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# Unexpected hypersurfaces with multiple fat points

Justyna Szpond

June 20, 2020

## Abstract

Starting with the ground-breaking work of Cook II, Harbourne, Migliore and Nagel, there has been a lot of interest in unexpected hypersurfaces. In the last couple of months a considerable number of new examples and new phenomena has been observed and reported on. All examples studied so far had just one fat point. In this note we introduce a new series of examples, which establishes for the first time the existence of unexpected hypersurfaces with multiple fat points. The key underlying idea is to study Fermat-type configurations of points in projective spaces.

## 1 Introduction

In the ground-breaking paper [4] the authors introduced the notion of unexpected curves.

**Definition 1.** We say that a reduced finite set of points  $Z \subset \mathbb{P}^2$  admits an unexpected curve of degree  $j + 1$  if there is an integer  $j > 0$  such that, for a general point  $P$ , the fat point scheme  $jP$  associated to the  $j$ -th power of the ideal of  $P$ , fails to impose the expected number of conditions on the linear system of curves of degree  $j + 1$  containing  $Z$ . That is,  $Z$  admits an unexpected curve of degree  $j + 1$  if

$$h^0(\mathcal{J}_{Z+jP}(j+1)) > \max \left\{ h^0(\mathcal{J}_Z(j+1)) - \binom{j+1}{2}, 0 \right\}.$$

This notion has been generalized to hypersurfaces in projective spaces of arbitrary dimension in the subsequent paper [7].

**Definition 2.** We say that a reduced set of points  $Z \subset \mathbb{P}^N$  admits an unexpected hypersurface of degree  $d$  for multiplicities  $m_1, \dots, m_s$ , if for general points  $P_1, \dots, P_s$  the zero-dimensional subscheme  $m_1P_1 + \dots + m_sP_s$  fails to impose the expected number of conditions on forms of degree  $d$  vanishing along  $Z$ .

A well known example of this kind is provided by 7 general double points in  $\mathbb{P}^4$ . There is a twisted quartic curve passing through these 7 points and its secant variety is a threefold of degree 3 vanishing to order 2 along the quartic curve, in particular in the 7 general points. This threefold is unexpected, see [1].

This example has two deficiencies. First, the singular points are not isolated. Second, the set  $Z$  is empty.

The first example of an unexpected surface  $Q_R$  in  $\mathbb{P}^3$  admitted by a non-empty set  $Z$  has been discovered and described in [2, Theorem 1]. In that example there is an isolated point  $R$

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of multiplicity 3 which is a general point. The surface  $Q_R$  has 4 additional double points, whose coordinates depend on  $R$ , and no other singularities. In the present note we study the following question.

**Problem.** *Does there exist a non-empty set  $Z$  in some projective space  $\mathbb{P}^N$ , which admits an unexpected hypersurface with at least 2 general fat points?*

We show that this actually happens. We have the following result in  $\mathbb{P}^5$ .

**Main Theorem.** *There exists a non-empty set  $Z$  in  $\mathbb{P}^5$  which admits an unexpected hypersurface of degree 4 for multiplicities 3 and 2. This hypersurface has only isolated singularities.*

We prove the Main Theorem in Section 4. In general, we have the following conjecture.

**Conjecture.** *For each  $N = 2k + 1$ ,  $k \geq 2$ , there exists a non-empty set  $Z$  in  $\mathbb{P}^N$  which admits an unexpected hypersurface of degree 4 for multiplicities  $3, \underbrace{2, \dots, 2}_{k-1}$ .*

Main Theorem verifies the Conjecture for  $N = 5$ . We were able to verify it by computer for  $N = 7$  and  $N = 9$ . Since we give an explicit equation of the unexpected hypersurface, the computer verification is restricted only by the capacity of computer hardware and software.

## 2 Fermat-type point configurations

By the way of warm-up we begin with points in  $\mathbb{P}^2$ . In Theorem 5 we show a peculiar family of unexpected curves, whose degree grows while the multiplicity in the general point remains fixed, equal 4. Passing then to the general case of  $\mathbb{P}^N$  we identify generators of the ideal of Fermat-type configuration of points. This is crucial for the subsequent part.

