCL} are highly correlated to the volatility of the temperature of the
 795 TCLs. The less noisy their temperature are, the more price sensitive and flexible their
 796 consumption profiles are.

797 The TCLs impact on system commitment decisions and consequent energy/FR
 798 dispatch levels is also analyzed and displayed in Figure 7.3.3 and 7.3.4, comparing the
 799 case when TCL flexibility is enabled (with different volatilities) with a no-flexibility
 800 scenario. In the no-flexibility scenario we impose $R_{TCL}(t) = 0$ and we consider that
 801 the TCLs operate exclusively according to their internal temperature X^{i,u^i} . They
 802 switch ON ($u^i(t) = P_{ON,i}$) when they reach their maximum feasible temperature
 803 X_{max}^i and they switch back OFF again ($u^i(t) = 0$) when they reach the minimum
 804 temperature X_{min}^i . Figure 7.3.8 shows the evolution across time of the internal tem-
 805 perature of two TCLs in the no-flexibility scenario. As explained in [4], the TCLs
 806 demand is much smaller than inelastic demand and its flexibility across the day is
 807 limited. The most noticeable variations occur between $t = 0\text{h}$ and $t = 6\text{h}$, where
 808 without TCL support, the optimal solution envisages a further curtailment of wind
 809 output in favor of an increase in OCGT and CCGT generation, as wind does not
 810 provide FR. As expected, the influence of the TCL on the system is larger when the
 811 temperature volatility is lower.

812

813

814

815

816 The system costs (i.e. UC solution) obtained with the coordination framework
 817 (CF) are now compared with the Business-as-usual (BAU) framework ones (the TCLs
 818 do not exploit their flexibility and they operate exclusively according to their internal

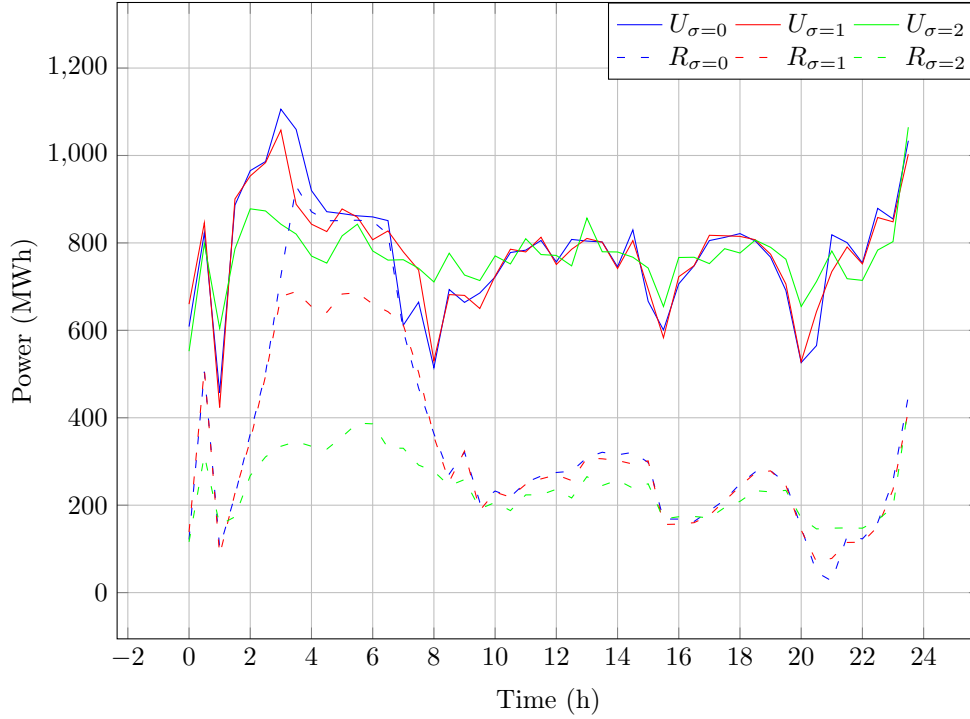


FIGURE 7.3.1. Total power consumption U (solid lines) and allocated response R of TCLs (dashed lines) at different volatility σ after 75 iterations of the algorithm

