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INTERPOLATION WITHOUT COMMUTANTS

OLEG SZEHR AND RACHID ZAROUF

ABSTRACT. We introduce a “dual-space approach” to mixed Nevanlinna-Pick/Carathéodory-Schur interpolation in Banach spaces X of holomorphic functions on the disk. Our approach can be viewed as complementary to the well-known commutant lifting approach of D. Sarason and B. Nagy-C.Foiaş. We compute the norm of the minimal interpolant in X by a version of the Hahn-Banach theorem, which we use to extend functionals defined on a subspace of kernels without increasing their norm. This Functional extensions lemma plays a similar role as Sarason’s Commutant lifting theorem but it only involves the predual of X and no Hilbert space structure is needed. As an example, we present the respective Pick-type interpolation theorems for Beurling-Sobolev spaces.

1. INTRODUCTION

1.1. The commutant lifting approach to interpolation theory. Given a finite sequence of distinct points $\lambda = (\lambda_i)_{i=1}^n$ in \mathbb{D} and another finite sequence $w = (w_i)_{i=1}^n$ in \mathbb{C} , the Nevanlinna-Pick interpolation problem is to find necessary and sufficient conditions for the existence of $f \in \mathcal{H}ol(\mathbb{D})$ that is bounded by 1 and that interpolates the data, i.e. $\|f\|_{H^\infty} := \sup_{z \in \mathbb{D}} |f(z)| \leq 1$ and $f(\lambda_i) = w_i$. The classical solution of G. Pick [16] and (later) R. Nevanlinna [8, 9] asserts that such f exists if and only if the “Pick-matrix”

$$\left(\frac{1 - w_i \overline{w_j}}{1 - \lambda_i \overline{\lambda_j}} \right)_{1 \leq i, j \leq n}$$

is positive-semidefinite. The celebrated commutant-lifting approach of D. Sarason [19] and B. Nagy-C. Foiaş [11, 1] established an operator-theoretic perspective on the interpolation problem. The main point is to view H^∞ as a multiplier algebra of the Hardy space H^2 (of holomorphic functions, whose Taylor coefficients are square-summable) and to identify $f \in H^\infty$ with the respective multiplication operator $Mult_f : H^2 \rightarrow H^2$, $\phi \mapsto f\phi$. In other words the norm of $f \in H^\infty$ equals to the operator norm of $Mult_f$. The Nevanlinna-Pick problem asks for conditions on the restriction M_f of $Mult_f$ to a subspace corresponding to $f(\lambda_i) = w_i$ such that M_f can be extended to the whole of H^2 maintaining $\|Mult_f\| \leq 1$. More precisely, for a Blaschke product

$$B = \prod_{\lambda_i \in \lambda} b_{\lambda_i}, \quad b_{\lambda_i} = \frac{z - \lambda_i}{1 - \overline{\lambda_i} z}$$

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the range BH^2 of the multiplication operator $Mult_B$ is a linear subspace of H^2 of functions that vanish at λ . The orthogonal complement

$$K_B := H^2 \ominus BH^2,$$

consists of rational functions whose poles are $1/\overline{\lambda_i}$, $i = 1 \dots n$. Let M_f be the compression of the multiplication operator $Mult_f$ from H^2 to K_B , i.e.

$$\begin{aligned} M_f : K_B &\rightarrow K_B \\ \phi &\mapsto (P_B \circ Mult_f)(\phi) = P_B(f\phi), \end{aligned}$$

where P_B is the orthogonal projection from H^2 to K_B . It is clear that $\|M_f\| \leq \|f\|_{H^\infty}$ and that M_f only depends on the values $\{f(\lambda_i) = w_i\}_{i=1}^n$. Sarason's main result [19, 1] asserts that if M_g is any operator that commutes with M_f on K_B then M_g can be extended to an operator $Mult_g$ that commutes with $Mult_f$ on H^2 *without increasing the operator's norm*. In conclusion one finds

$$\begin{aligned} \|M_f\| &= \inf \{ \|Mult_g\| : P_B \circ Mult_g = M_f \} \\ &= \inf \{ \|g\|_{H^\infty} : g \in H^\infty, g(\lambda_i) = f(\lambda_i) = w_i, \forall i \} \end{aligned}$$

where the first equality is a consequence of Sarason's result and the second holds by the construction of K_B . It is elementary to check that $\|M_f\| \leq 1$ is equivalent to the Pick-matrix being positive-semidefinite [14].

This new perspective allowed for various generalizations of the classical Nevanlinna-Pick problem. To mention a few, it has been noticed that the assumption of non-degeneracy of the λ_i is not principal. The purely degenerate case $\lambda_1 = \dots = \lambda_n = 0$ corresponds to the prescription of the first n Taylor-coefficients of f . The respective interpolation theory has been studied long before as the Carathéodory-Schur interpolation problem. Thus the commutant lifting approach provided a unified framework with a simultaneous discussion of two classical problems of interpolation theory. Second, Sarason's result was generalized by B. Nagy-C. Foiaş [11, 1], whose commutant lifting theorem asserts that any operator commuting with A can be lifted to an operator commuting with any unitary dilation of A without increasing its norm. Sarason's lemma is the special case that A is the multiplication operator $Mult_z$ on H^2 . A more general result is the intertwining lifting theorem of Nagy-Foiaş [6, 10]. As a consequence the H^2 -specific discussion has been generalized to *Reproducing Kernel Hilbert Space* (RKHS) to study interpolation problems of the respective commutant algebras: Let a positive-definite function $(z, \zeta) \mapsto \kappa(z, \zeta)$ on $\mathbb{D} \times \mathbb{D}$ be given (i.e. $\sum_{i,j} a_i \overline{a_j} \kappa(\lambda_i, \lambda_j) > 0$ for all finite subsets $(\lambda_i) \subset \mathbb{D}$ and all non-zero families of complex numbers $\{a_i\}$) and let $\kappa_\zeta = \kappa(\cdot, \zeta)$. Following Aronszajn [2] there exists a unique Hilbert space of functions $\mathcal{H}(\kappa)$, such that κ enjoys the *reproducing kernel property*, i.e. for all $f \in \mathcal{H}(\kappa)$ it holds

$$f(\zeta) = \langle f, \kappa_\zeta \rangle_{\mathcal{H}(\kappa)}.$$

When κ is holomorphic in the first variable and antiholomorphic in the second this yields a RKHS $\mathcal{H}(\kappa)$ of holomorphic functions on \mathbb{D} . The algebra of multipliers \mathbb{M}_κ is a Banach algebra of functions ϕ for which $f\phi \in \mathcal{H}(\kappa)$ for each $f \in \mathcal{H}(\kappa)$ and the norm is the norm of the corresponding multiplication operator on $\mathcal{H}(\kappa)$. For the Cauchy kernel $\kappa_\zeta(z) = \frac{1}{1-\zeta z}$

