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# Stochastic modeling of mesoscale eddies in oceanic dynamics

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Workshop on Frontiers of Uncertainty Quantification in Fluid Dynamics  
Pisa(Italy), September 12, 2019

- Better small-scale representation : have explicit formulations and interpretations (*v.s. parametrization with ad hoc tuning*)
- Physical consistency : respect a set of conservation laws (*e.g. energy, circulation*) (*v.s. arbitrary Gaussian forcing*)
- Useful in uncertainty quantification (UQ) : provide more reliable ensemble forecasts (EF) and more efficient spread for ensemble data assimilation (*v.s. perturbations of initial condition (PIC)* )

- 1 Location Uncertainty (LU) Principles
- 2 Stochastic Barotropic Vorticity Equation (SBVE)
- 3 Parametrizations of Noise
- 4 Long-term Diagnosis of Time-Statistics
- 5 Short-term Verification of Ensemble Forecasts

# Location Uncertainty (LU) Principles

- Stochastic flow :

$$d\mathbf{X}_t = \underbrace{\mathbf{w}(\mathbf{X}_t, t)dt}_{\text{large-scale / resolved}} + \underbrace{\boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t}_{\text{small-scale / unresolved}}$$

- Functional process :

$$\boldsymbol{\sigma}(\mathbf{x}, t)d\mathbf{B}_t = \int_{\Omega} \check{\boldsymbol{\sigma}}(\mathbf{x}, \mathbf{y}, t)d\mathbf{B}_t(\mathbf{y})d\mathbf{y}$$

$\mathbf{B}_t$  is a cylindrical Wiener process of infinite dimension (in some Hilbert space) and  $\check{\boldsymbol{\sigma}}$  is deterministic symmetric kernel

# Location Uncertainty (LU) Principles

- Covariance operator :

$$\begin{aligned} Q(\mathbf{x}, \mathbf{y}, t, s) &= \mathbb{E} \left[ \boldsymbol{\sigma}(\mathbf{x}, t) d\mathbf{B}_t (\boldsymbol{\sigma}(\mathbf{y}, s) d\mathbf{B}_s)^T \right] \\ &= \delta(t - s) dt \int_{\Omega} \check{\boldsymbol{\sigma}}(\mathbf{x}, \mathbf{z}, t) \check{\boldsymbol{\sigma}}^T(\mathbf{y}, \mathbf{z}, s) dz \end{aligned}$$

- Variance tensor (per unit of time) :

$$\mathbf{a}(\mathbf{x}, t) = \frac{Q(\mathbf{x}, t)}{dt} = \boldsymbol{\sigma} \boldsymbol{\sigma}^T(\mathbf{x}, t)$$

- Turbulent Kinetic Energy (TKE) :

$$\text{TKE} = \frac{1}{2} \frac{\text{tr}(\mathbf{a})}{dt} \quad (m^2 \cdot s^{-2})$$

# Stochastic Reynolds Transport Theorem (SRTT)

We assume that  $\nabla \cdot \boldsymbol{\sigma} = 0$  in the following.

- Rate of change of a scalar process  $\theta$  within a volume transported by the stochastic flow :

$$d \int_{\mathcal{V}(t)} \theta(\mathbf{x}, t) d\mathbf{x} = \int_{\mathcal{V}(t)} (\mathbf{D}_t \theta + \theta \nabla \cdot \mathbf{w}^*) d\mathbf{x}$$

- Stochastic transport operator :

$$\mathbf{D}_t \theta \triangleq d_t \theta + \underbrace{\left( \mathbf{w} - \frac{1}{2} \nabla \cdot \mathbf{a} \right)}_{\mathbf{w}^*} \cdot \nabla \theta dt + \underbrace{\boldsymbol{\sigma} dB_t \cdot \nabla \theta}_{\text{multiplicative noise}} - \underbrace{\nabla \cdot \left( \frac{\mathbf{a}}{2} \nabla \theta \right)}_{\text{subgrid diffusion}} dt$$

$\mathbf{w}^*$  : corrected drift – effect of statistical inhomogeneity of the small-scale flow component; generalization of the Stokes drift

[Bauer, Chandramouli, Chapron, Li & Mémin 2019a]

# Conservation Laws

We assume that  $\nabla \cdot \mathbf{w}^* = 0$  in the following.