### 2.1 Fermat point configurations in $\mathbb{P}^2$

Let  $n \geq 1$  be a positive integer and let  $I_{2,n}$  be the complete intersection ideal in  $\mathbb{C}[x, y, z]$ , viewed as the graded ring of  $\mathbb{P}^2$ , generated by

$$x^n - y^n, y^n - z^n.$$

Let  $F_{2,n}$  be the ideal

$$F_{2,n} = I_{2,n} \cap (x, y) \cap (x, z) \cap (y, z).$$

The support of  $I_{2,n}$  is the set of  $n^2$  points

$$(1 : \varepsilon^\alpha : \varepsilon^\beta)$$

where  $\varepsilon$  is a primitive root of unity of order  $n$  and  $1 \leq \alpha, \beta \leq n$ . The support of  $F_{2,n}$  is the union of the support of  $I_{2,n}$  and the 3 coordinate points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$  and  $(0 : 0 : 1)$ .

The ideals  $F_{2,n}$  have appeared recently in various guises, most notably in the context of the containment problem, see [11] for an introduction to this circle of ideas and [9], [8] for specific applications. Their algebraic properties have been studied in depth by Nagel and Secleanu [10].

For  $n = 3$  the set of points defined by  $F_{2,3}$  is the set of singular points (i.e. points where two or more arrangement lines meet) of the dual Hesse arrangement of 9 lines with 12 triple points at the zeroes of  $F_{2,3}$ . This arrangement is depicted in Figure 1. The lines are indicated by curved segments as it is not possible to embed this configuration in the real projective plane due to the celebrated Sylvester-Gallai Theorem, see [3]. It has been proved already in [6, Lemma 2.1] that the ideal  $F_{2,3}$  is generated in degree 4. More precisely we have

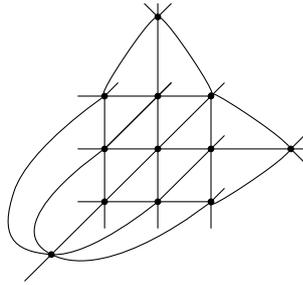


Figure 1: Dual Hesse arrangement

**Lemma 3.** *The ideal  $F_{2,3}$  is generated by the binomials*

$$x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3).$$

In fact exactly the same proof works for any  $n \geq 3$ , so that we have

**Lemma 4.** *The ideal  $F_{2,n}$  is generated by the binomials*

$$x(y^n - z^n), y(z^n - x^n), z(x^n - y^n).$$

This ideal is supported on  $n^2$  points, which form a complete intersection, and the 3 coordinate points. It turns out that this set of points admits an unexpected curve (Definition 1) for any  $n \geq 3$ .

**Theorem 5** (Unexpected curves with a point of multiplicity 4). *Let  $Z$  be the configuration of points in  $\mathbb{P}^2$  defined by  $F_{2,n}$  for  $n \geq 3$ . Let  $R = (a : b : c)$  be a general point in  $\mathbb{P}^2$ . With  $u = \binom{n}{2} - 1$ ,  $v = \binom{n-1}{2}$ ,  $w = \binom{n+1}{2}$ , the polynomial*

$$\begin{aligned} Q_P(x : y : z) = & - cxy((ub^n + vc^n)(z^n - x^n) + (ua^n + vc^n)(y^n - z^n)) \\ & - bxz((ua^n + vb^n)(y^n - z^n) + (uc^n + vb^n)(x^n - y^n)) \\ & - ayz((ub^n + va^n)(z^n - x^n) + (uc^n + va^n)(x^n - y^n)) \\ & + wa^{n-1}bcx^2(y^n - z^n) + wab^{n-1}cy^2(z^n - x^n) \\ & + wabc^{n-1}z^2(x^n - y^n) \end{aligned} \quad (1)$$

- vanishes at all points of  $Z$ ,
- vanishes to order 4 at  $R$ ,
- defines an unexpected curve of degree  $n + 2$  for  $Z$  with respect to  $R$ .

*Proof.* This claim is easily verified by direct (computer aided) calculations, which we omit.  $\square$

Theorem 5 is of interest as it exhibits a first family of unexpected curves, where the degree of curves grows but the multiplicity at the general point remains constant.

