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Abstract. This paper supersedes the published article [7]. We show that a divergence-free measure on the plane is a continuous sum of unit tangent vector fields on rectifiable Jordan curves. This *loop* decomposition is more precise than the general decomposition in terms of elementary solenoids given by S.K. Smirnov when applied to the planar case, and is made fairly concrete in terms of measure-theoretic components of the level sets of the function whose gradient is the $\pi/2$-rotation of the original divergence-free measure. The proof rests on a version of the co-area formula for homogeneous BV functions, and on the approximate continuity of measure theoretic connected components of suplevel sets of such functions with respect to the level. We apply these results to inverse potential problems whose source term is the divergence of some unknown (vector-valued) measure; e.g., inverse magnetization problems when magnetizations are modeled by $\mathbb{R}^3$-valued Borel measures. We investigate methods for recovering a magnetization $\boldsymbol{\mu}$ by penalizing the measure theoretic total variation norm $\|\boldsymbol{\mu}\|_{TV}$. In particular, if a magnetization is supported in a plane, then TV-regularization schemes always have a unique minimizer, even in the presence of noise. It is further shown that TV-norm minimization (among magnetizations generating the same field) uniquely recovers planar magnetizations in the following cases: when the magnetization is carried by a collection of sufficiently separated line segments and a set that is purely 1-unrectifiable, or when a superset of the support is tree-like. We note that such magnetizations can be recovered via TV-regularization schemes in the zero noise limit by taking the regularization parameter to zero. This suggests definitions of sparsity in the present infinite dimensional context, that generate results akin to compressed sensing.

1. Introduction

This paper supersedes the article [7]. The main addition is Proposition 4.6, that requires establishing Propositions 3.6 and 4.4 together with item (*iii*) in Theorem 4.5, as well as Lemma 4.3. Theorem 3.7 may be of independent interest. The purpose of Proposition 4.6 is to give a detailed, and inasmuch as possible concrete description of the loop deccomposition of planar divergence-free $\mathbb{R}^2$-valued measures. Measurability issues are not completely straightforward here, and they rely on approximate continuity of measure-theoretic connected components of suplevel sets of homogeneous BV-functions, at almost every level. This approximate continuity property, summarized it in Theorem 3.7, is perhaps new.

After they completed the present research, the authors became aware of the interesting manuscript [8] where the loop decomposition of planar divergence-free measures is proven by different, less explicit methods. Still, the result of [8] cannot be substituted for Theorem 4.5 of this paper, because we establish specific properties of the representing measure ρ in (61) that are used in

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Section 5. We also mention that an earlier version of Theorem 4.5 already appears in the 2019 thesis [30].

The present work deals with the structure of finite divergence-free measures in the plane, and applications thereof to inverse magnetization problems on thin plates. These are prototypical of inverse potential problems with source term in divergence form, and have been the main motivation of the authors to develop a purely measure-geometric result like Theorem 4.5. The latter asserts that a planar divergence-free measure can be decomposed as a superposition of elementary "loops"; *i.e.*, unit tangent vector fields on rectifiable Jordan curves. This result is more precise than the general structure theorem for solenoids given by Smirnov in [29] (valid in any dimension), and is hinted at on page 843 of that reference. Because divergence-free distributions in the plane are rotations by $\pi/2$ of distributional gradients, one is quickly left to decompose gradients of "homogeneous" BV-functions; *i.e.*, locally integrable functions whose partial derivatives are finite measures. To do this, we combine a version of the co-area formula for homogeneous BV-functions (Theorem 3.4) with a decomposition into Jordan curves of the measure-theoretic boundary of planar sets of finite perimeter given in [1]. The latter is a special case of the decomposition of 1-dimensional integral currents into indecomposable elements [18, 4.2.25], in which the pattern of orientations has special structure. To handle measurability issues in the integral expressing the decomposition of a divergence-free measure as a superposition of loops, and to relate the tangent fields of the loops to the polar decomposition of the measure, we also establish (in any dimension) an approximate continuity property of measure-theoretic connected components of suplevel sets for homogeneous BV-functions (Theorem 3.7), which is interesting in its own right.

The loop decomposition of planar divergence-free measures has interesting applications to inverse magnetization problems for thin plates, when magnetizations are modelled by $\mathbb{R}^3$-valued measures supported on a set S (in the thin plate case, $S \subset \mathbb{R}^2$). Then, the inverse magnetization problem consists in recovering such a measure, say $\boldsymbol{\mu}$, from knowledge of the magnetic field $\mathbf{b}(\boldsymbol{\mu})$ that it generates, see Section 1.1 for details. Magnetizations supported in a plane generate the zero magnetic field if and only if they are tangent to that plane and divergence-free there (see Lemma 2.1). Thus, the kernel of the forward operator mapping $\boldsymbol{\mu}$ to $\mathbf{b}(\boldsymbol{\mu})$ consists precisely of planar divergence-free measures in this case. The loop decomposition gives insight on the structure of this kernel, enabling us to give sufficient conditions for a magnetization to be *TV-minimal on S*; *i.e.*, the magnetization has minimum total variation among those magnetizations supported on S that generate the same field. When a TV-minimal magnetization on S is unique among magnetizations generating the same field, we call it *strictly TV-minimal on S*. By standard regularization theory, strictly TV-minimal magnetizations can be recovered by solving a sequence of minimization problems for the so-called regularizing functional, which is the sum of the quadratic residuals and a penalty term consisting of the product of a regularization parameter $\lambda > 0$ and the total variation of the unknown, see (4). Then, any sequence of minimizers of the regularizing functional converges weak-$*$ to the strictly TV-minimal measure generating the data (when it exists), as the regularizing parameter and the noise tend jointly to zero in a suitable manner, see *e.g.* [11]. In short: regularizing schemes that penalize the total variation are consistent to recover strictly TV-minimal magnetizations, and thus, any assumption ensuring strict TV-minimality gives rise to a consistency result. For the larger class of magnetizations supported on a slender set S (see Section 1.1 for a definition), such a consistency result is obtained in [6, Theorem 2.6] by showing, using reference [29], that magnetizations supported on a purely 1-unrectifiable set are strictly TV-minimal. Specializing to the case of planar S and appealing to the loop decomposition will allow us to obtain more general conditions, proving for instance that magnetizations carried by the union of a purely 1-unrectifiable set and a collection of sufficiently separated line segments are strictly TV-minimal (Corollary 5.4 and Theorem 5.2).

The results just mentioned are reminiscent of compressed sensing, where underdetermined systems of linear equations in $\mathbb{R}^n$ are approximately solved by minimizing the residuals while penalizing the l^1-norm. This favors the recovery of sparse solutions (*i.e.* solutions having a large number of zero components) when they exist, see *e.g.* [19] and [9]. In this connection, the gist of [6, Theorem 2.6] and its sharpening described above for the planar case is to define notions of "sparsity" in the present, infinite-dimensional context. Our results warrant the use of regularizing schemes that penalize the total variation (a natural analog of the l^1-norm), in order to recover magnetizations which are sparse according such definitions.

Our second application of the loop decomposition to inverse magnetization problems on thin plates is to prove that, for each value of the regularization parameter, the minimizer of the regularizing functional is unique (Theorem 5.7). This result is important for algorithmic approaches to the inverse magnetization problem, because it tells us that for every choice of the regularization parameter there is a unique estimate of the unknown magnetization based on the regularization scheme (5). It is also surprising, for in the case that a magnetization is TV-minimal, but not strictly TV minimal, one would rather expect the regularizing functional to have several minimizers, at least for small values of the regularizing parameter.

To conclude this introduction, let us stress that magnetizations supported in a plane are commonly considered in paleomagnetic studies, where thin slabs of rock are modeled by planar regions [5, 24, 31, 25]. It would be interesting to carry over the contents of the present paper to more general slender surfaces in $\mathbb{R}^3$ than the plane, as the results could apply to other situations in geosciences or medical imaging. In practice, the development of numerically effective algorithms for these inverse problem raises delicate issues of discretization. Such considerations are not addressed in this paper, but will be taken up in future work.

1.1. **Background and Overview of Results.** Let us first describe the inverse magnetization problem, which serves as a motivation for the results to come. For a closed subset $S \subset \mathbb{R}^3$, let $\mathcal{M}(S)$ denote the space of finite signed Borel measures supported on S. We shall use the space $\mathcal{M}(S)^3$ of $\mathbb{R}^3$-valued measures supported on S to model physical magnetizations distributed on S and shall often use "magnetization on S" interchangeably with "element of $\mathcal{M}(S)^3$". For $\boldsymbol{\mu} \in \mathcal{M}(S)^3$, we let $|\boldsymbol{\mu}|$ denote the *total variation measure* of $\boldsymbol{\mu}$. The latter is a positive measure, and we put $\|\boldsymbol{\mu}\|_{TV} := |\boldsymbol{\mu}|(\mathbb{R}^3)$ for the *total variation* of $\boldsymbol{\mu}$, see Section 1.2.

The *magnetic field* $\mathbf{b}(\boldsymbol{\mu})$ generated by a magnetization $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ is defined, at a point x not in the support of $\boldsymbol{\mu}$, in terms of the *scalar magnetic potential* $\Phi(\boldsymbol{\mu})$ by (see [22]):

$$\mathbf{b}(\boldsymbol{\mu})(x) = -\mu_0 \nabla \Phi(\boldsymbol{\mu})(x), \qquad x \notin \operatorname{supp} \boldsymbol{\mu}, \tag{1}$$

where μ_0 is the *magnetic constant* and ∇ indicates the gradient. Here, $\Phi(\boldsymbol{\mu})(x)$ is given by

$$\Phi(\boldsymbol{\mu})(x) := \frac{1}{4\pi} \int \nabla_y \frac{1}{|x-y|} \cdot d\boldsymbol{\mu}(y) = \frac{1}{4\pi} \int \frac{x-y}{|x-y|^3} \cdot d\boldsymbol{\mu}(y), \tag{2}$$

where, for $x, y \in \mathbb{R}^3$, $x \cdot y$ and $|x|$ denote the Euclidean scalar product and norm and ∇_y the gradient with respect to y. Clearly, $\Phi(\boldsymbol{\mu})$ and the components of $\mathbf{b}(\boldsymbol{\mu})$ are harmonic functions on $\mathbb{R}^3 \setminus S$. Moreover, formula (2) defines $\Phi(\boldsymbol{\mu})$ on the whole of $\mathbb{R}^3$ as a member of $L^2(\mathbb{R}^3) + L^1(\mathbb{R}^3)$ (see [6, Proposition 2.1]) so that $\mathbf{b}(\boldsymbol{\mu})$, initially defined on $\mathbb{R}^3 \setminus S$, extends to a $\mathbb{R}^3$-valued divergence-free distribution on $\mathbb{R}^3$. Indeed, we may write

$$\Delta \Phi = \nabla \cdot \boldsymbol{\mu} \quad \text{and} \quad \mathbf{b}(\boldsymbol{\mu}) = \mu_0 \left(\boldsymbol{\mu} - \nabla \Phi(\boldsymbol{\mu}) \right), \tag{3}$$

where $\nabla\cdot\boldsymbol{\mu}$ indicates the divergence of $\boldsymbol{\mu}$. Note that (3) yields a Helmholtz-Hodge decomposition of $\boldsymbol{\mu}$, as the sum of a gradient and a divergence-free distribution. However, neither term is a measure in general but rather a distribution of order -1.

The inverse magnetization problem is to recover $\boldsymbol{\mu}$ from measurements of $\mathbf{b}(\boldsymbol{\mu})$ taken on a set $Q \subset \mathbb{R}^3 \setminus S$ which, due to the oriented nature of sensors (coils), are usually observed in one direction only, say along some unit vector $v \in \mathbb{R}^3$. We assume for simplicity that v is the same at each measurement point. For instance, it is so in usual Scanning Magnetic Microscopy experiments (SMM) where data consist of point-wise values of the normal component of the magnetic field on a planar region not intersecting S, see [24, 31, 25]. Geometric conditions on Q, S and v, ensuring that such measurements suffice to determine $\mathbf{b}(\boldsymbol{\mu})$ in the entire region $\mathbb{R}^3 \setminus S$, are given in [6, Lemma 2.3], and recalled for convenience when S is planar in Section 5.2 further below. In the remainder of this introduction, we assume that these assumptions are satisfied.

Still, the mapping $\boldsymbol{\mu} \to \mathbf{b}(\boldsymbol{\mu})$ is generally not injective, which is a major difficulty with this inverse problem. In this connection, we say that $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{M}(S)^3$ are *S-equivalent* if $\mathbf{b}(\boldsymbol{\mu})$ and $\mathbf{b}(\boldsymbol{\nu})$ agree on $\mathbb{R}^3 \setminus S$. A magnetization $\boldsymbol{\mu}$ is said to be *S-silent* if $\boldsymbol{\mu}$ is S-equivalent to the zero magnetization; i.e., if $\mathbf{b}(\boldsymbol{\mu})$ vanishes on $\mathbb{R}^3 \setminus S$.

Since no nonzero harmonic function lies in $L^2(\mathbb{R}^3)+L^1(\mathbb{R}^3)$, it follows from (3) that a divergence-free magnetization is S-silent. A partial converse is given in [6, Theorem 2.2], namely a S-silent magnetization is divergence-free provided that S is *slender*, meaning it has Lebesgue measure zero and each connected component of $\mathbb{R}^3 \setminus S$ has infinite Lebesgue measure. The slenderness assumption is a strong one: for instance it rules out the case where S is a volumic sample or a closed surface. However, it is satisfied in important special cases, for example in paleomagnetic studies, as mentioned already, or in Geomagnetism where some regions of the Earth's crust are assumed to be non-magnetic (or much less magnetic) than the others [20], or even in Electro-Encephalography where sources of primary current are often considered to lie on the surface of the encephalon (which is closed and therefore not slender) but their support should arguably leave out the brain stem connecting to the spinal cord (therefore the support is contained in a slender set).

In [29], Smirnov describes divergence-free measures in $\mathbb{R}^n$, also known as *solenoids*, in terms of integrals of elementary components that are absolutely continuous with respect to 1-dimensional Hausdorff measure $\mathcal{H}^1$. Consequently, if S is slender and $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ is such that there is a *purely 1-unrectifiable* set (*i.e.*, whose intersection with any 1-rectifiable set has $\mathcal{H}^1$- measure zero, see [26]) of full $|\boldsymbol{\mu}|$ measure, then $\boldsymbol{\mu}$ is mutually singular to every S-silent magnetization and so has minimum total variation amongst all magnetizations that are S-equivalent to $\boldsymbol{\mu}$. This observation led the authors in [6] to consider the following extremal problem involving the quantity $M_S(\boldsymbol{\mu})$, defined for $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ by

$$M_S(\boldsymbol{\mu}) := \inf\{\|\boldsymbol{\nu}\|_{TV} : \boldsymbol{\nu} \text{ is } S\text{-equivalent to } \boldsymbol{\mu}\}.$$

Extremal Problem 1. Given $\boldsymbol{\mu}_0 \in \mathcal{M}(S)^3$, find $\boldsymbol{\mu}$ that is S-equivalent to $\boldsymbol{\mu}_0$ satisfying

$$\|\boldsymbol{\mu}\|_{TV} = M_S(\boldsymbol{\mu}_0).$$

A solution to Extremal Problem 1 is, by definition, TV-minimal on S and is strictly TV-minimal on S if this solution is unique. When $S \subset \mathbb{R}^3$ is slender and $\boldsymbol{\mu}_0 \in \mathcal{M}(S)^3$, we find that $\boldsymbol{\mu}_0$ is strictly TV-minimal on S for the three cases listed below. Here case (a) is essentially [6, Theorem 2.6] and a special case of Theorem 5.2 to come, while (b) is contained in [6, Theorem 2.11] and (c) follows from Corollary 5.4 further below.

(a) there is a purely 1-unrectifiable set of full $|\boldsymbol{\mu}_0|$ measure;

(b) the set S is a finite disjoint union of compact sets $S_1, \dots S_k$ and

$$\boldsymbol{\mu}_0\lfloor_{S_i} = \mathbf{u}_i|\boldsymbol{\mu}_0|\lfloor_{S_i},$$

for some set of unit vectors $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{R}^3$, in which case we say $\boldsymbol{\mu}_0$ is *piecewise unidirectional*;

(c) $\boldsymbol{\mu}_0$ has a carrier contained in a countable union of coplanar disjoint line segments L_k such that the distance from any L_k to any L_j, $j \neq k$, is greater than or equal to $\mathcal{H}^1(L_k)$.

Corollary 5.4 also implies that (a) can be combined (c), namely if a measure satisfies (c) and we add to it a measure on S carried by a purely 1-unrectifiable set, then we get a measure which is strictly TV-minimal again.

Now, for ρ a positive measure on Q, let $A : \mathcal{M}(S)^3 \to L^2(Q, \rho)$ be the *forward operator* mapping $\boldsymbol{\mu}$ to the restriction of $\mathbf{b}(\boldsymbol{\mu}) \cdot v$ on Q (see (74)). The measure ρ does not play a significant role in what follows (e.g., it could be chosen to be Lebesgue measure on Q), but it is important for practical applications. To recover solutions of Extremal Problem 1 knowing the restriction f of $\mathbf{b}(\boldsymbol{\mu}_0) \cdot v$ to Q, the theory of regularization for convex problems [11] suggests to minimize with respect to $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ the functional

$$\mathcal{F}_{f,\lambda}(\boldsymbol{\mu}) := \|f - A\boldsymbol{\mu}\|^2_{L^2(Q,\rho)} + \lambda\|\boldsymbol{\mu}\|_{TV} \tag{4}$$

for some suitable value of the *regularization parameter* $\lambda > 0$. That is, we consider:

Extremal Problem 2. Given $f \in L^2(Q)$ and $\lambda > 0$, find $\boldsymbol{\mu}_\lambda \in \mathcal{M}(S)^3$ such that

$$\mathcal{F}_{f,\lambda}(\boldsymbol{\mu}_\lambda) = \inf_{\boldsymbol{\mu}\in\mathcal{M}(S)^3} \mathcal{F}_{f,\lambda}(\boldsymbol{\mu}). \tag{5}$$

When Q and S are positively separated, the existence of at least one minimizer is a consequence of the weak-$*$ compactness of the unit ball in $\mathcal{M}(S)^3$ see *e.g.* [10, Proposition 3.6]. Solving Extremal Problem 2 is a particular *regularization scheme* for the Inverse Magnetization Problem, namely one that penalizes the total variation of the unknown.

It is standard that if $f = A\boldsymbol{\mu}_0$ and $\lambda_n \to 0$, then any subsequence of $\boldsymbol{\mu}_{\lambda_n}$ has a subsequence converging weak-$*$ to a solution of Extremal Problem 1. To account for measurement noise, one usually replaces f by $f_n = A\boldsymbol{\mu}_0 + e_n$, and then the same result holds for a sequence $\boldsymbol{\mu}_n$ minimizing (4) with $f = f_n$ and $\lambda = \lambda_n$, provided that both λ_n and $\|e_n\lambda_n^{-1/2}\|_{L^2(Q,\rho)}$ tend to 0, see [11, Theorems 2&5] or [21, Theorems 3.5&4.4]. In particular, if there is a unique solution $\boldsymbol{\mu}_0$ of Extremal Problem 1, then we get weak-$*$ convergence of $\boldsymbol{\mu}_n$ to $\boldsymbol{\mu}_0$. A stronger result, involving weak-$*$ convergence of the total variation measure $|\boldsymbol{\mu}_n|$, can be found in [6, Theorem 4.3]. To recap, we have a consistency property asserting that a magnetization meeting a certain assumptions (*e.g.* either (a), (b) or (c) above) can be approximately recovered *via* the regularization scheme (5), when the noise is small and the regularization parameter λ is chosen small but still larger than the square of the noise (the so-called Morozov discrepancy principle). Note that (5) may *a priori* have several minimizers, for the total variation norm is not strictly convex and the kernel of A is nontrivial, whence the objective function (4) is not strictly convex either as is easy to see.

In Section 5, we analyze Extremal Problems 1 and 2 further in the case where S is contained in a plane. We prove that $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ is the unique solution to Extremal Problem 1 in case (c) listed above (Theorem 5.3), and also that Extremal Problem 2 has a unique solution for any data (Theorem 5.7).

Both results depend on Theorem 4.5, asserting that a two-dimensional divergence-free measure $\boldsymbol{\nu}$ can be decomposed into loops, *i.e.* contour integrations along rectifiable Jordan curves, in such a way that the Radon-Nykodim derivative $d\boldsymbol{\nu}/d|\boldsymbol{\nu}|(x)$ is essentially the unit tangent to any of these curves through x. The proof of the latter occupies Section 4, after some preparation in Section 3 where we recall the co-area formula and show approximate continuity of suplevel sets of homogeneous BV-functions. Section 2 describes relevant results from [29], while Appendix A gathers technical facts connected to the latter.

1.2. **Notation.** We conclude this section with some notation and definitions regarding measures and distributions. For a vector x in the Euclidean space $\mathbb{R}^n$ (we mainly deal with $n = 2$ or 3), we denote the j-th component of x by x_j and the partial derivative with respect to x_j by ∂_{x_j}. By default, we consider vectors as column vectors; e.g., for $x \in \mathbb{R}^3$ we write $x = (x_1, x_2, x_3)^T$ where "T" denotes "transpose". We write $\mathbb{N}$ for the nonnegative integers, $\mathbb{N}^*$ for the positive integers, and $\mathbb{R}^+$ for the nonnegative real numbers. We use bold symbols to represent vector-valued functions and measures, and the corresponding nonbold symbols with subscripts to denote the respective components; e.g., $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T$ or $\mathbf{b}(\boldsymbol{\mu}) = (b_1(\boldsymbol{\mu}), b_2(\boldsymbol{\mu}), b_3(\boldsymbol{\mu}))^T$. For $x \in \mathbb{R}^n$ and $R > 0$, we let $\mathbb{B}(x, R)$ indicate the open ball centered at x with radius R, and $\mathbb{S}(x, R)$ the boundary sphere. This notation does not show dependence on n, but no confusion should arise. We denote by $\mathcal{M}(E)$ the space of finite signed measures on $E \subset \mathbb{R}^n$.

We write χ_E for the characteristic function of a set E and δ_x for the Dirac delta measure at x. Given a $\mathbb{R}^m$-valued measure in $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^n)^m$ and a Borel set $E \subset \mathbb{R}^n$, we denote by $\boldsymbol{\mu}\lfloor E$ the measure obtained by restricting $\boldsymbol{\mu}$ to E (*i.e.* for every Borel set $B \subset \mathbb{R}^n$, $\boldsymbol{\mu}\lfloor E(B) := \boldsymbol{\mu}(E \cap B)$).

For $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^n)^m$, the *total variation measure* $|\boldsymbol{\mu}|$ is defined on Borel sets $B \subset \mathbb{R}^n$ by

$$|\boldsymbol{\mu}|(B) := \sup_{\mathcal{P}} \sum_{P \in \mathcal{P}} |\boldsymbol{\mu}(P)|, \tag{6}$$

where the supremum is taken over all finite Borel partitions $\mathcal{P}$ of B. The *total variation norm of* $\boldsymbol{\mu}$ is then defined as

$$\|\boldsymbol{\mu}\|_{TV} := |\boldsymbol{\mu}|(\mathbb{R}^n). \tag{7}$$

The support of $\boldsymbol{\mu}$ (*i.e.* the complement of the largest open set U such that $|\boldsymbol{\mu}|(U) = 0$) is denoted as $\operatorname{supp} \boldsymbol{\mu}$. Since $|\boldsymbol{\mu}|$ is a Radon measure, the Radon-Nikodym derivative $\mathbf{u}_{\boldsymbol{\mu}} := d\boldsymbol{\mu}/d|\boldsymbol{\mu}|$ exists as a $\mathbb{R}^m$-valued $|\boldsymbol{\mu}|$-integrable function and it satisfies $|\mathbf{u}_{\boldsymbol{\mu}}| = 1$ a.e. with respect to $|\boldsymbol{\mu}|$.

For $\Omega \subset \mathbb{R}^n$ an open set, we denote by $C_c(\Omega, \mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued continuous functions with compact support on Ω, equiped with the sup-norm. When $m = 1$, we drop the dependence on m and simply write $C_c(\Omega)$. A similar notational simplification is used for other functional spaces introduced below.

We shall identify $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^n)^m$ with the linear form on $C_c(\mathbb{R}^n, \mathbb{R}^m)$ given by

$$\langle \boldsymbol{\mu}, \mathbf{f} \rangle := \int \mathbf{f} \cdot d\boldsymbol{\mu}, \qquad \mathbf{f} \in C_c(\mathbb{R}^n, \mathbb{R}^m). \tag{8}$$

The norm of the functional (8), is $\|\boldsymbol{\mu}\|_{TV}$. More generally, for $\Omega \subset \mathbb{R}^n$ an open set, it follows from Lusin's theorem [27, Cor. to Theorem 2.23], applied to the restriction of $\mathbf{u}_{\boldsymbol{\mu}}$ to "large" compact sets in Ω, and from the dominated convergence theorem that

$$|\boldsymbol{\mu}|(\Omega) = \sup\{\langle \boldsymbol{\mu}, \boldsymbol{\varphi} \rangle,\ \boldsymbol{\varphi} \in C_c(\Omega, \mathbb{R}^m),\ |\boldsymbol{\varphi}| \leq 1\}. \tag{9}$$

The functional (8) extends naturally with the same norm to the Banach space $C_0(\mathbb{R}^n, \mathbb{R}^m)$ of $\mathbb{R}^m$-valued continuous functions on $\mathbb{R}^n$ vanishing at infinity.

At places, we also identify $\boldsymbol{\mu}$ with the restriction of (8) to $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$, the space of C^∞-smooth functions with compact support, equiped with the usual topology of test functions [28]. We refer to a continuous linear functional on $C_c^\infty(\mathbb{R}^n,\mathbb{R}^m)$ as being a distribution, and put ∂_{x_i} to mean distributional derivative with respect to the variable x_i.

