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Oceanic fluid dynamics under location uncertainty

Part.I : Towards a stochastic modeling for the Quasi-Geostrophic system

Long Li¹,
Étienne Mémin¹, Bruno Deremble²

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Brest, June 12, 2018

Objectives

- better small-scale representation

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- better small-scale representation
- identify regously subgrid effects

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- **correct false numerical dissipation**

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- **better ensemble spreading**

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Plan

- fluid flows under location uncertainty

Introduction

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- fluid flows under location uncertainty
- **stochastic QG equations**

Introduction

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- better small-scale representation
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- better ensemble spreading

Plan

- fluid flows under location uncertainty
- stochastic QG equations
- multi-layer stochastic shallow water system

Principes

- Lagrangian displacement :

$$d\mathbf{X}(\mathbf{x}, t) = \mathbf{w}(\mathbf{X}(\mathbf{x}, t), t)dt + \boldsymbol{\sigma}(\mathbf{X}(\mathbf{x}, t), t)d\mathbf{B}_t, \quad \forall (\mathbf{x}, t) \in D \times \mathbb{R}^+, D \subset \mathbb{R}^3$$

$$\mathbf{X}(\mathbf{x}, 0) = \mathbf{x}, \quad \forall \mathbf{x} \in D$$

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- Eulerian velocity :

$$U(\mathbf{x}, t) = \underbrace{\mathbf{w}(\mathbf{x}, t)}_{\text{large-scale}} + \underbrace{\boldsymbol{\sigma}(\mathbf{x}, t)\dot{\mathbf{B}}_t}_{\text{small-scale}}, \quad \forall (\mathbf{x}, t) \in D \times \mathbb{R}^+$$

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Multiplicative noise

- $(\mathbf{B}_t)_{t \in \mathbb{R}^+}$ is a cylindrical Wiener process in $L^2(D, \mathbb{R}^3)$

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$$\boldsymbol{\sigma}(\mathbf{x}, t)d\mathbf{B}_t = \int_D \tilde{\boldsymbol{\sigma}}(\mathbf{x}, \mathbf{y}, t)d\mathbf{B}_t(\mathbf{y})d\mathbf{y}, \quad \forall (\mathbf{x}, t) \in D \times \mathbb{R}^+$$

Principes

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- Subgrid tensor :

$$\boldsymbol{\alpha} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T = \frac{1}{dt}\mathbb{E}\left[(\boldsymbol{\sigma}d\mathbf{B}_t)(\boldsymbol{\sigma}d\mathbf{B}_t)^T\right]$$

SRRT for solenoidal turbulence

- Volumetric rate of change of a scalar for $\mathbf{0} = \nabla \cdot \boldsymbol{\sigma} d\mathbf{B}_t = \nabla \cdot \boldsymbol{\sigma}$:

$$d \int_{\mathcal{V}(t)} q(\mathbf{x}, t) d\mathbf{x} = \int_{\mathcal{V}(t)} \left[d_t q + (\nabla \cdot (q\mathbf{w}) - \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} (q a_{ij})) dt + \nabla q \cdot \boldsymbol{\sigma} d\mathbf{B}_t \right] d\mathbf{x}$$

Stochastic Reynolds Transport Theorem (SRRT)

SRRT for solenoidal turbulence

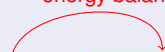
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- Conservation of extensive scalar :

$$d_t q + \underbrace{\mathbf{w}^* dt \cdot \nabla q}_{\text{advection}^*} + \underbrace{\sigma d\mathbf{B}_t \cdot \nabla q}_{\text{forcing}} - \underbrace{\nabla \cdot \frac{1}{2} (\mathbf{a} \nabla q) dt}_{\text{diffusion}} = -q \nabla \cdot \mathbf{w}^* dt$$

energy balance



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energy balance

- Effective drift :

$$\mathbf{w}^* = \mathbf{w} - \frac{1}{2} \nabla \cdot \mathbf{a} \quad \text{Stokes drift ?}$$

Stochastic conservation laws

Navier Stokes equations under location uncertainty

- Momentum equation :

$$\begin{aligned}d_t \mathbf{w} + (\mathbf{w}^* dt + \boldsymbol{\sigma} d\mathbf{B}_t) \cdot \nabla \mathbf{w} - \frac{1}{\rho} \nabla \cdot \left(\frac{1}{2} \rho \mathbf{a} \nabla \mathbf{w} \right) dt + \mathbf{f} \times (\mathbf{w} dt + \boldsymbol{\sigma} d\mathbf{B}_t) \\= -\frac{1}{\rho} \nabla (p dt + dp'_t) - \rho \mathbf{k} dt + \nu \nabla^2 (\mathbf{w} dt + \boldsymbol{\sigma} d\mathbf{B}_t),\end{aligned}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity and dp'_t is a centered random process.