## 2.2 Fermat-type configurations of points in $\mathbb{P}^N$

For a positive integer  $n$  let  $I_{N,n}$  be the complete intersection ideal defined by the binomials

$$x_0^n - x_1^n, x_1^n - x_2^n, \dots, x_{N-1}^n - x_N^n. \quad (2)$$

The set  $Z_{N,n}$  of zeroes of  $I_{N,n}$  is the set of  $n^N$  points of the form

$$(1 : \varepsilon^{\alpha_1} : \varepsilon^{\alpha_2} : \dots : \varepsilon^{\alpha_N}),$$

where  $\varepsilon$  is a primitive root of unity of order  $n$  and

$$1 \leq \alpha_1, \alpha_2, \dots, \alpha_N \leq n.$$

Adding to this set of points the coordinate points of  $\mathbb{P}^N$ , we obtain the set  $W_{N,n}$ , and we define the ideal  $F_{N,n}$  to be the ideal of  $W_{N,n}$ .

In the sequel we will deal with the index  $n = 3$ , so we drop the degree from the notation. We also use the convention that the indices are understood modulo  $N + 1$ . For example  $x_{N+2} = x_1$ . Moreover it is convenient to introduce the following notation

$$[a, b] = a^3 - b^3,$$

which we use for numbers and for monomials. Note that the symbol  $[a, b]$  is anti-symmetric and satisfies a Jacobi-type identity

$$a^3[b, c] + b^3[c, a] + c^3[a, b] = 0 \quad \text{for all } a, b, c. \quad (3)$$

Lemma 3 generalizes in the following way.

**Lemma 6.** *The ideal  $F_N$  is generated by the binomials of the form*

$$x_i[x_{i+1}, x_j], \quad (4)$$

where  $i \in \{0, 1, \dots, N\}$  and  $j \in \{0, 1, \dots, N\} \setminus \{i, i + 1\}$ .

*Proof.* Let  $J$  be the ideal generated by binomials in (4). It is clear that elements in  $J$  vanish in all points of  $W_N$ , so there is  $J \subset F_N$ .

It is also easy to see that  $J$  contains all binomials of the form  $x_i[x_j, x_k]$ , where the indices are mutually distinct. Indeed, we have

$$x_i[x_j, x_k] = x_i[x_{i+1}, x_k] - x_i[x_{i+1}, x_j].$$

It is also clear that, for  $n = 3$ , the ideal  $I_N$  defined in (2) is generated also by binomials

$$[x_0, x_1], [x_0, x_2], \dots, [x_0, x_N],$$

which are slightly more convenient to work with in this proof.

Let  $f \in F_N$  be an arbitrary polynomial. We want to show the containment  $f \in J$  or equivalently  $f = 0 \pmod{J}$ . Since  $f$  vanishes in particular in all points of  $Z_N$ , there are homogeneous polynomials  $g_1, \dots, g_N$  such that

$$f = \sum_{i=1}^N g_i[x_0, x_i]. \quad (5)$$

From now on, we work  $\pmod{J}$ . Since for  $j \neq 0, i$  the binomials  $x_j[x_0, x_i]$  are in  $J$ , we may assume that each polynomial  $g_i$  depends only on  $x_0$  and  $x_i$ . Moreover, for a fixed  $j \neq 0, i$  we have

$$x_0 x_i[x_0, x_i] = x_i \cdot x_0[x_j, x_i] - x_0 \cdot x_i[x_j, x_0] \in J.$$

Thus it must be

$$g_i = a_i x_0^d + b_i x_i^d \pmod{J}$$

for some  $d \geq 0$  and some scalars  $a_1, \dots, a_N, b_1, \dots, b_N \in \mathbb{C}$ . Evaluating (5) in the coordinate points we obtain

$$a_1 + a_2 + \dots + a_N = 0 \quad \text{and} \quad b_1 = b_2 = \dots = b_N = 0. \quad (6)$$

Hence (5) reduces to

$$f = a_1 x_0^d [x_0, x_1] + a_2 x_0^d [x_0, x_2] + \dots + a_N x_0^d [x_0, x_N] \pmod{J}. \quad (7)$$

If  $d = 0$ , then evaluating (7) in the coordinate points, we obtain

$$a_1 = a_2 = \dots = a_N = 0$$

and we are done.