	$\sigma = 0$	$\sigma = 1$	$\sigma = 2$
BAU	2.770×10^7	2.770×10^7	2.772×10^7
CF	2.719×10^7	2.725×10^7	2.740×10^7

TABLE 1
Minimized system costs in (£)

819 temperature as previously explained) in Tab. 1. As expected the costs are lower in the
 820 CF where TCLs participate in reducing the system generation costs. The reduction
 821 is higher for $\sigma = 0$, where the reduction is about 1.9%, than for $\sigma = 1$ or $\sigma = 2$, where
 822 the the reduction is respectively about 1.6% and 1.2%. This relies on the tendency of
 823 the TCLs to be more flexible when their volatility is low. The reduction observed in
 824 the CF scenario is due to the smaller use of OCGT and CCGT generation technologies
 825 for the benefit of wind.

826 The behaviour of the single loads is now analyzed in Figures 7.3.5, 7.3.6 and 7.3.7.
 827 These figures show for different volatility the evolution across time of the internal
 828 temperature X^i of 5 individual TCLs when the optimal power schedule is applied. As
 829 desired, in each case the temperature ranges between -21°C and -14°C . The more the
 830 volatility is high, the more the temperature profile is noisy and bounded in a small
 831 interval.

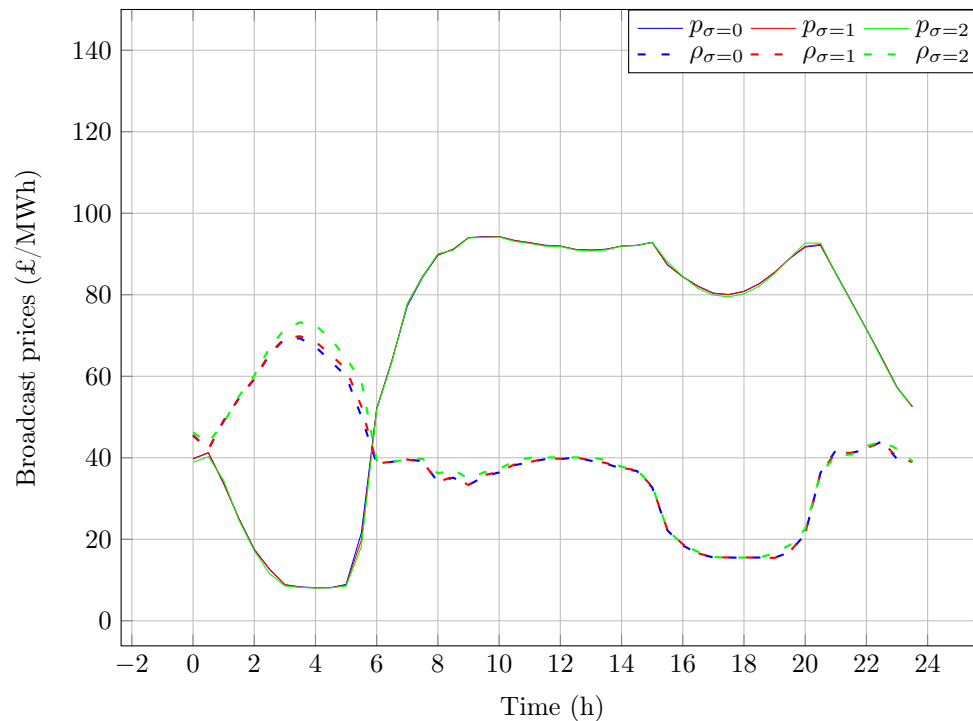


FIGURE 7.3.2. Electricity price p (solid lines) and response availability price ρ (dashed lines) at different volatility σ after 75 iterations of the algorithm