we obtain $\mathcal{H}(\kappa) = H^2$ and $\mathbb{M}_\kappa = H^\infty$. It is a natural question to ask for which kernels κ apart from the Cauchy kernel a “Nevanlinna-Pick theorem” holds. Assuming that κ is a so-called complete Nevanlinna-Pick kernel, i.e. it satisfies the identity

$$\kappa(z, \zeta) - \frac{\kappa(z, \mu)\kappa(\mu, \zeta)}{\kappa(\mu, \mu)} = F_\mu(z, \zeta)\kappa(z, \zeta)$$

for some $\mu \in \mathbb{D}$, $\kappa(\mu, \mu) \neq 0$ and some positive semidefinite function F_μ on $\mathbb{D} \times \mathbb{D}$ such that $|F_\mu(z, \zeta)| < 1$, it is shown in [17] that the Nevanlinna-Pick theorem holds mutatis mutandis: There exists a multiplier f of norm at most 1 which satisfies the interpolation condition $f(\lambda_i) = w_i$ if and only if

$$[\kappa(\lambda_i, \lambda_j)(1 - w_i \overline{w_j})]_{1 \leq i, j \leq n} \geq 0.$$

The interesting article [20] contains a detailed discussion of complete Nevanlinna-Pick kernels in the context of Dirichlet spaces. A general commutant lifting theorem for spaces with Nevanlinna-Pick kernels is proved in [3].

1.2. Our approach and its motivation. In this article we study interpolation problems beyond the context of RKHS. Let X be a Banach space that is continuously embedded into $Hol(\mathbb{D})$. Our goal is to obtain information on the interpolation quantity

$$I_X(\lambda, w) = \inf \left\{ \|f\|_X : f \in X, f^{(j)}(\lambda_i) = w_i^{(j)}, 1 \leq i \leq n, 0 \leq j < n_i \right\},$$

where λ_i carries degeneracy n_i ,

$$w = \left(w_1^{(0)}, w_1^{(1)}, \dots, w_1^{(n_1-1)}, \dots, w_s^{(0)}, w_s^{(1)}, \dots, w_s^{(n_s-1)}, \dots \right)$$

and $f^{(j)}$ stands for the j -th derivative of f . This definition covers mixed problems of Nevanlinna-Pick and Carathéodory-Schur type [14]. Our approach is closely related to the established commutant lifting theory. A major common point will lie in the role of the space K_B and the compressions M_f of the multiplication operator $Mult_f : X \rightarrow X$. Our main conceptual insight might be seen in the observation that no RKHS structure is needed to identify the space K_B and that the formulation of the solution in terms of the multiplication operator (and with it the occurrence of the multiplier algebra) can be done in an independent step. Thus we conceptually split the Hilbert space specific commutant lifting theorem, into a “Functional extension lemma” and a formulation of the solution in terms of the multiplication operator. Just as the commutant lifting theorem allows one to extend an operator from K_B to H^2 without increasing the operator norm this lemma allow us to extend functionals from K_B to a Banach space of holomorphic functions without increasing the norm of the functional. Our motivation is twofold,

1) on the theoretical side: The commutant lifting approach has generated significant impact on interpolation theory, operator theory, functional analysis and beyond. We see our method as complementary to this approach, but for certain types of spaces it is simpler. Instead of studying the commutant algebra of a RKHS, we will work with duals of a class of Banach spaces. In many cases the dual space turns out to be more tangible than the commutant algebra.

2) on the practical side: Our result is interesting from a practical standpoint because it dramatically simplifies the numerical computation of the quantity $I_X(\lambda, w)$. The search

domain for the minimization is an infinite-dimensional Banach space, and therefore the search does not admit implementation on finite-memory and finite-precision computers. In contrast the representation afforded by the functional extension lemma reduces the original (infinite-dimensional) minimization problem to a search in K_B for an optimal n -dimensional vector of coefficients $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^n$, which can be obtained by applying any standard function minimization algorithm. Consequently, we provide the theoretical basis for the reduction of the general optimization problem to one that can actually be implemented in practice.

1.3. Outline of the paper. In Section 2 we explain how our approach applies to a large class of Banach spaces and we compare our methods in more detail with those of D. Sarason. Section 3 contains our main results. Lemma 1 provides an expression for $I_X(\lambda, w)$ in terms of the norm of a functional on the space K_B . Assuming that X is a unital Banach algebra Theorem 2 relates this expression to the compressed multiplication operator. This theorem extends Sarason's original result to any unital algebra X whose predual contains $K_B = K_\lambda$. Corollary 4 applies interpolation theory to derive sharp estimates on norms of functions of algebraic operators admitting an X -functional calculus. We conclude Section 3 with an application of Theorem 2 to the case $X = W$, the *Wiener algebra* of absolutely convergent Taylor series. Section 2 contains the details regarding W , its definition and our motivation to study it. Section 4 shows applications of Lemma 1 to the so-called *Beurling-Sobolev spaces* $X = l_A^q(\beta)$ and $X = H^\infty$. Section 5 discusses the situation in which the data λ has an accumulation point inside the disk. In this case there is a unique solution f to the interpolation problem in X . Finally Section 6 contains the proof of Corollary 7 where we recover Pick's usual criterion.

2. INTERPOLATION IN BANACH SPACES

Our approach can be seen as a “duality based” discussion of interpolation theory in the Banach space X . We endow $Hol(\mathbb{D})$ with a formal Cauchy-type scalar product

$$\langle f, g \rangle = \sum_{k \geq 0} \hat{f}(k) \overline{\hat{g}(k)},$$

where $f = \sum_{k \geq 0} \hat{f}(k) z^k$ denotes the Taylor expansion of f . For $f, g \in H^2$ this coincides with the usual H^2 scalar product $(\cdot, \cdot)_{H^2}$ inherited from L^2 , i.e. $\langle f, g \rangle = (f, g)_{H^2}$. We assume that X is a dual space $X = Y'$ w.r.t. $\langle \cdot, \cdot \rangle$. Y will always denote the *exact* predual of X which means that the norm of g in Y can be computed by the classical Hahn-Banach formula

$$\|g\|_Y := \sup \left\{ \frac{|\langle g, f \rangle|}{\|f\|_X} : f \in X \right\}.$$

We will also assume that the predual Y contains the set of all analytic polynomials as a dense subset. Any function $f \in X$ can be interpreted as a functional on Y , $f \mapsto \tilde{f} := \langle \cdot, f \rangle$, and for the norm we have by Hölder's inequality

$$\|f\|_X = \|\tilde{f}\|_{Y'} := \sup \left\{ \frac{|\langle g, f \rangle|}{\|g\|_Y} : g \in Y \right\}.$$