If  $d > 0$ , then, using the first part of (6), and noting that

$$a_i x_0^d [x_0, x_i] - a_i x_0^d [x_0, x_N] = a_i x_0^d [x_N, x_i],$$

we can rewrite (7) as

$$f = a_1 x_0^d [x_N, x_1] + a_2 x_0^d [x_N, x_2] + \dots + a_{N-1} x_0^d [x_N, x_{N-1}],$$

which is clearly an element of  $J$  and we are done again.  $\square$

### 3 Unexpected quartic surfaces in $\mathbb{P}^3$ with a triple point

The story of unexpected hypersurfaces begins with unexpected curves discovered by Di Gennaro, Ilardi and Vallès in [5] and studied systematically by Cook II, Harbourne, Migliore and Nagel in [4]. In [2] the first example of a higher dimensional unexpected hypersurface (a surface in  $\mathbb{P}^3$ ) has been described by Bauer, Malara, Szemberg and the present author. Shortly after that Harbourne, Migliore, Nagel and Teitler in [7] constructed examples of unexpected hypersurfaces in projective spaces of arbitrary dimension. All these examples have just one singular general point.

Here we recall the construction of the unexpected quartic in  $\mathbb{P}^3$ . This example has not only motivated our construction of unexpected hypersurfaces with multiple fat points but it serves as a building block for this construction.

According to Lemma 6, in  $\mathbb{P}^3$  the ideal  $F_{3,3}$  has 8 generators:

$$g_{0,2} = x_0[x_1, x_2], \quad g_{0,3} = x_0[x_1, x_3], \quad g_{1,3} = x_1[x_2, x_3], \quad g_{1,0} = x_1[x_2, x_0],$$

$$g_{2,0} = x_2[x_3, x_0], \quad g_{2,1} = x_2[x_3, x_1], \quad g_{3,1} = x_3[x_0, x_1], \quad g_{3,2} = x_3[x_0, x_2].$$

It is convenient to introduce the following notation. Given mutually distinct numbers  $i, j \in \{0, 1, 2, 3\}$ , we denote by  $k$  the index  $\{0, 1, 2, 3\} \setminus \{i, i+1, j\}$ . As usually, the indices are considered modulo 4. With this notation we have (compare [2, Theorem 6]).

**Theorem 7.** *For a general point  $R = (a_0 : a_1 : a_2 : a_3)$ , the quartic*

$$Q_R(x_0 : x_1 : x_2 : x_3) = \sum_{i=0}^3 \sum_{j=i+2}^{i+3} (-1)^k a_i^2 [a_{i+1}, a_k] \cdot g_{i,j}$$

*has a triple point at  $R$  and satisfies  $Q_R \in F_{3,3}$ .*

*Proof.* The second property is obvious. Checking that  $Q_R$  vanishes to order 3 at  $R$  is a straightforward computation. However for further reference we will present some computations which make heavy use of available symmetries.

To begin with, we check that the first order derivatives of  $Q_R$  vanish at  $R$ . By symmetry it is enough to check just one derivative. We compute the one with respect to  $x_0$  and obtain

$$\frac{\partial}{\partial x_0} Q_R = ([a_1, a_2][x_1, x_3] - [a_1, a_3][x_1, x_2]) a_0^2 + 3x_0^2 (a_1^2 x_1 [a_2, a_3] + a_2^2 x_2 [a_3, a_1] + a_3^2 x_3 [a_1, a_2]).$$

$$\begin{pmatrix} 0 & 0 & -a_1 & 0 & -a_2 & 0 & a_3 & a_3 \\ a_1^2 & a_1^2 & -a_0^2 & 0 & 0 & 0 & 0 & 0 \\ a_2^2 & 0 & 0 & 0 & a_0^2 & 0 & 0 & 0 \\ 0 & -a_3^2 & 0 & 0 & 0 & 0 & a_0^2 & a_0^2 \\ a_0 & a_0 & 0 & 0 & 0 & -a_2 & -a_3 & 0 \\ 0 & 0 & a_2^2 & a_2^2 & 0 & -a_1^2 & 0 & 0 \\ 0 & 0 & 0 & a_3^2 & 0 & 0 & a_1^2 & 0 \\ -a_0 & 0 & a_1 & a_1 & 0 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & 0 & a_3^2 & a_3^2 & 0 & -a_2^2 \\ 0 & -a_0 & 0 & -a_1 & a_2 & a_2 & 0 & 0 \end{pmatrix}$$

Table 1: Interpolation matrix for a point of multiplicity 3 in  $\mathbb{P}^3$ .