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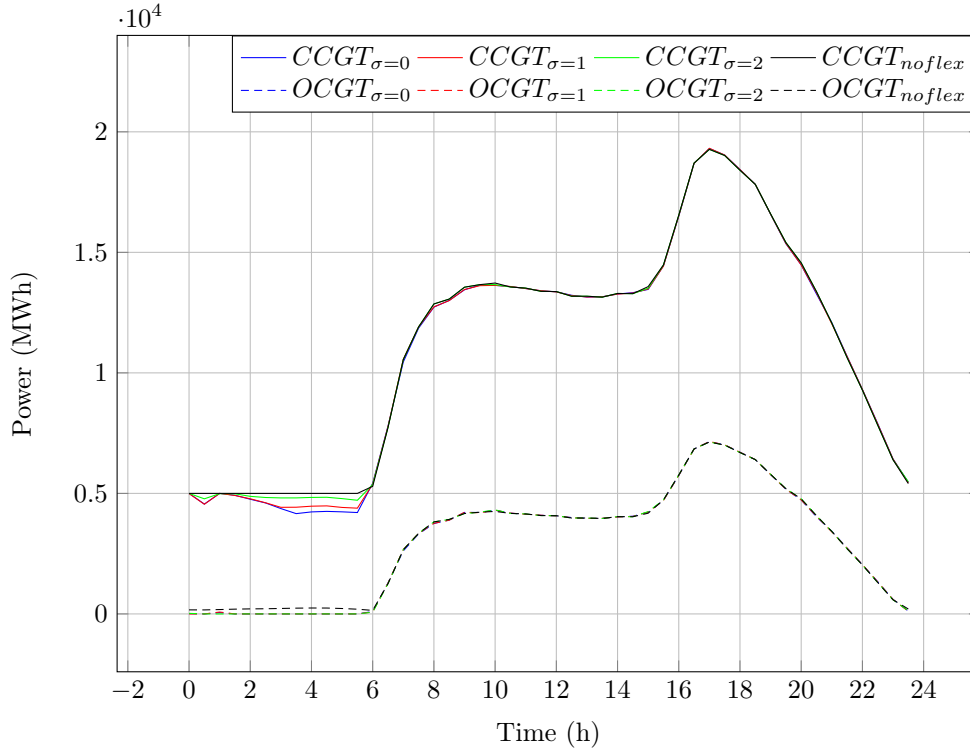


FIGURE 7.3.3. Breakdown of frequency response allocated by generators at different volatility σ and for no-flexibility scenario

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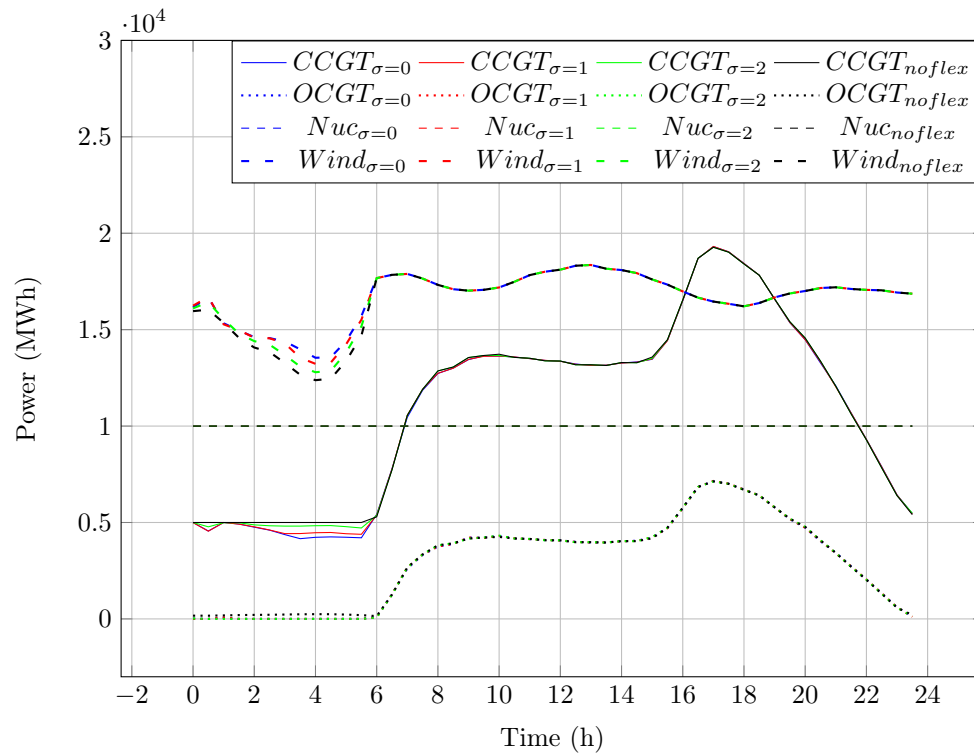