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- Mass equation :

$$\begin{aligned}d_t \rho + (\mathbf{w}^* dt + \boldsymbol{\sigma} d\mathbf{B}_t) \cdot \nabla \rho - \nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla \rho \right) dt = -\rho \nabla \cdot \mathbf{w}^* dt \\ \nabla \cdot \boldsymbol{\sigma} d\mathbf{B}_t = 0\end{aligned}$$

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- Continuity equation (a sufficient constraint) :

$$0 = \nabla \cdot \mathbf{w} = \nabla \cdot (\nabla \cdot \mathbf{a}) = \nabla \cdot \boldsymbol{\sigma}$$

Simple Boussinesq equations under location uncertainty

- Boussinesq approximations :

$$\rho(\mathbf{x}, t) = \rho_b + \delta\rho(\mathbf{x}, t), \text{ with } |\delta\rho| \ll \rho_b$$

$$p(\mathbf{x}, t) = p_0(z) + \delta p(\mathbf{x}, t), \text{ with } |\delta p| \ll |p_0|$$

- Hydrostatic balance :

$$\frac{\partial p_0}{\partial z}(z) = -g\rho_b$$

Stochastic governing equations for stratified ocean

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- Boussinesq thermodynamic equation :

▷ Applies SRRT with $\nabla \cdot \boldsymbol{\sigma} d\mathbf{B}_t = 0$ for $b(\mathbf{x}, t) = b_0(z) + b'(\mathbf{x}, t)$:

$$d_t b' + (\mathbf{w}^* dt + \boldsymbol{\sigma} d\mathbf{B}_t) \cdot \nabla b' + N^2 (\mathbf{w}^* dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_z) = \nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla b' \right) dt + \nabla \cdot \left(\frac{1}{2} \mathbf{a}_{.z} N^2 \right) dt$$

Geostrophic scaling assumptions

1. Classical scalings :

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Geostrophic scaling assumptions

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- A small Rossby number : $Ro \ll 1$
- A small variation of f : $|\beta L| \ll f_0$
- The scale of motion is not significantly larger than the deformation scale :

$$\frac{Ro}{Bu} = \mathcal{O}(Ro) \Rightarrow \frac{\partial b'}{\partial z} \ll N^2$$

Continuously stratified QG system

Geostrophic scaling assumptions

2. Uncertainties scalings :

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- The vertical uncertainty is small compared with the horizontal uncertainties :

$$\frac{(\sigma d\mathbf{B}_t)_z}{\|(\sigma d\mathbf{B}_t)_H\|} \sim \frac{Ro}{Bu} D, \quad D = \frac{h}{L} \ll 1$$

Continuously stratified QG system

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- A moderate uncertainty such that the energy dissipated by the horizontal small-scale flow is the same order than the large-scale kinetic energy :

$$\alpha_H \sim UL \Leftrightarrow MKE \sim U^2, \quad TKE \sim A_H/T\sigma$$

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$$\mathbf{a}_H \sim UL \iff MKE \sim U^2, \quad TKE \sim A_H/T_\sigma$$

- Results :

$$(\sigma dB_t)_z \frac{\partial}{\partial z} = \mathcal{O}\left(\frac{Ro}{Bu}\right)$$

$$\frac{\mathbf{a}_{Hz}}{\mathbf{a}_H} \sim \frac{Ro}{Bu} D, \quad \frac{a_{zz}}{\mathbf{a}_H} \sim \left(\frac{Ro}{Bu}\right)^2 D^2$$

$$\forall i \in H, \mathbf{a}_{Hz} \frac{\partial^2}{\partial x_i \partial z} dt = \mathcal{O}\left(\frac{Ro}{Bu}\right), \quad a_{zz} \frac{\partial^2}{\partial z^2} dt = \mathcal{O}\left(\left(\frac{Ro}{Bu}\right)^2\right)$$