We denote Lebesgue measure on $\mathbb{R}^n$ by $\mathcal{L}_n$ and d-dimensional Hausdorff measure by $\mathcal{H}^d$, see [15] for the definitions. We normalize $\mathcal{H}^d$ for $d=1$ and 2 so that it coincides with arclength and surface area for smooth curves and surfaces, and more generally that it agrees with d-dimensional volume for nice d-dimensional subsets of $\mathbb{R}^n$. We denote the Hausdorff dimension of a set E by $\dim_{\mathcal{H}}(E)$. We say that $E\subset R^n$ is *m-rectifiable* if it is the countable union of images of Lipschitz functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, up to a set of $\mathcal{H}^m$-measure zero, see [26, Def. 15.3].

For $E\subset\mathbb{R}^n$ a measurable set and $1\leq p\leq\infty$, we write $L^p(E)$ for the familiar Lebesgue space of (equivalence classes of $\mathcal{L}_n$-a.e. coinciding) real-valued measurable functions on E whose p-th power is integrable, with norm $\|g\|_{L^p(E)}=(\int_E|g|^pd\mathcal{L}_n)^{1/p}$ (ess. $\sup_E|g|$ if $p=\infty$). If E is open, we set $L^1_{loc}(E)$ to consist of functions f whose restriction $f_{|K}$ to K lies in $L^1(K)$, for every compact $K\subset E$. Since $E=\cup_nK_n$ with K_n compact, $L^1_{loc}(E)$ is a Fréchet space for the distance $d_1(f,g)=\sum_n 2^{-n}\|f-g\|_{L^1(K_n)}/(1+\|f-g\|_{L^1(K_n)})$. For $\nu\in\mathcal{M}(\mathbb{R}^n)$ a positive measure different from $\mathcal{L}_n$, we put $L^1[d\nu]$ for the space of real-valued integrable functions against ν.

We are particularly concerned with magnetizations supported on $\mathbb{R}^2\times\{0\}\subset\mathbb{R}^3$ and hence, with a slight abuse of notation, given $S\subset\mathbb{R}^2$ and $\boldsymbol{\mu}\in\mathcal{M}(S\times\{0\})^3$, we shall identify S with $S\times\{0\}\subset\mathbb{R}^3$ and $\boldsymbol{\mu}$ with $\boldsymbol{\mu}\lfloor(\mathbb{R}^2\times\{0\})$. In addition, we let $\mathfrak{R}$ denote the rotation by $\pi/2$ in $\mathbb{R}^2$; i.e., $\mathfrak{R}((x_1,x_2)^T)=(-x_2,x_1)^T$.

For an open set $\Omega\subset\mathbb{R}^n$, recall the space $BV(\Omega)$ of functions of *bounded variation* comprised of functions in $L^1(\Omega)$ whose distributional derivatives are signed measures on Ω (see, [32]). We let $BV_{loc}(\Omega)$ denote the space of functions whose restriction to any relatively compact open subset Ω_1 of Ω lies in $BV(\Omega_1)$. We define the space $\dot{BV}(\Omega)$ of "homogeneous" BV-functions to consist of locally integrable functions whose distributional derivatives are finite signed measures on Ω. Note that $\phi\in\dot{BV}(\Omega)$ if and only if it is a distribution on Ω such that $\nabla\phi\in\mathcal{M}(\Omega)^n$, by [14, Theorem 6.7.7]. If $\phi\in\dot{BV}(\Omega)$, we see from (9) by mollification that

$$\|\nabla\phi\|_{TV}=\sup_{\varphi\in C_c^1(\Omega,\mathbb{R}^n),|\varphi|\leq1}\int\varphi\cdot d(\nabla\phi)=\sup_{\varphi\in C_c^1(\Omega,\mathbb{R}^n),|\varphi|\leq1}\int\phi\,\nabla\cdot\varphi\,d\mathcal{L}_2, \tag{10}$$

where $C_c^1(\Omega,\mathbb{R}^n)$ denotes the space of $\mathbb{R}^n$-valued continuously differentiable functions with compact support in Ω, see [15, Ch. 5].

2. Divergence-free measures on $\mathbb{R}^n$

We recall in this section the decomposition of divergence-free measures into elementary components obtained in [29]. We also point at additional properties of the elementary components, the proofs of which are appended in Appendix A to streamline the exposition.

2.1. **Curves as measures.** For $a<b$ two real numbers, we call a Lipschitz mapping $\boldsymbol{\gamma}:[a,b]\to\mathbb{R}^n$ a *parametrized rectifiable curve*, while the image $\Gamma:=\boldsymbol{\gamma}([a,b])$ is simply termed a (non-parametrized) *rectifiable curve.* By Rademacher's Theorem (see [15]), $\boldsymbol{\gamma}$ is differentiable a.e. on $[a,b]$. Note that $\boldsymbol{\gamma}$ needs not be injective, *i.e.* the curve needs not be simple. If we let $N(\boldsymbol{\gamma},x)$ be the cardinality (finite or infinite) of the preimage $\boldsymbol{\gamma}^{-1}(x)$, then the length $\ell(\boldsymbol{\gamma})$ of $\boldsymbol{\gamma}$ is

$$\ell(\boldsymbol{\gamma}):=\int_a^b|\boldsymbol{\gamma}'(t)|\,dt=\int N(\boldsymbol{\gamma},x)\,d\mathcal{H}^1(x), \tag{11}$$

where the second equality follows from the area formula [18, 3.2.3]. In particular, $\mathcal{H}^1(\Gamma) < \infty$ and $\mathcal{H}^1$-almost every $x \in \Gamma$ is attained only finitely many times by $\boldsymbol{\gamma}$. Observe that $\ell(\boldsymbol{\gamma}) \neq \mathcal{H}^1(\Gamma)$ in general. When $|\boldsymbol{\gamma}'(t)| = 1$ a.e. on $[a, b]$, we call $\boldsymbol{\gamma}$ a unit speed parametrization. This means that $\boldsymbol{\gamma}$ parametrizes Γ (non injectively perhaps) by percursed arclength.

If $\boldsymbol{\gamma}$ is injective on $[a, b)$ and $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$, we say that $\boldsymbol{\gamma}$ is a parametrized rectifiable Jordan curve and Γ a rectifiable Jordan curve; in this case $\ell(\boldsymbol{\gamma}) = \mathcal{H}^1(\Gamma)$. Given a Jordan curve Υ (*i.e.* the image of a circle by an injective continuous map) such that $\mathcal{H}^1(\Upsilon) < \infty$, one can easily construct a unit speed parametrization $\boldsymbol{\gamma} : [0, \mathcal{H}^1(\Upsilon)] \to \Upsilon$ which is injective on $[0, \mathcal{H}^1(\Upsilon))$ with $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(\mathcal{H}^1(\Upsilon))$. Thus, a Jordan curve Υ is rectifiable if and only if $\mathcal{H}^1(\Upsilon) < \infty$.

For $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ a parametrized rectifiable curve, we define $\mathbf{R}_{\boldsymbol{\gamma}} \in \mathcal{M}(\mathbb{R}^n)^n$ by

$$(12) \qquad \langle \mathbf{R}_{\boldsymbol{\gamma}}, \mathbf{g} \rangle := \int_a^b \mathbf{g}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) dt = \int_\Gamma \left(\sum_{t \in \boldsymbol{\gamma}^{-1}(x)} \mathbf{g}(x) \cdot \boldsymbol{\gamma}'(t) \right) d\mathcal{H}^1(x), \qquad \mathbf{g} \in C_0(\mathbb{R}^n)^n,$$

where the second equality follows from the area formula. Clearly, $\mathbf{R}_{\boldsymbol{\gamma}}$ is supported on Γ and $\|\mathbf{R}_{\boldsymbol{\gamma}}\|_{TV} \leq \ell(\boldsymbol{\gamma})$. If we define $\psi : [a, b] \to [0, \ell(\boldsymbol{\gamma})]$ by $\psi(t) = \int_a^t |\boldsymbol{\gamma}'(\tau)| d\tau$, then ψ is Lipschitz with $\psi'(t) = |\boldsymbol{\gamma}'(t)|$ a.e. and there is a unit speed parametrization $\widetilde{\boldsymbol{\gamma}} : [0, \ell(\boldsymbol{\gamma})] \to \Gamma$ such that $\boldsymbol{\gamma} = \widetilde{\boldsymbol{\gamma}} \circ \psi$, by the chain rule and Sard's theorem for Lipschitz functions (see [26, Theorem 7.4]). Moreover, we see from the area formula that $\mathbf{R}_{\boldsymbol{\gamma}} = \mathbf{R}_{\widetilde{\boldsymbol{\gamma}}}$, so we assume unless otherwise stated that parametrized rectifiable curves are unit speed parametrizations.

By Lemma A.1, $\mathbf{R}_{\boldsymbol{\gamma}}$ is absolutely continuous with respect to $\mathcal{H}^1\lfloor\Gamma$ and has Radon-Nykodim derivative $d\mathbf{R}_{\boldsymbol{\gamma}}/d(\mathcal{H}^1\lfloor\Gamma)(x) = \sum_{t \in \boldsymbol{\gamma}^{-1}(x)} \boldsymbol{\gamma}'(t)$ at $\mathcal{H}^1$-a.e. $x \in \boldsymbol{\Gamma}$. Hence, for every Borel set $B \subset \mathbb{R}^n$, we have that

$$(13) \qquad \mathbf{R}_{\boldsymbol{\gamma}}(B) = \int_{\Gamma \cap B} \left(\sum_{t \in \boldsymbol{\gamma}^{-1}(x)} \boldsymbol{\gamma}'(t) \right) d\mathcal{H}^1(x), \qquad |\mathbf{R}_{\boldsymbol{\gamma}}|(B) = \int_{\Gamma \cap B} \left| \sum_{t \in \boldsymbol{\gamma}^{-1}(x)} \boldsymbol{\gamma}'(t) \right| d\mathcal{H}^1(x).$$

It may happen that $\|\mathbf{R}_{\boldsymbol{\gamma}}\|_{TV} < \ell(\boldsymbol{\gamma})$, because cancellation can occur in (12). To discard such cases, we consider for each $\ell > 0$ the collection $\mathcal{C}_\ell$ of those $\mathbf{R}_{\boldsymbol{\gamma}}$ associated to a parametrized rectifiable curve $\boldsymbol{\gamma}$ of length ℓ that satisfy $\|\mathbf{R}_{\boldsymbol{\gamma}}\|_{TV} = \ell$. By Lemma A.2, we have that $\mathbf{R}_{\boldsymbol{\gamma}} \in \mathcal{C}_\ell$ if and only if Γ has a well defined (oriented) unit tangent $\boldsymbol{\tau}(x)$ at $\mathcal{H}^1$-a.e. x, given by $\boldsymbol{\gamma}'(t)$ for any t such that $\boldsymbol{\gamma}(t) = x$. In this case, we note that (13) can be rewritten as

$$(14) \qquad \mathbf{R}_{\boldsymbol{\gamma}}(B) = \int_{\Gamma \cap B} N(\boldsymbol{\gamma}, x) \boldsymbol{\tau}(x) \, d\mathcal{H}^1(x), \qquad |\mathbf{R}_{\boldsymbol{\gamma}}|(B) = \int_{\Gamma \cap B} N(\boldsymbol{\gamma}, x) \, d\mathcal{H}^1(x).$$

2.2. **Decomposition of solenoids into curves.** Since $\mathcal{M}(\mathbb{R}^n)^n$ is dual to $C_c(\mathbb{R}^n, \mathbb{R}^n)$ which is separable, the closed ball $\mathcal{B}_\ell \subset \mathcal{M}(\mathbb{R}^n)^n$ centered at 0 of radius ℓ is a compact metrizable space for the weak-$*$ topology. In particular, $\mathcal{C}_\ell$ equipped with the weak-$*$ topology is a (non complete) metric space. Now, suppose that $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^n)^n$ is a solenoid, *i.e.* that $\nabla \cdot \boldsymbol{\mu} = 0$ (as a distribution). Then, it follows from [29, Theorem A] that $\boldsymbol{\mu}$ can be decomposed into elements from $\mathcal{C}_\ell$, meaning there is a positive finite Borel measure ρ on $\mathcal{C}_\ell$ such that, for ρ-a.e. $\boldsymbol{\gamma}$, the measure $\mathbf{R}_{\boldsymbol{\gamma}}$ is supported in $\operatorname{supp} \boldsymbol{\mu}$ and

$$(15) \qquad \langle \boldsymbol{\mu}, \mathbf{g} \rangle = \int_{\mathcal{C}_\ell} \langle \mathbf{R}_{\boldsymbol{\gamma}}, \mathbf{g} \rangle d\rho(\mathbf{R}_{\boldsymbol{\gamma}}), \qquad \langle |\boldsymbol{\mu}|, \varphi \rangle = \int_{\mathcal{C}_\ell} \langle |\mathbf{R}_{\boldsymbol{\gamma}}|, \varphi \rangle d\rho(\mathbf{R}_{\boldsymbol{\gamma}}),$$

for all $\mathbf{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. Of course, by mollification, it is clear that (15) more generally holds for $\mathbf{g} \in C_c(\mathbb{R}^n, \mathbb{R}^n)$ and $\varphi \in C_c(\mathbb{R}^n)$. By Lemma A.3, the two equalities in Equation

(15) amount to say that, for each Borel set $B \subset \mathbb{R}^n$,

$$\boldsymbol{\mu}(B) = \int_{\mathcal{C}_\ell} \mathbf{R}_{\boldsymbol{\gamma}}(B)\, d\rho(\mathbf{R}_{\boldsymbol{\gamma}}), \qquad |\boldsymbol{\mu}|(B) = \int_{\mathcal{C}_\ell} |\mathbf{R}_{\boldsymbol{\gamma}}|(B)\, d\rho(\mathbf{R}_{\boldsymbol{\gamma}}). \tag{16}$$

We note that (16) was used in the proof of [6, Theorem 2.6] without further justification.

The representation (15) is far from unique: for instance $\ell > 0$ was arbitrary. Moreover, the $\mathbf{R}_{\boldsymbol{\gamma}}$ need not be divergence-free even though $\boldsymbol{\mu}$ is; *i.e.*, the solenoid $\boldsymbol{\mu}$ gets decomposed *via* (15) into elementary components $\mathbf{R}_{\boldsymbol{\gamma}}$ that may not be solenoids. In this connection, observe that $\nabla\cdot\mathbf{R}_{\boldsymbol{\gamma}} = \delta_{\boldsymbol{\gamma}(b)} - \delta_{\boldsymbol{\gamma}(a)}$ which vanishes if only if $\boldsymbol{\gamma}$ is a closed parametrized curve. In the next section, we discuss a more subtle decomposition of $\boldsymbol{\mu}$, this time into divergence-free components, which is established in [29, Theorem B]. In a sense, it is obtained by letting $\ell \to \infty$ in (15).

2.3. **Decomposition of solenoids into elementary solenoids.** In the terminology of [29], an *elementary solenoid* $\mathbf{T_f}$ is a $\mathbb{R}^n$-valued measure associated to a Lipschitz function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ with $|\mathbf{f}'(t)| \le 1$, acting on $\boldsymbol{\varphi} \in C_c(\mathbb{R}^n)^n$ by the formula:

$$\mathbf{T_f}(\boldsymbol{\varphi}) = \lim_{s\to+\infty} \frac{1}{2s} \int_{-s}^{s} \boldsymbol{\varphi}(\mathbf{f}(t)) \cdot \mathbf{f}'(t)\, dt, \tag{17}$$

where the existence of the limit is *assumed* for every $\boldsymbol{\varphi}$ (for instance, it will exist if $\mathbf{f}$ is periodic or quasi-periodic). In addition, it is required that $\mathbf{f}(\mathbb{R}) \subset \operatorname{supp} \mathbf{T_f}$ and that $\|\mathbf{T_f}\|_{TV} = 1$. Letting $\mathbf{f}_s := \mathbf{f}|_{[-s,s]}$, we get with the notation of Section 2.1 that $\mathbf{T} = *\lim \mathbf{R}_{\mathbf{f}_s}/(2s)$ as $s \to +\infty$, where $*\lim$ indicates the weak-$*$ limit. It is clear from (17) that $\operatorname{supp} \mathbf{T_f} \subset \overline{\mathbf{f}(\mathbb{R})}$, therefore the condition that $\mathbf{f}(\mathbb{R}) \subset \operatorname{supp} \mathbf{T_f}$ really means that $\operatorname{supp} \mathbf{T_f} = \overline{\mathbf{f}(\mathbb{R})}$. By Lemma A.4, we may assume without loss of generality that $|\mathbf{f}'(t)| = 1$ a.e. on $\mathbb{R}$ in the definition of $\mathbf{T_f}$. It is straightforward to check that $\nabla\cdot\mathbf{T_f} = 0$, since $\mathbf{T_f}(\nabla\Psi) = \lim_s (\Psi(\mathbf{f}(s)) - \Psi(\mathbf{f}(-s))/s = 0$ for any $\Psi \in C^1_c(\mathbb{R}^n)$. Hence, $\mathbf{T_f}$ is indeed a solenoid. We denote by $\mathfrak{S}(\mathbb{R}^n)$ the set of elementary solenoids on $\mathbb{R}^n$. Since it is contained in $\mathcal{B}_1$, the set $\mathfrak{S}(\mathbb{R}^n)$ is a metric space when endowed with the weak-$*$ topology.

It is more difficult to describe members of $\mathfrak{S}(\mathbb{R}^n)$ than members of $\mathcal{C}_\ell$, but still their structure is reminiscent of (13) as we now indicate. Indeed, putting $\Gamma_s = \mathbf{f}([-s,s])$ and $N(\mathbf{f},x,s)$ for the cardinality (finite or infinite) of those $t \in [-s,s]$ such that $\mathbf{f}(t) = x$, let us define the *normalized arclength* of the parametrization $\mathbf{f}_s : [-s,s] \to \mathbb{R}^n$ to be the measure on $\mathbb{R}^n$ given by

$$d\nu_s(x) := \frac{N(\mathbf{f},x,s)}{2s} d(\mathcal{H}^1 \lfloor \Gamma_s)(x). \tag{18}$$

From (11), we see that ν_s is a probability measure for each $s > 0$, and by Lemma A.5, the family $(\boldsymbol{\nu}_s)_{s>0}$ converges weak-$*$, as $s \to +\infty$, to the probability measure $|\mathbf{T_f}|$. Moreover, the Radon Nykodim derivative $\mathbf{u}_{\mathbf{T_f}}$ extrapolates, in a sense made precise in that lemma, a limit of averaged tangents to $\mathbf{f}(\mathbb{R})$. For instance, if $\mathbf{g}_k$ is a sequence in $C_c(\mathbb{R}^n)$ such that $|\mathbf{g}_k| \le 1$ and $\lim_k \mathbf{g}_k(x) = \mathbf{u}_{\mathbf{T_f}}(x)$ for $|\mathbf{T_f}|$-a.e. $x \in \mathbb{R}^n$ (such a sequence exists by Lusin's theorem), then to any real sequence $s_k \to +\infty$ there is a subsequence $s_{j(k)}$ such that (compare (84)):

$$\lim_{k\to\infty} \int \left| \mathbf{g}_k(x) - \frac{\sum_{t\in\mathbf{f}^{-1}(x),\, |t|\le s_{j(k)}} \mathbf{f}'(t)}{N(\mathbf{f},x,s_{j(k)})} \right|^2 d\boldsymbol{\nu}_s(x) = 0.$$

A typical example is obtained when $\mathbf{f}$ is a line winding on a torus with irrational slope. Then $|\mathbf{T_f}|$ is the normalized area measure and $\mathbf{u}_{\mathbf{T_f}}$ is a continuous tangential vector field on the torus.

It is shown in [29, Theorem B] that each $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^k)^k$ with $\nabla\cdot\boldsymbol{\mu} = 0$ can be expressed as

$$\langle \boldsymbol{\mu}, \boldsymbol{\varphi}\rangle = \int_{\mathfrak{S}(\mathbb{R}^k)} \langle \mathbf{T}, \boldsymbol{\varphi}\rangle \, d\rho(\mathbf{T}), \qquad \boldsymbol{\varphi} \in C_c(\mathbb{R}^n, \mathbb{R}^n), \tag{19}$$

for some positive Borel measure $\rho = \rho(\boldsymbol{\mu})$ on $\mathfrak{S}(\mathbb{R}^k)$, in such a way that

$$\langle |\boldsymbol{\mu}|, \boldsymbol{\varphi}\rangle = \int_{\mathfrak{S}(\mathbb{R}^k)} \langle |\mathbf{T}|, \boldsymbol{\varphi}\rangle \, d\rho(\mathbf{T}). \tag{20}$$

Arguing as in Lemma A.3, one sees that (19) and (20) together are equivalent to

$$\boldsymbol{\mu}(B) = \int_{\mathfrak{S}(\mathbb{R}^k)} \mathbf{T}(B)\, d\rho(\mathbf{T}), \qquad |\boldsymbol{\mu}|(B) = \int_{\mathfrak{S}(\mathbb{R}^k)} |\mathbf{T}|(B)\, d\rho(\mathbf{T}) \tag{21}$$

for every Borel set B, in particular $\operatorname{supp}\mathbf{T} \subset \operatorname{supp}\boldsymbol{\mu}$ for ρ-a.e. $\mathbf{T} \in \mathfrak{S}(\mathbb{R}^k)$. In [29], the relations (19) and (20) are summarized by saying that a divergence-free measure can be completely decomposed into elementary solenoids.

In dimension 3 already, the functions $\mathbf{f}$ giving rise to a well-defined measure $\mathbf{T_f}$ *via* (17) can have rather complex behaviour, see examples in [29, Sec. 1.3]. However, in dimension 2, the decomposition (19) can be achieved using periodic $\mathbf{f}$ parametrizing rectifiable Jordan curves: this follows from Theorem 4.5 in Section 4. In this connection, we note that if $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ satisfies $|\mathbf{f}'| = 1$ a.e. and is periodic of period $L > 0$, then the limit in (17) does exist and in fact $\mathbf{T_f} = \mathbf{R}_{\boldsymbol{\gamma}}/L$, where $\boldsymbol{\gamma} : [0, L] \to \mathbb{R}^n$ is the restriction $\mathbf{f}_{|[0,L]}$. Clearly then, we have that $\operatorname{supp}\mathbf{T_f} = \boldsymbol{\gamma}([0, L]) = \mathbf{f}(\mathbb{R})$, and in order that $\mathbf{T_f}$ be an elementary solenoid it is necessary and sufficient that $\|\mathbf{T_f}\|_{TV} = 1$. This amounts to require that $\|\mathbf{R}_{\boldsymbol{\gamma}}\|_{TV} = L$ or, equivalently, that $\mathbf{R}_{\boldsymbol{\gamma}} \in \mathcal{C}_L$. By the discussion after (13), this is the case when $\boldsymbol{\gamma}([0, L])$ is a rectifiable Jordan curve.

$\mathbb{R}^3$-valued solenoids with planar support are of particular significance for our applications. The following elementary lemma, essentially contained in [5], gives simple characterizations of such solenoids. We include a proof for the convenience of the reader. Recall the definition of $\dot{BV}$ and the notation $\mathfrak{R}$ for the rotation by $\pi/2$ in $\mathbb{R}^2$.

Lemma 2.1. *Let $S \subset \mathbb{R}^2 \times \{0\}$ be closed, $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T \in \mathcal{M}(S)^3$, and $\boldsymbol{\mu}_T = (\mu_1, \mu_2)^T$. The following are equivalent:*

(a) *$\nabla\cdot\boldsymbol{\mu} = 0$ in the distributional sense on $\mathbb{R}^3$.*
(b) *$\mu_3 = 0$ and $\nabla\cdot\boldsymbol{\mu}_T = 0$ in the distributional sense on $\mathbb{R}^2$.*
(c) *$\mu_3 = 0$ and $\boldsymbol{\mu}_T = \mathfrak{R}\nabla\phi = (-\partial_{x_2}\phi, \partial_{x_1}\phi)^T$ for some $\phi \in \dot{BV}(\mathbb{R}^2)$.*

Proof. Since $\boldsymbol{\mu}$ has support contained in $\mathbb{R}^2 \times \{0\}$, it can be written in tensor product form as $\boldsymbol{\mu} = (\boldsymbol{\mu}\lfloor\mathbb{R}^2) \otimes \delta_{x_3=0}$ and thus $\nabla\cdot\boldsymbol{\mu} = (\nabla\cdot\boldsymbol{\mu}_T) \otimes \delta_{x_3=0} + \mu_3 \otimes \delta'_{x_3=0}$, where $\delta_{x_3=0}$ is the Dirac mass at zero on $\mathbb{R}$ in the variable x_3 and $\delta'_{x_3=0}$ its distributional derivative. Hence, (b) implies that $\nabla\cdot\boldsymbol{\mu} = 0$ and therefore (b)⇒(a). Next, for any $\phi \in C_c^\infty(\mathbb{R}^3)$, let $\phi_0, \phi_1 \in C_c^\infty(\mathbb{R}^2)$ be given by $\phi_0(x_1, x_2) = \phi(x_1, x_2, 0)$ and $\phi_1(x_1, x_2) = \partial_{x_3}\phi(x_1, x_2, 0)$. Then, it holds that

$$\langle \nabla\cdot\boldsymbol{\mu}, \phi\rangle = -\langle \mu_1, \partial_{x_1}\phi_0\rangle - \langle \mu_2, \partial_{x_2}\phi_0\rangle - \langle \mu_3, \phi_1\rangle. \tag{22}$$

Pick ϕ of the form $\phi(x_1, x_2, x_3) = \psi(x_1, x_2)\eta(x_3)$ where $\psi \in C_c^\infty(\mathbb{R}^2)$ and $\eta \in C_c^\infty(\mathbb{R})$. First, letting η be such that $\eta(0) = 1$ and $\eta'(0) = 0$, we deduce from (22) that if $\nabla\cdot\boldsymbol{\mu} = 0$ then $\nabla\cdot\boldsymbol{\mu}_T = 0$. Second, letting η be such that $\eta(0) = 0$ and $\eta'(0) = 1$, we deduce from (22) again that if $\nabla\cdot\boldsymbol{\mu} = 0$ then $\mu_3 = 0$, whence (a)⇒(b).