- Conservation of extensive scalar :

$$D_t \theta = 0$$

- Conservation of tracer energy : [Resseguier, Memin & Chapron, 2017a]

$$\begin{aligned} d \int_{\Omega} \frac{1}{2} \theta^2 &= \int_{\Omega} \theta d_t \theta + \frac{1}{2} d \langle \theta \rangle_t \\ &= \int_{\Omega} \frac{1}{2} \theta^2 \nabla \cdot (\mathbf{w}^* dt + \boldsymbol{\sigma} d\mathbf{B}_t) = 0 \end{aligned}$$

- Energy decomposition of mean and variance fields :

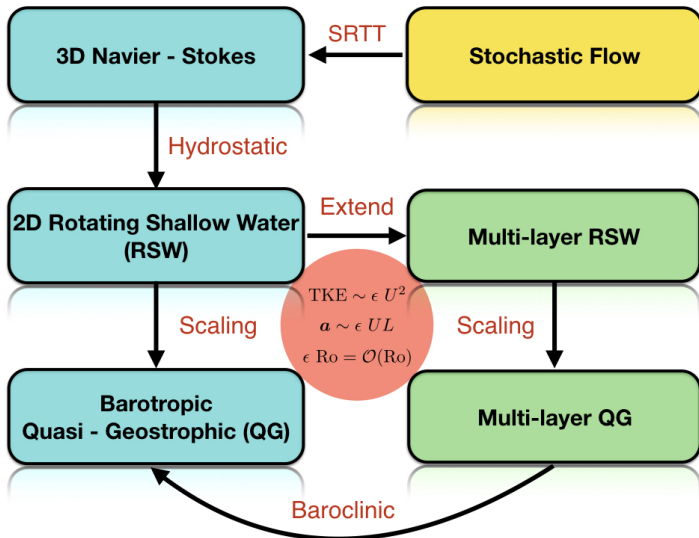
$$0 = \frac{d}{dt} \int_{\Omega} \frac{1}{2} (\mathbb{E}[\theta])^2 + \frac{d}{dt} \int_{\Omega} \frac{1}{2} \text{Var}[\theta]$$



# Outline

- 1 Location Uncertainty (LU) Principles
- 2 Stochastic Barotropic Vorticity Equation (SBVE)**
- 3 Parametrizations of Noise
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# Stochastic Barotropic Vorticity Equation (SBVE)



# Stochastic Barotropic Vorticity Equation (SBVE)

- Forced–dissipative advection of the potential vorticity (PV)  $q$  with source process :

$$D_t q = (\mathbf{S}_1 + F + D)dt + \mathbf{S}_2 d\mathbf{B}_t$$

$$S_1 = \frac{1}{2} \sum_{i,j=1,2} \partial_{ij}^2 (\nabla^\perp a_{ij} \cdot \mathbf{u}), \quad S_2 d\mathbf{B}_t = -\text{tr} \left( \nabla^\perp (\boldsymbol{\sigma} d\mathbf{B}_t)^T \nabla \mathbf{u}^T \right)$$

- Kinematic relationship between PV and stream function  $\psi$  :

$$q = \nabla^2 \psi + f, \quad \mathbf{u} = \nabla^\perp \psi$$

- Conservation of kinetic energy in the absence of  $D$  and  $F$  : [Bauer, Chandramouli, Chapron, Li & Mémin 2019a]

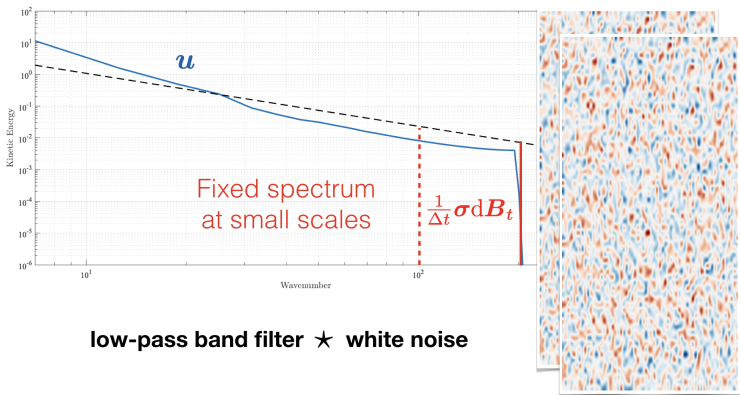
$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} \|\nabla \psi\|^2 = 0$$

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# Parametrizations of Noise

- **Homogeneous** in space **Stationary** in time : [Resseguier, Memin & Chapron, 2017b]

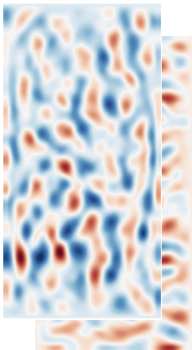


- Homogeneous Non-stationary : [Resseguier, Li, Jouan, Derian, Mémin & Chapron, 2019]  
Based on estimation of the absolute diffusivity spectral density (ADSD).

