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Quasi-geostrophic flow under location uncertainty

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Centre Bretagne, ZI de la Pointe du Diable, 29280 Plouzané, France

- 1 Location Uncertainty (LU) Principles
- 2 Barotropic quasi-geostrophic (QG) flow under LU
- 3 Multi-layered QG under LU

Stochastic flow

$$d\mathbf{X}_t = \mathbf{u}(\mathbf{X}_t, t)dt + \underbrace{\sigma(\mathbf{X}_t, t)d\mathbf{B}_t}_{\text{Uncertainty or Noise}} \quad (1)$$

Spatial structure of noise

- Correlation operator

$$\sigma[f](\mathbf{x}, t) = \int_{\Omega} \check{\sigma}(\mathbf{x}, \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y} \quad (2)$$

$\check{\sigma}$ is assumed to be bounded

Spatial structure of noise

- Correlation operator

$$\sigma[f](x, t) = \int_{\Omega} \check{\sigma}(x, y, t) f(y) dy \quad (2)$$

$\check{\sigma}$ is assumed to be bounded

- Covariance and variance

$$\begin{aligned} Q(x, y, t, s) &= \mathbb{E} \left[(\sigma(x, t) dB_t) (\sigma(y, s) dB_s)^T \right] \\ &= \delta(t - s) dt \int_{\Omega} \check{\sigma}(x, z, t) \check{\sigma}^T(y, z, s) dz \end{aligned} \quad (3)$$

$$a \triangleq Q/dt = \sigma \sigma^T \quad (4)$$

Spectral representation

- Noise

$$\sigma(\mathbf{x}, t) d\mathbf{B}_t = \sum_{n \in \mathbb{N}} \phi_n(\mathbf{x}, t) d\beta_t^n \quad (5)$$

ϕ_n : orthogonal eigenfunctions of the covariance

β^n : 1D standard Brownian motions

Spectral representation

- Noise

$$\sigma(\mathbf{x}, t) d\mathbf{B}_t = \sum_{n \in \mathbb{N}} \phi_n(\mathbf{x}, t) d\beta_t^n \quad (5)$$

- Variance

$$\mathbf{a}(\mathbf{x}, t) = \sum_{n \in \mathbb{N}} \phi_n(\mathbf{x}, t) \phi_n^T(\mathbf{x}, t) \quad (6)$$

ϕ_n : orthogonal eigenfunctions of the covariance

β^n : 1D standard Brownian motions

Stochastic Reynolds transport theorem (Mémin, 2014)

$$d \int_{\mathcal{V}(t)} \theta(\mathbf{x}, t) d\mathbf{x} = \int_{\mathcal{V}(t)} (\mathbf{D}_t \theta + \theta \nabla \cdot (\mathbf{u} - \mathbf{u}_s)) d\mathbf{x} \quad (7)$$

- Stochastic transport operator (Resseguier et al., 2017)

$$\mathbf{D}_t \theta \triangleq d_t \theta + (\mathbf{u} - \mathbf{u}_s) \cdot \nabla \theta dt + \sigma d\mathbf{B}_t \cdot \nabla \theta - \frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \theta) dt \quad (8)$$

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- Itô–Stokes drift (Bauer et al., 2020)

$$\mathbf{u}_s \triangleq \frac{1}{2} \nabla \cdot \mathbf{a} \quad (9)$$

Transport under LU

- Conservation of extensive tracer

$$D_t\theta = 0 \quad (10)$$

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- Conservation of tracer energy (Resseguier et al., 2017)

$$d_t \int_{\Omega} \frac{1}{2} \theta^2 d\mathbf{x} = 0 \quad (11)$$

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Barotropic quasi-geostrophic (QG) flow under LU

- Potential vorticity (PV)

$$q = \nabla \times \mathbf{u} - \psi/R_d^2 + f \quad (\mathbf{u} = \nabla^\perp \psi) \quad (12)$$

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$$D_t q = (\mathcal{D} + \mathcal{F})dt + dS_t^\sigma \quad (13)$$

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$$D_t q = (\mathcal{D} + \mathcal{F})dt + d\mathcal{S}_t^\sigma \quad (13)$$

- PV sources and sinks

$$d\mathcal{S}_t^\sigma = \sum_i -J(\sigma dB_t^i - u_s^i dt, u^i) + \frac{1}{2} \nabla \cdot (\partial_i^\perp \mathbf{a} \nabla u^i) dt \quad (14)$$