Continuously stratified QG system

Non-dimensional primitive equations under location uncertainty

- Momentum :

$$Ro \left[d_t \mathbf{u} + \left(\mathbf{u}^* dt + (\sigma d\mathbf{B}_t)_H \right) \cdot \nabla_H \mathbf{u} - \nabla_H \cdot \left(\frac{1}{2} \mathbf{a}_H \nabla_H \mathbf{u} \right) dt + \mathcal{O} \left(\frac{Ro}{Bu} \right) \right] \\ + \left(f_0 + Ro\beta(y - y_0) \right) \mathbf{k} \times \left(\mathbf{u} dt + (\sigma d\mathbf{B}_t)_H \right) = -\nabla_H \left(\tilde{p} dt + d\tilde{p}'_t \right)$$

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- Hydrostasy :

$$b' dt + \mathcal{O}(RoD^2) = \frac{\partial}{\partial z} \left(\tilde{p} dt + d\tilde{p}'_t \right)$$

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- Continuity :

$$\nabla_H \cdot \mathbf{u} + \frac{\partial w}{\partial z} = 0$$

$$\nabla_H \cdot (\nabla_H \cdot \mathbf{a}_H) + \frac{Ro}{Bu} \frac{\partial}{\partial z} (\nabla_H \cdot \mathbf{a}_{Hz}) = \nabla_H \cdot (\boldsymbol{\sigma} d\mathbf{B}_t)_H + \frac{Ro}{Bu} \frac{\partial (\boldsymbol{\sigma} d\mathbf{B}_t)_z}{\partial z} = 0 \quad (1)$$

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- Thermodynamic :

$$Ro \left[d_t b' + \left(\mathbf{u}^* dt + (\sigma d\mathbf{B}_t)_H \right) \cdot \nabla_H b' - \nabla_H \cdot \left(\frac{1}{2} \mathbf{a}_H \nabla_H b' \right) dt + \frac{\partial b'}{\partial z} w dt \right] \\ + Bu w dt + Ro \left[(\sigma d\mathbf{B}_t)_z - \frac{1}{2} \nabla_H \cdot \mathbf{a}_{Hz} dt + \mathcal{O} \left(\frac{Ro}{Bu} \right) \right] = 0$$

Continuously stratified QG system

Zeroth order relations

- Pressure balances rotation :

$$f_0 \mathbf{k} \times (\mathbf{u}_0 dt + (\boldsymbol{\sigma} dB_t)_H) = -\nabla_H (p_0 dt + d\tilde{p}'_t) \Leftrightarrow \begin{cases} f_0 v_0 = \frac{\partial p_0}{\partial x}, f_0 u_0 = -\frac{\partial p_0}{\partial y} \\ f_0 (\boldsymbol{\sigma} dB_t)_y = \frac{\partial d\tilde{p}'_t}{\partial x}, f_0 (\boldsymbol{\sigma} dB_t)_x = -\frac{\partial d\tilde{p}'_t}{\partial y} \end{cases}$$

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- Horizontal incompressibilities :

$$0 = \nabla_H \cdot \mathbf{u}_0 = \nabla_H \cdot (\boldsymbol{\sigma} dB_t)_H \xrightarrow{(1)} \frac{\partial (\boldsymbol{\sigma} dB_t)_z}{\partial z} \approx 0, \frac{\partial}{\partial z} (\nabla_H \cdot \mathbf{a}_{Hz}) \approx 0 \quad (2)$$

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$$0 = \nabla_H \cdot \mathbf{u}_0 = \nabla_H \cdot (\boldsymbol{\sigma} d\mathbf{B}_t)_H \xrightarrow{(1)} \frac{\partial (\boldsymbol{\sigma} d\mathbf{B}_t)_z}{\partial z} \approx 0, \frac{\partial}{\partial z} (\nabla_H \cdot \mathbf{a}_{Hz}) \approx 0 \quad (2)$$

- Relative vorticity :

$$\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = \frac{\nabla_H^2 p_0}{f_0}$$

Continuously stratified QG system

Zerth order relations

- Pressure balances rotation :

$$f_0 \mathbf{k} \times (\mathbf{u}_0 dt + (\sigma d\mathbf{B}_t)_H) = -\nabla_H(p_0 dt + d\tilde{p}'_t) \Leftrightarrow \begin{cases} f_0 v_0 = \frac{\partial p_0}{\partial x}, f_0 u_0 = -\frac{\partial p_0}{\partial y} \\ f_0 (\sigma d\mathbf{B}_t)_y = \frac{\partial d\tilde{p}'_t}{\partial x}, f_0 (\sigma d\mathbf{B}_t)_x = -\frac{\partial d\tilde{p}'_t}{\partial y} \end{cases}$$