Vanishing of the first bracket when evaluating at  $R$  is straightforward. The vanishing of the second bracket follows from the Jacobi-type identity (3).

For the second order derivative with respect to  $x_0$  we have

$$\frac{\partial^2}{\partial x_0^2} Q_R = -6x_0 \cdot \sum_{k=1}^3 (-1)^k a_j^2 a_l^2 (a_l x_j - a_j x_l),$$

where the indices  $j, l$  are chosen in dependence on  $k$  so that  $j < l$  and  $\{j, k, l\} = \{1, 2, 3\}$ .

For the mixed derivative of order 2 we may again use the symmetry and look at just one other variable, here  $x_1$ . We have

$$\frac{\partial^2}{\partial x_1 \partial x_0} Q_R = 3[a_2, a_3](a_1^2 x_0^2 - a_0^2 x_1^2).$$

Clearly, in both cases we obtain zero, when evaluating at  $R$ .

Instead of computing derivatives one by one, we can also use the whole interpolation matrix at once. The rank of this matrix is the number of conditions which vanishing to order 3 imposes on generators of  $F_{3,3}$ .

We present this matrix in Table 1, since this is a baby case for our argument in the proof of Theorem 10, see Table 2. The columns of the table correspond to the generators of  $F_{3,3}$ , whereas the rows stand for order 2 partial derivatives, evaluated at  $R$ . Taking into account that the point  $R$  is general, we have divided out common factors appearing in some rows. This is the reason why not all rows have entries of the same degree. Nevertheless, the matrix as entire, remains of course homogeneous. Also its rank does not depend on the performed divisions. It is elementary, but tedious, to check that the rank of the interpolation matrix is 7.  $\square$

#### 4 Unexpected quartic 4-folds in $\mathbb{P}^5$ with a triple and double point

In this section we present in detail unexpected hypersurfaces in  $\mathbb{P}^5$ , which have 2 general fat points, one of multiplicity 3 and the other one of multiplicity 2.

According to Lemma 6 the ideal  $F_{5,3}$  has 24 generators of degree 4 and its zero locus consists of  $3^5$  Fermat points and 6 coordinate points, so that there are altogether 249 points in  $W_{5,3}$ . The linear system  $V_0$  of quartics in  $\mathbb{P}^5$  has (affine) dimension 126, so the points in  $W_{5,3}$  do not impose independent conditions on quartics. Of course there exists a subset  $\Omega$  of  $102 = 126 - 24$  points in  $W_{5,3}$  which impose independent conditions on quartics in  $\mathbb{P}^5$ . Indeed, take any point  $P_1$  in  $W_{3,3}$ . The linear system  $V_1$  of quartics vanishing at  $P_1$  has codimension 1 in  $V_0$ . Then take the second point  $P_2$  in  $W_{3,3}$  away from the base locus of  $V_1$ . The linear system  $V_2$  of quartics vanishing at  $P_2$  (and  $P_1$ ) has codimension 1 in  $V_1$ , hence codimension 2 in  $V_0$ . Proceeding by induction, we determine the set  $\Omega$ . It is not unique and, in fact, it is not relevant for our considerations here.

Since vanishing to order 3 imposes 21 conditions, it is expected that the system of quartics in  $\mathbb{P}^5$  vanishing along  $W_{5,3}$  and having a triple point in a general point  $R = (a_0 : a_1 : \dots : a_5)$  has (vector space) dimension 3. It turns out however that this system has in fact higher dimension.

**Proposition 8.** *Let  $I(R)$  denote the ideal of the point  $R$ . For*

$$V_{5,3} = H^0(\mathbb{P}^5; \mathcal{O}_{\mathbb{P}^5}(4) \otimes F_{5,3} \otimes I(R)^3)$$

*we have  $\dim V_{5,3} = 6$ .*

*Proof.* We verify our claim computing the interpolation matrix at  $R$ . More precisely, we compute all partial derivatives of order 2 (there are 21 of them) of all 24 generators of  $F_{5,3}$ . The matrix we get has a lot of zeros and thus its rank is easy to determine. The rank is the number of *independent* conditions imposed on quartics in  $F_{5,3}$  by vanishing at  $R$  to order 3.