The main point on which our interpolation theory is footed is that Y contains the space K_λ of rational functions whose poles are located at $1/\overline{\lambda_i}$, with possible multiplicity n_i :

$$K_\lambda =: \text{span}\{k_{\lambda_i, j} : 1 \leq i \leq n, 0 \leq j < n_i\},$$

where for $\lambda_i \neq 0$, $k_{\lambda_i, j} = \left(\frac{d}{d\lambda_i}\right)^j k_{\lambda_i}$ and $k_{\lambda_i} = \frac{1}{1-\overline{\lambda_i}z}$ is the Cauchy kernel at λ_i while $k_{0, i} = z^i$. In other words we only consider X such that the scalar products $\langle f, k_{\lambda_i} \rangle$ are finite, which is a very mild assumption as long as n is finite, i.e. the boundary behaviour of the kernels $k_{\lambda_i, j}$ is regular. Under this assumption we have for any $f \in X$ that

$$\langle f, k_{\lambda_i} \rangle = \sum_{k \geq 0} \hat{f}(k) \lambda_i^k = f(\lambda_i)$$

and similarly

$$\langle f, k_{\lambda_i, j} \rangle = f^{(j)}(\lambda_i).$$

To compare to the setting of commutant lifting observe that no matter what X and Y are it still holds that $K_\lambda = K_B = H^2 \cap (BH^2)^\perp$, where B is the finite Blaschke product corresponding to λ . If $\lambda_i \neq \lambda_j$ for $i \neq j$ we have $K_\lambda = \text{span}\{k_{\lambda_i}\}$, which corresponds to the Nevanlinna-Pick problem. The purely degenerate kernels $K_\lambda = \text{span}\{z^i\}$ correspond to the Carathéodory-Schur problem. Notice that

- Sarason computes $I_{H^\infty}(\lambda, w)$ by lifting an operator commuting with M_z on K_B to an operator commuting with $Mult_z$ on the whole of H^2 without increasing its norm,
- we compute $I_X(\lambda, w)$ by using a version of the Hahn-Banach theorem extending a functional from the subspace K_λ of Y to the whole space Y without increasing its norm. Subsequently we rewrite the norm of the functional on K_λ in terms of the compressed multiplication operator and thereby extend Sarason's original result.

In view of applications we are particularly interested in the case that $X = W \subsetneq H^\infty$ is the *Wiener algebra* of absolutely convergent Taylor series

$$W := \left\{ f = \sum_{j \geq 0} \hat{f}(j) z^j \in \mathcal{H}ol(\mathbb{D}) : \|f\|_W := \sum_{j \geq 0} |\hat{f}(j)| < \infty \right\}.$$

To the best of our knowledge neither the Nevanlinna-Pick nor the Carathéodory-Schur interpolation problem have been studied in this setup before. By von Neumann's inequality Hilbert space contractions admit an H^∞ functional calculus. As a consequence Sarason's H^∞ -interpolation theory has contributed significant insight to the study of such operators [12]. Similarly *Banach space contractions* are related to a Wiener algebra functional calculus. Our interest in W comes from developing an analogous theory for contractions on Banach space.

More generally we will discuss the *Beurling-Sobolev spaces* of functions $f \in \mathcal{H}ol(\mathbb{D})$ whose sequence of Taylor coefficients $\{\hat{f}(j)\}_{j \geq 0}$ are contained in the weighted sequence spaces $l^q(\beta)$, $q \in [1, \infty]$, $\beta \in \mathbb{R}$:

$$X = l_A^q(\beta) := \left\{ f = \sum_{j \geq 0} \hat{f}(j) z^j \in \mathcal{H}ol(\mathbb{D}) : \|f\|_{l_A^q(\beta)} := \left(\sum_{j \geq 0} |\hat{f}(j)|^q \omega_j^q \right)^{1/q} < \infty \right\},$$

where $w_0 = 1$ and $w_j = j^\beta$ for $j \geq 1$. Again the general interpolation problem in such spaces has not been previously investigated. Notice that $W = l_A^1(0)$ and that the Hilbert space $l_A^2(0)$ is just the standard Hardy space H^2 . It is easily verified that those spaces satisfy our assumptions and that the norm on the predual of $l_A^q(\beta)$ is $\|\cdot\|_{l_A^p(-\beta)}$ where p is the conjugate exponent of q : $\frac{1}{p} + \frac{1}{q} = 1$. We also show how our method applies to $X = H^\infty$, whose predual is given by [7, Chapter VII],

$$H^\infty = (L^1/\overline{H_0^1})',$$

and we recover Pick's classical result. (Here $L^1 = L^1(\partial\mathbb{D})$ is the usual L^1 space of the unit circle and $H_0^1 = zH^1$ is a subspace of the respective Hardy space H^1 , see [14].)

3. MAIN RESULTS

Let X, Y and K_λ be given. Our goal is to express $I_X(\lambda, w)$ in terms of a quantity that can be determined from K_λ alone. We suppose for notational convenience that $\lambda_i \neq \lambda_j$ for $i \neq j$. In case that λ_i carries degeneracy n_i the below argumentation can be immediately extended by considering the kernels $k_{\lambda_i, j}$, $0 \leq j \leq n_i$. Plugging in definitions we have

$$\begin{aligned} I_X(\lambda, w) &= \inf \{ \|f\|_X : f \in X, f(\lambda_i) = w_i, \forall i = 1 \dots n \} \\ &= \inf \left\{ \left\| \tilde{f} \right\|_{Y'} : f \in X, \tilde{f}(k_{\lambda_i}) = \overline{w_i}, \forall i = 1 \dots n \right\}, \end{aligned}$$

where we have used that $\|f\|_X = \left\| \tilde{f} \right\|_{Y'}$, and $\tilde{f}(k_{\lambda_i}) = \langle k_{\lambda_i}, f \rangle = \overline{\langle f, k_{\lambda_i} \rangle} = \overline{w_i}$. The condition $\tilde{f}(k_{\lambda_i}) = \overline{w_i}$ means that the restriction $\tilde{f}|_{K_\lambda}$ coincides with the functional

$$\begin{aligned} \tilde{k} : K_\lambda &\rightarrow \mathbb{C} \\ \sum_{i=1}^n \alpha_i k_{\lambda_i} &\mapsto \tilde{k} \left(\sum_{i=1}^n \alpha_i k_{\lambda_i} \right) = \sum_{i=1}^n \alpha_i \overline{w_i}, \end{aligned}$$

which is tantamount to

$$I_X(\lambda, w) = \inf \left\{ \left\| \tilde{f} \right\|_{Y'} : f \in X, \tilde{f}|_{K_\lambda} = \tilde{k} \right\}.$$

Consequently

$$I_X(\lambda, w) \geq \left\| \tilde{k} \right\|_{(K_\lambda, \|\cdot\|_Y) \rightarrow \mathbb{C}},$$

where

$$\left\| \tilde{k} \right\|_{(K_\lambda, \|\cdot\|_Y) \rightarrow \mathbb{C}} := \sup_{g \in K_\lambda, g \neq 0} \frac{|\tilde{k}(g)|}{\|g\|_Y}.$$