Suppose now that (b) holds. Then $(-\mu_2, \mu_1)^T$ satisfies the Schwartz rule when viewed as a $\mathbb{R}^2$ valued distribution on $\mathbb{R}^2$; i.e, $\partial_{x_2}(-\mu_2) = \partial_{x_1}\mu_1$. Therefore, $\mathfrak{R}\,\boldsymbol{\mu}_T = (-\mu_2, \mu_1)^T$ is the gradient of a scalar valued distribution Ψ (see, [28]). Since the components of $\nabla\Psi$ are finite signed measures, $\Psi \in BV_{\rm loc}$ [14, Theorem 6.7.7] so that in fact $\Psi \in \dot{BV}(\mathbb{R}^2)$. Thus, (c) holds with $\phi = -\Psi$ and we get that (b)⇒(c). In the other direction if $\boldsymbol{\mu}_T = (-\partial_{x_2}\phi, \partial_{x_1}\phi)^T$ for some distribution ϕ, then $\nabla\cdot\boldsymbol{\mu}_T = -\partial_{x_1}\partial_{x_2}\phi + \partial_{x_2}\partial_{x_1}\phi = 0$ so that (c)⇒(b). □

Lemma 2.1 entails that decomposing solenoids in the plane is equivalent, up to a rotation, to decomposing gradients. As surmised in [29], the latter can be achieved *via* the co-area formula and the decomposition of the measure-theoretical boundary of sets of finite perimeter in $\mathbb{R}^2$ into rectifiable Jordan curves. In Section 3 to come, we record a version of the co-area formula for $\dot{BV}$-functions, and we establish approximate continuity of M-connected components of sup-level sets of such functions (see Proposition 3.6 and Theorem 3.7). The latter is needed to handle measurability issues in the loop decomposition of planar divergence-free measures (see Proposition 4.6), but is also of independent interest. Though we later lean on the planar case, it would be artificial to restrict to $\mathbb{R}^2$ in Section 3 and we shall present the material in $\mathbb{R}^n$.

3. Sup-level sets of functions in $\dot{BV}(\mathbb{R}^n)$ and the co-area formula

We first record a summability property of homogeneous BV-functions.

Lemma 3.1. [2, Theorem 3.47] *If $\phi \in \dot{BV}(\mathbb{R}^n)$, there is $p \in \mathbb{R}$ such that $\phi - p \in L^{n/(n-1)}(\mathbb{R}^n)$.*

Next, we collect several definitions and properties that are central to what follows. For $E \subset \mathbb{R}^n$ a Borel set, the *measure-theoretical boundary* of E is the set $\partial_M E$ defined by

$$\partial_M E := \left\{ x \in \mathbb{R}^n : \limsup_{\rho\to 0} \frac{\mathcal{L}_n(\mathbb{B}(x,\rho)\cap E)}{\mathcal{L}_n(\mathbb{B}(x,\rho))} > 0 \text{ and } \limsup_{\rho\to 0} \frac{\mathcal{L}_n(\mathbb{B}(x,\rho)\setminus E)}{\mathcal{L}_n(\mathbb{B}(x,\rho))} > 0 \right\}. \tag{23}$$

Note that for any set E, $\partial_M E$ is a subset of the topological boundary of E.

A measurable set $E \subset \mathbb{R}^n$ such that $\nabla\chi_E \in \mathcal{M}(\mathbb{R}^n)^n$ is said to be *of finite perimeter*[1]. For such a set it holds that

$$|\nabla\chi_E| = \mathcal{H}^{n-1}\lfloor\partial_M E, \tag{24}$$

and $\|\nabla\chi_E\|_{TV} = \mathcal{H}^{n-1}(\partial_M E)$ is called the perimeter of E, denoted as $\mathcal{P}(E)$. The identity (24) can be obtained by combining [15, Theorem 5.15 (iii)], saying that (24) holds when $\partial_M E$ is replaced by the so-called reduced boundary of E, with [15, Lemma 5.5], asserting that $\partial_M E$ differs from the reduced boundary by a set of $\mathcal{H}^{n-1}$-measure zero (see also [4, Theorem 10.3.2]).

It follows from (24) that a set of finite perimeter has a measure-theoretical boundary of finite $\mathcal{H}^{n-1}$-measure. In contrast, its Euclidean boundary can be much larger and even have positive $\mathcal{L}_n$-measure, as the following example shows.

Example 3.1. Let $E_1 = \overline{\mathbb{B}}(0,1) \subset \mathbb{R}^2$ and $\{q_j\}_{j\in\mathbb{N}}$ enumerate all points in E_1 with rational coordinates. Having defined inductively a closed set E_n for $n \geq 1$, let j_n be the smallest integer such that q_{j_n} lies interior to E_n and B_n the largest open ball centered at q_{j_n} contained in E_n, with radius $r_n \leq 2^{-n}$ (at some steps B_n could be empty). Then, define $E_{n+1} = E_n \setminus B_n$ which must be a closed set with nonempty interior, otherwise a finite union of balls of total $\mathcal{L}_2$-measure less than $\pi/3$ would cover $\mathbb{B}(0,1)$. Hence, the process can continue indefinitely, and we let $E = \bigcap E_n$ which is a closed set.

Clearly E has no interior, for all the q_j have been excised out in the process; therefore its Euclidean boundary is E itself. Moreover, $\mathcal{L}_2(E) \geq \pi - \pi\sum_{n=1}^{\infty} r_n^2 \geq \pi(1 - \sum_{n=1}^{\infty} 4^{-n}) > 0$.

[1]In [4, 15, 32], the definition is that $\chi_E \in BV(\mathbb{R}^n)$. The present definition means that $\chi_E \in \dot{BV}(\mathbb{R}^n)$ and, in view of Lemma 3.1, amounts to requiring that either χ_E or $\chi_{\mathbb{R}^n\setminus E}$ lies in $BV(\mathbb{R}^n)$.

Now, by the standard Green formula, each E_n is of finite perimeter, because it is a finitely connected set with piecewise smooth boundary. Thus, $\{\chi_{E_n}\}$ is a nonincreasing sequence of BV-functions and their point-wise limit χ_E is integrable. Also, by (24), it holds that $\|\nabla\chi_{E_n}\|_{TV} \leq 2\pi\sum_{n=0}^{\infty} r_n \leq 4\pi$, therefore we can use [32, Remark 5.2.2] to the effect that $\chi_E \in BV(\mathbb{R}^2)$, i.e. E is a set of finite perimeter with Euclidean boundary of positive $\mathcal{L}_2$-measure, as announced.

For any $E \subset \mathbb{R}^n$ of finite perimeter, we define the *generalized unit inner normal* $\boldsymbol{\nu}_E$ to $\partial_M E$ as the Radon-Nikodym derivative $\mathbf{u}_{\nabla\chi_E}$ which is but $d\nabla\chi_E/d(\mathcal{H}^{n-1}\lfloor\partial_M E)$, by (24). Then, the Radon Nikodym Theorem entails the following version of the *Gauss-Green formula*:

Lemma 3.2. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. Then, for each Borel set $B \subset \mathbb{R}^n$,*

$$\nabla\chi_E(B) = \int_B \boldsymbol{\nu}_E \, d\left(\mathcal{H}^{n-1}\lfloor\partial_M E\right) \tag{25}$$

or, equivalently, $d\nabla\chi_E = \boldsymbol{\nu}_E d(\mathcal{H}^{n-1}\lfloor\partial_M E)$ as measures on $\mathbb{R}^n$.

The connection with the classical Gauss-Green formula is more transparent on the distributional version of (25), namely:

$$\int \chi_E \, \nabla\cdot\boldsymbol{\varphi} \, d\mathcal{L}_n = -\int \boldsymbol{\varphi}\cdot\boldsymbol{\nu}_E \, d\left(\mathcal{H}^{n-1}\lfloor\partial_M E\right), \quad \boldsymbol{\varphi} \in C_c^1(\mathbb{R}^n, \mathbb{R}^n). \tag{26}$$

The identity (26) was proven in [12, 13] and [16, 17]; see also [15, Theorem 5.16] and [4, Theorem 10.3.2]. Note that if E has finite perimeter, then so does $\mathbb{R}^n \setminus E$ and $\boldsymbol{\nu}_{\mathbb{R}^n\setminus E} = -\boldsymbol{\nu}_E$.

Remark. When $n = 2$, we see from (26) that $\boldsymbol{\nu}_E$ coincides with the usual, differential-geometric inner unit normal to the boundary of E when the latter is a rectifiable Jordan curve, for in this case Green's formula is valid for both definitions of the normal (see [3, Theorem 10–43] for a suitable version of the Green formula here). Actually, Lemma 4.3 entails that the measure-theoretical boundary of any planar set of finite perimeter is comprised of a countable union of rectifiable Jordan curves, up to a set of $\mathcal{H}_1$-measure zero. Thus, both notions of inner unit normal coincide $\mathcal{H}^1$-a.e. on the measure-theoretical boundary of such a set.

Whenever $\phi \in BV(\mathbb{R}^n)$, the suplevel sets

$$E_t := \{x \in \mathbb{R}^n \mid \phi(x) > t\} \tag{27}$$

have finite perimeter for a.e. $t \in \mathbb{R}$ [15, Theorem 5.9]. Of course, the set E_t, as well as a number of subsequent sets in $\mathbb{R}^n$ that we will consider, is defined up to a set of $\mathcal{L}_n$-measure zero only, but which representative is chosen will be irrelevant for our purposes. Hereafter, we abbreviate the sentence "up to a set of $\mathcal{L}_n$-measure zero" by "mod-$\mathcal{L}_n$", and similarly for $\mathcal{H}^{n-1}$. The sup-level sets are a key ingredient of the co-area (or Fleming-Rishel) formula for BV-functions. In Theorem 3.4 below, we record a version of this formula for functions in $\dot{BV}(\mathbb{R}^n)$ which suits our purposes.

Lemma 3.3. *Let $\phi \in \dot{BV}(\mathbb{R}^n)$ and E_t be as in (27) for $t \in \mathbb{R}$. Then, E_t has finite perimeter for a.e. $t \in \mathbb{R}$ and*

$$\int f d|\nabla\phi| = \int_{-\infty}^{\infty}\int f d|\nabla\chi_{E_t}| \, dt \quad \textit{for each } |\nabla\phi|\textit{-integrable Borel function } f. \tag{28}$$

Moreover, it holds that

$$\int \boldsymbol{\varphi}\cdot d(\nabla\phi) = \int_{-\infty}^{\infty}\int \boldsymbol{\varphi}\cdot d(\nabla\chi_{E_t}) \, dt \quad \textit{for each } \boldsymbol{\varphi} \in C_c(\mathbb{R}^n, \mathbb{R}^n). \tag{29}$$

Proof. The lemma follows easily from the proof of [15, Theorem 5.9] and that $\phi_{|U} \in BV(U)$ whenever $\phi \in \dot{BV}(\mathbb{R}^n)$ and U is a bounded open set. Specifically, putting $E_t^k := \{x \in \mathbb{B}(0,k) : \phi(x) > t\}$, we get from [15, Theorem 5.9 (i)] that $\int_{-\infty}^{\infty} \|\nabla\chi_{E_t^k}\|_{TV} dt \leq \|\nabla\phi\|_{TV}$ for all $k \in \mathbb{N}$. For a.e. t, it is clear that $\nabla\chi_{E_t^k}$ converges weak-* to $\nabla\chi_{E_t}$ when $k \to \infty$, therefore $\liminf_k \|\nabla\chi_{E_t^k}\|_{TV} \leq \|\nabla\chi_{E_t}\|_{TV}$ and it follows from the Fatou lemma that $\int_{-\infty}^{\infty} \|\nabla\chi_{E_t}\|_{TV} dt \leq \|\nabla\phi\|_{TV}$. Hence, E_t has finite perimeter for a.e. t, and when $\boldsymbol{\varphi} \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ then (29) follows from claim 1 in the proof of [15, Theorem 5.9]. Thus, by mollification, (29) holds as stated. Moreover, [15, Theorem 5.9 (i)] and (10) together imply that (28) holds for simple functions $f = \sum_{j=1}^N a_j \chi_{U_j}$, where the U_j are bounded open sets, therefore it holds for all simple functions by regularity of finite Borel measures on $\mathbb{R}^n$. The case of $|\nabla\Phi|$-integrable Borel functions follows from this. $\square$

From Lemma (3.3), we deduce the following version of the co-area formula for $\dot{BV}(\mathbb{R}^n)$-functions that meets our needs.

Theorem 3.4. *Suppose $\phi \in \dot{BV}(\mathbb{R}^n)$ and let E_t be as in (27). Then, for any Borel set $B \subset \mathbb{R}^n$, $\mathbf{g} \in L^1[d|\nabla\phi|]^n$ and $h \in L^1[d|\nabla\phi|]$, it holds that*

$$\text{(a)}\ \ |\nabla\phi|(B) = \int_{-\infty}^{\infty} |\nabla\chi_{E_t}|(B)\, dt = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\partial_M E_t \cap B)\, dt,$$

$$\text{(b)}\ \ \int h\, d(|\nabla\phi|) = \int_{-\infty}^{\infty} \int h\, d(|\nabla\chi_{E_t}|)\, dt = \int_{-\infty}^{\infty} \int h\, d\left(\mathcal{H}^{n-1} \lfloor \partial_M E_t\right)\, dt,$$

$$\text{(c)}\ \ \nabla\phi(B) = \int_{-\infty}^{\infty} \nabla\chi_{E_t}(B)\, dt = \int_{-\infty}^{\infty} \int_B \boldsymbol{\nu}_{E_t}\, d\left(\mathcal{H}^{n-1} \lfloor \partial_M E_t\right)\, dt,$$

$$\text{(d)}\ \ \int \mathbf{g} \cdot d(\nabla\phi) = \int_{-\infty}^{\infty} \int \mathbf{g} \cdot d(\nabla\chi_{E_t})\, dt = \int_{-\infty}^{\infty} \int \mathbf{g} \cdot \boldsymbol{\nu}_{E_t}\, d\left(\mathcal{H}^{n-1} \lfloor \partial_M E_t\right)\, dt,$$

where in (b) the function h lies in both $L^1[d|\nabla\chi_{E_t}|]$ and $L^1[d\mathcal{H}^{n-1}\lfloor \partial_M E_t]$ for a.e. t and in (d) the functions $\mathbf{g}$ and $\mathbf{g} \cdot \boldsymbol{\nu}_{E_t}$ lie in $L^1[d|\nabla\chi_{E_t}|]^n$ and $L^1[d\mathcal{H}^{n-1}\lfloor \partial_M E_t]$, respectively, for a.e. t.

Proof. Taking $f = \chi_B$ in (28) implies the first equality in (a), and the second one is just the combination with (24). To show the first equality in (c), we apply Lusin's theorem to the effect that χ_B is the bounded pointwise limit of a sequence $\varphi_k \in C_c(\mathbb{R}^n)$ except on a Borel set E of $|\nabla\phi|$-measure zero. From the first equality in (a) we get $|\nabla\chi_{E_t}|(E) = 0$ for a.e. t, and for such t it holds by dominated convergence that if we pick $v \in \mathbb{R}^n$, then $\lim_k \int \varphi_k v \cdot d(\nabla\chi_{E_t}) = v \cdot \nabla(\chi_{E_t})(B)$. Since v is arbitrary in $\mathbb{R}^n$, the first equality in (c) now follows from (29), applied with $\boldsymbol{\varphi} = \varphi_k v$, by invoking the dominated convergence theorem when $k \to \infty$ in $L^1[d|\nabla\phi|]$ on the left hand side, and in $L^1(\mathbb{R})$ on the right hand side. The second equality in (c) ensues from (25).

Next, (a) yields (b) for simple functions, and the case of $C_c(\mathbb{R}^n)$-functions follows by uniform approximation, using (a). The case of bounded $|\nabla\phi|$-measurable functions can now be obtained from Lusin's Theorem and dominated convergence, using (a) to ascertain that a Borel set B such that $|\nabla\phi|(B) = 0$ has $|\nabla\chi_{E_t}|(B) = 0$ and $\mathcal{H}^{n-1}\lfloor \partial_M E_t(B) = 0$ for a.e. t. The general case follows by monotone convergence. That (c) implies (d) follows similarly, proceeding componentwise to pass from continuous $\mathbf{g}$ to the case where $\mathbf{g} \in L^1[d|\nabla\phi|]^n$ (compare the proof of (82) in Lemma A.3). $\square$

One can also give a description of the "measure theoretical discontinuities" of $\dot{BV}$-functions similar to the one of BV-functions. For a Borel function f on $\mathbb{R}^n$ and any $x \in \mathbb{R}^n$ we define (see

[15, Def. 5.8, 5.9]):

$$
\begin{aligned}
f^{\sup}(x) &:= \operatorname{ap}\limsup_{y\to x} f(y) = \inf\left\{ t \,\middle|\, \lim_{r\to 0}\frac{\mathcal{L}_n(\mathbb{B}(x,r)\cap\{\phi>t\})}{\mathcal{L}_n(\mathbb{B}(x,r))}=0\right\},\\
f^{\inf}(x) &:= \operatorname{ap}\liminf_{y\to x} f(y) = \sup\left\{ t \,\middle|\, \lim_{r\to 0}\frac{\mathcal{L}_n(\mathbb{B}(x,r)\cap\{\phi<t\})}{\mathcal{L}_n(\mathbb{B}(x,r))}=0\right\}\\
&\text{and } J(f) := \left\{ x \,\middle|\, f^{\inf}(x) < f^{\sup}(x)\right\}.
\end{aligned} \tag{30}
$$

Lemma 3.5. *Given $\phi \in \dot{BV}(\mathbb{R}^n)$, the set $J(\phi)$ is $(n-1)$-rectifiable. Furthermore, $\nabla\phi\lfloor J(\phi)$ is absolutely continuous with respect to $\mathcal{H}^1$ and, with E_t as in (27), its Radon-Nykodim derivative satisfies for a.e. $t \in \mathbb{R}$ and $\mathcal{H}^{n-1}$-a.e. $x \in \partial_M E_t \cap J$: $d\nabla\phi/d\mathcal{H}^{n-1} = (\phi^{\sup} - \phi^{\inf})\boldsymbol{\nu}_{E_t}$.*

Proof. The first assertion of the lemma follows by arguing as in the proof of [15, Theorem 5.17], using the co-area formula from Theorem 3.4. For $\Omega \subset \mathbb{R}^n$ an arbitrary bounded open set, the restriction $\phi|_\Omega$ lies in $BV(\Omega)$. By [4, Remark 10.3.4, Theorem 10.4.1] we obtain the second assertion when ϕ gets replaced with $\phi|_\Omega$ and E_t by $E_t' := E_t \cap \Omega$ (the t-suplevel set of $\phi|_\Omega$). As Ω is arbitrary, the result for ϕ now follows by noticing that $(\nabla\phi)\lfloor\Omega = \nabla(\phi|\Omega)$ and that for each t such that $\partial_M E_t$ has finite perimeter in $\mathbb{R}^n$ and intersects Ω, then $\boldsymbol{\nu}_{E_t} = \boldsymbol{\nu}_{E_t'}$ on Ω. □

Our next result elaborates on the work in [1]. Recall that a set $E \subset \mathbb{R}^n$ with finite perimeter is called *indecomposable* if it cannot be partitioned as $E = F_1 \cup F_2$ with $\mathcal{L}_n(F_i) > 0$ for $i = 1, 2$ and $\mathcal{P}(F_1) + \mathcal{P}(F_2) = \mathcal{P}(E)$. Every set E of finite perimeter can be partitioned as a countable union $\cup_i C_i$, where the C_i are indecomposable with $\mathcal{L}_n(C_i) > 0$ for each i and $\sum_i \mathcal{P}(C_i) = \mathcal{P}(E)$. Such a partition is unique mod-$\mathcal{L}_n$, and the C_i are called *the M-connected components of E*; moreover, if $F \subset E$ and F is indecomposable, then $F \subset C_i$ mod-$\mathcal{L}_n$ for some i, see [1, Theorem 1].

There is no natural way to order the M-connected components of a set E of finite perimeter, but we can enumerate them so that their $\mathcal{L}_n$-measures are nonincreasing; of course, several orderings with this property will exist if distinct components have the same measure. Also, if E has finitely many M-connected components, it is convenient to append to them a countable infinity of spurious components having $\mathcal{L}_n$-measure zero (therefore also zero perimeter). This will allow us to consistently index the M-connected components over $\mathbb{N}$, regardless whether the set under consideration has finitely many nontrivial components or not.

Formally, let $\mathcal{S}$ be the set of sequences $(F_i)_{i\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ mod-$\mathcal{L}_n$ such that $\mathcal{L}_n(F_i) \geq \mathcal{L}_n(F_{i+1})$ and $\lim_i \mathcal{L}_n(F_i) = 0$. We say that two elements $(F_i)_{i\in\mathbb{N}}$, $(F_i')_{i\in\mathbb{N}}$ of $\mathcal{S}$ are equivalent if there is bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $F_{\sigma(i)} = F_i'$ for all i mod $\mathcal{L}_n$. We denote by $\dot{\mathcal{S}}$ the set of equivalence classes. For E a set of finite perimeter and $C_0, C_1, C_2, \cdots$ a list of its M-connected components, arranged so that their $\mathcal{L}_n$ measures are nonincreasing, we consider $(C_i)_{i\in\mathbb{N}}$ as (a representative of) an element of $\dot{\mathcal{S}}$. If $\mathcal{L}_n(E) < \infty$, then clearly $\mathcal{L}_n(C_i) < \infty$ for all i, and if $\mathcal{L}_n(E) = \infty$, then C_0 is the only component with infinite $\mathcal{L}_n$-measure [1, Rem. 1]. In particular, since $\sum_i \mathcal{P}(C_i) = \mathcal{P}(E)$, we have indeed that $\lim_i \mathcal{L}_n(C_i) = 0$, by the isoperimetric inequality (see *e.g.* [15, Theorem 5.11]). Of course, $(C_i)_{i\in\mathbb{N}}$ is a rather special element of $\mathcal{S}$, because the C_i are pairwise disjoint mod-$\mathcal{L}_n$ and the $\partial_M C_i$ are pairwise disjoint mod-$\mathcal{H}^{n-1}$ (see [1, Proposition 3]).

We now recall an extremal property of M-connected components. Fix $\alpha \in (1, n/(n-1))$ and, for any measurable set $F \subset \mathbb{R}^n$, set $G(F) := (\int_F e^{-|x|^2}dx)^{1/\alpha}$. If E has finite perimeter, then its

M-connected components are the unique solution of

$$\max\left\{\sum_{i\in\mathbb{N}} G(F_i):\ (F_i)_{i\in\mathbb{N}}\in\dot{\mathcal{S}},\ \text{the } F_i \text{ partition } \ E,\ \sum_{i\in\mathbb{N}}\mathcal{P}(F_i)\leq\mathcal{P}(E)\right\}, \tag{31}$$

see the proof of [1, Theorem 1].

We will also need the notion of local convergence in measure for sets of finite perimeter, which is just the L^1_{loc}-convergence of their characteristic function. Any sequence of sets with uniformly bounded perimeters has a subsequence converging locally in measure, and the perimeter is lower semi-continuous for this type of convergence, see *e.g.* [23, Proposition 3.6 & Theorem 3.7].

Proposition 3.6. *Let $\phi\in\dot{BV}(\mathbb{R}^n)$ and E_t be as in (27). For t such that E_t has finite perimeter, let $(C_0^t, C_1^t, C_2^t,\cdots)\in\mathcal{S}$ be (a representative of) the M-connected components of E_t. To each $\eta>0$, there is a σ-compact set $\Sigma_\eta\subset\mathbb{R}$, with $\mathcal{L}_1(\mathbb{R}\setminus\Sigma_\eta)<\eta$, having the following properties.*

(i) For each $t\in\Sigma_\eta$, it holds that E_t has finite perimeter.

(ii) If $(t_m)_{m\geq1}$ is a sequence in Σ_η converging to $t_0\in\Sigma_\eta$, there is a subsequence t_{m_j} such that $C_i^{t_{m_j}}$ converges locally in measure, for fixed i as $j\to\infty$, to a set $F_i\subset\mathbb{R}^n$ of finite perimeter, and the sequence $(F_0,F_1,F_2,\cdots)$ is equivalent to $(C_0^{t_0}, C_1^{t_0}, C_2^{t_0},\cdots)$ in $\dot{\mathcal{S}}$.

(iii) it holds that $\lim_j\mathcal{L}_n((C_i^{t_{m_j}}\setminus F_i)\cup(F_i\setminus C_i^{t_{m_j}}))=0$ and $\lim_j\mathcal{P}(C_i^{t_{m_j}})=\mathcal{P}(F_i)$ for each i.