0	0	0	0	$-a_1$	0	0	0	$-a_2$	0	0	0	$-a_3$	0	0	0	$-a_4$	0	0	0	$a_5$	$a_5$	$a_5$	$a_5$	
$a_1^2$	$a_1^2$	$a_1^2$	$a_1^2$	$-a_0^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$a_2^2$	0	0	0	0	0	0	0	$a_0^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	$a_3^2$	0	0	0	0	0	0	0	0	0	0	$a_0^2$	0	0	0	0	0	0	0	0	0	0	0	
0	0	$a_4^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	$a_0^2$	0	0	0	0	0	0	0	
0	0	0	$-a_5^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$a_0^2$	$a_0^2$	$a_0^2$	$a_0^2$	
$a_0$	$a_0$	$a_0$	$a_0$	0	0	0	0	0	$-a_2$	0	0	0	$-a_3$	0	0	0	$-a_4$	0	0	$-a_5$	0	0	0	
0	0	0	0	$a_2^2$	$a_2^2$	$a_2^2$	$a_2^2$	0	$-a_1^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	$a_3^2$	0	0	0	0	0	0	0	$a_1^2$	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	$a_4^2$	0	0	0	0	0	0	0	0	0	0	$a_1^2$	0	0	0	0	0	0	
0	0	0	0	0	0	0	$a_5^2$	0	0	0	0	0	0	0	0	0	0	0	0	$a_1^2$	0	0	0	
$-a_0$	0	0	0	$a_1$	$a_1$	$a_1$	$a_1$	0	0	0	0	0	0	$-a_3$	0	0	0	$-a_4$	0	0	$-a_5$	0	0	
0	0	0	0	0	0	0	0	$a_3^2$	$a_3^2$	$a_3^2$	$a_3^2$	0	0	$-a_2^2$	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	$a_4^2$	0	0	0	0	0	0	0	$a_2^2$	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	$a_5^2$	0	0	0	0	0	0	0	0	$a_2^2$	0	0	
0	$-a_0$	0	0	0	$-a_1$	0	0	$a_2$	$a_2$	$a_2$	$a_2$	0	0	0	0	0	0	0	0	$-a_4$	0	0	$-a_5$	0
0	0	0	0	0	0	0	0	0	0	0	0	$a_4^2$	$a_4^2$	$a_4^2$	$a_4^2$	0	0	0	$-a_3^2$	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$a_5^2$	0	0	0	0	0	0	$a_3^2$	0	
0	0	$-a_0$	0	0	0	$-a_1$	0	0	0	$-a_2$	0	$a_3$	$a_3$	$a_3$	$a_3$	0	0	0	0	0	0	0	$-a_5$	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$a_5^2$	$a_5^2$	$a_5^2$	$a_5^2$	0	0	0	$-a_4^2$	
0	0	0	$-a_0$	0	0	0	$-a_1$	0	0	0	$-a_2$	0	0	0	$-a_3$	$a_4$	$a_4$	$a_4$	$a_4$	0	0	0	0	

Table 2: Interpolation matrix for a point of multiplicity 3 in  $\mathbb{P}^5$ .

□

Since a double point in  $\mathbb{P}^5$  imposes 6 conditions on forms of arbitrary degree, we do not expect that there is a non-zero section in  $V_{5,3}$  vanishing to order 2 at an additional general point. However such unexpected hypersurface does exist!

We define first the following 6 cones in  $\mathbb{P}^5$ . For indices  $i, j \in \{0, \dots, 5\}$ , with  $i < j$ , we denote by  $s, t, u, v$  the remaining 4 indices in the growing order. Then, for  $i, j \in \{0, 1, 2, 3\}$  with  $i < j$  we set

$$R_{i,j} = (a_s : a_t : a_u : a_v)$$

and

$$J_{i,j} = Q_{R_{i,j}}(x_s : x_t : x_u : x_v),$$

where  $Q_R$  is taken from Theorem 7. Note that this is a cone.

**Remark 9.** It can be shown that  $J_{0,1}, \dots, J_{2,3}$  generate  $V_{5,3}$  but this is irrelevant for the further assertions and we omit the verification of this claim.

**Theorem 10.** *Let  $P = (b_0 : b_1 : \dots : b_5)$  be a general point in  $\mathbb{P}^5$ . Then there exists a unique quartic  $Q_{R,P} \in V_{5,3}$  vanishing at*

- all points of the Fermat-type configuration  $W_5$ ,
- the point  $R = (a_0 : a_1 : \dots : a_5)$  to order 3,
- the point  $P$  to order 2.