- Horizontal incompressibilities :

$$0 = \nabla_H \cdot \mathbf{u}_0 = \nabla_H \cdot (\sigma d\mathbf{B}_t)_H \stackrel{(1)}{\rightarrow} \frac{\partial (\sigma d\mathbf{B}_t)_z}{\partial z} \approx 0, \frac{\partial}{\partial z} (\nabla_H \cdot \mathbf{a}_{Hz}) \approx 0 \quad (2)$$

- Relative vorticity :

$$\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = \frac{\nabla_H^2 p_0}{f_0}$$

- Hydrostasy :

$$\frac{\partial p_0}{\partial z} = b_0, \frac{\partial \tilde{p}'_t}{\partial z} = \mathcal{O}(RoD^2)$$

Continuously stratified QG system

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- Hydrostasy :

$$\frac{\partial p_0}{\partial z} = b_0, \frac{\partial \tilde{p}'_t}{\partial z} = \mathcal{O}(RoD^2)$$

- Thermal wind balance :

$$\frac{\partial \mathbf{u}_0}{\partial z} \cdot \nabla_H b_0 = 0, \frac{\partial (\sigma d\mathbf{B}_t)_H}{\partial z} = \mathcal{O}(RoD^2), \frac{\partial \mathbf{a}_H}{\partial z} = \mathcal{O}(Ro^2 D^4) \quad (3)$$

Continuously stratified QG system

First order equations

- Momentum equations :

$$\begin{aligned}d_t u_0 + \left(\mathbf{u}_0^* dt + (\boldsymbol{\sigma} dB_t)_H \right) \cdot \nabla_H u_0 - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H u_0 \right) dt - f_0 v_1 dt \\ - \beta (y - y_0) \left(v_0 dt + (\boldsymbol{\sigma} dB_t)_y \right) = - \frac{\partial p_1}{\partial x} dt - A_4 \nabla_H^4 u_0 dt\end{aligned}\quad (4)$$

$$\begin{aligned}d_t v_0 + \left(\mathbf{u}_0^* dt + (\boldsymbol{\sigma} dB_t)_H \right) \cdot \nabla_H v_0 - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H v_0 \right) dt + f_0 u_1 dt \\ + \beta (y - y_0) \left(u_0 dt + (\boldsymbol{\sigma} dB_t)_x \right) = - \frac{\partial p_1}{\partial y} dt - A_4 \nabla_H^4 v_0 dt\end{aligned}\quad (5)$$

Continuously stratified QG system

First order equations

- Momentum equations :

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- Continuity equation :

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0\quad (6)$$

Continuously stratified QG system

First order equations

- Momentum equations :

$$\begin{aligned} d_t u_0 + \left(\mathbf{u}_0^* dt + (\sigma d\mathbf{B}_t)_H \right) \cdot \nabla_H u_0 - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H u_0 \right) dt - f_0 v_1 dt \\ - \beta(y - y_0) \left(v_0 dt + (\sigma d\mathbf{B}_t)_y \right) = - \frac{\partial p_1}{\partial x} dt - A_4 \nabla_H^4 u_0 dt \end{aligned} \quad (4)$$

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- Continuity equation :

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \quad (6)$$

- Cross-differentiating (4)-(5) combining with (6) :

$$\begin{aligned} d_t \left[\frac{\nabla_H^2 p_0}{f_0} + \beta(y - y_0) \right] + \left(\mathbf{u}_0^* dt + (\sigma d\mathbf{B}_t)_H \right) \cdot \nabla_H \left[\frac{\nabla_H^2 p_0}{f_0} + \beta(y - y_0) \right] \\ - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H \left[\frac{\nabla_H^2 p_0}{f_0} + \beta(y - y_0) \right] \right) dt = \left(f_0 \frac{\partial w_1}{\partial z} - \frac{A_4}{f_0} \nabla_H^6 p_0 \right) dt + R \end{aligned}$$

Continuously stratified QG system

First order equations

- Nonlinear source-sink terms :

$$R_1 = \left(-\frac{\partial u_0^*}{\partial x} \frac{\partial v_0}{\partial x} + \frac{\partial u_0^*}{\partial y} \frac{\partial u_0}{\partial x} - \frac{\partial v_0^*}{\partial x} \frac{\partial v_0}{\partial y} + \frac{\partial v_0^*}{\partial y} \frac{\partial u_0}{\partial y} \right) dt$$
$$= -tr\left(\mathcal{D}[\mathbf{u}]\mathcal{J}\mathcal{D}\left[-\nabla_H \cdot \frac{\mathbf{a}_H}{2}\right]\right) dt$$