(iv) One has the limiting relations:

$$\lim_{p\to\infty}\limsup_j\sum_{i\geq p}\mathcal{P}_n(C_i^{t_{m_j}})=0,\quad\text{and}\quad\lim_{p\to\infty}\limsup_j\sup_{i\geq p}\mathcal{L}_n(C_i^{t_{m_j}})=0. \tag{32}$$

Proof. By Lemma 3.1, we may assume that $\phi\in L^{n/(n-1)}(\mathbb{R}^n)$. For $t\in\mathbb{R}$, let us define $M(t):=\lim_{\epsilon\to0}\mathcal{L}_n(\{x:\,t-\epsilon<\phi(x)\leq t+\epsilon\})$. If we fix $k\in\mathbb{N}^*$, every finite sequence $t_1,\cdots,t_\ell$ with $1/k<t_1<t_2<\cdots<t_\ell$ is such that $\sum M(t_i)\leq k^{n/(n-1)}\|\phi\|^{n/(n-1)}_{L^{n/(n-1)}(\mathbb{R}^n)}$. Hence, the set of $t>0$ such that $M(t)>0$ is at most countable, and the same holds for $t<0$. Let $N\subset\mathbb{R}$ be a countable set with $0\in N$ such that $M(t)=0$ for $t\notin N$. Let further $Z\subset\mathbb{R}$ be a Borel set of measure zero such that E_t has finite perimeter for $t\notin Z$, see Lemma 3.3. It follows from Theorem 3.4 (a) that the map $t\mapsto\mathcal{P}(E_t)$ is integrable on $\mathbb{R}$ and therefore, by Lusin's theorem and the regularity of $\mathcal{L}_1$, for each integer $k\geq1$ we can find a compact set $K_k\subset[-k,k]$, with $K_k\cap(Z\cup N)=\emptyset$ and $\mathcal{L}_1([-k,k]\setminus K_k)<1/k^2$, such that $t\mapsto\mathcal{P}(E_t)$ is continuous $K_k\to\mathbb{R}$. Define $\Sigma:=\cup_{k\neq k'}(K_k\cap K_{k'})$, and observe that it is a σ-compact set such that $\mathcal{L}_1(\mathbb{R}\setminus\Sigma)=0$, because a.e. $t\in\mathbb{R}$ belongs to only finitely many sets $[-k,k]\setminus K_k$, by the Borel-Cantelli lemma; hence, a.e. t belongs to all but finitely many K_k, and therefore also to Σ.

We claim that the restriction of $t\mapsto\mathcal{P}(E_t)$ to Σ is continuous. Otherwise indeed, there would be a sequence t_m in Σ, converging to $t_0\in\Sigma$, such that

$$|\mathcal{P}(E_{t_m})-\mathcal{P}(E_{t_0})|>\varepsilon>0\quad\text{for all } m. \tag{33}$$

Since by construction $t\mapsto\mathcal{P}(E_t)$ is continuous on K_k which is compact, it would imply that each K_k contains at most finitely many t_m, for if not a subsequence t_{m_j} would converge in some K_{k_0} to a number $\tilde{t}\in K_{k_0}$ which can be none but t_0, and $\mathcal{P}(E_{t_{m_j}})$ would converge to $\mathcal{P}(E_{t_0})$ which is impossible by (33). Hence, replacing (t_m) with a subsequence if necessary, we may assume that each t_m belongs to at most one K_k. However, by definition of Σ, t_m must belong to two of them at least, a contradiction *which proves the claim.*

Let N_1 denote the norm of $t \mapsto \mathcal{P}(E_t)$ in $L^1(\mathbb{R})$, and set $\Sigma_\eta := \{t \in \Sigma, \mathcal{P}(E_t) \le N_1/\eta\}$. By construction, Σ_η is σ-compact and $\mathcal{L}_1(\mathbb{R} \setminus \Sigma_\eta) < \eta$. Note that $\Sigma_\eta \cap Z = \emptyset$, therefore (i) holds.

Now, let $t_m \to t_0$ in Σ_η. As $t_0 \neq 0$ (for $0 \notin \Sigma_\eta$) and $\phi \in L^{n/(n-1)}(\mathbb{R}^n)$, either $t_0 > 0$ in which case $\mathcal{L}_n(E_{t_0}) < \infty$, or else $t_0 < 0$ in which case $\mathcal{L}_n(E_{t_0}) = \infty$. In the former (resp. latter) case, we may assume that $t_m > 0$ (resp. $t_m < 0$), and then $\mathcal{L}_n(E_{t_m}) < \infty$ (resp. $\mathcal{L}_n(E_{t_m}) = \infty$) for all m. By the boundednes of $t_m \mapsto \mathcal{P}(E_{t_m}) = \sum_i \mathcal{P}(C_i^{t_m})$ (since $t \mapsto \mathcal{P}(E_t)$ is bounded on Σ_η by construction), we get that $\mathcal{P}(C_i^{t_m})$ is bounded independently of i and m, hence for each i some subsequence $C_i^{t^{(i)}_{m_j}}$ converges locally in measure to a set F_i of finite perimeter. Using a diagonal argument, we may assume that $t^{(i)}_{m_j} = t_{m_j}$ is independent of i, and that $C_i^{t_{m_j}}$ converges locally in measure to F_i for each $i \ge 0$. Next, recall from (31) the definition of G and let us prove that

$$\lim_{p\to\infty} \limsup_j \sum_{i=p}^{\infty} G(C_i^{t_{m_j}}) = 0. \tag{34}$$

For this, we adapt the argument of [1, proof of Eqn. (12)]: from the isoperimetric inequality (recall $\mathcal{L}_n(C_i^{t_{m_j}}) < \infty$ for $i \ge 1$) and the subadditivity of perimeter, we get for each $p \ge 1$ and some dimensional constant γ_n that

$$p^{\frac{n-1}{n}} \mathcal{L}_n^{\frac{n-1}{n}}(C_p^{t_{m_j}}) \le \mathcal{L}_n^{\frac{n-1}{n}}\left(\bigcup_{i=1}^{p} C_i^{t_{m_j}}\right) \le \gamma_n \sum_{i=1}^{p} \mathcal{P}(C_i^{t_{m_j}}) \le \gamma_n \mathcal{P}(E_{t_{m_j}}), \tag{35}$$

where the first inequality is because $\mathcal{L}_n(C_i^{t_{m_j}})$ does not increase with i and the $C_i^{t_{m_j}}$ are disjoint mod-$\mathcal{L}_n$. Since $e^{-|x|^2} \le 1$ and $\alpha < n/(n-1)$, we deduce from (35) and the isoperimetric inequality again that

$$\begin{aligned}\sum_{i=p}^{\infty} G(C_i^{t_{m_j}}) \le \sum_{i=p}^{\infty} \mathcal{L}_n^{\frac{1}{\alpha}}(C_i^{t_{m_j}}) &\le \frac{\left(\gamma_n \mathcal{P}(E_{t_{m_j}})\right)^{\frac{n}{\alpha(n-1)}-1}}{p^{\frac{1}{\alpha}-\frac{(n-1)}{n}}} \sum_{i=p}^{\infty} \mathcal{L}_n^{\frac{(n-1)}{n}}(C_i^{t_{m_j}}) \\ &\le \frac{\left(\gamma_n \mathcal{P}(E_{t_{m_j}})\right)^{\frac{n}{\alpha(n-1)}-1}}{p^{\frac{1}{\alpha}-\frac{(n-1)}{n}}} \sum_{i=p}^{\infty} \gamma_n \mathcal{P}(C_i^{t_{m_j}}) \le \frac{\left(\gamma_n \mathcal{P}(E_{t_{m_j}})\right)^{\frac{n}{\alpha}(n-1)}}{p^{\frac{1}{\alpha}-\frac{(n-1)}{n}}} \quad ,\end{aligned}$$

from which (34) follows because $\mathcal{P}(E_{t_{m_j}})$ is bounded independently of m. Observe also that $G(C_i^{t_{m_j}}) \to G(F_i)$ for fixed i as $m \to \infty$, because $x \mapsto e^{-|x|^2}$ is summable and so a 3-ε argument reduces the issue to L^1_{loc}-convergence of $e^{-|x|^2}\chi_{C_i^{t_{m_j}}}(x)$ to $e^{-|x|^2}\chi_{F_i}(x)$, which follows from local convergence in measure of $C_i^{t_{m_j}}$ to F_i. Now, by (34), for every $\epsilon > 0$ there is a $p > 0$ such that $\limsup_j \sum_{i=p}^{\infty} G(C_i^{t_{m_j}}) < \epsilon$. Thus

$$\sum_i G(F_i) \le \liminf_{j\to\infty} \sum_i G(C_i^{t_{m_j}}) \le \lim_{j\to\infty} \sum_{i=0}^{p} G(C_i^{t_{m_j}}) + \epsilon = \sum_{i=0}^{p} G(F_i) + \epsilon \le \sum_i G(F_i) + \epsilon,$$

where the first inequality follows from Fatou's lemma (for series). Since ϵ was arbitrary, we get

$$\lim_{j\to\infty} \sum_i G(C_i^{t_{m_j}}) = \sum_i G(F_i). \tag{36}$$

Because the $C_i^{t_{m_j}}$ are pairwise disjoint mod-$\mathcal{L}_n$, so are the F_i. Moreover, since $t_0 \notin N$ by definition of Σ_η, we have that

$$\lim_{t\to t_0} \mathcal{L}_n\left((E_t \setminus E_{t_0})\bigcup(E_{t_0} \setminus E_t)\right) = 0, \tag{37}$$

implying by local convergence in measure that $F_i \subset E_{t_0}$ mod-$\mathcal{L}_n$ for each i. In addition, as $\alpha > 1$, we see that (34) *a fortiori* implies

$$\sum_i \int_{F_i} e^{-|x|^2} dx, = \lim_{j\to\infty} \sum_i \int_{C_i^{t_{m_j}}} e^{-|x|^2} dx = \lim_{j\to\infty} \int_{E_{t_{m_j}}} e^{-|x|^2} dx = \int_{E_{t_0}} e^{-|x|^2} dx,$$

where the last equality follows from (37). Thus, as $e^{-|x|^2} > 0$ for all x, we get $\mathcal{L}_n(E_{t_0} \setminus \cup_i F_i) = 0$, whence the F_i partition E_{t_0} mod-$\mathcal{L}_n$. Also, by the lower semi-continuity of perimeter with respect to local convergence in measure, we get that

$$\sum_i \mathcal{P}(F_i) \le \lim_j \sum_i \mathcal{P}(C_i^{t_{m_j}}) = \lim_j \mathcal{P}(E_{t_{m_j}}) = \mathcal{P}(E_{t_0}), \tag{38}$$

where the last equality comes from the continuity of $t \mapsto \mathcal{P}(E_t)$ on Σ_η. Therefore, by the maximizing property (31) of M-connected components, it holds that

$$\sum_i G(F_i) \le \sum_i G(C_i^{t_0}). \tag{39}$$

We claim that in fact $\sum_i G(F_i) = \sum_i G(C_i^{t_0})$. To show this, it is enough to consider separately the two cases where $t_{m_j} \to t_0$ from above and from below. Assume first that $t_{m_j} > t_0$ for all j, whence $E_{t_{m_j}} \subset E_{t_0}$. Set $F_i^{t_{m_j}} := E_{t_{m_j}} \cap C_i^{t_0}$ and observe that the $(F_i^{t_{m_j}})_{i\in\mathbb{N}}$ are disjoint mod-$\mathcal{L}_n$ and form a partition of $E_{t_{m_j}}$ mod-$\mathcal{L}_n$. As $\partial_M F_i^{t_{m_j}} \subset \partial_M E_{t_{m_j}} \cup \partial_M C_i^{t_0}$ by definition (23), and because each point of $\partial_M F_i^{t_{m_j}} \setminus \partial_M C_i^{t_0}$ is clearly a density point of $C_i^{t_0}$, we get since the sets of density points of the $C_i^{t_0}$ are pairwise disjoint while $\mathcal{H}^{n-1}(\partial_M C_{i_1}^{t_0} \cap \partial_M C_{i_2}^{t_0}) = 0$ for $i_1 \ne i_2$ (see [1, Proposition 3]) that the $\partial_M F_i^{t_{m_j}}$ are pairwise disjoints mod-$\mathcal{H}^{n-1}$. Hence, by [1, Proposition 3] again, it holds that $\mathcal{P}(E_{t_{m_j}}) = \sum_i \mathcal{P}(F_i^{t_{m_j}})$ and so the $F_i^{t_{m_j}}$ are candidate maximizers in (31) if we put $E = E_{t_{m_j}}$ there. However, as $\mathcal{L}_n(E_{t_0} \setminus E_{t_{m_j}}) \to 0$ by (37), it holds that $\sum_i \mathcal{L}_n(C_i^{t_0} \setminus F_i^{t_{m_j}}) \to 0$ when $j \to \infty$, and since $e^{-|x|^2}$ is summable we get by dominated convergence that

$$\sum_i G(C_i^{t_0}) = \lim_j \sum_i G(F_i^{t_{m_j}}) \le \lim_j \sum_i G(C_i^{t_{m_j}}), \tag{40}$$

where the last inequality comes from the maximizing character of the $(C_i^{t_{m_j}})$ in (31) when $E = E_{t_{m_j}}$. The claim in this case now follows from (40), (39) and (36). Assume next that $t_{m_j} < t_0$ for all j, whence $E_{t_{m_j}} \supset E_{t_0}$. Since $C_i^{t_0}$ is indecomposable and $C_i^{t_0} \subset E_{t_{m_j}}$, it holds that $C_i^{t_0} \subset C_{\ell_i}^{t_{m_j}}$ mod-$\mathcal{L}_n$ for some ℓ_i, by [1, Theorem 1]. Obviously then, $\sum_i G(C_i^{t_0}) \le \sum_i G(C_i^{t_{m_j}})$, and in view of (36), (39) *this proves the claim in all cases.*

From the claim, we deduce by uniqueness of a maximizer in (31) that $(F_i)_{i\in\mathbb{N}}$ and $(C_i^{t_0})_{i\in\mathbb{N}}$ are equivalent in $\dot{\mathcal{S}}$, thereby proving (ii). In particular $\sum_i \mathcal{P}(F_i) = \mathcal{P}(E_{t_0})$, and since $\lim_j \mathcal{P}(C_i^{t_{m_j}}) \ge \mathcal{P}(F_i)$ for each i by lower semi-continuity of the perimeter under local convergence in measure, we deduce from (38) that $\lim_j \mathcal{P}(C_i^{t_{m_j}}) = \mathcal{P}(F_i)$, thereby proving the second half of (iii). To prove the first half, observe that if $t_{m_j} > t_0$ then $E_{t_{m_j}} \subset E_{t_0}$. Therefore $C_i^{t_{m_j}}$, which is indecomposable, must be included in $C_\ell^{t_0}$ for some $\ell = \ell(i,j)$. But for j large enough $C_\ell^{t_0}$ can be none but F_i, and so $\lim_j \mathcal{L}_n(F_i \setminus C_i^{t_{m_j}}) \le \lim_j \mathcal{L}_n(E_{t_0} \setminus E_{t_{m_j}}) = 0$, by (37). If on the contrary $t_{m_j} < t_0$, then $E_{t_{m_j}} \supset E_{t_0}$ and each $C_\ell^{t_0}$, which is indecomposable, must be included in $C_i^{t_{m_j}}$ for some $i = i(\ell, j)$. Necessarily then, it holds that $C_\ell^{t_0} = F_i$, and so $\lim_j \mathcal{L}_n(C_i^{t_{m_j}} \setminus F_i) \le \lim_j \mathcal{L}_n(E_{t_{m_j}} \setminus E_{t_0}) = 0$, by (37) again. Since every F_i is a $C_\ell^{t_0}$ for some $\ell = \ell(i)$, this proves (iii).

To establish (iv), note since $\sum_{i=0}^\infty \mathcal{P}(C_i^{t_0}) < \infty$ that to each $\varepsilon > 0$ there is $i_0 \ge 1$ with $\sum_{i=i_0}^\infty \mathcal{P}(C_i^{t_0}) < \varepsilon$. Then, by lower-semi continuity of the perimeter with respect to local convergence in measure, there is $j_0 = j_0(i_0)$ so large that

$$\sum_{i=0}^{i_0-1} \mathcal{P}(C_i^{t_{m_j}}) > \sum_{i=0}^{i_0-1} \mathcal{P}(C_i^{t_0}) - \varepsilon, \qquad j \ge j_0,$$

and since $\lim_j \sum_i \mathcal{P}(C_i^{t_{m_j}}) = \sum_i \mathcal{P}(C_i^{t_0})$ by (38), we get for j large enough that $\sum_{i=i_0}^\infty \mathcal{P}(C_i^{t_{m_j}}) \le \varepsilon$. As ε was arbitrary, this gives us the first limit in (32), which implies the second by the isoperimetric inequality because $\mathcal{L}_n(C_i^{t_{m_j}}) < \infty$ for $i \ge 1$. □

We equip $\mathcal{S}$ with the distance $d_{\mathcal{S}}((E_i), (E'_i)) = \sup_i d_1(\chi_{E_i}, \chi_{E'_i})$, where d_1 is a distance function on $L^1_{loc}(\mathbb{R}^n)$, and we endow $\dot{\mathcal{S}}$ with the quotient topology (*i.e.* the coarsest topology such that the canonical map $\mathcal{S} \to \dot{\mathcal{S}}$ is continuous). Then, Proposition 3.6 may be construed as an approximate continuity result of the M-connected components of the suplevel sets of a homogeneous BV-function with respect to the level. Recall that a map $\psi : \mathbb{R} \to \mathcal{E}$, with $\mathcal{E}$ a topological space, is approximately contiuous at $t_0 \in \mathbb{R}$ if, for every neighborhood $V \subset \mathcal{E}$ of $\psi(t_0)$, it holds that

$$\lim_{r \to 0} \frac{\mathcal{L}_1\left(\{t : |t - t_0| < r,\ \psi(t) \notin V\}\right)}{r} = 0. \tag{41}$$

Theorem 3.7. *Let $\phi \in \dot{BV}(\mathbb{R}^n)$ and E_t its suplevel set at level t, cf.* (27)*. Then, the map $\psi : \mathbb{R} \to \dot{\mathcal{S}}$ sending t to the M-connected components of E_t is approximately continuous $\mathcal{L}_1$-a.e.*

Proof. It follows from assertions (ii), (iv) of Proposition 3.6 and from the definition of the quotient topology that ψ is continuous on Σ_η for each $\eta > 0$. So, when t_0 is a density point of Σ_η for some $\eta > 0$, then (41) holds. But if D_η denotes the set of such density points, then $\mathbb{R} \setminus (\cup_{k \ge 1} D_{1/k^2})$ has measure zero, by the Borel-Cantelli lemma. Hence (41) holds a.e. □

4. Loop decomposition of divergence-free planar measures

In this section, we make use of Theorem 3.4 and Proposition 3.6 when $n = 2$ to decompose gradients of functions in $\dot{BV}(\mathbb{R}^2)$ as a continuous sum of measures of the form (25), with $\partial_M E$ a rectifiable Jordan curve. The results in this section, up to and including Proposition 4.4, could be developed in an analogous way for $n \ge 3$, replacing Jordan curves with Jordan boundaries (see [1]). However, we stick with $n = 2$ since our main application, stated in Theorem 4.5, is to describe divergence-free vector fields whereas the connection with gradients, stated in Lemma 2.1, only works in the plane.

Lemma 4.1. *Let $E, F \subset \mathbb{R}^2$ be sets of finite perimeter such that $\mathcal{L}_2(E \setminus F) = 0$. Then for $\mathcal{H}^1$-a.e. $x \in \partial_M E \cap \partial_M F$, it holds that $\boldsymbol{\nu}_F(x) = \boldsymbol{\nu}_E(x)$.*

Proof. Given $\epsilon > 0$, $x, v \in \mathbb{R}^2$ with $v \neq 0$ and $G \subset \mathbb{R}^2$, define the half-disk

$$H_\epsilon(x, v) := \{y \in \mathbb{B}(\epsilon, x) \ : \ (y - x) \cdot v > 0\}, \tag{42}$$

and let

$$L_G(x, v) := \lim_{\epsilon \to 0} \frac{\mathcal{L}_2(H_\epsilon(x, v) \cap G)}{\mathcal{L}_2(H_\epsilon(x, v))} = \lim_{\epsilon \to 0} \frac{2\mathcal{L}_2(H_\epsilon(x, v) \cap G)}{\pi \epsilon^2}$$

whenever the limit exists. Assume G has finite perimeter. Then, for $\mathcal{H}^1$-a.e. $x \in \partial_M G$, $\boldsymbol{\nu}_G(x)$ is the unique unit vector that satisfies

$$L_G(x, \boldsymbol{\nu}_G(x)) = 1 \qquad \text{and} \qquad L_G(x, -\boldsymbol{\nu}_G(x)) = 0,$$

(see [4, Proposition 10.3.4 and Theorem 10.3.2] or [32, Thm. 5.6.5]). Since E is included in F except for a set of $\mathcal{L}_2$-measure zero, clearly $L_E(x, -\boldsymbol{\nu}_F(x)) = 0$ for $\mathcal{H}^1$-a.e. $x \in \partial_M F$. Let $Z \subset \partial_M F$ be the set consisting of such x. Moreover, $L_E(x, \boldsymbol{\nu}_E(x)) = 1$ for $\mathcal{H}^1$-a.e. $x \in \partial_M E$, and we let $Y \subset \partial_M F$ be the set consisting of such x. Now, if for $x \in X \cap Y$ we had $\boldsymbol{\nu}_E \neq \boldsymbol{\nu}_F$, the truncated positive cone $C_\epsilon := H_\epsilon(x, -\boldsymbol{\nu}_F) \cap H_\epsilon(x, \boldsymbol{\nu}_E)$ would have strictly positive angle, say θ, and since

$$\limsup_{\epsilon \to 0} \frac{2\mathcal{L}_2(H_\epsilon(x, \boldsymbol{\nu}_E) \cap E \cap C_\epsilon)}{\pi \epsilon^2} = \limsup_{\epsilon \to 0} \frac{2\mathcal{L}_2(E \cap C_\epsilon)}{\pi \epsilon^2} \leq L_E(x, -\boldsymbol{\nu}_F) = 0,$$

we would have that

$$L_E(x, \boldsymbol{\nu}_E) \ = \ \lim_{\epsilon \to 0} \frac{\mathcal{L}_2(H_\epsilon(x, \boldsymbol{\nu}_E) \cap (E \setminus C_\epsilon))}{\mathcal{L}_2(H_\epsilon(x, \boldsymbol{\nu}_E))} \ \leq \ \limsup_{\epsilon \to 0} \frac{\mathcal{L}_2(H_\epsilon(x, \boldsymbol{\nu}_E) \setminus C_\epsilon)}{\mathcal{L}_2(H_\epsilon(x, \boldsymbol{\nu}_E))} \ \leq \ 1 - \frac{\theta}{\pi},$$

a contradiction. □

Let us make one more piece of notation: for $\Gamma \subset \mathbb{R}^2$ a Jordan curve, we denote by $\mathrm{int}(\Gamma)$ (resp. $\mathrm{ext}(\Gamma)$) the bounded (resp. unbounded) connected component of $\mathbb{R}^2 \setminus \Gamma$.

Lemma 4.2. *If $\Gamma \subset \mathbb{R}^2$ is a rectifiable Jordan curve, then $\partial_M(\mathrm{int}(\Gamma)) = \Gamma$ mod-$\mathcal{H}^1$.*

Proof. Clearly $\partial_M(\mathrm{int}(\Gamma))$ is a subset of the topological boundary of $\mathrm{int}(\Gamma)$ which is Γ. Now, by [1, Proposition 2 & Theorem 7], $\partial_M(\mathrm{int}(\Gamma))$ is equal to a rectifiable Jordan curve $\tilde{\Gamma}$ mod-$\mathcal{H}^1$.

Thus, $\mathcal{H}^1(\tilde{\Gamma} \setminus \Gamma) = 0$ whence $\tilde{\Gamma} \cap \Gamma$ is dense in $\tilde{\Gamma}$, and so $\tilde{\Gamma} \subset \Gamma$ by compactness of Γ. Therefore, by the Jordan curve theorem, $\tilde{\Gamma} = \Gamma$ which implies our lemma. □

The next lemma elaborates on [1, Corollary 1].

Lemma 4.3. *The measure-theoretical boundary of a set $E \subset \mathbb{R}^2$ of finite perimeter decomposes mod-$\mathcal{H}^1$ as the union of two countable families of rectifiable Jordan curves $\{\Gamma_k^+\}_{k \in K}$ and $\{\Gamma_j^-\}_{j \in J}$, with $K, J \subset \{1, 2, 3 \cdots\}$, such that*

$$\nabla \chi_E = \sum_{k \in K} \nabla \chi_{\mathrm{int}(\Gamma_k^+)} - \sum_{j \in J} \nabla \chi_{\mathrm{int}(\Gamma_j^-)} \tag{43}$$

$$\mathcal{H}^1 \lfloor (\partial_M E) = \sum_{k \in K} \mathcal{H}^1 \lfloor \Gamma_k^+ + \sum_{j \in J} \mathcal{H}^1 \lfloor \Gamma_j^-. \tag{44}$$

Moreover, if we let

$$I_k := \{j \in J : \mathrm{int}(\Gamma_j^-) \subset \mathrm{int}(\Gamma_k^+)\} \qquad \text{and} \qquad Y_k = \mathrm{int}(\Gamma_k^+) \setminus \cup_{j \in I_k} \mathrm{int}(\Gamma_j^-), \tag{45}$$

as well as

$$Y_0 := \bigcap_{j\in J} \mathrm{ext}(\Gamma_j^-) \ \textit{if}\ \mathcal{L}_2(E)=\infty \text{ and } Y_0 := \emptyset \ \ \textit{otherwise}, \tag{46}$$

then the Y_k for $k \in K$, together with Y_0 if nonempty, are the M-connected components of E. In particular, it holds that

$$E = \left(\bigcup_{k\in K} Y_k\right) \cup Y_0 \quad \text{mod-}\mathcal{L}_2. \tag{47}$$

In addition, if we put

$$\tilde{I}_k := \{j \in I_k : \ \textit{there is no } k' \in K \ \textit{such that}\ \mathrm{int}(\Gamma_k^+) \supsetneq \mathrm{int}(\Gamma_{k'}^+) \supset \mathrm{int}(\Gamma_j^-)\} \tag{48}$$

along with

$$I_\infty := \{j \in J : \ \textit{there is no } k \in K \ \textit{such that}\ \mathrm{int}(\Gamma_k^+) \supset \mathrm{int}(\Gamma_j^-)\}, \tag{49}$$

then $I_\infty \neq \emptyset$ if and only if $\mathcal{L}_2(E) = \infty$ and each $j \in J$ belongs to $\tilde{I}_k$ for some unique k or else to I_∞. Furthermore, for each $k \in K$, the sets $\{\mathrm{int}(\Gamma_i^-)\}_{i\in\tilde{I}_k}$ together with $\mathrm{ext}(\Gamma_k^+)$ are the M-connected components of $\mathbb{R}^2 \setminus Y_k$, and if $\mathcal{L}_2(E) = \infty$ then the $\{\mathrm{int}(\Gamma_j^-)\}_{j\in I_\infty}$ are the M-connected components of $\mathbb{R}^2 \setminus Y_0$.