J: Jacobi operator

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- Energy conservation ($\mathcal{D} = \mathcal{F} = 0$) (Bauer et al., 2020)

$$d_t \int_\Omega \frac{1}{2} \|\nabla \psi\|^2 + \frac{1}{2} (\psi/R_d)^2 = 0 \quad (15)$$

Test case: monochromatic Rossby wave

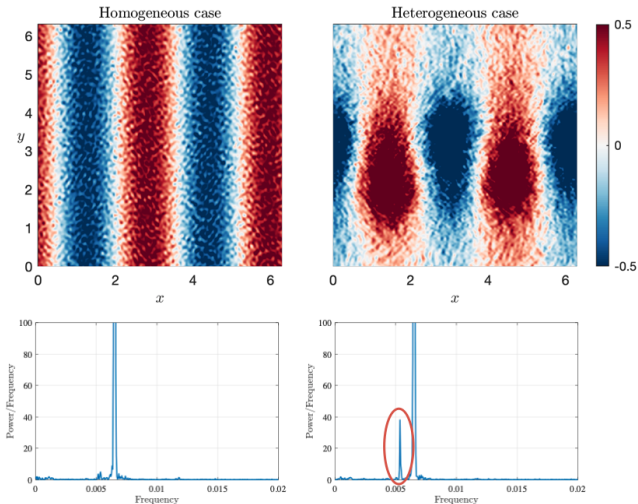
- Aim: study effect of small-scale inhomogeneity on structuration of large-scale flow

Test case: monochromatic Rossby wave

- Aim: study effect of small-scale inhomogeneity on structuration of large-scale flow
- Configurations: initialized by a single Rossby wave; doubly periodic domain; under both homogeneous ($u_s = 0$) and heterogeneous noise with same amplitude

Numerical results: effect of small-scale inhomogeneity

Relative vorticity at $t = 1000$



Click for video

Test case: wind-driven circulation

- Aim: study low-frequency variability of LU coarse models

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- Configurations: double gyre wind forcing $\mathcal{F} = \frac{1}{f_0} \nabla \times \boldsymbol{\tau}$; Biharmonic dissipation of vorticity $\mathcal{D} = -A_4 \nabla^6 \psi$; shallow basin ($R_d = \infty$) without topography; free-slip boundary condition; LU coarse model at 16×32 , 32×64 and 64×128

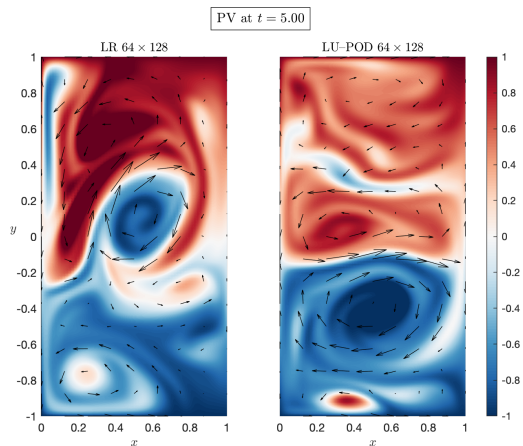
Test case: wind-driven circulation

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- Configurations: double gyre wind forcing $\mathcal{F} = \frac{1}{f_0} \nabla \times \boldsymbol{\tau}$; Biharmonic dissipation of vorticity $\mathcal{D} = -A_4 \nabla^6 \psi$; shallow basin ($R_d = \infty$) without topography; free-slip boundary condition; LU coarse model at 16×32 , 32×64 and 64×128
- POD noise

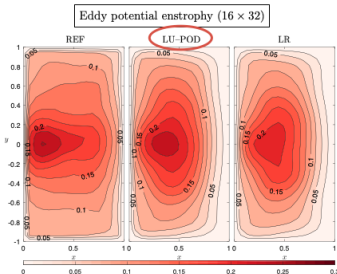
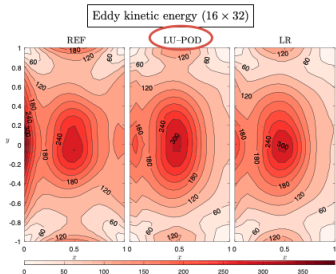
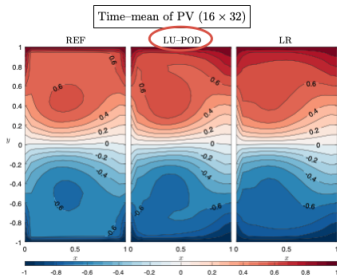
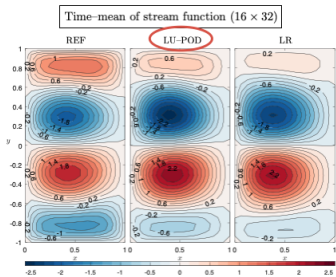
$$\boldsymbol{\sigma} d\mathbf{B}_t = \sum_{n=N_1}^{N_2} \phi_n d\beta_t^n, \quad \mathbf{a} = \sum_{n=N_1}^{N_2} \phi_n \phi_n^T \quad (16)$$

where the EOFs ϕ_n are learned from eddy-resolving (256×512) data.