According to Hahn-Banach theorem [18, Theorem 5.16, p. 104] since K_λ is a subspace of the normed linear space Y and \tilde{k} is a bounded linear functional on K_λ , \tilde{k} can be extended to a bounded linear functional on the whole of Y having the same norm as \tilde{k} . In other

words there exists $\tilde{f}^* \in Y'$ such that $\tilde{f}^*|_{K_\lambda} = \tilde{k}$ and $\|\tilde{f}^*\|_{Y'} = \|\tilde{k}\|_{(K_\lambda, \|\cdot\|_Y) \rightarrow \mathbb{C}}$. Thus the infimum in $I_X(\lambda, w)$ is achieved and

$$I_X(\lambda, w) = \|\tilde{k}\|_{(K_\lambda, \|\cdot\|_Y) \rightarrow \mathbb{C}}.$$

We note that with this simple formula the interpolation problem is, in principle, solved. We have written $I_X(\lambda, w)$ exclusively as a function of the interpolation data, which is encoded in K_λ . What remains is to write the interpolation problem in a more familiar form, e.g. in terms of the Pick matrix.

Lemma 1 (Functional extension lemma). *Let X be a Banach space of holomorphic functions on \mathbb{D} whose exact predual is Y . Let $\lambda = (\lambda_i)_{i=1}^n$ be a sequence in \mathbb{D} such that $K_\lambda \subset Y$. Defining the functional \tilde{k} on K_λ by*

$$\tilde{k}(k_{\lambda_i, j}) = \overline{w_i^{(j)}}, \quad \forall i = 1 \dots n, \quad \forall j = 0 \dots n_i - 1$$

the following equality holds

$$I_X(\lambda, w) = \|\tilde{k}\|_{(K_\lambda, \|\cdot\|_Y) \rightarrow \mathbb{C}}.$$

The result asserts that given $C > 0$ there exists $f \in X$ such that $f^{(j)}(\lambda_i) = w_i^{(j)}$ and $\|f\|_X \leq C$ iff

$$(3.1) \quad \left| \sum_{i=1}^n \sum_{j=0}^{n_i} \alpha_{i,j} \overline{w_i^{(j)}} \right| \leq C \left\| \sum_{i=1}^n \sum_{j=0}^{n_i} \alpha_{i,j} k_{\lambda_i, j} \right\|_Y$$

holds for any sequence of complex numbers $(\alpha_{i,j})_{i,j}$. Notice that when no degeneracy is present in the data condition (3.1) is structurally reminiscent to the positivity of the Pick matrix

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\overline{\alpha_i} \alpha_j (C^2 - \overline{w_i} w_j)}{1 - \overline{\lambda_i} \lambda_j} \geq 0.$$

The main difference is that the Pick condition is quadratic in the α_i , while (3.1) is linear. Indeed the positivity of the Pick matrix is equivalent to a bound on the H^2 operator norm of the compressed multiplication operator, which explains the occurrence of quadratic terms. Our next goal is thus to rewrite (3.1) in terms of the compressed shift operator. To work with multiplication operators it is clear that we must endow the spaces X, Y with additional structure. We shall assume that:

- (1) X is a unital Banach algebra (i.e. $1 \in X$ and for all $f_1, f_2 \in X$, X contains the product $f_1 f_2 \in X$ and $\|f_1 f_2\|_X \leq \|f_1\|_X \|f_2\|_X$).
- (2) The exact predual Y of X has a division property (i.e. $g \in Y \implies \frac{g-g(0)}{z} \in Y$).

To keep the notation simple we write briefly $S = Mult_z$ for the multiplication operator on X , S is commonly called the shift operator. The adjoint of S with respect to our Cauchy-type duality is S^* defined on Y . S^* is also known as the backward shift operator and satisfies

$$S^* f = \frac{f - f(0)}{z}, \quad f \in Y.$$

It can be checked simply that K_λ is invariant with respect to S^* . Moreover for any $\lambda \in \mathbb{D}$ and for any analytic polynomial g we have

$$S^*k_\lambda = \bar{\lambda}k_\lambda, \quad g(S)^*k_\lambda = \overline{g(\lambda)}k_\lambda.$$

Theorem 2. *Let $X \subset \mathcal{H}ol(\mathbb{D})$ be a unital Banach algebra satisfying the division property and $\lambda = (\lambda_i)_{i=1}^n$ be a finite sequence in \mathbb{D} whose associated Blaschke product is denoted by B . For any analytic polynomial g it holds that*

$$I_X(\lambda, g(\lambda)) = \|g(S)^*|K_\lambda\|_{Y \rightarrow Y}$$

where Y is the exact predual of X .

Remark 3. Regarding the case $X = H^\infty$: The proof of Corollary 7 below shows that

$$\|g(S)^*|K_\lambda\|_{L^1/\overline{H_0^1} \rightarrow L^1/\overline{H_0^1}} = \|g(S)^*|K_\lambda\|_{H^2 \rightarrow H^2},$$

which is Sarason's formulation of Pick's theorem [16]. This way Lemma 1 applies to $X = H^\infty = (L^1/\overline{H_0^1})'$ and yields Pick's criterion.

The main point of Theorem 2 is that it provides a representation of the interpolation quantity in terms of the norm of an operator. This allows an elementary comparison with Pick's classical criterion of positive-semidefiniteness but it also opens the doors to interesting applications in matrix analysis. We apply the theorem to obtain sharp estimates on norms of functions of matrices with given spectrum. It is known that the interpolation quantity can itself be seen as an upper estimate to the norm of a function of a matrix. We show that the converse also holds. We begin by reviewing the known upper bound:

An operator T is *algebraic* if there exists an analytic polynomial $g \neq 0$ such that $g(T) = 0$. We denote by m_T its *minimal polynomial*, i.e. the unique monic polynomial annihilating T whose degree $|m_T|$ is minimal. Given an algebraic operator T with spectrum in \mathbb{D} we put $m = m_T = \prod_{i=1}^{|m|} (z - \lambda_i)$ with $|\lambda_i| < 1$ for $i = 1 \dots |m|$. We assume that there exists a unital algebra $X \subset \mathcal{H}ol(\mathbb{D})$ on which T admits a functional calculus with constant $c > 0$, that is

$$(3.2) \quad \|g(T)\| \leq c \|g\|_X$$

for any polynomial g . In addition to assumptions (1) and (2) we assume furthermore that X satisfies the *division property*

$$[f \in X, \lambda \in \mathbb{D}, \text{ and } f(\lambda) = 0] \Rightarrow \left[\frac{f}{z - \lambda} \in X \right].$$

Following [13] instead of considering g directly in inequality (3.2), we add multiples of m to this function and consider $h = g + mf$ with $f \in X$. This leads to

$$\|g(T)\| \leq c I_X(\lambda, g(\lambda)).$$

Following [22, Lemma III.6] we extend this inequality to any rational function Ψ whose set of poles $\{\xi_i\}_{i=1}^p$ is separated from the eigenvalues of T considering the analytic polynomial

$$g(z) = \Psi \prod_{i=1}^p \left(\frac{m(\xi_i) - m(z)}{m(\xi_i)} \right),$$

where all singularities are lifted and observe that $g(T) = \Psi(T)$. This gives

$$\|\Psi(T)\| \leq c \|g(S)^*|K_\lambda\|_{Y \rightarrow Y} = c \|\Psi(S)^*|K_\lambda\|_{Y \rightarrow Y}$$

because $m(S)^*|K_\lambda = 0$.