Proof. By [1, Corollary 1], there exists two families $\{\Gamma_k^+\}_{k\in K}$ and $\{\Gamma_j^-\}_{j\in J}$ of countably many rectifiable Jordan curves (we can always take $K, J \subset \{1, 2, 3 \cdots\}$), satisfying:

(a) $\partial_M E = \bigcup_k \Gamma_k^+ \cup \bigcup_j \Gamma_j^-$ mod-$\mathcal{H}^1$,
(b) For any two $\mathrm{int}(\Gamma_k^+)$ and $\mathrm{int}(\Gamma_l^+)$ either one is contained in the other or they are disjoint. Similarly, for any two $\mathrm{int}(\Gamma_j^-)$ and $\mathrm{int}(\Gamma_i^-)$ either one is contained in the other or they are disjoint.
(c) $\mathcal{H}^1(\partial_M E) = \sum_k \mathcal{H}^1(\Gamma_k^+) + \sum_j \mathcal{H}^1(\Gamma_j^-)$, in particular the curves are disjoint mod-$\mathcal{H}^1$.
(d) If $l \neq k$ and $\mathrm{int}(\Gamma_k^+) \subset \mathrm{int}(\Gamma_l^+)$ then there exists a $\mathrm{int}(\Gamma_j^-)$ with the property that $\mathrm{int}(\Gamma_k^+) \subset \mathrm{int}(\Gamma_j^-) \subset \mathrm{int}(\Gamma_l^+)$. Analogously, if $j \neq i$ and $\mathrm{int}(\Gamma_j^-) \subset \mathrm{int}(\Gamma_i^-)$ then there exists a $\mathrm{int}(\Gamma_k^+)$ such that $\mathrm{int}(\Gamma_j^-) \subset \mathrm{int}(\Gamma_k^+) \subset \mathrm{int}(\Gamma_i^-)$.
(e) The Y_k defined in (45), along with Y_0 defined in (46) if nonempty[2], are the M-connected components of E, in particular (47) holds. Note that if $\mathcal{L}_2(E) = \infty$, then Y_0 is the M-connected component of infinite $\mathcal{L}_2$-measure. Note also that $\mathcal{L}_2(E) = \infty$ (equivalently: $Y_0 \neq \emptyset$) if and only if there exists a $\mathrm{int}(\Gamma_j^-)$ not contained in any $\mathrm{int}(\Gamma_k^+)$, that is: if and only if $I_\infty \neq \emptyset$.

It remains for us to show that this decomposition satisfies (43) and that the last two assertions after (49) do hold. In view of (26) and (44), it is enough for (43) to hold that

(i) for any $k \in K$, $\nabla\chi_E \lfloor \Gamma_k^+ = \nabla\chi_{\mathrm{int}(\Gamma_k^+)}$,
(ii) for any $j \in J$, $\nabla\chi_E \lfloor \Gamma_j^- = -\nabla\chi_{\mathrm{int}(\Gamma_j^-)}$.

To obtain (i) and (ii), we will prove that for each $k_0 \in K$ (resp. $j_0 \in J$) and $\mathcal{H}^1$-a.e. $x \in \Gamma_{k_0}^+$ (resp. $\Gamma_{j_0}^-$), we have $\boldsymbol{\nu}_E(x) = \boldsymbol{\nu}_{\mathrm{int}\Gamma_{k_0}^+}(x)$ (resp. $\boldsymbol{\nu}_E(x) = -\boldsymbol{\nu}_{\mathrm{int}\Gamma_{j_0}^-}(x)$).

[2] In [1, Cor. 1], the set Y_0 is not introduced, but an abstract "Jordan curve" Γ_∞^+, reducing to the point at ∞ (*i.e.* having zero length and interior $\mathbb{R}^2$), is allowed in case $\mathcal{L}_2(E) = \infty$, so that Y_0 corresponds to $\mathrm{int}(\Gamma_\infty^+) \setminus \cup_j \mathrm{int}(\Gamma_j^-)$.

Fix $k_0 \in K$ and let $F_{k_0} := \mathrm{int}(\Gamma^+_{k_0}) \cap E$. Define $\tilde{K} := \{k \in K : \mathrm{int}(\Gamma^+_k) \subset \mathrm{int}(\Gamma^+_{k_0})\}$ and $\tilde{J} := \bigcup_{k\in\tilde{K}} I_k$. The pair of families of rectifiable Jordan curves $\{\Gamma^+_k\}_{k\in\tilde{K}}$, $\{\Gamma^-_j\}_{j\in\tilde{J}}$ *a fortiori* meets properties (b) and (d) above when the indices k, l and j, i range over $\tilde{K}$ and $\tilde{J}$, respectively. Also, by (c), these families are such that

(f) each two different Jordan curves are disjoint mod-$\mathcal{H}^1$,

(g) $\sum_k \mathcal{H}^1(\Gamma_k) + \sum_j \mathcal{H}^1(\Gamma^-_j) < \infty$, $k \in \tilde{K}$, $j \in \tilde{J}$.

Moreover, we get from (b) and (47) that

(h) $F_{k_0} = \bigcup_{k\in\tilde{K}} Y_k$ mod-$\mathcal{L}_2$.

Properties (b), (d), (f), (g) and (h) show that F_{k_0}, $\{\Gamma^+_k\}_{k\in\tilde{K}}$ and $\{\Gamma^-_j\}_{j\in\tilde{J}}$ satisfy the assumptions of [1, Theorem 5]. The latter implies that F_{k_0} has finite perimeter and that $\partial_M F_{k_0} = \bigcup_{k\in\tilde{K}} \Gamma^+_k \cup \bigcup_{j\in\tilde{J}} \Gamma^-_j$ mod-$\mathcal{H}^1$. Applying Lemma 4.1 twice, we now get that $\boldsymbol{\nu}_E(x) = \boldsymbol{\nu}_{F_{k_0}}(x) = \boldsymbol{\nu}_{\mathrm{int}(\Gamma^+_{k_0})}(x)$ for $\mathcal{H}^1$-a.e. $x \in (\partial_M F_{k_0} \cap \partial_M E \cap \partial_M \mathrm{int}(\Gamma^+_{k_0}))$, and by Lemma 4.2 this intersection reduces to $\Gamma^+_{k_0}$ mod-$\mathcal{H}^1$. This proves (i).

To prove (ii), pick $j_0 \in J$ and assume first that $j_0 \notin I_\infty$, so there is $k_0 \in K$ such that $\mathrm{int}(\Gamma^+_{k_0}) \supset \mathrm{int}(\Gamma^-_{j_0})$. As there is no infinite sequence $\mathrm{int}(\Gamma^+_{\ell_1}) \supsetneq \mathrm{int}\ (\Gamma^+_{\ell_2}) \supsetneq \cdots$ each element of which contains $\mathrm{int}(\Gamma^-_{j_0})$ (otherwise the isoperimetric inequality would imply that $\pi^{1/2}\mathcal{H}^1(\Gamma^+_{\ell_i}) \geq \mathcal{L}_2^{1/2}(\mathrm{int}(\Gamma^-_{j_0})) > 0$ for all i and this would contradict (g)), we may choose k_0 so that $\mathrm{int}(\Gamma^+_{k_0})$ is smallest with the property that $\mathrm{int}(\Gamma^+_{k_0}) \supset \mathrm{int}(\Gamma^-_{j_0})$ or, equivalently, such that $j_0 \in \tilde{I}_{k_0}$ defined in (48). Note that such a k_0 is unique, by (b), thereby proving in passing the next-to-last assertion after (49).

Now, the sets $\{\mathrm{int}(\Gamma^-_j)\}_{j\in\tilde{I}_{k_0}}$ are disjoint, by (b) and (d). Moreover, for each $i \in I_{k_0}$, there is $j \in \tilde{I}_{k_0}$ such that $\mathrm{int}(\Gamma^-_i) \subset \mathrm{int}(\Gamma^-_j)$, because of (d) and the fact that there is no infinite sequence $\mathrm{int}(\Gamma^-_{j_1}) \subsetneq \mathrm{int}(\Gamma^+_{k_1}) \subsetneq \mathrm{int}(\Gamma^-_{j_2}) \subsetneq \mathrm{int}(\Gamma^+_{k_2}) \cdots$, by (c) and the isoperimetric inequality again. In particular, we have that

$$Y_{k_0} = \mathrm{int}(\Gamma^+_{k_0}) \setminus \bigcup_{j\in\tilde{I}_{k_0}} \mathrm{int}(\Gamma^-_j). \tag{50}$$

Thus, the set Y_{k_0} and the pair of families of curves $\{\Gamma^+_{k_0}\}$, $\{\Gamma^-_j, j \in \tilde{I}_{k_0}\}$ (the first family has only one element) satisfy the assumptions of [1, Theorem 5], to the effect that

$$\partial_M Y_{k_0} = \Gamma^+_{k_0} \cup \bigcup_{j\in\tilde{I}_{k_0}} \Gamma^-_j \quad \text{mod-}\mathcal{H}^1. \tag{51}$$

In another connection, if we define F_{k_0} as before, we get from the first part of the proof and Lemma 4.1 that

$$\Gamma^-_{j_0} \subset \partial_M F_{k_0} \cap \partial_M E \qquad \text{and} \qquad \boldsymbol{\nu}_E(x) = \boldsymbol{\nu}_{F_{k_0}}(x), \quad \mathcal{H}^1\text{-a.e. } x \in \Gamma^-_{j_0}. \tag{52}$$

Moreover, (h) implies that $F_{k_0} \supset Y_{k_0}$ mod-$\mathcal{L}_2$, and (52), (51) that $\Gamma^-_{j_0} \subset \partial_M F_{k_0} \cap \partial_M Y_{k_0}$ mod-$\mathcal{H}^1$, therefore we conclude from Lemma 4.1 that

$$\boldsymbol{\nu}_{F_{k_0}}(x) = \boldsymbol{\nu}_{Y_{k_0}}(x), \qquad \mathcal{H}^1\text{-a.e. } x \in \Gamma^-_{j_0}. \tag{53}$$

Besides, since $Y_{k_0} \subset \mathrm{ext}(\Gamma^-_{j_0})$ by (50), while $\Gamma^-_{j_0} \subset \partial_M Y_{k_0} \cap \partial_M \mathrm{ext}(\Gamma^-_{j_0})$ mod-$\mathcal{H}^1$ by (51) and Lemma 4.2, we get from Lemma 4.1 again that

$$\boldsymbol{\nu}_{Y_{k_0}}(x) = \boldsymbol{\nu}_{\mathrm{ext}(\Gamma^-_{j_0})}(x) = -\boldsymbol{\nu}_{\mathrm{int}(\Gamma^-_{j_0})}(x), \qquad \mathcal{H}^1\text{-a.e. } x \in \Gamma^-_{j_0}. \tag{54}$$

The conjunction of (52), (53) and (54) proves (ii) when $j_0 \notin I_\infty$. Next, assume that $j_0 \in I_\infty$; in particular $I_\infty \neq \emptyset$ so that $Y_0 \neq \emptyset$, where Y_0 was defined in (46). If we define

$$\tilde{I} := \{i \in J : \text{ there is no } j \in J \text{ such that } \mathrm{int}(\Gamma_j^-) \supsetneq \mathrm{int}(\Gamma_i^-)\}, \tag{55}$$

we obviouly have that $Y_0 = \bigcap_{i\in\tilde{I}} \mathrm{ext}(\Gamma_j^-)$. Note that the sets $\{\mathrm{int}(\Gamma_i^-)\}_{i\in\tilde{I}}$ are disjoint, by (b). Thus, if we let $\Upsilon_i^+ := \Gamma_i^-$, we get in view of (c) that the set $\mathbb{R}^2 \setminus Y_0 = \bigcup_{i\in\tilde{I}} \mathrm{int}(\Upsilon_i^+)$ together with the pair of families of rectifiable Jordan curves $\{\Upsilon_i^+,\, i \in \tilde{I}\}$, $\emptyset$ (*i.e.* the second family is empty), satisfy the assumptions of [1, Theorem 5]. The latter implies that

$$\partial_M(\mathbb{R}^2 \setminus Y_0) = \bigcup_{i\in\tilde{I}} \Gamma_i^-, \tag{56}$$

and since $j_0 \in \tilde{I}$, by (d), we get from Lemma 4.1 that $\boldsymbol{\nu}_{\mathrm{int}(\Gamma_{j_0}^-)}(x) = \boldsymbol{\nu}_{\mathbb{R}^2\setminus Y_0}(x) = -\boldsymbol{\nu}_{Y_0}(x)$ for $\mathcal{H}^1$-a.e. $x \in \Gamma_{j_0}^-$. As $Y_0 \subset E$ and $\Gamma_{j_0}^- \subset \partial_M E \cap \partial_M Y_0$, by (56), another application of Lemma 4.1 yields that $\boldsymbol{\nu}_{Y_0}(x) = \boldsymbol{\nu}_E(x)$ for $\mathcal{H}^1$-a.e. $x \in \Gamma_{j_0}^-$, thereby establishing (ii) in this case as well.

To prove the last assertion after (49), pick $k \in K$ and observe from (b) and (d) that the sets $\mathrm{ext}(\Gamma_k^+)$ and $\{\mathrm{int}(\Gamma_i^-)\}_{i\in\tilde{I}_k}$ are pairwise disjoint, while $\mathbb{R}^2 \setminus Y_k$ is their union. These sets are indecomposable, by Lemma 4.2 and [1, Theorem 2], and since their measure-theoretical boundaries are pairwise disjoint mod-$\mathcal{H}^1$, because of (c), we deduce from [1, Propostion 3] that their perimeters add up to $\mathcal{P}(\mathbb{R}^2 \setminus Y_k)$. Hence, they are indeed the M-connected components of $\mathbb{R}^2 \setminus Y_k$. If $\mathcal{L}_2(E) = \infty$, so that $Y_0 \neq \emptyset$, a similar reasoning on (56) shows that the $\{\mathrm{int}(\Gamma_i^-)\}_{i\in\tilde{I}}$ are the M-connected components of $\mathbb{R}^2 \setminus Y_0$, and it remains for us to prove that $\tilde{I} = I_\infty$. From (d), we know that $I_\infty \subset \tilde{I}$. Conversely, if $j \in J$ and $j \notin I_\infty$, we showed earlier there is a unique $k_0 \in K$ such that $j \in \tilde{I}_{k_0}$. We also know that Y_{k_0} is a M-connected component of E, therefore it is indecomposable and disjoint mod-$\mathcal{L}_2$ from Y_0 which is another such component. Consequently, by (56), we have that $Y_{k_0} \subset \bigcup_{i\in\tilde{I}} \mathrm{int}(\Gamma_i^-)$ mod-$\mathcal{L}_2$. As the $\{\mathrm{int}(\Gamma_i^-)\}_{i\in\tilde{I}}$ are the M-connected components of $\mathbb{R}^2 \setminus Y_0$ and Y_{k_0} is indecomposable, we get that $Y_{k_0} \subset \mathrm{int}(\Gamma_{i_0}^-)$ mod-$\mathcal{L}_2$ for some $i_0 \in \tilde{I}$. It implies easily that $\mathcal{H}^1$-a.e. point of $\partial_M Y_{k_0}$ is not a density point of $\mathrm{ext}(\Gamma_{i_0}^-)$. *A fortiori* then, by (51), $\Gamma_j^- \subset \overline{\mathrm{int}(\Gamma_{i_0}^-)}$ mod-$\mathcal{H}^1$ where the bar indicates Euclidean closure. Since Γ_j^- is a closed curve we get in fact that $\Gamma_j^- \subset \overline{\mathrm{int}(\Gamma_{i_0}^-)}$, and by the Jordan curve theorem it follows that $\mathrm{int}(\Gamma_j^-) \subset \mathrm{int}(\Gamma_{i_0}^-)$, whence $j \notin \tilde{I}$. The proof is now complete. □

Lemma 4.3 tells us that the measure-theoretical boundary of a set E of finite perimeter consists of two countable families of Jordan curves, namely $\{\Gamma_k^+\}_{k\in K}$ and $\{\Gamma_j^-\}_{j\in J}$, such that the $\mathrm{int}\,\Gamma_j^-$ and the $\mathrm{ext}\Gamma_k^+$ are the M-connected components of the complements of the M-connected components of E. This will allow us to put a structure on these Jordan curves. More precisely, recall from Section 3 (we put $n = 2$) the set $\mathcal{S}$ of sequences of subsets of $\mathbb{R}^2$ mod-$\mathcal{L}_2$ whose $\mathcal{L}_2$-measures are non increasing and tend to zero, as well as the set $\dot{\mathcal{S}}$ of equivalence classes modulo permutations. As stressed in that section, the M-connected components of a set of finite perimeter may be regarded as a member of $\dot{\mathcal{S}}$, a representative of which is obtained in $\mathcal{S}$ by arranging the M-connected components in nonincreasing measure, and appending to them infinitely many copies of the emptyset if these components are finite in number. For $S \in \mathcal{S}$, say $S = (F_0, F_1, F_2, \cdots)$, we let for simplicity $\mathfrak{U}_S = \cup_j F_j$, and we let $\mathcal{T}$ be the subset of $\mathcal{S}^{\mathbb{N}}$ consisting of sequences $(S_0, S_1, S_2, \cdots)$ such that $(\mathbb{R}^2 \setminus \mathfrak{U}_{S_0}, \mathbb{R}^2 \setminus \mathfrak{U}_{S_1}, \mathbb{R}^2 \setminus \mathfrak{U}_{S_2}, \cdots)$ also lies in $\mathcal{S}$. We say that two elements $(S_i)_{i\in\mathbb{N}}$ and

$(S'_i)_{i\in\mathbb{N}}$ of $\mathcal{T}$ are equivalent if there is a permutation $\sigma:\mathbb{N}\to\mathbb{N}$ such that S_i and $S'_{\sigma(i)}$ represent the same element in $\dot{\mathcal{S}}$. We call $\dot{\mathcal{T}}$ the set of equivalence classes.

With the notation of Lemma 4.3, let K be ordered so that the $\mathcal{L}_2(Y_k)$, $k\in K$, are nonincreasing, and append to the sequence Y_k infinitely many copies of the empty set if K is finite. We define a particular element $S=(S_0,S_1,S_2,\cdots)$ of $\mathcal{T}$ as follows. Let $S_0=(\emptyset,\emptyset,\cdots)$ if $\mathcal{L}_2(E)<\infty$, otherwise let S_0 be a representative in $\mathcal{S}$ of the M-connected components of $\mathbb{R}^2\setminus Y_0$. Let further S_k, for $k\geq 1$, be a representative in $\mathcal{S}$ of the M-connected components of $\mathbb{R}^2\setminus Y_k$. Note that $(\mathbb{R}^2\setminus\mathfrak{U}_{S_0},\mathbb{R}^2\setminus\mathfrak{U}_{S_1},\mathbb{R}^2\setminus\mathfrak{U}_{S_2},\cdots)$ is equal to $(Y_0,Y_1,\cdots)$ if $\mathcal{L}_2(E)=\infty$ and to $(\mathbb{R}^2,Y_1,\cdots)$ if $\mathcal{L}_2(E)<\infty$, so it is an element of $\mathcal{S}$. Hence, $S:=(S_0,S_1,S_2,\cdots)$ belongs to $\mathcal{T}$, and if for $k\geq 0$ we write $S_k=(S_{k,0},S_{k,1},\cdots)$, where the $S_{k,j}$ are sets of finite perimeter mod-$\mathcal{L}_2$ constitutive of $S_k\in\mathcal{S}$, then: (i) for $k\geq 1$ we have $S_{k,0}=\mathrm{ext}(\Gamma_k^+)$ while $(S_{k,j})_{j\geq 1}$ enumerates the $(\mathrm{int}(\Gamma_j^-))_{j\in\tilde{I}_k}$ in nonincreasing $\mathcal{L}_2$-measure, with infinitely many copies of the empty set appended when $\tilde{I}_k$ is finite; (ii) if $\mathcal{L}_2(E)=\infty$ then $(S_{0,j})_{j\in\mathbb{N}}$ enumerates the $(\mathrm{int}(\Gamma_j^-))_{j\in I_\infty}$ in nonincreasing $\mathcal{L}_2$-measure, with infinitely many copies of the empty set appended when I_∞ is finite, and if $\mathcal{L}_2(E)<\infty$ then $S_{0,j}=\emptyset$ for all j. Altogether, the families $\{(\mathrm{ext}(\Gamma_k^+))_{k\in K}\}$, $\{(\mathrm{int}(\Gamma_j^-))_{j\in J}\}$, padded with copies of the empty set if needed and arranged in the previously described structure as entries of the infinite array $(S_{k,j})$, $0\leq k,j\leq\infty$, define some $S\in\mathcal{T}$. Of course, S depends on the ordering we chose to enumerate the Y_k and the M-connected components of the $\mathbb{R}^2\setminus Y_k$, if there are several orderings making their $\mathcal{L}_2$-measures nonincreasing. However, the equivalence class $\dot{S}\in\dot{\mathcal{T}}$ is independent of such choices.

We orient the Γ_k^+ counterclockwise and the Γ_j^- clockwise. This allows us to regard Γ_k^+ (resp. Γ_j^-) as the image of a unique parametrized Jordan curve $\boldsymbol{\gamma}_k^+$ (resp. $\boldsymbol{\gamma}_j^-$). We shall identify $\mathrm{ext}(\Gamma_k^+)$ (resp. $\mathrm{int}(\Gamma_j^-)$) with $\boldsymbol{\gamma}_k^+$ (resp. $\boldsymbol{\gamma}_j^-$), and we regard the emptyset as a degenerate curve reducing to a point. This way, the sets $S_{k,j}$ defined above can be viewed as parametrized rectifiable Jordan curves, and the latter can in turn be considered as measures if we regard a parametrized Jordan curve $\boldsymbol{\gamma}$ as the member $\mathbf{R}_{\boldsymbol{\gamma}}$ of $\mathcal{M}(\mathbb{R}^2)^2$ defined in (12). Here, a degenerate curve has constant parametrization and therefore corresponds to the zero measure. Recall also from Section 2.3 that if $\boldsymbol{\gamma}$ is a parametrized rectifiable Jordan curve of length $L>0$ and $\tilde{\boldsymbol{\gamma}}:\mathbb{R}\to\mathbb{R}^2$ is the periodic extension of $\boldsymbol{\gamma}$, then $\tilde{\boldsymbol{\gamma}}$ defines *via* (17) the elementary solenoid $\mathbf{T}_{\tilde{\boldsymbol{\gamma}}}=\mathbf{R}_{\boldsymbol{\gamma}}/L$, and in the degenerate case where $\boldsymbol{\gamma}$ reduces to a point, we define $\mathbf{T}_{\tilde{\boldsymbol{\gamma}}}=0$.

Proposition 4.4. *Let $\phi\in\dot{BV}(\mathbb{R}^2)$ and E_t be as in (27). For t such that E_t has finite perimeter, let $S^t:=(S_0^t,S_1^t,S_2^t,\cdots)\in\mathcal{T}$ be constructed as indicated above from the curves $\{(\Gamma_k^+)_{k\in K}\}$, $\{(\Gamma_j^-)_{j\in J}\}$ obtained by applying Lemma 4.3 to E_t. Write $S_k^t=(S_{k,0}^t,S_{k,1}^t,\cdots)$ for the components of $S_k^t\in\mathcal{S}$. As we just explained, each $S_{k,j}^t$ identifies with a parametrized Jordan curve $\boldsymbol{\gamma}_{k,j}^t$ with image $\Gamma_{k,j}^t$. To each $\eta>0$, there is a σ-compact set $\Sigma_\eta\subset\mathbb{R}$, with $\mathcal{L}_1(\mathbb{R}\setminus\Sigma_\eta)<\eta$, such that:*

(i) For each $t\in\Sigma_\eta$, it holds that E_t has finite perimeter.

(ii) For each sequence $(t_m)_{m\geq 1}$ in Σ_η converging to $t_0\in\Sigma_\eta$, there is a subsequence t_{m_ℓ} such that $\mathbf{R}_{\boldsymbol{\gamma}_{k,j}^{t_{m_\ell}}}$ converges weak-$*$, as $\ell\to\infty$ for fixed k,j, to $\mathbf{R}_{\boldsymbol{\gamma}_{k,j}}$ for some parametrized Jordan curve $\boldsymbol{\gamma}_{k,j}$ with image $\Gamma_{k,j}$. Moreover, $(\boldsymbol{\gamma}_{k,j})_{k,j\in\mathbb{N}}$ is equivalent to S^{t_0} in $\dot{\mathcal{T}}$.

(iii) We have the limiting relation $\lim_\ell\mathcal{H}^1(\Gamma_{k,j}^{t_{m_\ell}})=\mathcal{H}^1(\Gamma_{k,j})$ for each (k,j).

(iv) It holds that $\mathbf{T}_{\tilde{\boldsymbol{\gamma}}_{k,j}^{t_{m_\ell}}}$ converges weak-$*$, as $\ell\to\infty$ for fixed k,j, to $\mathbf{T}_{\tilde{\boldsymbol{\gamma}}_{k,j}}$.