Numerical results: instantaneous variability



Numerical results: qualitative accuracy of statistics



Numerical results: quantitative accuracy of statistics

Model	Mean	EKE	Skewness	Kurtosis
LR	0.108	0.073	0.190	0.218
LU	0.094	0.064	0.161	0.182
LU-NS	0.100	0.067	0.185	0.191
LU-CP	0.102	0.068	0.185	0.208

Table: Normalized RMSE of statistics for stream function (32×64)

Model	Mean	EKE	Skewness	Kurtosis
LR	0.061	0.122	0.166	0.155
LU	0.056	0.116	0.146	0.122
LU-NS	0.056	0.116	0.148	0.130
LU-CP	0.056	0.115	0.156	0.138

Table: Normalized RMSE of statistics for PV (32×64)

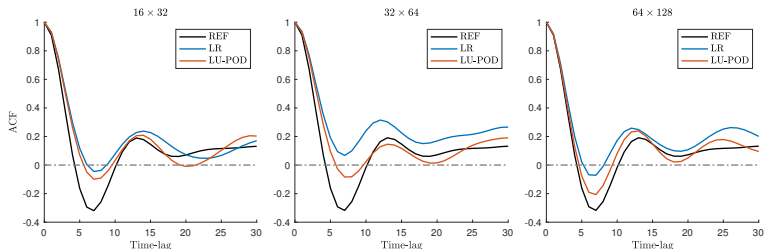
LU-NS: $\overline{u_s}$

LU-CP: $\overline{u_s}$ & dS_t^σ

Autocorrelation function for total circulation

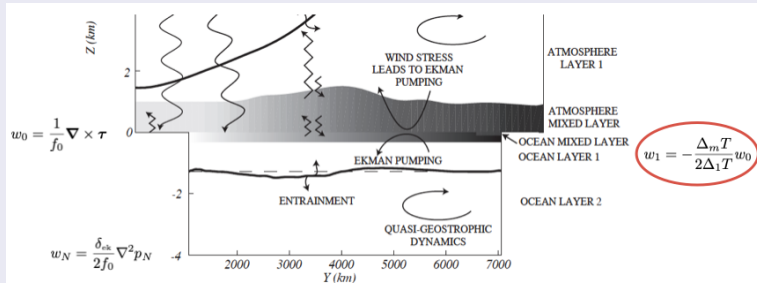
$$\text{ACF}(\tau) = \frac{1}{\sigma_{\Psi}^2} \overline{(\Psi(t) - \bar{\Psi}^t)(\Psi(t + \tau) - \bar{\Psi}^t)}^t, \quad (17)$$

where $\Psi(t) \triangleq \frac{1}{\Omega} \int_{\Omega} \psi(\mathbf{x}, t) d\mathbf{x}$ with its standard derivation σ_{Ψ} .



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Quasi-geostrophic coupled model (Q-GCM) (Hogg et al., 2003)



Governing equations

- Evolution of mixed-layer temperature

$$D_t T_m = (\mathcal{D}_m + \mathcal{F}_m) dt \quad (18)$$

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$$D_t q_k = (\mathcal{D}_k + \mathcal{F}_k) dt + dS_{t,k}^\sigma, \quad \mathcal{F}_k = -\frac{f_0}{H_k} (w_k - w_{k-1}) \quad (19)$$

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w : vertical entrainment

- Evolution of k -th layer pressure

$$q_k = \frac{1}{f_0} \nabla^2 p_k + \frac{f_0}{H_k} (\eta_k - \eta_{k-1}) + \beta y \quad (20)$$

η : perturbed interface height

LU of type Gent–McWilliams (GM)

- Aim: constraint noise along “isopycnal” surfaces

$$\tilde{\sigma}_k d\mathbf{B}_t \perp \nabla \Theta_k, \quad \Theta_k \triangleq \frac{f_0}{H_k} (\eta_k - \eta_{k-1}) \quad (21)$$