$$R_2 = -\frac{\partial(\sigma dB_t)_x}{\partial x} \frac{\partial v_0}{\partial x} + \frac{\partial(\sigma dB_t)_x}{\partial y} \frac{\partial u_0}{\partial x} - \frac{\partial(\sigma dB_t)_y}{\partial x} \frac{\partial v_0}{\partial y} + \frac{\partial(\sigma dB_t)_y}{\partial y} \frac{\partial u_0}{\partial y}$$
$$= -tr\left(\mathcal{D}[\mathbf{u}]\mathcal{J}\mathcal{D}\left[(\sigma d\mathbf{B}_t)_H\right]\right)$$

$$R_3 = \nabla_H \cdot \left(\frac{1}{2} \frac{\partial \mathbf{a}_H}{\partial x} \nabla_H v_0 \right) dt - \nabla_H \cdot \left(\frac{1}{2} \frac{\partial \mathbf{a}_H}{\partial y} \nabla_H u_0 \right) dt$$

$$R_4 = -\beta \nabla_H \cdot \mathbf{a}_y dt$$

Continuously stratified QG system

First order equations

- Nonlinear source-sink terms :

$$R_1 = \left(-\frac{\partial u_0^*}{\partial x} \frac{\partial v_0}{\partial x} + \frac{\partial u_0^*}{\partial y} \frac{\partial u_0}{\partial x} - \frac{\partial v_0^*}{\partial x} \frac{\partial v_0}{\partial y} + \frac{\partial v_0^*}{\partial y} \frac{\partial u_0}{\partial y} \right) dt$$
$$= -tr\left(\mathcal{D}[\mathbf{u}]\mathbf{J}\mathcal{D}\left[-\nabla_H \cdot \frac{\mathbf{a}_H}{2}\right]\right)dt$$

$$R_2 = -\frac{\partial(\sigma dB_t)_x}{\partial x} \frac{\partial v_0}{\partial x} + \frac{\partial(\sigma dB_t)_x}{\partial y} \frac{\partial u_0}{\partial x} - \frac{\partial(\sigma dB_t)_y}{\partial x} \frac{\partial v_0}{\partial y} + \frac{\partial(\sigma dB_t)_y}{\partial y} \frac{\partial u_0}{\partial y}$$
$$= -tr\left(\mathcal{D}[\mathbf{u}]\mathbf{J}\mathcal{D}\left[(\sigma d\mathbf{B}_t)_H\right]\right)$$

$$R_3 = \nabla_H \cdot \left(\frac{1}{2} \frac{\partial \mathbf{a}_H}{\partial x} \nabla_H v_0 \right) dt - \nabla_H \cdot \left(\frac{1}{2} \frac{\partial \mathbf{a}_H}{\partial y} \nabla_H u_0 \right) dt$$

$$R_4 = -\beta \nabla_H \cdot \mathbf{a}_y dt$$

▷ \mathbf{a} homogeneous $\Rightarrow R = R_2, \mathbb{E}[R] = 0$

N.B. $\mathcal{D}[\mathbf{u}] = \frac{1}{2}(\nabla_H \mathbf{u} + \nabla_H^T \mathbf{u})$: deformation tensor, $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: 90° rotation matrix

Continuously stratified QG system

First order equations

- Thermodynamic equation :

$$d_t b_0 + \left(\mathbf{u}_0^* dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H \right) \cdot \nabla_H b_0 - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H b_0 \right) dt + Bu \left[\left(w_1 - \nabla_H \cdot \frac{\mathbf{a}_H z}{2} \right) dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_z \right] = 0 \quad (7)$$

Continuously stratified QG system

First order equations

- Thermodynamic equation :

$$d_t b_0 + \left(\mathbf{u}_0^* dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H \right) \cdot \nabla_H b_0 - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H b_0 \right) dt + Bu \left[\left(w_1 - \nabla_H \cdot \frac{\mathbf{a}_H z}{2} \right) dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_z \right] = 0 \quad (7)$$

- Derivating (7) along z :

$$\begin{aligned} d_t \frac{\partial b_0}{\partial z} + \left(\mathbf{u}_0^* dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H \right) \cdot \nabla_H \frac{\partial b_0}{\partial z} - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H \frac{\partial b_0}{\partial z} \right) dt \\ + Bu \left[\left(\frac{\partial w_1}{\partial z} - \underbrace{\frac{\partial}{\partial z} \nabla_H \cdot \frac{\mathbf{a}_H z}{2}}_{0 \leftarrow (2)} \right) dt + \underbrace{\frac{\partial (\boldsymbol{\sigma} d\mathbf{B}_t)_z}{\partial z}}_{0 \leftarrow (2)} \right] + S = 0 \end{aligned}$$