Proof. We adopt the notation of Lemma 4.3 for the decomposition of E^t, only with an extra-superscript t to keep track of the level; *e.g.*, as in Y_k^t. By Lemma 3.1 we may assume that $\phi \in L^2(\mathbb{R}^2)$, so that $\mathcal{L}_2(E_t) = \infty$ when $t < 0$ and $\mathcal{L}_2(E_t) < \infty$ when $t > 0$. To avoid bookkeeping with indices, we give the proof when $t_0 < 0$ only, as the case where $t_0 > 0$ is similar but simpler. Thus, we may assume that $t_m < 0$ for all m. With Σ_η as in Proposition 3.6, we know from the latter that (i) holds and that, for some subsequence t_{m_i}, the $Y_k^{t_{m_i}}$ converge, locally in measure for fixed k as $i \to \infty$, to some F_k such that $(F_k)_{k\geq 0}$ is equivalent to $(Y_k^{t_0})_{k\geq 0}$ in $\dot{\mathcal{S}}$. Moreover, we know from (iii) of this proposition that $\lim_i \mathcal{L}_n((Y_k^{t_{m_i}} \setminus F_k) \cup (F_k \setminus Y_k^{t_{m_i}})) = 0$ and that $\lim_i \mathcal{P}(Y_k^{t_{m_i}}) = \mathcal{P}(F_k)$ for each k. Equivalently, the $\mathbb{R}^2 \setminus Y_k^{t_{m_i}}$ converge locally in measure to $\mathbb{R}^2 \setminus F_k$ as $i \to \infty$ and $\lim_i \mathcal{L}_n((\mathbb{R}^2 \setminus Y_k^{t_{m_i}}) \setminus (\mathbb{R}^2 \setminus F_k) \cup ((\mathbb{R}^2 \setminus F_k) \setminus (\mathbb{R}^2 \setminus Y_k^{t_{m_i}}))) = 0$, while $\lim_i \mathcal{P}(\mathbb{R}^2 \setminus Y_k^{t_{m_i}}) = \mathcal{P}(\mathbb{R}^2 \setminus F_k)$ for each k. This is all we need to apply the proof of Proposition 3.6 to $\mathbb{R}^2 \setminus Y_k^{t_{m_i}}$ instead of $E^{t_{m_i}}$, to the effect that for each $k \geq 0$ there is a subsequence $t_{m_{i_\ell}}^{(k)}$ of t_{m_i} such that $S_{k,j}^{t_{m_{i_\ell}}^{(k)}}$ converges locally in measure to some $C_{k,j}$, where $(C_{k,j})_{j\in\mathbb{N}}$ is equivalent to $S_k^{t_0}$ in $\dot{\mathcal{S}}$. Using a diagonal argument, we can make $t_{m_{i_\ell}}^{(k)}$ independent of k and we rename it as t_{m_ℓ} for simplicity. By construction, we may write for $k = 0$ or $j \geq 1$ that $C_{k,j} = \operatorname{int}(\Gamma_{k,j})$ mod-$\mathcal{L}_2$ with $\Gamma_{k,j} = \Gamma_l^{-,t_0}$ for some $l = l(k,i)$, while for $k \geq 1$ we have $C_{k,0} = \operatorname{ext}(\Gamma_{k,0})$ mod-$\mathcal{L}_2$ with $\Gamma_{k,0} = \Gamma_k^{+,t_0}$. Moreover, we know from the proof of Proposition 3.6 point (iii) that $\lim_\ell \mathcal{P}(S_{k,j}^{t_{m_\ell}}) = \mathcal{P}(C_{k,j})$ or, equivalently, that $\lim_\ell \mathcal{H}^1(\Gamma_{k,j}^{t_{m_\ell}}) = \mathcal{H}^1(\Gamma_{k,j})$, which proves (iii). Now, if we let $\boldsymbol{\gamma}_{k,j}$ be a parametrization of $\Gamma_{k,j}$ and $\boldsymbol{\gamma}_{k,j}^{t_{m_\ell}}$ be a parametrization of $\Gamma_{k,j}^{t_{m_\ell}}$, oriented clockwise for $j \geq 1$ or $k = 0$ and counterclockwise when $j = 0$ and $k \geq 1$, it follows from (26) and a mollification argument, since $\mathcal{H}^1(\Gamma_{k,j}^{t_{m_\ell}})$ is bounded for fixed k, j as $\ell \to \infty$, that $\boldsymbol{\gamma}_{k,j}^{t_{m_\ell}}$ converges weak-$*$ to $\boldsymbol{\gamma}_{k,j}$. Applying pointwise a rotation by $\pi/2$, this is tantamount to say that $\mathbf{R}_{\boldsymbol{\gamma}_{k,j}^{t_{m_\ell}}}$ converges weak-$*$ to $\mathbf{R}_{\boldsymbol{\gamma}_{k,j}}$, thereby proving (ii). Note that when $\mathcal{H}^1(\Gamma_{k,j}) > 0$, then the assertion of item (iv) follows immediately from items (ii) and (iii). Now suppose $\mathcal{H}^1(\Gamma_{k,j}) = 0$. Let $\mathbf{f} \in C_c(\mathbb{R}^2)^2$ and $\epsilon > 0$. By uniform continuity, there is some $\delta > 0$ such that $|\mathbf{f}(x) - \mathbf{f}(y)| < \epsilon$ whenever $|x - y| < \delta$. Let L_ϵ be such that $\operatorname{diam}(\Gamma_{k,j}^{t_{m_\ell}}) < \delta$ for $\ell \geq L_\epsilon$. Since $\mathbf{R}_{\boldsymbol{\gamma}_{k,j}^{t_{m_\ell}}}$ is divergence free for all j, k, ℓ, it annihilates constant functions. Thus, for $x_\ell \in \Gamma_{k,j}^{t_{m_\ell}}$, we have

$$|\langle \mathbf{f}, \mathbf{R}_{\boldsymbol{\gamma}_{k,j}^{t_{m_\ell}}} \rangle| = |\langle \mathbf{f} - \mathbf{f}(x_\ell), \mathbf{R}_{\boldsymbol{\gamma}_{k,j}^{t_{m_\ell}}} \rangle| \leq \epsilon \mathcal{H}^1(\Gamma_{k,j}^{t_{m_\ell}}),$$

which verifies (iv) in this case. □

In the discussion before Proposition 4.4, we identified the curves $\{\Gamma_k^+\}_{k\in K}$ and $\{\Gamma_j^-\}_{j\in J}$ forming the measure-theoretical boundary of a set of finite perimeter with (the equivalence classes of) an element of $\mathcal{T}$ of the form $S = (S_{k,j})_{k,j\in\mathbb{N}}$ where $S_{k,j}$ is (the interior of) a (possibly degenerate) Jordan curve oriented clockwise for $j \geq 1$ or $k = 0$, while $S_{k,0}$ is (the exterior of) a Jordan curve oriented counterclockwise when $k \geq 1$. We let $\mathcal{C} \subset \mathcal{T}$ denote the set of such elements, and $\dot{\mathcal{C}}$ the set of equivalence classes. Recalling that $\mathcal{M}(\mathbb{R}^2)^2$ equipped with the weak-$*$ topology is a metric space, say with distance d_w, we endow $\mathcal{C}$ with the distance $d_{\mathcal{C}}((S_{k,j}), (S'_{k,j})) := \sup_{k,j} d_w(S_{k,j}, S'_{k,j})$ and $\dot{\mathcal{C}}$ with the quotient topology. We also find it more convenient to enumerate with a single index the curves $S_{k,j}$ constitutive of $S \in \mathcal{C}$: for this, we choose a bijection $\sigma : \mathbb{N}^2 \to \mathbb{N}$ and we write $\Gamma_{\sigma(k,j)} := S_{i,j}$. The orientation of the corresponding parametrized curve $\boldsymbol{\gamma}_{\sigma(i,j)}$ will depend on the choice of σ, and so do the permutations defining equivalence classes in $\dot{\mathcal{C}}$, but our results will not. We can now state the representation theorem for divergence-free measures in the plane:

Theorem 4.5. *Let $\boldsymbol{\nu} \in \mathcal{M}(S)^2$ be divergence-free in $\mathbb{R}^2$. Then, there exists $G \subset \mathbb{R}$ with $\mathcal{L}_1(\mathbb{R}\backslash G) = 0$ such that, for $t \in G$, there is a countable collection of (possibly degenerate) parametrized rectifiable Jordan curves $\{\boldsymbol{\gamma}_n^t\}_{n\in\mathbb{N}}$ with images Γ_n^t such that:*

(i) the $(\Gamma_n^t)_{n\in\mathbb{N}}$ are disjoint up to a set of $\mathcal{H}^1$-measure zero and $\Gamma_n^t \subset \operatorname{supp}\boldsymbol{\nu}$ for each n;
(ii) the union $\bigcup_n \Gamma_n^t$ is, up to a set of $\mathcal{H}^1$-measure zero, the measure-theoretical boundary $\partial_M\Omega(t)$ of a set $\Omega(t) \subset \mathbb{R}^2$ of finite perimeter;
(iii) $\Omega(t_1) \supset \Omega(t_2)$ if $t_1 < t_2$, and the mapping $t \mapsto (\boldsymbol{\gamma}_n^t)_{n\in\mathbb{N}}$ from $\mathbb{R}$ to $\dot{\mathcal{C}}$ is approximately continuous for a.e. t;
(iv) For any Borel set $B \subset \mathbb{R}^2$, $\mathbf{g} \in L^1[d|\boldsymbol{\nu}|]^2$ and $h \in L^1[d|\boldsymbol{\nu}|]$, it holds that

$$\boldsymbol{\nu}(B) = \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \left(\int_B \boldsymbol{\tau}_n^t \, d\left(\mathcal{H}^1 \lfloor \Gamma_n^t\right) \right) dt, \tag{57}$$

where $\boldsymbol{\tau}_n^t = (\boldsymbol{\gamma}_n^t)'/|(\boldsymbol{\gamma}_n^t)'|$ is the unit tangent vector field to Γ_n^t oriented by $\boldsymbol{\gamma}_n^t$,

$$|\boldsymbol{\nu}|(B) = \int_{\mathbb{R}} \mathcal{H}^1(\partial_M\Omega(t) \cap B) dt = \int_{\mathbb{R}} \left(\sum_{n\in\mathbb{N}} \mathcal{H}^1(\Gamma_n^t \cap B) \right) dt, \tag{58}$$

$$\int \mathbf{g} \cdot d\boldsymbol{\nu} = \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \left(\int \mathbf{g} \cdot \boldsymbol{\tau}_n^t \, d\left(\mathcal{H}^1 \lfloor \Gamma_n^t\right) \right) dt, \tag{59}$$

and

$$\int h d|\boldsymbol{\nu}| = \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \left(\int h \, d\left(\mathcal{H}^1 \lfloor \Gamma_n^t\right) \right) dt, \tag{60}$$

where the inner integrals on the right handsides of (59) *and* (60) *are well defined for a.e. $t \in \mathbb{R}$.*
(v) The set $J := \bigcup_{\substack{t_1\neq t_2\in G\\ n_1,n_2\in\mathbb{N}}} \Gamma_{n_1}^{t_1} \cap \Gamma_{n_2}^{t_2}$ is 1-rectifiable in $\mathbb{R}^2$ and $\boldsymbol{\nu}\lfloor_J$ is absolutely continuous with respect to $\mathcal{H}^1$; for a.e. $t \in G$, $\mathbf{u}_{\boldsymbol{\nu}}(x) = \boldsymbol{\tau}_n^t(x)$ for $\mathcal{H}^1$-a.e. $x \in J \cap \partial_M\Omega(t)$. More generally, it holds for a.e. $t \in G$ and every $n \in \mathbb{N}$ that $\mathbf{u}_{\boldsymbol{\nu}}(x) = \boldsymbol{\tau}_n^t(x)$ for $\mathcal{H}^1$-a.e. $x \in \Gamma_n^t$.

Proof. By Lemma 2.1, we have $\boldsymbol{\nu}(B) = \Re\nabla\phi(B)$ for some $\phi \in \dot{BV}(\mathbb{R}^2)$. Defining E_t as in (27), we get from Lemma 3.3 that it has finite perimeter for a.e. t. We let G be the set of such t, and for $t \in G$ we let $\{\boldsymbol{\gamma}_n^t\}_{n\in\mathbb{N}}$ be a representative in $\mathcal{C}$ of the element of $\dot{\mathcal{C}}$ corresponding to the family of curves $(\boldsymbol{\gamma}_{k,j}^t) \in \mathcal{T}$ appearing in Proposition 4.4, see discussion after the proof of that proposition. If we set $\Omega(t) = E_t$, then (ii) and the first assertion in (i) come from Lemma 4.3, the first assertion in (iii) is obvious and the second on approximate continuity follows from Proposition 4.4 much like Theorem 3.7 did from Proposition 3.6. Recalling definition (13), we see that Theorem 3.4 and the remark after Lemma 3.2 together imply (iv), where it should be noted that equations (57) through (60) only depend on the equivalence class of $\{\boldsymbol{\gamma}_n^t\}_{n\in\mathbb{N}}$ in $\dot{\mathcal{C}}$. Since (58) implies that $\mathcal{H}^1(\Gamma_n(t) \setminus \operatorname{supp}\boldsymbol{\nu}) = 0$ for a.e. $t \in \mathbb{R}$ the second half of (i) holds.

Observing that $\bigcup_{n\in\mathbb{N}} \Gamma_n^t = \partial_M E_t$ mod-$\mathcal{H}^1$, we see for each $t \in G$ that every $x \in J$ lies in $\partial_M(\mathbb{R}^2 \setminus E_{t_1}) \cap \partial_M E_{t_2}$ for some $t_1 < t_2$. Remembering the definitions in (30), this implies that, for every $x \in J$, $\phi^{\inf}(x) \leq t_1 < t_2 \leq \phi^{\sup}(x)$. Hence, by Lemma 3.5, $J \subset J(\phi)$ and the first two assertions of (v) follow. Now, evaluating $\|\boldsymbol{\nu}\|$ with (58) and integrating (59) against $\mathbf{u}_{\boldsymbol{\nu}}$ we get,

$$\int_{\mathbb{R}} \left(\sum_{n\in\mathbb{N}} \mathcal{H}^1(\Gamma_n^t) \right) dt = \|\boldsymbol{\nu}\| = \int \mathbf{u}_{\boldsymbol{\nu}} \cdot d\boldsymbol{\nu} = \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \left(\int \mathbf{u}_{\boldsymbol{\nu}} \cdot \boldsymbol{\tau}_n^t \, d\left(\mathcal{H}^1 \lfloor \Gamma_n^t\right) \right) dt,$$

and noting that $\mathbf{u}_{\boldsymbol{\nu}} \cdot \boldsymbol{\tau}_n^t \leq 1$, with equality only when $\mathbf{u}_{\boldsymbol{\nu}} = \boldsymbol{\tau}_n^t$, gives us the last assertion of (v). □

Decomposition (57)-(58) is a special case of (21), as we now show.

Proposition 4.6. *Let $\boldsymbol{\nu} \in \mathcal{M}(S)^2$ be divergence-free in $\mathbb{R}^2$, with G, $\{\boldsymbol{\gamma}_n^t\}_{n\in\mathbb{N}}$ and Γ_n^t as in Theorem 4.5. Take $\tilde{\boldsymbol{\gamma}}_n^t$ to be the periodic extension to $\mathbb{R}$ of $\boldsymbol{\gamma}_n^t$. If we set*

$$\rho(\mathfrak{B}) := \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \mathcal{H}^1(\Gamma_n^t)\delta_{\mathbf{T}_{\tilde{\boldsymbol{\gamma}}_n^t}}(\mathfrak{B})dt \quad \text{for every Borel } \mathfrak{B} \subset \mathfrak{S}(\mathbb{R}^2), \tag{61}$$

then the integral exists and ρ defines a Borel measure on $\mathfrak{S}(\mathbb{R}^2)$ such that (21) holds with $\boldsymbol{\mu} = \boldsymbol{\nu}$.

Proof. As in Section 2.2, let $\mathcal{B}_1$ denote the unit ball in $\mathcal{M}(\mathbb{R}^2)^2$ with the weak-$*$ topology. Let $\mathfrak{B} \subset \mathcal{B}_1$ be Borel, and $F : \mathbb{R} \to \mathbb{R}$ denote the integrand in (61). Recall from Proposition 4.4 the σ-compact sets Σ_η such that $\mathcal{L}_1(\mathbb{R} \setminus \Sigma_\eta) < \eta$ for $\eta > 0$. By the Borel-Cantelli lemma, $\Sigma_0 := \bigcup_{j\in\mathbb{N}^*} \Sigma_{1/j^2}$ is σ-compact such that $\mathcal{L}_1(\mathbb{R}\setminus\Sigma_0) = 0$. Hence, if $F|_{\Sigma_\eta}$ is a Borel function, then F is also Borel. We will show that $F|_{\Sigma_\eta}$ is Borel by writing it as a composition of Borel functions.

Let $\mathcal{Q} := \ell_1(\mathbb{N}) \times \mathcal{B}_1^{\mathbb{N}}$ where $\mathcal{B}_1^{\mathbb{N}}$ is given the product topology, and $\dot{\mathcal{Q}}$ denote the quotient space under the relation $(a_n, \boldsymbol{\mu}_n) \sim (b_n, \boldsymbol{\nu}_n)$ if and only if there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_{\sigma(n)} = a_n$ and $\boldsymbol{\nu}_{\sigma(n)} = \boldsymbol{\mu}_n$. We endow $\dot{\mathcal{Q}}$ with the quotient topology. Define $f_1 : \Sigma_\eta \to \dot{\mathcal{Q}}$ by $f_1(t) := [(\mathcal{H}^1(\Gamma_n^t), \mathbf{T}_{\tilde{\boldsymbol{\gamma}}_n^t})]$, where the bracket represents the equivalence class; note that indeed $\sum_n \mathcal{H}^1(\Gamma_n^t) < \infty$, because this sum is $\mathcal{P}(E_t)$ which is uniformly bounded on Σ_η by construction, see proof of Proposition 3.6. By points (iii) and (iv) of Proposition 4.4, f_1 is continuous (observe that $\sim$ takes quotient by all permutations, not just those used to define $\dot{\mathcal{T}}$, which does not affect continuity). Now let $\tilde{f}_2 : \mathcal{Q} \to \mathbb{R}$ be defined by $\tilde{f}_2(a_n, \boldsymbol{\mu}_n) := \sum_n a_n \chi_{\mathfrak{B}}(\boldsymbol{\mu}_n)$. Clearly, $\tilde{f}_2$ is Borel since it is the limit of Borel functions, and since it is invariant under permutations on n the quotient map $f_2 : \dot{\mathcal{Q}} \to \mathbb{R}$ is well-defined and Borel.

Altogether, $F|_{\Sigma_\eta} = f_2 \circ f_1$, is Borel and so is F. Hence, since F is nonnegative and its integral is bounded by $\int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \mathcal{H}^1(\Gamma_n^t) = \|\boldsymbol{\nu}\|$, the set function ρ given by (61) defines a Borel measure on $\mathcal{B}_1$. By restriction ρ defines a Borel measure on $\mathfrak{S}(\mathbb{R}^2)$. Finally we will show that the left equation of (21) holds, the proof for the right one is similar. Let $B \subset \mathbb{R}^2$ be Borel, $\{a_i\}_{i=0}^n$ be a partition of $[-1, 1]$, (T_1, T_2) be the components of $\mathbf{T}$, for $i < n$ and $j = 1, 2$, $\mathfrak{A}_i^j := \{\mathbf{T} \in \mathfrak{S}(\mathbb{R}^2) | a_{i-1} \leq T_j(B) < a_i\}$, $\mathfrak{A}_n^j := \{\mathbf{T} \in \mathfrak{S}(\mathbb{R}^2) | a_{n-1} \leq T_j(B) \leq 1\}$, $M_j = \sum_i a_i \rho(\mathfrak{A}_i^j)$ and $m_j = \sum_i a_{i-1}\rho(\mathfrak{A}_i^j)$. Then

$$m_j = \sum_i a_{i-1} \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \mathcal{H}^1(\Gamma_n^t)\delta_{\mathbf{T}_{\tilde{\boldsymbol{\gamma}}_n^t}}(\mathfrak{A}_i^j)dt \leq \int_{\mathbb{R}} \sum_i \sum_{n:\mathbf{T}_{\tilde{\boldsymbol{\gamma}}_n^t}\in\mathfrak{A}_i} \mathcal{H}^1(\Gamma_n^t)(\mathbf{T}_{\tilde{\boldsymbol{\gamma}}_n^t})_j(B)dt,$$

where the right hand-side of this equation is equal to $(\boldsymbol{\nu}(B))_j$ in view of (57), Fubini's theorem and the fact that the $\mathfrak{A}_i^j$'s form a partition of $\mathfrak{S}(\mathbb{R}^2)$. Analogously $(\boldsymbol{\nu}(B))_j \leq M_j$, hence, taking the limit as $\max\{a_i - a_{i-1}\} \to 0$ and using $\rho(\mathfrak{S}(\mathbb{R}^2)) = \|\boldsymbol{\nu}\| < \infty$, we get (21). □

Theorem 4.5 (iii) asserts approximate continuity of $\partial_M\Omega(t)$ with respect to t in the weak-$*$ sense. Still, the $\Omega(t)$ could all have different topologies as can be seen from the following example.

Example 4.1. We will generate a BV function φ_∞, valued in $[0,1]$, whose suplevel sets E_t all have different topologies. Then, $\boldsymbol{\nu} := \Re\nabla\varphi_\infty$ is divergence-free and $\Omega(t) = E_t$ in Theorem 4.5, thereby yielding an example with the aforementioned property.

We construct φ_∞ as the limit of a bounded increasing sequence (ϕ_m) of BV functions. Let us first define a family of sets of finite perimeter that we will use to construct the ϕ_m. For any two integers m and n such that $m \geq 0$ and $1 \leq n \leq 2^m$, define the set $b(n,m) \subset \mathbb{R}^2$ to be the closed ball around the point (n,m) with perimeter 2^{-2m-1} (thus, radius $2^{-2m-2}/\pi$) minus 2^m pairwise disjoint nonempty open balls contained in this closed ball. We pick the sum of the perimeters of this 2^m open balls to be strictly less than 2^{-2m-1}. Note that the $b(n,m)$ are pairwise disjoint. Define $\varphi_0 := \frac{1}{2}\chi_{b(1,0)}$ and, for $m > 0$, $\varphi_m := \varphi_{m-1} + \sum_{k=1}^{2^m} \frac{2k-1}{2^{m+1}}\chi_{b(k,m)}$. Then $\|\nabla\varphi_0\|_{TV} < 1/2$, moreover for $m > 0$:

$$\begin{aligned}\|\nabla\varphi_m\|_{TV} &= \|\nabla\varphi_{m-1}\|_{TV} + \sum_{k=1}^{2^m}\frac{2k-1}{2^{m+1}}\|\nabla\chi_{b(k,m)}\|_{TV}\\ &< \|\nabla\varphi_{m-1}\|_{TV} + \sum_{k=1}^{2^m}\frac{2k-1}{2^{m+1}}(2^{-2m-1}+2^{-2m-1})\\ &= \|\nabla\varphi_{m-1}\|_{TV} + \frac{2^{2m}}{2^{3m+1}},\end{aligned}$$

and hence, $\|\nabla\varphi_m\|_{TV} < 1$ for every m. Thus, φ_∞, the pointwise limit of the nondecreasing sequence of functions $\{\varphi_m\}_m$, is a BV function (see [32, Theorem 5.2.1]).

Now, for m, n, p and q some integers such that $1 \leq n \leq 2^m$ and $1 \leq p \leq 2^q$, it is clear that $b(n,m)$ is topologically equivalent to $b(p,q)$ if and only if $q = m$. Hence, with the notation of Theorem 4.5, we see that given $s,t \in (0,1)$, the sets $\Omega(t)$ and $\Omega(s)$ can be topologically equivalent only if they contain, for each fixed m, the same number of sets from the family $\{b(n,m)\}_{n=1}^{2^m}$. However if $s < t$ then there exist two positive integers m and n such that $s < \frac{2n-1}{2^{m+1}} < t$, thus $b(n,m) \subset \Omega(s)\setminus\Omega(t)$ and therefore $\Omega(t)$ is not topologically equivalent to $\Omega(s)$.

5. Applications to Inverse Magnetization Problems

5.1. **Solutions to Extremal Problem 1.** For $\boldsymbol{\mu},\boldsymbol{\nu} \in \mathcal{M}(\mathbb{R}^3)$ with $\mathbf{f}_{\boldsymbol{\mu}}$ to denote the Radon-Nikodym derivative of $\boldsymbol{\mu}$ with respect to $|\boldsymbol{\nu}|$, we define for $|\boldsymbol{\nu}|$-a.e. x:

$$\mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}}(x) := \begin{cases}\frac{\mathbf{f}_{\boldsymbol{\mu}}(x)}{|\mathbf{f}_{\boldsymbol{\mu}}(x)|}, & \mathbf{f}_{\boldsymbol{\mu}}(x) \neq 0,\\ \mathbf{u}_{\boldsymbol{\nu}}(x), & \mathbf{f}_{\boldsymbol{\mu}}(x) = 0.\end{cases} \tag{62}$$

We put $E = \mathbf{f}_{\boldsymbol{\mu}}^{-1}(0)$ and observe that

$$\int \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot d\boldsymbol{\nu} = \int_{E^c} \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot \mathbf{u}_{\boldsymbol{\nu}}\, d|\boldsymbol{\nu}| + |\boldsymbol{\nu}|(E). \tag{63}$$

The next lemma provides a variational characterization of solutions to Extremal Problem 1.

Lemma 5.1. *Let $S \subset \mathbb{R}^3$ be closed and suppose $\boldsymbol{\mu},\boldsymbol{\nu} \in \mathcal{M}(S)^3$, with $\mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}}$ and E as above. Then*

$$\|\boldsymbol{\mu}\|_{TV} \leq \|\boldsymbol{\mu} + t\boldsymbol{\nu}\|_{TV}, \text{ for every } t > 0, \tag{64}$$

if and only if

$$\int \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot d\boldsymbol{\nu} \geq 0. \tag{65}$$

Hence, $\|\boldsymbol{\mu}\|_{TV} = M_S(\boldsymbol{\mu})$ if and only if (65) holds for every S-silent $\boldsymbol{\nu} \in \mathcal{M}(S)^3$. The inequality (64) is strict for every $t > 0$ if the inequality (65) is strict.