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- Projected noise

$$\tilde{\sigma}_k d\mathbf{B}_t = \mathbf{P}_k (\sigma_k d\mathbf{B}_t), \quad \mathbf{P}_k = \mathbf{I} - \frac{\nabla \Theta_k (\nabla \Theta_k)^T}{\|\nabla \Theta_k\|^2} \quad (22)$$

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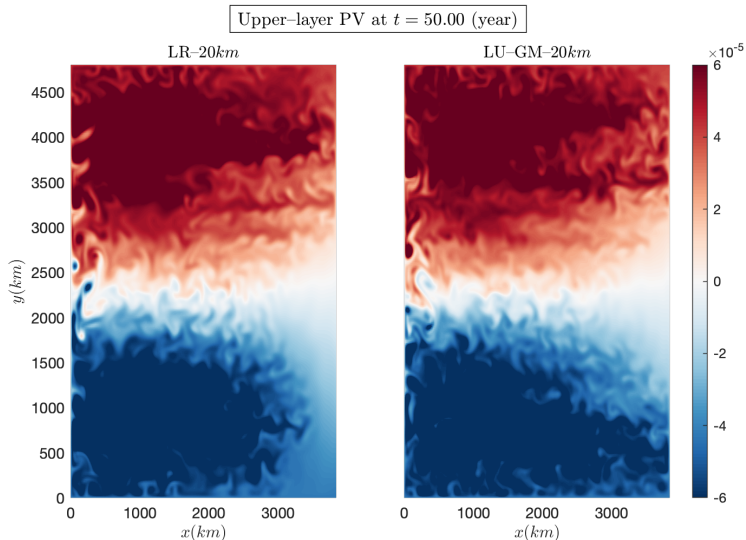
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- LU–GM diffusion tensor

$$\tilde{\mathbf{a}}_k = \mathbf{P}_k \mathbf{a}_k \mathbf{P}_k^T \quad (23)$$

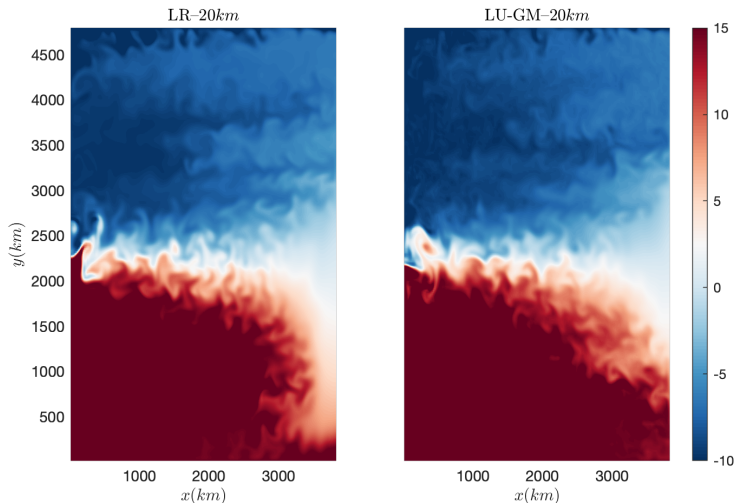
Numerical results: instantaneous variability



Click for video

Numerical results: instantaneous variability

Mixed-layer temperature anomaly at $t = 50.00$ (year)



[Click for video](#)

Measures of statistics (Grooms et al., 2015)

- Pattern correlation (PC)

$$PC = \frac{\int_{\Omega} \sigma_f^2 \sigma_{f_{\text{ref}}}^2 d\mathbf{x}}{(\int_{\Omega} \sigma_f^4 d\mathbf{x})(\int_{\Omega} \sigma_{f_{\text{ref}}}^4 d\mathbf{x})} \quad \nearrow \quad (24)$$

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- Gaussian approximation of relative entropy

$$\text{Dispersion} = \frac{1}{|\Omega|} \int_{\Omega} \left(\frac{\sigma_{f_{\text{ref}}}^2}{\sigma_f^2} - 1 - \log \left(\frac{\sigma_{f_{\text{ref}}}^2}{\sigma_f^2} \right) \right) d\mathbf{x} \quad \searrow \quad (25)$$

$$\text{Entropy} = \frac{1}{2} \left(\frac{1}{|\Omega|} \int_{\Omega} \frac{(\bar{f}^t - \bar{f}_{\text{ref}}^t)^2}{\sigma_f^2} d\mathbf{x} + \text{Dispersion} \right) \quad \searrow \quad (26)$$

Numerical results: comparison of measures

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.41	0.71	0.51	79	41
LU-POD	0.38	0.69	0.52	38	19
LU-GM	0.36	0.66	0.53	16	8

Table: Measures of skill for upper-layer pressure (40km).