Corollary 4. *In the setting of Theorem 2, if T admits a c functional calculus on X then for any rational function Ψ whose poles are distinct from the eigenvalues of T it holds that*

$$\|\Psi(T)\| \leq c \|\Psi(S)^*|K_\lambda\|_{Y \rightarrow Y}.$$

Notice that the right hand side only depends on the norm on Y and the minimal polynomial of T . The bound is optimal since equality is achieved for the compression of any multiplication operator to K_λ . We formulate a corollary of the theorem for matrices. This is achieved simply by identifying $\mathbb{C}^{|m|} \cong K_\lambda$ and introducing an orthonormal basis of K_λ .

Let $\mathcal{M}_n(\mathbb{C})$ be the set of $n \times n$ complex matrices and $M \in \mathcal{M}_n(\mathbb{C})$ with minimal polynomial $m = m_M$. Given any particular norm $|\cdot|$ on \mathbb{C}^n we consider the corresponding operator norm of M : $\|M\| = \|M\|_{(\mathbb{C}^n, |\cdot|) \rightarrow (\mathbb{C}^n, |\cdot|)}$. We introduce a norm $|\cdot|_*$ on $\mathbb{C}^{|m|} \cong K_\lambda$ by

$$|\vec{x}|_* := \left\| \sum_{j=1}^{|m|} x_j e_j \right\|_Y$$

where

$$e_1 = \frac{(1 - |\lambda_1|^2)^{1/2}}{1 - \bar{\lambda}_1 z}, \quad e_j := \frac{(1 - |\lambda_k|^2)^{1/2}}{1 - \bar{\lambda}_j z} \prod_{i=1}^{j-1} b_{\lambda_i}, \quad j = 1 \dots |m|$$

is the *Malmquist-Walsh family*: a particular orthonormal basis for K_λ [15, p. 137]. We denote by $\|\cdot\|_*$ the matrix norm induced by $|\cdot|_*$.

Corollary 5. *Let $M \in \mathcal{M}_n(\mathbb{C})$ be a complex $n \times n$ matrix, with minimal polynomial $m = \prod_{i=1}^{|m|} (z - \lambda_i)$, $\lambda_i \in \mathbb{D}$, and such that M admits a c functional calculus on X . If Ψ be any rational function whose poles are distinct from the zeroes of m then it holds*

$$\|\Psi(M)\| \leq c \|\Psi(\hat{M}_z)^*\|_*,$$

where

$$\left(\hat{M}_z\right)_{ij} = \begin{cases} 0 & \text{if } i < j \\ \lambda_i & \text{if } i = j \\ (1 - |\lambda_i|^2)^{1/2} (1 - |\lambda_j|^2)^{1/2} \prod_{\mu=j+1}^{i-1} (-\bar{\lambda}_\mu) & \text{if } i > j. \end{cases}$$

Observe that \hat{M}_z is the matrix of $M_z|K_\lambda$ with respect to the *Malmquist-Walsh basis* $(e_i)_{i=1}^{|m|}$. Its entries are computed in [22, Proposition III.5]. The above corollary provides the theoretical foundation for efficient computation of sharp upper estimates to norms of rational functions of matrices. It says that for given spectrum for any norm and any rational function a sharp upper estimate is given in terms of the $*$ -norm of the matrix $\Psi(\hat{M}_z)^*$. The quantity $\|\Psi(\hat{M}_z)^*\|_*$ can be computed with less effort as compared to the interpolation quantity I_X , which involves an optimization over an infinite set.

We recall that every Banach space contraction admits a functional calculus on the Wiener algebra W with constant $c = 1$. Indeed given $M \in \mathcal{M}_n(\mathbb{C})$ such that $\|M\| \leq 1$ for some induced matrix norm $\|\cdot\|$ and $g(z) = \sum_{j=0}^d \hat{g}(j)z^j$ we have

$$\|g(M)\| \leq \sum_{j=0}^d |\hat{g}(j)| \|M^j\| \leq \|g\|_W.$$

Corollary 5 applied to $X = W = l_A^1(0)$, whose exact predual is equipped with the norm $\|\cdot\|_{l_A^\infty}$, yields

$$\|\Psi(M)\| \leq \|\Psi(\hat{M}_z)^*\|_{l_A^\infty \rightarrow l_A^\infty}.$$

Based on this, the authors [23] have recently provided an explicit class of counterexamples to Schäffer's conjecture [21] about norms of inverses.

4. APPLICATIONS

We illustrate the application of Lemma 1 for some specific choices of X including the cases $X = H^2, H^\infty, W$ and more generally $X = l_A^q(-\beta)$.

Corollary 6. *In the setting of Lemma 1 there exists $f \in l_A^q(-\beta)$ such that $f(\lambda_i) = w_i$ and $\|f\|_{l_A^q(-\beta)} \leq C$ iff*

$$\left| \sum_{i=1}^n \alpha_i w_i \right| \leq C \left(\left| \sum_{i=1}^n \alpha_i \right|^p + \sum_{j \geq 1} j^{\beta p} \left| \sum_{i=1}^n \alpha_i \lambda_i^j \right|^p \right)^{1/p}$$

holds for any sequence of complex numbers $(\alpha_i)_{i=1}^n$.

This reduces the interpolation problem on the infinite-dimensional Beurling-Sobolev space to an optimization task over a finite-dimensional space of rational functions $K_\lambda = \text{span}\{k_{\lambda_i} : i = 1 \dots n\}$, which is much better accessible to computers. When X is the Hardy space $H^2 = l_A^2(0)$ this formula can be simplified. Then it holds that

$$\left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} \right\|_{H^2}^2 = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \bar{\alpha}_j \langle k_{\lambda_i}, k_{\lambda_j} \rangle = \sum_{i=1}^n \sum_{j=1}^n \frac{\alpha_i \bar{\alpha}_j}{1 - \bar{\lambda}_i \lambda_j}$$

and the above condition reduces to

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\bar{\alpha}_i \alpha_j (C^2 - \bar{w}_i w_j (1 - \bar{\lambda}_i \lambda_j))}{1 - \bar{\lambda}_i \lambda_j} \geq 0.$$

If in addition X is a Banach algebra the optimization task on K_λ is a consequence of the representation of $I_X(\lambda, w)$ in terms of the backward shift operator S^* in Theorem 2.