Proof. Let $\boldsymbol{\mu}_s$ denote the singular part of $\boldsymbol{\mu}$ with respect to $|\boldsymbol{\nu}|$. Then, for $\epsilon > 0$,

$$\begin{aligned}
\|\boldsymbol{\mu} + \epsilon\boldsymbol{\nu}\|_{TV} &= \int |\mathbf{f}_{\boldsymbol{\mu}} + \epsilon \mathbf{u}_{\boldsymbol{\nu}}|\, d|\boldsymbol{\nu}| + \|\boldsymbol{\mu}_s\|_{TV} \\
&= \int_{E^c} |\mathbf{f}_{\boldsymbol{\mu}} + \epsilon \mathbf{u}_{\boldsymbol{\nu}}|\, d|\boldsymbol{\nu}| + \epsilon|\boldsymbol{\nu}|(E) + \|\boldsymbol{\mu}_s\|_{TV} \\
&= \|\boldsymbol{\mu}\|_{TV} + \epsilon \left(\int_{E^c} \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot \mathbf{u}_{\boldsymbol{\nu}}\, d|\boldsymbol{\nu}| + |\boldsymbol{\nu}|(E) \right) + o(\epsilon) \\
&= \|\boldsymbol{\mu}\|_{TV} + \epsilon \int \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot d\boldsymbol{\nu} + o(\epsilon),
\end{aligned} \tag{66}$$

where the above used that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $\mathbf{a} \neq 0$ and $|\mathbf{b}| = 1$ (with $\mathbf{a} = \mathbf{f}_{\boldsymbol{\mu}}$ and $\mathbf{b} = \mathbf{u}_{\boldsymbol{\nu}}$),

$$|\mathbf{a} + \epsilon\mathbf{b}| = |\mathbf{a}| \left(1 + 2\epsilon \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2} + \epsilon^2 \frac{|\mathbf{b}|^2}{|\mathbf{a}|^2} \right)^{1/2} = |\mathbf{a}| + \epsilon \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b} + \frac{1}{|\mathbf{a}|} \mathcal{O}(\epsilon^2),$$

together with $|\boldsymbol{\nu}|(\{\mathbf{x} : 0 < |\mathbf{f}_{\boldsymbol{\mu}}(\mathbf{x})| < \epsilon\}) = o(1)$ as $\epsilon \to 0$. Using the convexity of the TV-norm we have for $0 < \epsilon \leq 1$ and $t > 0$:

$$\|\boldsymbol{\mu} + t\epsilon\boldsymbol{\nu}\|_{TV} = \|(1-\epsilon)\boldsymbol{\mu} + \epsilon(\boldsymbol{\mu} + t\boldsymbol{\nu})\|_{TV} \leq (1-\epsilon)\|\boldsymbol{\mu}\|_{TV} + \epsilon\|\boldsymbol{\mu} + t\boldsymbol{\nu}\|_{TV},$$

which implies

$$t \frac{\|\boldsymbol{\mu} + \epsilon t\boldsymbol{\nu}\|_{TV} - \|\boldsymbol{\mu}\|_{TV}}{t\epsilon} \leq \|\boldsymbol{\mu} + t\boldsymbol{\nu}\|_{TV} - \|\boldsymbol{\mu}\|_{TV}. \tag{67}$$

If (65) holds, then it follows in view of (66) (with $t\epsilon$ instead of ϵ) that the limit of the left-hand side of (67) is nonnegative when $\epsilon \to 0^+$, which implies (64). Conversely, if (64) holds then the left hand side of (67) is nonnegative and using (66) we can take the limit as $\epsilon \to 0^+$ to obtain (65). That the inequality (64) is strict for every $t > 0$ when the inequality (65) is strict follows immediately from the above computations. □

We say that $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ is *carried by* a set if that set has full $|\boldsymbol{\mu}|$-measure; i.e., the complement has $|\boldsymbol{\mu}|$-measure zero. Recall that a set $B \subset \mathbb{R}^n$ is *purely 1-unrectifiable* if $\mathcal{H}^1(E \cap B) = 0$ for every 1-rectifiable set E. Clearly a set of $\mathcal{H}^1$-measure zero is purely 1-unrectifiable.

Theorem 5.2. *Let $S \subset \mathbb{R}^3$ be slender and closed and suppose $\widetilde{\boldsymbol{\mu}} \in \mathcal{M}(S)^3$ is carried by a purely 1-unrectifiable set. Then $\widetilde{\boldsymbol{\mu}}$ is strictly TV-minimal. Moreover, if $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ is TV-minimal on S, then so is $\boldsymbol{\mu} + \widetilde{\boldsymbol{\mu}}$.*

Proof. Since S is slender, any S-silent magnetization $\boldsymbol{\nu}$ is divergence-free. From the decomposition (16), we then have that $\boldsymbol{\nu}$ and $\widetilde{\boldsymbol{\mu}}$ are mutually singular since the latter is carried by a purely 1-unrectifiable set, showing that $\widetilde{\boldsymbol{\mu}}$ is strictly TV-minimal.

Next suppose $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ satisfies $\|\boldsymbol{\mu}\|_{TV} = M_S(\boldsymbol{\mu})$ and $\boldsymbol{\nu} \in \mathcal{M}(S)^3$ be S-silent. Since $\boldsymbol{\nu}$ and $\widetilde{\boldsymbol{\mu}}$ are mutually singular, $d\widetilde{\boldsymbol{\mu}}/d|\boldsymbol{\nu}| = 0$ and thus, recalling definition (62), we see that $\mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} = \mathbf{w}_{\boldsymbol{\mu}+\widetilde{\boldsymbol{\mu}}}^{\boldsymbol{\nu}}$, $|\boldsymbol{\nu}|$-a.e. Lemma 5.1 then implies $\|\boldsymbol{\mu} + \widetilde{\boldsymbol{\mu}}\|_{TV} = M_S(\boldsymbol{\mu} + \widetilde{\boldsymbol{\mu}})$. □

The first assertion of Theorem 5.2 sharpens Theorem 2.6 of [6] stating that a magnetization supported on a purely 1-unrectifiable set is strictly TV-minimal. In the case that S is planar, this result can be strengthened by the following theorem.

Theorem 5.3. *Let $S \subset \mathbb{R}^2 \times \{0\}$ be closed and suppose $\boldsymbol{\mu}$ is a magnetization carried by a Borel set $Z \subset S$ that satisfies*

$$\mathcal{H}^1(\Gamma \cap Z) \leq \mathcal{H}^1(\Gamma \setminus Z), \tag{68}$$

for any rectifiable Jordan curve $\Gamma \subset S$. Then $\boldsymbol{\mu}$ is TV-minimal on S. If $\boldsymbol{\nu} \in \mathcal{M}(S)^3$ is S-silent and $\|\boldsymbol{\mu}+\boldsymbol{\nu}\|_{TV} = \|\boldsymbol{\mu}\|_{TV}$, then equality holds in (68) when $\Gamma = \Gamma_n^t$ for almost every t and every $n \in \mathbb{N}$ in the loop decomposition of $\boldsymbol{\nu}$. In particular, $\boldsymbol{\mu}$ is strictly TV-minimal on S if the inequality (68) is strict for every nondegenerate $\Gamma \subset S$, and then $\boldsymbol{\mu} + \tilde{\boldsymbol{\mu}}$ is also strictly TV-minimal when $\tilde{\boldsymbol{\mu}}$ is carried by a purely 1-unrectifiable set.

Proof. Let $\boldsymbol{\nu}$ be an S-silent magnetization with $\mathbf{f}_{\boldsymbol{\mu}}$, $\mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}}$ as in (62), and loop decompositions $\{\Gamma_n^t\}$ and recall $E = \mathbf{f}_{\boldsymbol{\mu}}^{-1}(0)$. Also let $\boldsymbol{\mu}_s$ denote the singular part of $\boldsymbol{\mu}$ with respect to $|\boldsymbol{\nu}|$. By Lemma 2.1 $\boldsymbol{\nu} = (\boldsymbol{\nu}_T, 0)$ where $\boldsymbol{\nu}_T \in \mathcal{M}(S)^2$ is divergence-free. For $t \in \mathbb{R}$ and $n \in \mathbb{N}$, let Γ_n^t and $\boldsymbol{\tau}_n^t$ be as in Theorem 4.5 from the decomposition of $\boldsymbol{\nu}_T$.

By assertion (v) of Theorem 4.5, we know for a.e. $t \in \mathbb{R}$ and for every $n \in \mathbb{N}$ that $\mathbf{u}_{\boldsymbol{\nu}}(x) = (\boldsymbol{\tau}_n^t(x), 0)$ for $\mathcal{H}^1$-a.e. $x \in \Gamma_n^t$. Note also that by (iv) of Theorem 4.5, $\mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot (\boldsymbol{\tau}_n^t, 0)$ is $\mathcal{H}^1$-integrable on Γ_n^t for every $n \in \mathbb{N}$ and a.e. $t \in \mathbb{R}$. Now, for every such t,

$$\begin{aligned}\int_{\Gamma_n^t} \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot (\boldsymbol{\tau}_n^t, 0)\, d\mathcal{H}^1 &= \int_{\Gamma_n^t \cap E^c} \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot (\boldsymbol{\tau}_n^t, 0)\, d\mathcal{H}^1 + \int_{\Gamma_n^t \cap E} \mathbf{u}_{\boldsymbol{\nu}} \cdot (\boldsymbol{\tau}_n^t, 0)\, d\mathcal{H}^1 \\ &= \int_{\Gamma_n^t \cap E^c} \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot (\boldsymbol{\tau}_n^t, 0)\, d\mathcal{H}^1 + \mathcal{H}^1(\Gamma_n^t \cap E) \\ &\geq -\mathcal{H}^1(\Gamma_n^t \cap E^c) + \mathcal{H}^1(\Gamma_n^t \cap E).\end{aligned} \tag{69}$$

From (60) we have

$$0 = \int_{Z^c} |\mathbf{f}_{\boldsymbol{\mu}}| d|\nu| = \int_{T_0} \sum_{n\in\mathbb{N}} \left(\int_{Z^c} |\mathbf{f}_{\boldsymbol{\mu}}|\, d\left(\mathcal{H}^1 \lfloor \Gamma_n^t\right)\right) dt.$$

Observing that $|\mathbf{f}_{\boldsymbol{\mu}}(x)| > 0$ for $x \in E^c$, the above equation implies that the $\mathcal{L}_1$-measure of

$$T_0 := \{t \in \mathbb{R} \mid \exists n \in \mathbb{N} : \mathcal{H}^1(\Gamma_n^t \cap E^c \cap Z^c) \neq 0\}$$

is zero; that is, $\mathcal{H}^1(\Gamma_n^t \cap E^c \cap Z^c) = 0$ for a.e. t. Thus, by (69) we get

$$\int_{\Gamma_n^t} \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot (\boldsymbol{\tau}_n^t, 0)\, d\mathcal{H}^1 \geq -\mathcal{H}^1(\Gamma_n^t \cap Z) + \mathcal{H}^1(\Gamma_n^t \setminus Z) \geq 0, \tag{70}$$

where the last inequality follows from the condition (68). Therefore, by (59),

$$\int_{\mathbb{R}^2} \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot d\boldsymbol{\nu} = \int_{\mathbb{R}} \sum_{n\in N^t} \left(\int_{\mathbb{R}^2} \mathbf{w}_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \cdot (\boldsymbol{\tau}_n^t, 0)\, d\left(\mathcal{H}^1 \lfloor \Gamma_n^t\right)\right) dt \geq 0, \tag{71}$$

and, hence, Lemma 5.1 gives us $\|\boldsymbol{\mu}\|_{TV} \leq \|\boldsymbol{\mu}+\boldsymbol{\nu}\|_{TV}$. Moreover, if there is a set of positive measure $E \subset \mathbb{R}$ such that for every $t \in E$ there exists an n for which the rightmost inequality in (70) is strict, then the inequality in (71) is also strict. Finally, (68) is invariant upon adding a purely 1-unrectifiable set to Z. □

Corollary 5.4. *Let $S \subset \mathbb{R}^2 \times \{0\}$ be closed and suppose $\boldsymbol{\mu}$ is a magnetization carried by a Borel set $Z \subset S$ that is contained in a purely 1-unrectifiable set plus a countable union $\bigcup_{k \in K} L_k$ where the L_k are disjoint line segments such that the distance from any L_k to any L_j, $j \neq k$, is greater than or equal to the length of L_k. Then* (68) *holds for any rectifiable Jordan curve Γ, and thus $\boldsymbol{\mu}$ is TV-minimal on S. Moreover, if the distance from any L_k to any L_j, $j \neq k$, is strictly greater than the length of L_k, then* (68) *is strict and $\boldsymbol{\mu}$ is strictly TV-minimal on S.*

Proof. By the last assertion of Theorem 5.3, it is enough to assume Z is contained in a countable union of line segments with the aforementioned properties. Let Γ be a rectifiable Jordan curve oriented by a parametrization $\boldsymbol{\gamma}$. Without loss of generality we may assume that $Z \cap L_k \neq \emptyset$ for all $k \in K$. If $K = \{1\}$ is a singleton, then (since L_1 is a line segment)

$$\mathcal{H}^1(\Gamma \cap Z) \leq \mathcal{H}^1(\Gamma \cap L_1) < \mathcal{H}^1(\Gamma \setminus L_1) \leq \mathcal{H}^1(\Gamma \setminus Z).$$

Otherwise, for each $k \in K$ there is some directed sub-arc $\Gamma_k \subset \Gamma$ with initial point in L_k, end point in some L_j for $j \neq k$, and interior in the complement of $\bigcup_{\ell \neq k} L_\ell$. Note that for $j \neq k \in K$, the interiors of Γ_k and Γ_j are disjoint, and that $\mathcal{H}^1(\Gamma \cap L_k) \leq \mathcal{H}^1(\Gamma_k)$ by assumption. Also note that this inequality is strict under the final assumption. Thus,

$$\mathcal{H}^1(\Gamma \cap Z) \leq \sum_{k \in K} \mathcal{H}^1(\Gamma \cap L_k) \leq \sum_{k \in K} \mathcal{H}^1(\Gamma_k) \leq \mathcal{H}^1(\Gamma \setminus Z),$$

where the second inequality is strict under the last assumption. □

We next characterize the space of S-silent magnetizations when S contains only a finite number of Jordan curves. First we consider the class of closed $S \subset \mathbb{R}^2$ that contain no *rectifiable* Jordan curve at all, and hence, cannot hold nontrivial silent magnetizations. We call such S *tree-like*. Note that any closed purely 1-unrectifiable set is tree-like, but the converse is not true. We also note that a tree-like set may contain a Jordan curve, such as the Koch curve, which is not rectifiable. As a consequence of Theorem 4.5 we obtain the following result.

Lemma 5.5. *Let S be a closed subset of $\mathbb{R}^2 \times \{0\}$. If $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ is nonzero and S-silent, then the support of $\boldsymbol{\mu}$ contains a rectifiable Jordan curve. Hence, if S is tree-like the only S-silent magnetization is the zero magnetization.*

Proof. Since $S \subset \mathbb{R}^2 \times \{0\}$, it is slender and hence S-silent magnetizations are divergence free. The lemma now follows from Theorem 4.5. □

For a closed set $S \subset \mathbb{R}^2 \times \{0\}$, let $\Sigma(S)$ denote the linear subspace of $\mathcal{M}(S)^3$ consisting of S-silent sources. The previous lemma shows that $\Sigma(S)$ is the trivial subspace when S is tree-like. The next theorem provides sufficient conditions that $\Sigma(S)$ is finite dimensional and generalizes the second assertion of Lemma 5.5 when $\mathcal{H}^1(S)$ is finite.

Theorem 5.6. *Let $S \subset \mathbb{R}^2 \times \{0\}$ be closed with empty interior. If the number n of bounded connected components of $\mathbb{R}^2 \times \{0\} \setminus S$ is finite, then the dimension of $\Sigma(S)$ is less than or equal to n. Furthermore, the dimension is equal to n if $\mathcal{H}^1(S)$ is finite.*

Proof. Let $S' \subset S$ be the union of all rectifiable Jordan curves contained in S and let m be the number of bounded connected components of $\mathbb{R}^2 \setminus S'$. Since $(\mathbb{R}^2 \setminus S) \cup (S \setminus S') = (\mathbb{R}^2 \setminus S')$ and the set $S \setminus S'$ is a subset of the topological boundary of $\mathbb{R}^2 \setminus S$, then $n \geq m$. From Theorem 4.5 it follows that $\Sigma(S) = \Sigma(S')$, thus showing that $\dim \Sigma(S') = m$ will prove our theorem.

Let $\{E_i\}_{i=1}^m$ be the family of bounded connected components of $\mathbb{R}^2 \setminus S'$. Note that each E_i is of finite perimeter since $\mathcal{H}^1(S')$ is finite. Let $\boldsymbol{\ell}_i := \mathfrak{R}\nabla\chi_{E_i}$ for $i = 1, ...m$. By Lemma 2.1 each $\boldsymbol{\ell}_i$

is S'-silent. To show that $\{\boldsymbol{\ell}_i\}_{i=1}^m$ generates $\Sigma(S')$, it is sufficient by Theorem 4.5 to prove that for any rectifiable Jordan curve $\Gamma \subset S'$ with arclength parametrization $\boldsymbol{\gamma}$, the magnetization $\mathbf{R}_{\boldsymbol{\gamma}}$ defined by (12) is in the span of the $\boldsymbol{\ell}_i$'s.

Using the Jordan curve theorem we can see that for any E_i such that $\mathrm{int}(\Gamma) \cap E_i \neq \emptyset$ we have that $E_i \subset \mathrm{int}(\Gamma)$. Hence there exists a $J \subset \{1, ..., m\}$ such that $\bigcup_{i\in J} E_i \subset \mathrm{int}(\Gamma) \subset S' \cup \bigcup_{i\in J} E_i$ and since $\mathcal{L}_2(S') = 0$, then

$$\begin{aligned} \mathbf{R}_{\boldsymbol{\gamma}} &= \Re\nabla\chi_{\mathrm{int}(\Gamma)} = \Re\nabla\chi_{\bigcup_{i\in J} E_i} \\ &= \sum_{i\in J} \Re\nabla\chi_{E_i} = \sum_{i\in J} \boldsymbol{\ell}_i, \end{aligned}$$

where the first equality comes from the remark after Lemma 3.2, equation (25) and Lemma 4.2.

To show linearly independence, assume that $\sum_{i=1}^m c_i \boldsymbol{\ell}_i = 0$ where $c_i \in \mathbb{R}$, $i = 1, .., m$. Since $0 = \sum_{i=1}^m c_i \Re\nabla\chi_{E_i} = \Re\nabla\left(\sum_{i=1}^m c_i\chi_{E_i}\right)$, thus $\sum_{i=1}^m c_i\chi_{E_i}$ is a constant but since the E_i's are bounded and disjoint then each $c_i = 0$ and hence the $\boldsymbol{\ell}_i$'s are indeed linearly independent. ☐

5.2. **Regularization by penalizing the total variation.** Let $S \subset \mathbb{R}^2 \times \{0\}$ and $Q \subset \mathbb{R}^3$ be closed and positively separated. For $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ and v a unit vector in $\mathbb{R}^3$, the component of the magnetic field $\mathbf{b}(\boldsymbol{\mu})$ in the direction v at $x \notin S$ is given, in view of (1), by

$$b_v(\boldsymbol{\mu})(x) := v \cdot \mathbf{b}(\boldsymbol{\mu})(x) = -\frac{\mu_0}{4\pi}\int \mathbf{K}_v(x-y)\cdot\, d\boldsymbol{\mu}(y), \tag{72}$$

where

$$\mathbf{K}_v(x) = \frac{v}{|x|^3} - 3x\frac{v\cdot x}{|x|^5} = \nabla\left(\frac{v\cdot x}{|x|^3}\right). \tag{73}$$

Consider a finite, positive Borel measure ρ with support contained in Q and let $A : \mathcal{M}(S)^3 \to L^2(Q,\rho)$ be the so-called *forward operator* defined by

$$A(\boldsymbol{\mu})(x) := b_v(\boldsymbol{\mu})(x), \qquad x \in Q. \tag{74}$$

The adjoint operator A^* is then given by (see [6, Section 3])

$$A^*(\Psi)(x) := -\mu_0\nabla(\nabla U^{\rho,\psi}\cdot v)(x), \qquad U^{\rho,\psi}(x) = -\frac{1}{4\pi}\int \frac{\Psi(y)}{|x-y|}d\rho(y). \tag{75}$$

Since Q and S are positively separated it follows from the harmonicity of K_v that $A^*(\Psi) \in C_0(S)^3$ and thus $A^* : (L^2(Q,\rho))^* \sim L^2(Q,\rho) \to C_0(S)^3 \subset (\mathcal{M}(S)^3)^*$. Note the kernel of the forward operator A contains all S-silent magnetizations. In the case this kernel consists exactly of S-silent magnetizations, we say that A is *S-sufficient*. It follows from [6, Lemmma 2.3] and the discussion thereafter that A is S-sufficient when $S \subset \mathbb{R}^2 \times \{0\}$ and $Q \subset \mathbb{R}^3$ are positively separated closed sets and for some complete real analytic surface $\mathcal{A} \subset \mathbb{R}^3 \setminus S$ we have:

(a) S and $\mathcal{A}$ are positively separated;
(b) S lies entirely within one connected component of $\mathbb{R}^3 \setminus \mathcal{A}$;
(c) $Q \cap \mathcal{A}$ has Hausdorff dimension strictly greater than 1 in each connected component of $\mathbb{R}^3 \setminus S$;
(d) $\mathrm{supp}\,\rho = Q$.

For $\boldsymbol{\mu} \in \mathcal{M}(S)^3$, $f \in L^2(Q,\rho)$, and $\lambda > 0$, recall from (4) the definition of $\mathcal{F}_{f,\lambda}$, and from (5) the notation $\boldsymbol{\mu}_\lambda \in \mathcal{M}(S)^3$ to designate a minimizer of $\mathcal{F}_{f,\lambda}$. As a second application of our results in Section 4, we prove:

Theorem 5.7. *Let S be a closed subset of $\mathbb{R}^2 \times \{0\}$, $Q \subset \mathbb{R}^3$ be a closed set and $\rho \in \mathcal{M}(Q)$ be such that the forward operator A defined in* (74) *is S-sufficient. For $f \in L^2(Q, \rho)$ and $\lambda > 0$, the solution to* (5) *is unique.*

Proof. It is well known (see *e.g.* [10, Propostion 3.6]) that $\boldsymbol{\mu}_\lambda \in \mathcal{M}(S)^3$ is a minimizer of $\mathcal{F}_{f,\lambda}$ if and only if:

$$\begin{array}{lll} A^*(f - A\boldsymbol{\mu}_\lambda) & = \frac{\lambda}{2}\mathbf{u}_{\boldsymbol{\mu}_\lambda} & |\boldsymbol{\mu}_\lambda|\text{-a.e. and} \\ |A^*(f - A\boldsymbol{\mu}_\lambda)| & \leq \frac{\lambda}{2} & \text{everywhere on } S. \end{array} \tag{76}$$

Moreover, it follows from the strict convexity of the L^2-norm that $\boldsymbol{\mu}'_\lambda \in \mathcal{M}(S)^3$ is another solution if and only if $A(\boldsymbol{\mu}'_\lambda - \boldsymbol{\mu}_\lambda) = 0$.

Assume for a contradiction that $\boldsymbol{\mu}_\lambda$ and $\boldsymbol{\mu}'_\lambda$ are two distinct minimizers in (5) and let $\boldsymbol{\mu} := \boldsymbol{\mu}'_\lambda - \boldsymbol{\mu}_\lambda$. As $\boldsymbol{\mu}'_\lambda - \boldsymbol{\mu}_\lambda = \boldsymbol{\mu}$ is absolutely continuous with respect to $|\boldsymbol{\mu}|$, the Lebesgue decompositions of $\boldsymbol{\mu}_\lambda$ and $\boldsymbol{\mu}'_\lambda$ with respect to $|\boldsymbol{\mu}|$ must have the same singular term. That is, these decompositions are necessarily of the form

$$d\boldsymbol{\mu}_\lambda = \boldsymbol{\gamma} d|\boldsymbol{\mu}| + d\boldsymbol{\nu}, \qquad d\boldsymbol{\mu}'_\lambda = \boldsymbol{\gamma}' d|\boldsymbol{\mu}| + d\boldsymbol{\nu},$$

where $|\boldsymbol{\nu}|$ is singular with respect to $|\boldsymbol{\mu}|$ and $\boldsymbol{\gamma}$, $\boldsymbol{\gamma}'$ are $|\boldsymbol{\mu}|$-integrable $\mathbb{R}^3$-valued functions.

Put for simplicity $\psi = (2/\lambda)(f - A(\boldsymbol{\mu}_\lambda)) = (2/\lambda)(f - A(\boldsymbol{\mu}'_\lambda))$. Thanks to (76) we know that $\mathbf{u}_{\boldsymbol{\mu}_\lambda} = A^*\psi$ and $\mathbf{u}_{\boldsymbol{\mu}'_\lambda} = A^*\psi$, $\boldsymbol{\mu}_\lambda$ and $\boldsymbol{\mu}'_\lambda$-a.e. respectively. Now, since $d|\boldsymbol{\mu}_\lambda| = |\boldsymbol{\gamma}| d|\boldsymbol{\mu}| + d|\boldsymbol{\nu}|$ and $d|\boldsymbol{\mu}'_\lambda| = |\boldsymbol{\gamma}'| d|\boldsymbol{\mu}| + d|\boldsymbol{\nu}|$, we have that

$$\mathbf{u}_{\boldsymbol{\mu}} d|\boldsymbol{\mu}| = d\boldsymbol{\mu} = \mathbf{u}_{\boldsymbol{\mu}'_\lambda} d|\boldsymbol{\mu}'_\lambda| - \mathbf{u}_{\boldsymbol{\mu}_\lambda} d|\boldsymbol{\mu}_\lambda| = A^*\psi d|\boldsymbol{\mu}'_\lambda| - A^*\psi d|\boldsymbol{\mu}_\lambda| = A^*\psi(|\boldsymbol{\gamma}'| - |\boldsymbol{\gamma}|) d|\boldsymbol{\mu}|.$$

Therefore $\mathbf{u}_{\boldsymbol{\mu}} = A^*\psi(|\boldsymbol{\gamma}'| - |\boldsymbol{\gamma}|)$ at $|\boldsymbol{\mu}|$-a.e point, and since $|A^*\psi| = 1$ on the supports of $\boldsymbol{\mu}_\lambda$ and $\boldsymbol{\mu}'_\lambda$ it holds that $\mathbf{u}_{\boldsymbol{\mu}}(x) = \pm_x A^*\psi(x)$ for $|\boldsymbol{\mu}|$-a.e. x, where the choice of sign $\pm_x$ has a subscript x to indicate that it may vary with x.