Continuously stratified QG system

First order equations

- Thermodynamic equation :

$$dt b_0 + \left(\mathbf{u}_0^* dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H \right) \cdot \nabla_H b_0 - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H b_0 \right) dt + Bu \left[\left(w_1 - \nabla_H \cdot \frac{\mathbf{a}_H z}{2} \right) dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_z \right] = 0 \quad (7)$$

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$$S = \left[\left(\frac{\partial u_0}{\partial z} - \frac{1}{2} \nabla_H \cdot \frac{\partial \mathbf{a}_H}{\partial z} \right) dt + \frac{\partial (\boldsymbol{\sigma} d\mathbf{B}_t)_H}{\partial z} \right] \cdot \nabla_H b_0 - \nabla_H \cdot \left(\frac{1}{2} \frac{\partial \mathbf{a}_H}{\partial z} \nabla_H b_0 \right) dt = 0 \leftarrow (3)$$

Continuously stratified QG system

First order equations

- Thermodynamic equation :

$$dt b_0 + \left(\mathbf{u}_0^* dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H \right) \cdot \nabla_H b_0 - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H b_0 \right) dt + Bu \left[\left(w_1 - \nabla_H \cdot \frac{\mathbf{a}_H z}{2} \right) dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_z \right] = 0 \quad (7)$$

- Derivating (7) along z :

$$dt \frac{\partial b_0}{\partial z} + \left(\mathbf{u}_0^* dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H \right) \cdot \nabla_H \frac{\partial b_0}{\partial z} - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H \frac{\partial b_0}{\partial z} \right) dt + Bu \left[\left(\frac{\partial w_1}{\partial z} - \underbrace{\frac{\partial}{\partial z} \nabla_H \cdot \frac{\mathbf{a}_H z}{2}}_{0 \leftarrow (2)} \right) dt + \underbrace{\frac{\partial (\boldsymbol{\sigma} d\mathbf{B}_t)_z}{\partial z}}_{0 \leftarrow (2)} \right] + S = 0$$

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- Results :

$$-\frac{\partial w_1}{\partial z} = \frac{1}{Bu} \left[dt \frac{\partial b_0}{\partial z} + \left(\mathbf{u}_0^* dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H \right) \cdot \nabla_H \frac{\partial b_0}{\partial z} - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H \frac{\partial b_0}{\partial z} \right) dt \right]$$

Continuously stratified QG system

QG equations under location uncertainty

- Potential vorticity (PV) :

$$q_0 = \frac{\nabla_H^2 p_0}{f_0} + \beta(y - y_0) + \frac{\partial}{\partial z} \left(\frac{f_0}{Bu} b_0 \right), \quad b_0 = \frac{\partial p_0}{\partial z}$$

Continuously stratified QG system

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- Evolution of PV :

$$d_t q_0 + \left(\mathbf{u}_0^* dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H \right) \cdot \nabla_H q_0 - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H q_0 \right) dt = -\frac{A_4}{f_0} \nabla_H^6 p_0 dt + R$$

Continuously stratified QG system

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- Dimensional version :

$$q = \frac{\nabla_H^2 \tilde{p}}{f_0} + f + \frac{\partial}{\partial z} \left(\frac{f_0}{N^2} \frac{\partial \tilde{p}}{\partial z} \right)$$

$$d_t q + \left(\mathbf{u}^* dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H \right) \cdot \nabla_H q - \nabla_H \cdot \left(\frac{\mathbf{a}_H}{2} \nabla_H q \right) dt = -\frac{A_4}{f_0} \nabla_H^6 \tilde{p} dt + R$$

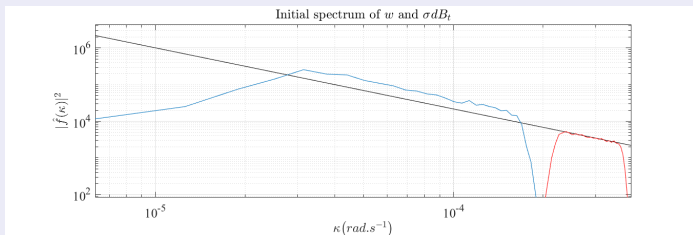
Parametrisation of noise

Some existing approaches

- Homogeneous and isotropic turbulence model :