FIGURE 7.3.4. Generation profiles at different volatility σ and for no-flexibility scenario

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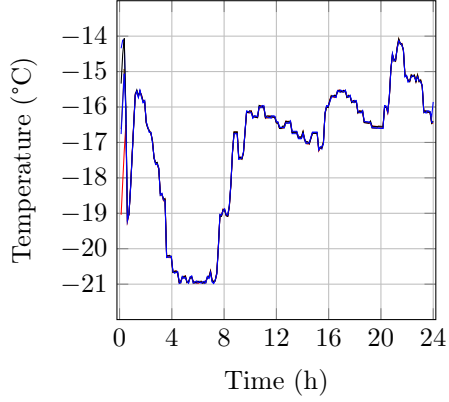


FIGURE 7.3.5. Internal temperature X^i of 5 individual TCLs when the optimal power schedule u^i is applied and $\sigma = 0$

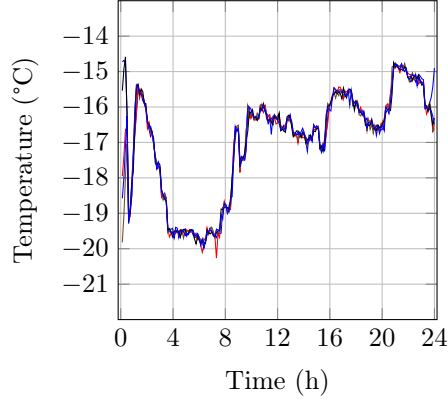


FIGURE 7.3.6. Internal temperature X^i of 5 individual TCLs when the optimal power schedule u^i is applied and $\sigma = 1$

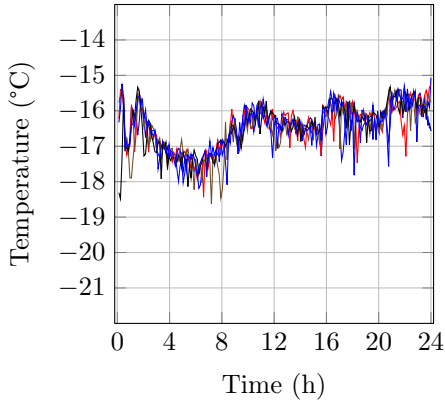


FIGURE 7.3.7. Internal temperature X^i of 5 individual TCLs when the optimal power schedule u^i is applied and $\sigma = 2$

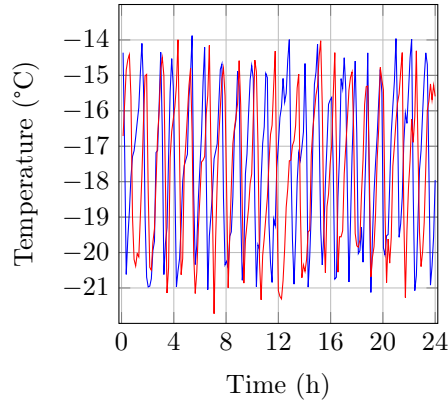


FIGURE 7.3.8. Internal temperature X^i of 2 individual TCLs when $\sigma = 2$ in the BAU framework