In the case $X = H^\infty$ classical theorem by G. Pick follows from Remark 3.

Corollary 7 (G. Pick [16], R. Nevanlinna [8, 9]). *Let $(\lambda_i)_{i=1}^n$ be a sequence of distinct points in \mathbb{D} and $(w_i)_{i=1}^n$ be a sequence in \mathbb{C} . There exists $f \in H^\infty$ such that $f(\lambda_i) = w_i$ and $\|f\|_{H^\infty} \leq C$ if and only if*

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\bar{\alpha}_i \alpha_j (C^2 - \bar{w}_i w_j)}{1 - \bar{\lambda}_i \lambda_j} \geq 0$$

for any sequence of complex numbers $(\alpha_i)_{i=1}^n$.

If X is the Wiener algebra W the following characterization follows directly from Theorem 2.

Corollary 8. *Let $(\lambda_i)_{i=1}^n$ be a sequence of distinct points in \mathbb{D} and $(w_i)_{i=1}^n$ be a sequence in \mathbb{C} . There exists $f \in W$ such that $f(\lambda_i) = w_i$ and $\|f\|_W \leq C$ if and only if*

$$\sup_{j \geq 0} \left| \sum_{i=1}^n \alpha_i w_i \lambda_i^j \right| \leq C \sup_{j \geq 0} \left| \sum_{i=1}^n \alpha_i \lambda_i^j \right|$$

for any sequence of complex numbers $(\alpha_i)_{i=1}^n$.

5. INTERPOLATION ON A SEQUENCE WITH ACCUMULATION POINT

For finite data Lemma 1 says that given $C > 0$ there exists $f \in X$ such that $f(\lambda_i) = w_i$ and $\|f\|_X \leq C$ iff

$$\left| \sum_{i=1}^n \alpha_i \overline{f(\lambda_i)} \right| \leq C \left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} \right\|_Y$$

for any sequence of complex numbers $(\alpha_i)_{i=1}^n$. If $\lambda = (\lambda_i)_{i \geq 1}$ is an infinite sequence of distinct points which contains an accumulation point in \mathbb{D} then f exists and is unique. Furthermore f satisfies the above inequality for any sequence $(z_i)_i$ in \mathbb{D} . We state this result in the case that X is a Beurling-Sobolev space $X = l_A^q(-\beta)$, $1 \leq q \leq \infty$, $\beta \in \mathbb{R}$. The proof, which we sketch below, is an adaptation of [5, Lemma 4.1] and of [5, Theorem 4.2].

Theorem 9. *Let $\lambda = (\lambda_i)_{i \geq 1}$ be a sequence of distinct points with an accumulation point in \mathbb{D} , $w = (w_i)_{i \geq 1}$ be a sequence in \mathbb{C} and $C > 0$. Let $X = l_A^q(-\beta)$ and p, q be conjugate exponents, $\frac{1}{p} + \frac{1}{q} = 1$. If*

$$(5.1) \quad \left| \sum_{i=1}^n \alpha_i w_i \right| \leq C \left(\left| \sum_{i=1}^n \alpha_i \right|^p + \sum_{j \geq 1} j^{\beta p} \left| \sum_{i=1}^n \alpha_i \lambda_i^j \right|^p \right)^{1/p}$$

for any $n \geq 1$ and any sequence of complex numbers $(\alpha_i)_{i \geq 1}$, then there exists a unique function $f \in l_A^q(-\beta)$ such that $f(\lambda_i) = w_i$ and $\|f\|_{l_A^q(-\beta)} \leq C$. Moreover the inequality

$$\left| \sum_{i=1}^n \alpha_i f(z_i) \right| \leq C \left(\left| \sum_{i=1}^n \alpha_i \right|^p + \sum_{j \geq 1} j^{\beta p} \left| \sum_{i=1}^n \alpha_i z_i^j \right|^p \right)^{1/p}$$

holds for all sequence of distinct points $(z_i) \subset \mathbb{D}$ and all sequence of complex numbers (α_i) .

The existence of f is a consequence of the following observations (see [5, Lemma 4.1] for details): Assuming first – as in [5, Lemma 4.1] – that the sequence $(\lambda_i)_{i \geq 1}$ is convergent, say to $\lambda_0 \in \mathbb{D}$, the sequence $(w_i)_i$ is bounded because $|w_i| \leq \|k_{\lambda_i}\|_{l_A^p(\beta)}$ for any $i \geq 1$. It

has at least one convergent subsequence $(w_{i'})$ whose limit we shall denote by w_0 . Using (5.1) we write

$$\left| \sum_{i=1}^n \alpha_i \overline{w_i} + \alpha_0 \overline{w_{i'}} \right| \leq \left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} + \alpha_0 k_{\lambda_{i'}} \right\|_{l_A^p(\beta)}$$

and observe that since $\lambda_{i'} \rightarrow \lambda_0$ as $i' \rightarrow \infty$ we can extend (5.1) by continuity to include the index value $i = 0$:

$$\left| \sum_{i=1}^n \alpha_i \overline{w_i} + \alpha_0 \overline{w_0} \right| \leq \left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} + \alpha_0 k_{\lambda_0} \right\|_{l_A^p(\beta)}.$$

Observe that for any $n \geq 1$ we have

$$\begin{aligned} |w_n - w_0|^p &\leq \|k_{\lambda_n} - k_{\lambda_0}\|_{l_A^p(\beta)}^p \\ &= \sum_{j \geq 1} j^{\beta p} |\lambda_n - \lambda_0|^{j p} \\ &= |\lambda_n - \lambda_0|^p \sum_{j \geq 0} (j+1)^{\beta p} |\lambda_n - \lambda_0|^{p j} \end{aligned}$$

but $\sum_{j \geq 0} (j+1)^{\beta p} |\lambda_n - \lambda_0|^{p j} \rightarrow_{n \rightarrow \infty} 1$ and the whole sequence (w_n) actually converges to w_0 . Similarly it is shown that the sequence of divided differences

$$\Delta(i, 1) = \frac{f(\lambda_i) - f(\lambda_0)}{\lambda_i - \lambda_0} = \frac{w_i - w_0}{\lambda_i - \lambda_0}$$

is also convergent. Denoting by w_{-1} its limit f must satisfy $f'(\lambda_0) = w_{-1}$. Repeated application of this process leads to assertions about higher-order divided differences

$$\Delta(i, j+1) = \frac{\Delta(i, j) - w_{-j}}{\lambda_i - \lambda_0}, \quad j \geq 1$$