From the S-sufficiency of A we know that $\boldsymbol{\mu}$ is S-silent. Also, by [6, Corollary 4.2] (take $\mathcal{B} = \mathbb{R}^2 \times \{0\}$ there), the supports of $\boldsymbol{\mu}_\lambda$ and $\boldsymbol{\mu}'_\lambda$ are contained in a finite collection of points and analytic arcs. In particular, there are only finitely many rectifiable Jordan curves contained in the support of $\boldsymbol{\mu}$ and they are all piecewise analytic. Thus, applying Theorem 4.5 to $\boldsymbol{\mu}$, we find there are finitely many piecewise analytic oriented Jordan curves $\Gamma_1, \cdots, \Gamma_N$ with respective unit tangent vector fields $\boldsymbol{\tau}_1, \cdots, \boldsymbol{\tau}_n$, and strictly positive real numbers $a_1, \cdots, a_N$ such that $\boldsymbol{\tau}_m = \boldsymbol{\tau}_n$ on $\Gamma_m \cap \Gamma_n$, $\mathcal{H}^1$-a.e. and

$$d\boldsymbol{\mu} = \sum_{n=1}^{N} a_n \boldsymbol{\tau}_n d\left(\mathcal{H}^1 \lfloor \Gamma_n\right).$$

In particular, $d|\boldsymbol{\mu}| = \sum_{n=1}^N a_n d\left(\mathcal{H}^1 \lfloor \Gamma_n\right)$ and $\boldsymbol{\tau}_n(x) = \mathbf{u}_{\boldsymbol{\mu}}(x) = \pm_x A^*\psi(x)$, for $|\boldsymbol{\mu}|$-a.e. x, hence $\mathcal{H}^1$-a.e., on Γ_n.

Fix n and let E be an analytic sub-arc of Γ_n. Being the unit tangent to an oriented analytic arc, $\boldsymbol{\tau}_n(x)$ must be an analytic function of $x \in E$, and so is $A^*\psi(x)$ by the real analyticity of $A^*\psi$, *cf.* (75). Hence, either $\boldsymbol{\tau}_n = A^*\psi$ or $\boldsymbol{\tau}_n = -A^*\psi$ everywhere on E. Therefore, E is a subset of a trajectory of the autonomous differential equation $\dot{x} = A^*\psi(x)$. Moreover, since E is bounded and percursed at unit speed, the corresponding trajectory extends beyond the endpoints of E, and since two distinct trajectories cannot intersect we conclude that Γ_n is smooth and constitutes a single, periodic trajectory. This, however, is impossible because $A^*\psi$ is a gradient vector field, by (75). $\square$

When S is planar and EP-1 has a unique solution, Theorem 4.3 from [6] and Theorem 5.7 together imply the following corollary.

Corollary 5.8. *Let $S \subset \mathbb{R}^2 \times \{0\}$ be closed, the forward operator A be S-sufficient, and $\boldsymbol{\mu}_0 \in \mathcal{M}(S)^3$. Set $f = A\boldsymbol{\mu}_0$ and, for $e \in L^2(Q,\rho)$, set $f_e := f + e$. For $\lambda > 0$, there is a unique minimizer $\boldsymbol{\mu}_{\lambda,e}$ of (4) where f gets replaced by f_e.*

If $\|\boldsymbol{\mu}\|_{TV} > \|\boldsymbol{\mu}_0\|_{TV}$ for any magnetization $\boldsymbol{\mu}$ that is S-equivalent to $\boldsymbol{\mu}_0$, then $\boldsymbol{\mu}_{\lambda,e}$ (resp. $|\boldsymbol{\mu}_{\lambda,e}|$) converges to $\boldsymbol{\mu}_0$ (resp. $|\boldsymbol{\mu}_0|$) in the narrow sense as $\lambda \to 0$ and $\|e\|_{L^2(Q)}/\sqrt{\lambda} \to 0$.

Theorems 5.2 and 5.3, Corollary 5.4, and Lemma 5.5 give sufficient conditions for the uniqueness of solutions to EP-1. Hence, if $\boldsymbol{\mu}_0 \in \mathcal{M}(S)^3$ is carried by a set $Z \subset S \subset \mathbb{R}^2 \times \{0\}$, then we may apply the above corollary under the following conditions:

(a) $\mathcal{H}^1(\Gamma \cap Z) < \mathcal{H}^1(\Gamma \setminus Z)$ for any rectifiable Jordan curve $\Gamma \subset S$, or
(b) $Z \subset W \cup \bigcup_{k \in K} L_k$ where $W \subset S$ is purely 1-unrectifiable and the L_k are disjoint line segments such that the distance from any L_k to any L_j, $j \neq k$, is greater than the length of L_k, or
(c) S is tree-like.

In particular, it follows from condition (b) that Corollary 5.8 applies when $\boldsymbol{\mu}_0$ is carried by a countable collection of points and sufficiently separated line segments.

We conclude with an example.

Example 5.1. Let $v_0 = v_4 = (0,0)$, $v_1 = (1,0)$, $v_2 = (1,1)$, and $v_3 = (0,1)$ denote the vertices of the unit square $[0,1]^2$ and let $\boldsymbol{\gamma}_i$ denote the arclength parametrization of the directed line segment from v_i to v_{i+1} for $i = 0,1,2,3$. Let $\boldsymbol{\mu}_0 = \mathbf{R}_{\boldsymbol{\gamma}_0} + \mathbf{R}_{\boldsymbol{\gamma}_2}$ and $\boldsymbol{\mu}_1 = -\mathbf{R}_{\boldsymbol{\gamma}_1} - \mathbf{R}_{\boldsymbol{\gamma}_3}$ and let S be any closed set that contains the unit square (e.g. $S = \mathbb{R}^2$). By Corollary 5.4 both $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_1$ are TV-minimal on S. However, $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_1$ are not strictly TV-minimal since $\boldsymbol{\mu}_0 - \boldsymbol{\mu}_1$ is the loop around $[0,1]^2$, showing that $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_1$ are S-equivalent. Clearly, any convex combination $(1-\alpha)\boldsymbol{\mu}_0 + \alpha\boldsymbol{\mu}_1$, $\alpha \in [0,1]$, is also S-equivalent to $\boldsymbol{\mu}_0$ and TV-minimal on S. In fact, any TV-minimal magnetization is of this form. Indeed, taking $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and $Z = \operatorname{supp} \boldsymbol{\mu}_0$, in (68), the only Γ that makes this inequality an equality is the boundary of $[0,1]$. Hence, by Theorem 5.3, any TV-minimal magnetization is of the form $\boldsymbol{\mu}_0 + s(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)$ for some $s \in \mathbb{R}$. Then minimality of the total variation forces $0 \leq s \leq 1$.

If we take $Q = [0,1]^2 \times \{1\}$ and $\rho = \mathcal{L}_2 \lfloor Q$ then the forward operator A is S-sufficient. With the notation of Corollary 5.8, we get since $\mathfrak{R}\boldsymbol{\mu}_0 = \boldsymbol{\mu}_1$ that if $e = \mathfrak{R}e$ then $\mathfrak{R}f_e = f_e$. In this case, we get from Theorem 5.7 that $\mathfrak{R}\boldsymbol{\mu}_{\lambda,e} = \boldsymbol{\mu}_{\lambda,e}$ for every $\lambda > 0$. Now, we know that any weak-$*$ limit of minimizers of EP-2 is TV-minimal, provided that both λ and $\|e\lambda^{-1/2}\|_{L^2(Q,\rho)}$ tend to 0 (see [11, Theorems 2&5]). Because the limit should also be invariant under $\mathfrak{R}$, it must be equal to $(\boldsymbol{\mu}_0 + \boldsymbol{\mu}_1)/2$. In particular, we get global weak-$*$ convergence of $\boldsymbol{\mu}_{\lambda,e}$ and $|\boldsymbol{\mu}_{\lambda,e}|$ for this example, as long as the noise e has the same symmetry as the data.

Appendix A.

In this appendix we gather several technical results (particularly Lemma A.3) concerning the Smirnov decomposition that are needed in Section 2.

Lemma A.1. *Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be a parametrized rectifiable curve, $\boldsymbol{\Gamma} = \boldsymbol{\gamma}([a,b])$ its image and $\mathbf{R}_{\boldsymbol{\gamma}}$ the $\mathbb{R}^n$-valued measure defined by (12). Then, $\mathbf{R}_{\boldsymbol{\gamma}}$ is absolutely continuous with respect to*

$\mathcal{H}^1\lfloor\mathbf{\Gamma}$, *and its Radon-Nikodym derivative is given by*

$$d\mathbf{R}_{\boldsymbol{\gamma}}/d(\mathcal{H}^1\lfloor\Gamma)(x) = \sum_{t\in\boldsymbol{\gamma}^{-1}(x)} \boldsymbol{\gamma}'(t), \qquad \mathcal{H}^1\text{-a.e. } x \in \mathbf{\Gamma}.$$

Proof. As $|\mathbf{R}_{\boldsymbol{\gamma}}|$ is regular (being a finite Borel measure on $\mathbb{R}^n$), for any open set $V \subset \mathbb{R}^n$ we have that

$$|\mathbf{R}_{\boldsymbol{\gamma}}|(V) = \sup\{|\langle \mathbf{R}_{\boldsymbol{\gamma}}, \boldsymbol{\varphi}\rangle|, \, \boldsymbol{\varphi} \in C_c(V, \mathbb{R}^n), \, |\boldsymbol{\varphi}| \leq 1\} \leq \int_{\Gamma\cap V} N(\boldsymbol{\gamma}, x)\, d\mathcal{H}^1(x). \tag{77}$$

Now, $\mathcal{H}^1\lfloor\Gamma$ is also regular, since it is finite and every open set in Γ is σ-compact, see [27, Theorem 2.18]. In particular, if $B \subset \mathbb{R}^n$ is a Borel set such that $\mathcal{H}^1(B\cap\Gamma) = 0$, then there is a decreasing sequence V_k of open sets in $\mathbb{R}^n$ with $V_k \supset B \cap \Gamma$ and $\mathcal{H}^1(\cap_k V_k \cap \Gamma) = 0$. Hence, we obtain from (77), (11) and the dominated convergence theorem that

$$|\mathbf{R}_{\boldsymbol{\gamma}}|(B) = |\mathbf{R}_{\boldsymbol{\gamma}}|(B\cap\Gamma) \leq \liminf_k |\mathbf{R}_{\boldsymbol{\gamma}}|(V_k) \leq \lim_k \int_{\Gamma\cap V_k} N(\boldsymbol{\gamma}, x)\, d\mathcal{H}^1(x) = 0.$$

Thus, $|\mathbf{R}_{\boldsymbol{\gamma}}|$ and *a fortiori* $\mathbf{R}_{\boldsymbol{\gamma}}$ are absolutely continuous with respect to $\mathcal{H}^1\lfloor\Gamma$. Next, it holds for any Borel set $B \subset \mathbb{R}^n$ that the characteristic function $\chi_{B|\Gamma}$ is the bounded pointwise limit $\mathcal{H}^1\lfloor\Gamma$-a.e. (and thus $|\mathbf{R}_{\boldsymbol{\gamma}}|$-a.e. by what precedes) of a sequence of continuous functions $g_k : \Gamma \to \mathbb{R}$, by Lusin's theorem. Since g_k is the restriction to Γ of some $f_k \in C_c(\mathbb{R}^n)$ with $\sup|f_k| = \sup|g_k|$ by the Tietze extension theorem (for Γ is compact), we get from (12) that for any $v \in \mathbb{R}^n$

$$\langle \mathbf{R}_{\boldsymbol{\gamma}}, f_k v\rangle = v \cdot \int_\Gamma f_k \left(\sum_{t\in\boldsymbol{\gamma}^{-1}(x)} \boldsymbol{\gamma}'(t) \right) d\mathcal{H}^1(x)$$

and, applying the dominated convergence theorem to both sides when $k \to \infty$, we conclude since v was arbitrary that

$$\mathbf{R}_{\boldsymbol{\gamma}}(B) = \int_{\Gamma\cap B} \left(\sum_{t\in\boldsymbol{\gamma}^{-1}(x)} \boldsymbol{\gamma}'(t) \right) d\mathcal{H}^1(x).$$

□

Lemma A.2. *Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be a unit speed parametrization, $\mathbf{\Gamma} = \boldsymbol{\gamma}([a,b])$ its image and $\mathbf{R}_{\boldsymbol{\gamma}}$ the $\mathbb{R}^n$-valued measure defined by* (12)*. Then, $\|\mathbf{R}_{\boldsymbol{\gamma}}\|_{TV} = \ell(\boldsymbol{\gamma})$ if and only if, for $\mathcal{H}^1$-a.e. $x \in \mathbf{\Gamma}$, we have that $\boldsymbol{\gamma}'(t)$ is independent of $t \in \boldsymbol{\gamma}^{-1}(x)$.*

Proof. If $\|\mathbf{R}_{\boldsymbol{\gamma}}\|_{TV} = \ell(\boldsymbol{\gamma})$, there is a sequence of continuous functions $\mathbf{g}_k \in C_c(\mathbb{R}^n, \mathbb{R}^n)$, with $|\mathbf{g}_k| \leq 1$, such that

$$\ell(\boldsymbol{\gamma}) = \lim_{k\to\infty} \langle \mathbf{R}_{\boldsymbol{\gamma}}, \mathbf{g}_k\rangle = \lim_{k\to\infty} \int_\Gamma \left(\sum_{t\in\boldsymbol{\gamma}^{-1}(x)} \mathbf{g}_k(x)\cdot\boldsymbol{\gamma}'(t) \right) d\mathcal{H}^1(x). \tag{78}$$

As $|\mathbf{g}_k(\boldsymbol{\gamma}(t))| \leq 1 = |\boldsymbol{\gamma}'(t)|$, we see from (11), (78) and the definition of $N(\boldsymbol{\gamma}, x)$ that for some subsequence $j(k)$ and $\mathcal{H}^1$-a.e. $x \in \Gamma$, we have $\lim_k \mathbf{g}_{j(k)}(x) \cdot \boldsymbol{\gamma}'(t) = 1$ for all t such that $\boldsymbol{\gamma}(t) = x$. In particular, $\boldsymbol{\gamma}'(t)$ is independent of $t \in \boldsymbol{\gamma}^{-1}(x)$ for $\mathcal{H}^1$-a.e. x. Conversely, if the latter property hold, we get from (13) and (11) that $|\mathbf{R}_{\gamma}|(\mathbb{R}^n) = \ell(\boldsymbol{\gamma})$. □

Lemma A.3. *Let $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^n)^n$ and ρ be a finite positive Borel measure on $\mathcal{C}_\ell$ for some $\ell > 0$. Then,* (15) *holds if and only if* (16) *does.*

Proof. Assume that (15) holds, and let $V \subset \mathbb{R}^n$ be open. Let $\varphi_k \in C_c(V)$ be a sequence of nonnegative functions increasing to χ_V; such a sequence is easily constructed using Urysohn's lemma and the σ-compactness of V. Applying the second identity in (15) to φ_k, we get by monotone convergence that

$$|\boldsymbol{\mu}|(V) = \lim_{k\to+\infty} \langle|\boldsymbol{\mu}|,\varphi_k\rangle = \lim_{k\to+\infty} \int \langle|\mathbf{R}_{\boldsymbol{\gamma}}|,\varphi_k\rangle d\rho(\mathbf{R}_{\boldsymbol{\gamma}}) = \int |\mathbf{R}_{\boldsymbol{\gamma}}|(V) d\rho(\mathbf{R}_{\boldsymbol{\gamma}}). \tag{79}$$

Hence, $|\boldsymbol{\mu}|$ and $\int |\mathbf{R}_{\boldsymbol{\gamma}}| d\rho$ coincide on open sets. In particular, we get for $V = \mathbb{R}^n$ that

$$\|\boldsymbol{\mu}\|_{TV} = \int_{\mathcal{C}_\ell} \|\mathbf{R}_{\boldsymbol{\gamma}}\|_{TV} d\rho(\mathbf{R}_{\boldsymbol{\gamma}}). \tag{80}$$

Moreover, as $|\boldsymbol{\mu}|$ is regular, we see from (79) that for any Borel set $B \subset \mathbb{R}^n$:

$$|\boldsymbol{\mu}|(B) = \inf\{|\boldsymbol{\mu}|(V),\ B \subset V \text{ open}\} = \inf_V \int_{\mathcal{C}_\ell} |\mathbf{R}_{\boldsymbol{\gamma}}|(V) d\rho(\mathbf{R}_{\boldsymbol{\gamma}}) \geq \int_{\mathcal{C}_\ell} |\mathbf{R}_{\boldsymbol{\gamma}}|(B) d\rho(\mathbf{R}_{\boldsymbol{\gamma}}). \tag{81}$$

The conjunction of (80) and (81) implies the second equality in (16).

To obtain the first equality in (16), apply Lusin's theorem to the effect that χ_B is the bounded pointwise limit of a sequence $f_k \in C_c(\mathbb{R}^n)$, except on a Borel set E of $|\boldsymbol{\mu}|$-measure zero. From the second equality in (16), it follows that $|\mathbf{R}_{\boldsymbol{\gamma}}|(E) = 0$ for ρ-a.e. $\mathbf{R}_{\boldsymbol{\gamma}} \in \mathcal{C}_\ell$. Thus, if we set $\mathbf{R}_{\boldsymbol{\gamma}} = (m_1, \cdots, m_n)^T$ to indicate the components of $\mathbf{R}_{\boldsymbol{\gamma}}$ in $\mathcal{M}(\mathbb{R}^n)^n$, we get *a fortiori* that $|m_j|(E) = 0$ for ρ-a.e. $\mathbf{R}_{\boldsymbol{\gamma}}$. So, picking $v = (v_1, \cdots, v_n)^T \in \mathbb{R}^n$, we deduce for such $\mathbf{R}_{\boldsymbol{\gamma}}$ on applying the dominated convergence theorem component-wise that

$$\lim_k \langle \mathbf{R}_{\boldsymbol{\gamma}}, f_k v\rangle = \sum_{j=1}^n v_j \lim_k \int f_k dm_j = \sum_{j=1}^n v_j \int \chi_B dm_j = v \cdot \mathbf{R}_{\boldsymbol{\gamma}}(B). \tag{82}$$

Since v was arbitrary, we can now show the first equality in (16) from the first equation in (15), applied with $\mathbf{g} = f_k v$, by invoking the dominated convergence theorem when $k \to \infty$, in $L^1[d|\boldsymbol{\mu}|]$ on the left hand side and in $L^1[d|\rho|]$ on the right hand side.

Conversely, if (16) holds, sets of $|\boldsymbol{\mu}|$-measure zero have $|\mathbf{R}_{\boldsymbol{\gamma}}|$-measure zero for ρ-a.e. $\mathbf{R}_{\boldsymbol{\gamma}}$, moreover $|\boldsymbol{\mu}|$ and $\int |\mathbf{R}_{\boldsymbol{\gamma}}| d\rho$ (resp. $\boldsymbol{\mu}$ and $\int \mathbf{R}_{\boldsymbol{\gamma}} d\rho$) have the same integral on simple functions, hence also on $L^1[d|\boldsymbol{\mu}|]$ (resp. $(L^1[d|\boldsymbol{\mu}|])^n$). This is logically stronger than (15). □

Lemma A.4. *Let $\mathbf{T_f}$ be an elementary solenoid as in* (17)*. Then, there is a Lipschitz map $\mathbf{g} : \mathbb{R} \to \mathbb{R}^n$ with $|\mathbf{g}'(t)| = 1$ a.e. such that $\mathbf{T_g}$ is an elementary solenoid with $\mathbf{T_f} = \mathbf{T_g}$.*

Proof. Recall from Section 2.3 that $\mathbf{T} = *\lim \mathbf{R}_{\mathbf{f}_s}/s$ as $s \to +\infty$, where we have set $\mathbf{f}_s = \mathbf{f}_{|[-s,s]}$. As the TV-norm of the weak-$*$ limit cannot exceed the limit of the TV-norms, we get since $|\mathbf{f}'(t)| \leq 1$ that

$$1 = \|\mathbf{T_f}\|_{TV} \leq \liminf_{s\to+\infty} \frac{1}{2s} \|\mathbf{R}_{\mathbf{f}_s}\|_{TV} \leq \liminf_{s\to+\infty} \frac{1}{2s} \int_{-s}^{s} |\mathbf{f}'(t)| dt \leq 1. \tag{83}$$

Thus, $\frac{1}{2s}\int_{-s}^{s} |\mathbf{f}'(t)| dt \to 1$ as $s \to +\infty$ and therefore, reparametrizing $\mathbf{f}$ by unit speed like we did for $\boldsymbol{\gamma}$ after (12), we obtain the desired function $\mathbf{g}$. □

Lemma A.5. *Let $\mathbf{T_f}$ be an elementary solenoid as in* (17) *and $\Gamma_s = \mathbf{f}([-s,s])$. Then, the family $\{\nu_s\}_{s>0}$ of normalized arclengths on Γ_s, defined in* (18)*, converges weak-$*$, when $s \to +\infty$, to the* *probability measure* $|\mathbf{T_f}|$. *Moreover, if* $\boldsymbol{\varphi}_j \in C_c(\mathbb{R}^n, \mathbb{R}^n)$ *is a sequence of continuous functions, with* $|\boldsymbol{\varphi}_j| \leq 1$, *such that* $\langle \mathbf{T_f}, \boldsymbol{\varphi}_j \rangle \to 1$ *as* $j \to \infty$, *then*

$$\lim_{j\to\infty} \limsup_{s\to+\infty} \int \left| \boldsymbol{\varphi}_j(x) - \frac{\sum_{t\in\mathbf{f}^{-1}(x),\,|t|\leq s} \mathbf{f}'(t)}{N(\mathbf{f},x,s)} \right|^2 d\nu_s = 0. \tag{84}$$

Proof. The family $\{\nu_s\}_{s>0}$ has at least one weak-$*$ accumulation point as $s \to +\infty$, say ν. Let s_k be a sequence of positive real numbers tending to $+\infty$ and such that ν_{s_k} converges weak-$*$ to ν. For $V \subset \mathbb{R}^n$ an open set, we get by (9) that

$$\begin{aligned} |\mathbf{T_f}|(V) &= \sup\{\langle \mathbf{T_f}, \boldsymbol{\varphi}\rangle,\ \boldsymbol{\varphi} \in C_c(V,\mathbb{R}^n),\ |\boldsymbol{\varphi}| \leq 1\} \\ &= \sup_{\boldsymbol{\varphi}} \lim_{s\to+\infty} \int_{\Gamma_s} \boldsymbol{\varphi}(x) \cdot \frac{\left(\sum_{t\in\mathbf{f}^{-1}(x),\,|t|\leq s} \mathbf{f}'(t)\right)}{2s} d\mathcal{H}^1(x) \\ &\leq \sup_{\boldsymbol{\varphi}} \liminf_{k\to\infty} \int_{\Gamma_{s_k}} |\boldsymbol{\varphi}(x)| \frac{\left|\sum_{t\in\mathbf{f}^{-1}(x),\,|t|\leq s_k} \mathbf{f}'(t)\right|}{2s_k} d\mathcal{H}^1(x) \\ &\leq \sup_{\boldsymbol{\varphi}} \lim_{k\to\infty} \int_{\Gamma_{s_k}} |\boldsymbol{\varphi}(x)| \frac{N(\mathbf{f},x,s_k)}{2s_k} d\mathcal{H}^1(x) = \sup_{\boldsymbol{\varphi}} \langle \nu, |\boldsymbol{\varphi}|\rangle \leq \nu(V). \end{aligned}$$

Thus, by regularity, $|\mathbf{T_f}|(B) \leq \nu(B)$ for any Borel set $B \subset \mathbb{R}^n$, and since $|\mathbf{T_f}|$ is a probability measure (by definition of an elementary solenoid) while $\|\nu\|_{TV} \leq 1$ by the Banach-Alaoglu theorem, we conclude that $|\mathbf{T_f}| = \nu$. This proves the first assertion.

Next, if $\boldsymbol{\varphi} \in C_c(\mathbb{R}^n, \mathbb{R}^n)$, $|\boldsymbol{\varphi}| \leq 1$, is such that $\langle \mathbf{T_f}, \boldsymbol{\varphi}\rangle > 1 - \varepsilon$ for some $\varepsilon \in (0,1)$, then it follows from (13) and the definition of $\mathbf{T_f}$ that for $s > s_0 = s_0(\boldsymbol{\varphi})$ large enough:

$$1 - \varepsilon < \int_{\Gamma_s} \boldsymbol{\varphi}(x) \cdot \frac{\left(\sum_{t\in\mathbf{f}^{-1}(x),\,|t|\leq s} \mathbf{f}'(t)\right)}{2s} d\mathcal{H}^1(x) = \int \boldsymbol{\varphi}(x) \cdot \frac{\left(\sum_{t\in\mathbf{f}^{-1}(x),\,|t|\leq s} \mathbf{f}'(t)\right)}{N(\mathbf{f},x,s)} d\nu_s(x).$$

Because $|\sum_{t\in\mathbf{f}^{-1}(x),\,|t|\leq s} \mathbf{f}'(t)| \leq N(\mathbf{f},x,s)$, the above inequality entails that

$$\int \left| \boldsymbol{\varphi}(x) - \frac{\sum_{t\in\mathbf{f}^{-1}(x),\,|t|\leq s} \mathbf{f}'(t)}{N(\mathbf{f},x,s)} \right|^2 d\nu_s < 2\varepsilon,$$

which implies (84). □

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