*Proof.* We are able to write down an explicit equation of the quartic

$$\begin{aligned} Q_{R,P}(x_0 : \dots : x_5) = & J_{2,3}(P) \cdot J_{0,1}(x_0 : \dots : x_5) - J_{1,3}(P) \cdot J_{0,2}(x_0 : \dots : x_5) \\ & + J_{0,3}(P) \cdot J_{1,2}(x_0 : \dots : x_5) + J_{1,2}(P) \cdot J_{0,3}(x_0 : \dots : x_5) \\ & - J_{0,2}(P) \cdot J_{1,3}(x_0 : \dots : x_5) + J_{0,1}(P) \cdot J_{2,3}(x_0 : \dots : x_5). \end{aligned}$$

The sign in the front of a summand of  $Q_{R,P}$  depends on whether there is a pair of consecutive numbers in the indices of involved  $J$ 's or not. If there isn't, i.e., for pairs (0, 2) and (1, 3) we get a minus.

Note that the first two assertions of the Theorem follow straightforwardly from the way we defined the cones  $J_{0,1}, \dots, J_{2,3}$ . The last assertion can be checked by direct computations as in the proof of Theorem 7.

The most tricky part of the proof was to find the coefficients in front of the cones. We don't have at the moment any theoretical explanation for their highly non-coincidental shape. This seems an interesting challenge to work on in the near future.  $\square$

## 5 Unexpected quartic $2k$ -folds in $\mathbb{P}^{2k+1}$ with a triple point and $k - 1$ double points

Proof of Theorem 10 suggests the following, inductive, approach towards the Conjecture stated in the Introduction. For  $N = 2k + 1$  let  $Q_N$  denote the unexpected quartic in  $\mathbb{P}^N$  with a triple point  $R$  and  $(k - 1)$  double points  $P_1, \dots, P_{k-1}$ . In  $\mathbb{P}^{N+2}$  we consider cones over all quartics  $Q_N$  contained in coordinate  $\mathbb{P}^N$ 's (i.e. 2 variables are set zero), singular at projections of points  $R, P_1, \dots, P_{k-1} \in \mathbb{P}^{N+2}$ . All these cones vanish to the right order at  $R$  and  $P_1, \dots, P_{k-1}$ , i.e. to order 3 at  $R$  and to order 2 at  $P_1, \dots, P_{k-1}$ . The final step of the construction is to find their linear combination, which additionally has a double point at  $P_k$ . However we were not able to guess the coefficients.

Fortunately, another approach works. Let, as above,  $Q_N$  be the unexpected quartic in  $\mathbb{P}^N$ . We start with  $Q_3$  in  $\mathbb{P}^3$ . Then we define  $Q_{N+2}$  as follows:

$$Q_{N+2}(x) = \sum_{i=0}^{N+2} a_i^2 \sum_{j=i+2}^{N+i+2} K(i, j) \cdot Q_N^{i,j}(P_k) \cdot x_i \left( x_{(i+1 \bmod (N+3))}^3 - x_{(j \bmod (N+3))}^3 \right),$$

where  $K(i, j) = \text{sgn}(j - i) \cdot (-1)^{j+i}$  is the sign function kindly provided to us by Jakub Kabat and  $Q_N^{i,j}$  is the unexpected quartic in variables  $x_0, \dots, x_{N+2}$  with  $x_i, x_{(j \bmod (N+3))}$  omitted.

**Statement.** Let  $Z = W_{N+2}$  and let  $R, P_1, \dots, P_{k-1}, P_k$  be  $(k+1)$  general points in  $\mathbb{P}^{N+2}$ . Then  $Q_{N+2}$

- vanishes at  $W_{N+2}$ ,
- vanishes at  $R = (a_0 : \dots : a_{N+2})$  to order 3,
- vanishes at  $P_i = (b_0^i : \dots : b_{N+2}^i)$  to order 2, for  $1 \leq i \leq k$ .

This is verified in [13] for  $N + 2 = 5, 7, 9$ . Whereas it is clear how to proceed, the capacity of Singular has been exceeded. Suffices it to say, that the unexpected quartic in  $\mathbb{P}^9$  requires ca. 250 000 000 characters.

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