α is diagonal and constant

- ▷ Through a pass-band spectral cutoff :



$$\text{In 2D, } (\sigma(\mathbf{x})d\mathbf{B}_t)_H = \nabla_H^\perp \psi_\sigma \star d\mathbf{B}_t, \hat{\psi}_\sigma(\boldsymbol{\kappa}) = A \mathbf{1}_{\kappa_1 \leq |\boldsymbol{\kappa}| \leq \kappa_2} |\boldsymbol{\kappa}|^{-\alpha}$$

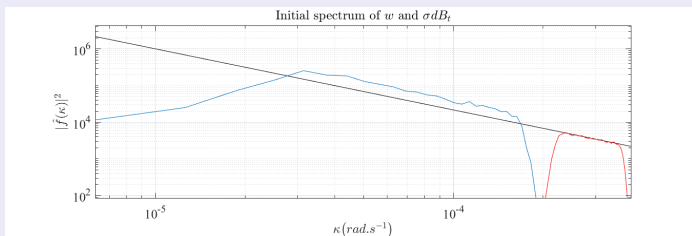
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- Reduced order model :

- ▷ POD approach using observations from velocity field

α is stationary

Parametrisation of noise

A new proposition

- A type of uncertainty living on the iso-surface of buoyancy, i.e. $\sigma dB_t \cdot \nabla b = 0$:

▷ Iso-surface projector :

$$\mathcal{P}_b = \begin{pmatrix} 1 & 0 & \alpha_x \\ 0 & 1 & \alpha_y \\ \alpha_x & \alpha_y & |\alpha|^2 \end{pmatrix}$$

$$\alpha = (\alpha_x, \alpha_y)^T = -\frac{\nabla_H b}{\partial b / \partial z} = -\frac{\nabla_H b' / N^2}{1 + \mathcal{O}(Ro)} = -\nabla_H \frac{\partial p / \partial z}{N^2}$$

Parametrisation of noise

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▷ Divergence-free projector :

$$(\sigma d\mathbf{B}_t)_H = \left(\underbrace{\mathbf{I}_2 - \Delta_H^{-1} \nabla_H \nabla_H^T}_{\delta_{ij} - \frac{\kappa_i \kappa_j}{|\boldsymbol{\kappa}|^2} \text{ in Fourier}} \right) (\boldsymbol{\alpha} \xi_t^z),$$

where ξ_t^z is the 3rd component of an original noise ξ_t , which may be the homogeneous and isotropic one, or Kraichnan turbulent model.

Parametrisation of noise

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where ξ_t^z is the 3rd component of an original noise ξ_t , which may be the homogeneous and isotropic one, or Kraichnan turbulent model.

α anisotropic, inhomogeneous and non-stationary

N.B. In a layered model, the derivative along z will be approximated by a finite difference between layer-averaged quantities.

A multi-layer model

Work in progress

- ([Hogg et al., 2003](#)) - A QG coupled model

A multi-layer model

Work in progress

- (Hogg et al., 2003) - A QG coupled model
 - ▷ only taken the ocean case in a double gyre basin, without heat flux

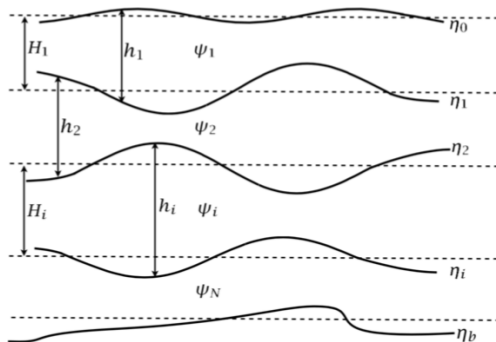


Figure: A multi-layer Shallow Water QG system

A multi-layer model

An N-layers QG shallow water system

- Evolution of $q^{(k)}$, $k = 1, \dots, N$:

$$d_t q^{(k)} + \frac{1}{f_0} J(p^{(k)}, q^{(k)}) dt - \left(\nabla_H \cdot \frac{\mathbf{a}_H^{(k)}}{2} \right) \cdot \nabla_H q^{(k)} dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H^{(k)} \cdot \nabla_H q^{(k)} \\ - \nabla_H \cdot \left(\frac{\mathbf{a}_H^{(k)}}{2} \nabla_H q^{(k)} \right) dt = -\frac{A_4}{f_0} \nabla_H^6 p^{(k)} dt + R^{(k)} + \underbrace{\frac{f_0}{H^{(k)}} w_{ek} \delta_{k1}}_{\text{surface pumping}} - \underbrace{\frac{h_{ek}}{2f_0} \nabla_H^2 p^{(N)} \delta_{kN}}_{\text{bottom drag}}$$