where w_{-j} is the limit of the sequence $(\Delta(i, j))_{i \geq 1}$. As before $(\Delta(i, j+1))_{i \geq 1}$ is convergent. Denoting by w_{-j-1} its limit we find that f must satisfy $f^{(j+1)}(\lambda_0) = (j+1)! w_{-j-1}$. At the end of this process Inequality (5.1) can be extended by continuity so that its left-hand side becomes

$$\left| \sum_{i=1}^n \alpha_i \overline{w_i} + \sum_{i \geq 0} \beta_i \frac{f^{(i)}(\lambda_0)}{i!} \right|,$$

where $(\beta_i)_{i \geq 1}$ is any arbitrary sequence of complex numbers. Expanding f in a Taylor series around λ_0 in terms of w_{-i} one can show that f solves the interpolation problem and express $f(z_n)$ as a linear combination of $\left(\frac{f^{(i)}(\lambda_0)}{i!} \right)_{i \geq 0}$ to generalize (5.1) such that it holds for any sequence (z_n) of distinct points in a neighborhood of λ_0 .

To complete the proof of Theorem 9 it remains to follow the steps of [5, Theorem 4.2]: Let λ_0 be an accumulation point in \mathbb{D} of the sequence (λ_i) , let $r > 0$ be such that the closure of $D(\lambda_0, r) =: \{z : |z - \lambda_0| < r\}$ is contained in \mathbb{D} , and let $(\lambda_{i'})$ be a convergent subsequence of (λ_i) with $\lambda_{i'} \rightarrow \lambda_0$ and $\{\lambda_{i'}\} \subset D(\lambda_0, r)$. We may use the above reasoning to conclude that there is an analytic function f on $D(\lambda_0, r)$ such that $f(\lambda_i) = w_i$ for all $\lambda_i \in D(\lambda_0, r)$ and which satisfies Inequality (5.1) for all sequence (z_i) in $D(\lambda_0, r)$. Using

the standard arguments of analytic continuation from [5, p. 566] it is shown that f can be continued to all of \mathbb{D} and that Inequality (5.1) holds throughout \mathbb{D} .

6. PROOF OF COROLLARY 7

Proof of Corollary 7. Step 1. We prove the lower bound

$$I_{H^\infty}(\lambda, w) \geq \sup_{(\alpha_i) \in \mathbb{C}^n} \frac{\sum_{i=1}^n \sum_{j=1}^n \frac{\overline{\alpha_i} \alpha_j \overline{w_i} w_j}{1 - \overline{\lambda_i} \lambda_j}}{\sum_{i=1}^n \sum_{j=1}^n \frac{\overline{\alpha_i} \alpha_j}{1 - \overline{\lambda_i} \lambda_j}}.$$

First we observe that

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i \overline{w_i} k_{\lambda_i} \right\|_{H^2}^2 &= \sum_{i=1}^n \sum_{j=1}^n \overline{\alpha_j} \alpha_i \overline{w_i} w_j \langle k_{\lambda_i}, k_{\lambda_j} \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{\alpha_i \overline{\alpha_j} \overline{w_i} w_j}{1 - \overline{\lambda_i} \lambda_j}. \end{aligned}$$

Second because of the Hahn-Banach formula for the norm in $H^2 = (H^2)'$ we have

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i \overline{w_i} k_{\lambda_i} \right\|_{H^2} &= \sup_{\|h\|_{H^2} \leq 1} \left\langle \sum_{i=1}^n \alpha_i \overline{w_i} k_{\lambda_i}, h \right\rangle \\ &= \sup_{\|h\|_{H^2} \leq 1} \left| \sum_{i=1}^n \alpha_i \overline{w_i} \overline{h(\lambda_i)} \right| \\ &= \sup_{\|h\|_{H^2} \leq 1} \left| \sum_{i=1}^n \alpha_i \overline{f(\lambda_i) h(\lambda_i)} \right| \\ &= \sup_{\|h\|_{H^2} \leq 1} \left\langle \sum_{i=1}^n \alpha_i k_{\lambda_i}, fh \right\rangle \end{aligned}$$

for any $f \in H^\infty$ such that $f(\lambda_i) = w_i$. Applying Cauchy-Schwarz inequality we find

$$\begin{aligned} \left| \left\langle \sum_{i=1}^n \alpha_i k_{\lambda_i}, fh \right\rangle \right| &\leq \left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} \right\|_{H^2} \|fh\|_{H^2} \\ &\leq \left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} \right\|_{H^2} \|f\|_{H^\infty} \|h\|_{H^2} \end{aligned}$$

which yields

$$\left\| \sum_{i=1}^n \alpha_i \overline{w_i} k_{\lambda_i} \right\|_{H^2} \leq \left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} \right\|_{H^2} \|f\|_{H^\infty}$$

for any $f \in H^\infty$ such that $f(\lambda_i) = w_i$. Therefore

$$I_{H^\infty}(\lambda, w) \geq \frac{\left\| \sum_{i=1}^n \alpha_i \overline{w_i} k_{\lambda_i} \right\|_{H^2}}{\left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} \right\|_{H^2}}$$

for all $(\alpha_i) \in \mathbb{C}^n$. It remains to replace α_i by $\overline{\alpha_i}$ to complete the proof of Step 1.

Step 2. We prove the upper bound

$$I_{H^\infty}(\lambda, w) \leq \sup_{(\alpha_i) \in \mathbb{C}^n} \frac{\sum_{i=1}^n \sum_{j=1}^n \frac{\overline{\alpha_i} \alpha_j \overline{w_i} w_j}{1 - \overline{\lambda_i} \lambda_j}}{\sum_{i=1}^n \sum_{j=1}^n \frac{\overline{\alpha_i} \alpha_j}{1 - \overline{\lambda_i} \lambda_j}}.$$

To this aim we observe that according to Lemma 1 the interpolation quantity $I_{H^\infty}(\lambda, w)$ is the smallest $C > 0$ such that

$$(6.1) \quad \left| \sum_{i=1}^n \alpha_i \overline{w_i} \right| \leq C \left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} \right\|_{L^1 / \overline{H_0^1}} \\ = C \inf \left\{ \left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} + \overline{z}g \right\|_{L^1} : g \in H^1 \right\}$$

for any sequence of complex numbers $(\alpha_i)_i$. We show that

$$C = \sup_{(\alpha_i) \in \mathbb{C}^n} \frac{\sum_{i=1}^n \sum_{j=1}^n \frac{\overline{\alpha_i} \alpha_j \overline{w_i} w_j}{1 - \overline{\lambda_i} \lambda_j}}{\sum_{i=1}^n \sum_{j=1}^n \frac{\overline{\alpha_i} \alpha_j}{1 - \overline{\lambda_i} \lambda_j}}$$