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.23	0.64	0.51	194	98
LU-POD	0.22	0.62	0.56	72	36
LU-GM	0.22	0.58	0.64	24	12

Table: Measures of skill for middle-layer pressure (40km).

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.12	0.67	0.57	1322	661
LU-POD	0.12	0.65	0.65	222	111
LU-GM	0.12	0.60	0.83	41	21

Table: Measures of skill for bottom-layer pressure (40km).

Numerical results: comparison of measures

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.44	0.71	0.44	851	26
LU-POD	0.44	0.69	0.46	36	18
LU-GM	0.41	0.66	0.50	15	8

Table: Measures of skill for upper-layer pressure (80km).

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.26	0.60	0.58	188	94
LU-POD	0.26	0.59	0.61	67	34
LU-GM	0.25	0.55	0.70	18	9

Table: Measures of skill for middle-layer pressure (80km).

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.12	0.67	0.56	1701	851
LU-POD	0.12	0.65	0.65	248	124
LU-GM	0.12	0.59	0.80	40	20

Table: Measures of skill for bottom-layer pressure (80km).

Energy pathways (Hogg and Blundell, 2006)

- KE decomposition

$$\text{KE}_k = \frac{\rho H_k}{2} \left(\underbrace{\langle u_k \rangle^2}_{\text{Zonal KE}} + \underbrace{(\overline{u_k^2} + \overline{v_k^2})}_{\text{Standing EKE}} + \underbrace{(u_k'^2 + v_k'^2)}_{\text{Transient EKE}} \right) \quad (27)$$

Numerical results: (eddy) kinetic energy budget

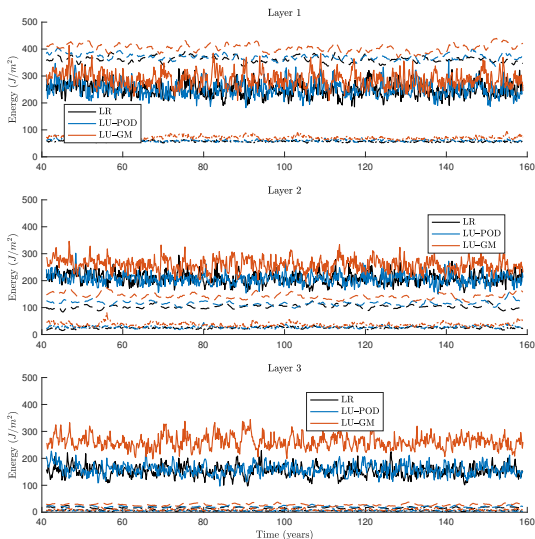


Figure: Comparison of zonal (dotted lines), standing eddy (dashed lines), and transient eddy kinetic energy (solid lines), for different coarse ($40km$) models, in (a) layer 1, (b) layer 2, and (c) layer 3.

Numerical results: energy exchange

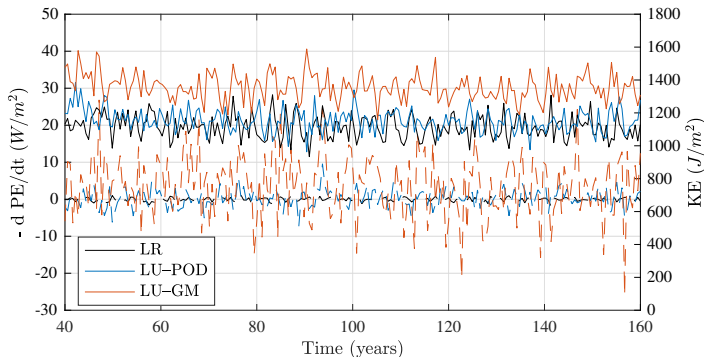


Figure: Time series of the negative rate of change of potential energy (dashed lines) and total kinetic energy (solid lines), for different coarse (40km) models.

- Development of jets

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- LU eddy-resolving simulation ($\sim 10km$)

Future works

- Development of jets
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- Southern channel ocean (Hogg and Blundell, 2006)

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- Implement LU-GM in primitive model (eg. NEMO)
- Application to ensemble data assimilation (Cotter et al., 2020)

Thank for Your Attention!

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