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- Evolution of $p^{(k)}$, $k = 1, \dots, N$:

$$q^{(k)} = \frac{\nabla_H^2 p^{(k)}}{f_0} + \beta(y - y_0) + \frac{f_0}{H^{(k)}} (\eta^{(k)} - \eta^{(k-1)})$$

A multi-layer model

An N-layers QG shallow water system

- Evolution of $q^{(k)}$, $k = 1, \dots, N$:

$$d_t q^{(k)} + \frac{1}{f_0} J(p^{(k)}, q^{(k)}) dt - \left(\nabla_H \cdot \frac{\mathbf{a}_H^{(k)}}{2} \right) \cdot \nabla_H q^{(k)} dt + (\boldsymbol{\sigma} d\mathbf{B}_t)_H^{(k)} \cdot \nabla_H q^{(k)} \\ - \nabla_H \cdot \left(\frac{\mathbf{a}_H^{(k)}}{2} \nabla_H q^{(k)} \right) dt = -\frac{A_4}{f_0} \nabla_H^6 p^{(k)} dt + R^{(k)} + \underbrace{\frac{f_0}{H^{(k)}} w_{ek} \delta_{k1}}_{\text{surface pumping}} - \underbrace{\frac{h_{ek}}{2f_0} \nabla_H^2 p^{(N)} \delta_{kN}}_{\text{bottom drag}}$$

- Evolution of $p^{(k)}$, $k = 1, \dots, N$:

$$q^{(k)} = \frac{\nabla_H^2 p^{(k)}}{f_0} + \beta(y - y_0) + \frac{f_0}{H^{(k)}} (\eta^{(k)} - \eta^{(k-1)})$$

- Perturbation interface height :

$$\eta^{(0)} = 0; \eta^{(k)} = \frac{p^{(k+1)} - p^{(k)}}{g^{(k)}}, k = 1, \dots, N-1; \eta^{(N)} = D(x, y)$$

$$g^{(k)} = \frac{g}{\rho_b} (\rho^{(k+1)} - \rho^{(k)}), k = 1, \dots, N-1$$

A multi-layer model

An N-layers QG shallow water system

- Evolution of $q^{(k)}$, $k = 1, \dots, N$:

$$d_t q^{(k)} + \frac{1}{f_0} J(p^{(k)}, q^{(k)}) dt - \left(\nabla_H \cdot \frac{\mathbf{a}_H^{(k)}}{2} \right) \cdot \nabla_H q^{(k)} dt + (\sigma d\mathbf{B}_t)_H^{(k)} \cdot \nabla_H q^{(k)} \\ - \nabla_H \cdot \left(\frac{\mathbf{a}_H^{(k)}}{2} \nabla_H q^{(k)} \right) dt = -\frac{A_4}{f_0} \nabla_H^6 p^{(k)} dt + R^{(k)} + \underbrace{\frac{f_0}{H^{(k)}} w_{ek} \delta_{k1}}_{\text{surface pumping}} - \underbrace{\frac{h_{ek}}{2f_0} \nabla_H^2 p^{(N)} \delta_{kN}}_{\text{bottom drag}}$$

- Evolution of $p^{(k)}$, $k = 1, \dots, N$:

$$q^{(k)} = \frac{\nabla_H^2 p^{(k)}}{f_0} + \beta(y - y_0) + \frac{f_0}{H^{(k)}} (\eta^{(k)} - \eta^{(k-1)})$$

- Perturbation interface height :

$$\eta^{(0)} = 0; \eta^{(k)} = \frac{p^{(k+1)} - p^{(k)}}{g^{(k)}}, k = 1, \dots, N-1; \eta^{(N)} = D(x, y)$$

$$g^{(k)} = \frac{g}{\rho_b} (\rho^{(k+1)} - \rho^{(k)}), k = 1, \dots, N-1$$

- Horizontal uncertainties :

$$(\sigma d\mathbf{B}_t)_H^{(k)} = \left(\mathbf{I}_2 - \Delta_H^{-1} \nabla_H \nabla_H^T \right) \left(\left(\nabla_H \eta^{(k-1)} - \nabla_H \eta^{(k)} \right) \xi_t^{(k)} \right), k = 1, \dots, N$$

Thanks for your attentions !

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