satisfies (6.1). We know from Lemma 1 that there exists $f \in H^\infty$ such that $f(\lambda_i) = w_i$ and $I_{H^\infty}(\lambda, w) = \|f\|_\infty$. We have

$$\sum_{i=1}^n \overline{\alpha_i} w_i = \sum_{i=1}^n \overline{\alpha_i} f(\lambda_i) \\ = \left\langle f, \sum_{i=1}^n \alpha_i k_{\lambda_i} \right\rangle = \left\langle f, \sum_{i=1}^n \alpha_i k_{\lambda_i} + \overline{z}g \right\rangle, \quad \forall g \in H^1$$

because $\langle f, \overline{z}g \rangle = \langle zfg, 1 \rangle = 0$. Let

$$B = \prod_{i=1}^n \frac{z - \lambda_i}{1 - \overline{\lambda_i} z}$$

be the Blaschke product corresponding to $\lambda = (\lambda_i)_{i=1}^n$ and consider the space

$$K = K_\lambda = \text{span}(k_{\lambda_i}, i = 1 \dots n) = H^2 \cap (BH^2)^\perp.$$

Observing that $B\overline{B} = 1$ on the unit circle $\partial\mathbb{D}$ we get for any $g \in H^1$

$$\sum_{i=1}^n \overline{\alpha_i} w_i = \left\langle f, B \left(\sum_{i=1}^n \alpha_i \overline{B} k_{\lambda_i} + \overline{z}B g \right) \right\rangle \\ = \left\langle f \left(\sum_{i=1}^n \overline{\alpha_i} B \overline{k_{\lambda_i}} + zB g \right), B \right\rangle.$$

It is easy to check that $\sum_{i=1}^n \overline{\alpha_i} B \overline{k_{\lambda_i}} \in zK$ and that $zB g \in zH^1$. We put

$$\varphi = \sum_{i=1}^n \overline{\alpha_i} B \overline{k_{\lambda_i}} + zB g = z\varphi_0$$

where $\varphi_0 \in K + H^1$ so that

$$\sum_{i=1}^n \overline{\alpha_i} w_i = \langle z f \varphi_0, B \rangle.$$

Following D. Sarason we transform the above right-hand side adapting his proof of [19, Lemma 2.1] for completeness. It is well-known [14, Corollary 3.7.4] that the zeroes $(\mu_i)_{i \geq 1}$ of $\varphi_0 \in H^1$ satisfy the Blaschke condition $\sum_i (1 - |\mu_i|) < \infty$. Thus defining the Blaschke product $B_0 = \prod_{i \geq 1} \frac{z - \mu_i}{1 - \overline{\mu_i} z}$ the function $\frac{\varphi_0}{B_0}$ belongs also to the Hardy space H^1 and does not vanish on \mathbb{D} . Therefore there exists $h \in \mathcal{H}ol(\mathbb{D})$ such that $\frac{\varphi_0}{B_0} = h^2$. We write $\varphi_0 = f_1 f_2$ where $f_1 = B_0 h$ and $f_2 = h$. We obviously have $|f_1(z)|^2 = |f_2(z)|^2 = |\varphi_0(z)|$ for any $z \in \partial\mathbb{D}$. Going back to the last expression of $\sum_{i=1}^n \overline{\alpha_i} w_i$ we denote by P the orthogonal projection from H^2 to K and write:

$$\begin{aligned} \sum_{i=1}^n \overline{\alpha_i} w_i &= \langle z f f_1 f_2, B \rangle \\ &= \langle z f (f_1 - P f_1 + P f_1) f_2, B \rangle \\ &= \langle z f (P f_1) f_2, B \rangle \end{aligned}$$

because $f_1 - P f_1 \in BH^2$ and $B \perp z f_2 BH^2$. For the same reason

$$\sum_{i=1}^n \overline{\alpha_i} w_i = \langle z f (P f_1) (P f_2), B \rangle = \left\langle z f (P f_1), B \overline{(P f_2)} \right\rangle.$$

It is easy to check that $B \overline{(P f_2)} \in zK$. Therefore we put $g_1 = P f_1$, $g_2 = \frac{B \overline{(P f_2)}}{z}$: g_1 and g_2 belong to K and for $i = 1, 2$ we have

$$\|g_i\|_{H^2} \leq \|P f_i\|_{H^2} \leq \|f_i\|_{H^2} = \sqrt{\|\varphi_0\|_{H^1}}.$$

To conclude we have

$$\sum_{i=1}^n \overline{\alpha_i} w_i = \langle f g_1, g_2 \rangle, \quad g_i \in K.$$

There exists (β_i) such that $g_2 = \sum_{i=1}^n \beta_i k_{\lambda_i}$ and therefore

$$\sum_{i=1}^n \overline{\alpha_i} w_i = \sum_{i=1}^n \overline{\beta_i} w_i g_1(\lambda_i) = \left\langle g_1, \sum_{i=1}^n \beta_i \overline{w_i} k_{\lambda_i} \right\rangle.$$

Applying Cauchy-Schwarz inequality we find

$$\begin{aligned}
\left| \sum_{i=1}^n \overline{\alpha_i} w_i \right|^2 &\leq \|g_1\|_{H^2}^2 \left\| \sum_{i=1}^n \beta_i \overline{w_i} k_{\lambda_i} \right\|_{H^2}^2 \\
&\leq \|\varphi_0\|_{H^1} \frac{\left\| \sum_{i=1}^n \beta_i \overline{w_i} k_{\lambda_i} \right\|_{H^2}^2}{\|g_2\|_{H^2}^2} \|g_2\|_{H^2}^2 \\
&\leq \|\varphi_0\|_{H^1}^2 \frac{\left\| \sum_{i=1}^n \beta_i \overline{w_i} k_{\lambda_i} \right\|_{H^2}^2}{\left\| \sum_{i=1}^n \beta_i k_{\lambda_i} \right\|_{H^2}^2} \\
&\leq \|\varphi_0\|_{H^1}^2 \sup_{(\alpha_i) \in \mathbb{C}^n} \frac{\sum_{i=1}^n \sum_{j=1}^n \frac{\overline{\alpha_i} \alpha_j \overline{w_i} w_j}{1 - \overline{\lambda_i} \lambda_j}}{\sum_{i=1}^n \sum_{j=1}^n \frac{\overline{\alpha_i} \alpha_j}{1 - \overline{\lambda_i} \lambda_j}}.
\end{aligned}$$

To complete the proof it remains to recall that

$$\begin{aligned}
\|\varphi_0\|_{H^1} &= \|z\varphi_0\|_{H^1} = \|\varphi\|_{H^1} \\
&= \left\| \sum_{i=1}^n \overline{\alpha_i} B \overline{k_{\lambda_i}} + zBg \right\|_{H^1} = \left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} + \overline{zg} \right\|_{L^1}
\end{aligned}$$

for any $g \in H^1$. □

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