# Parametrizations of Noise

- **Heterogeneous Stationary** : [Chandramouli, Mémin, Chapron & Heitz 2019b]

Based on snapshot proper orthogonal decomposition (POD) method from data



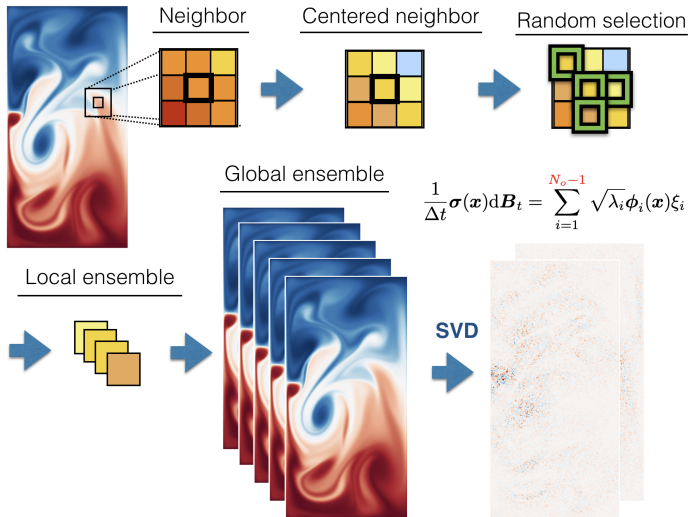
$$\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}) + \sum_{i=1}^{n-1} \alpha_i(t) \phi_i(\mathbf{x})$$

$$\frac{1}{\Delta t} \boldsymbol{\sigma}(\mathbf{x}) d\mathbf{B}_t = \sum_{i=n}^N \sqrt{\lambda_i} \phi_i(\mathbf{x}) \xi_i, \quad \xi_i \sim \mathcal{N}(0, 1)$$

$$\frac{1}{\Delta t} \mathbf{a}(\mathbf{x}) = \sum_{i=n}^N \lambda_i \phi_i(\mathbf{x}) \phi_i^T(\mathbf{x})$$

# Parametrizations of Noise

- **Heterogeneous Non-stationary** [Li, Bauer, & Mémin 2019] : On-line learning 'Pseudo-observations' from effective resolution



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# Long-term Diagnosis of Time-Statistics

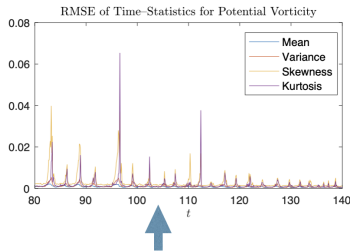
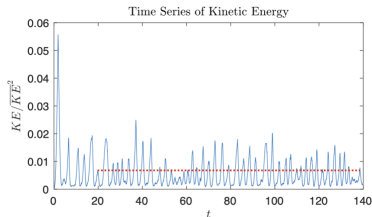
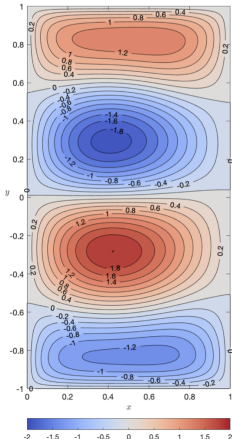
- Results predicted by DNS simulation :

**Instantaneous  
double-gyre  
circulation**



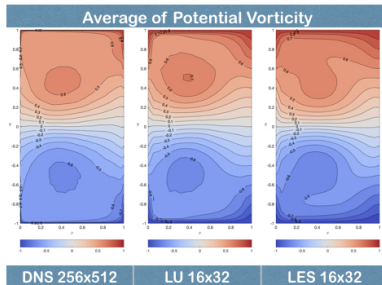
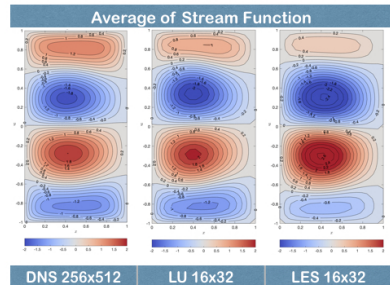
**highly variable**

**Time-Averaged  
four-gyre  
circulation**



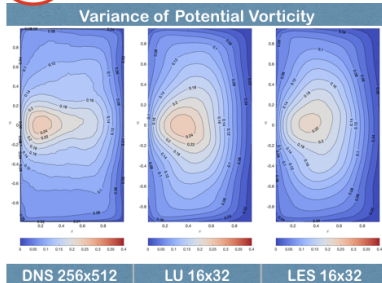
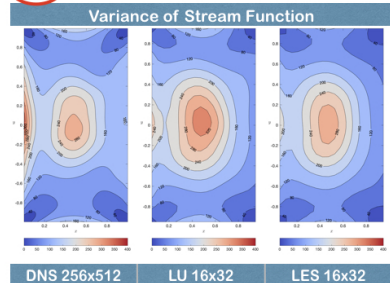
**by perturbation of time-intervals**

# Long-term Diagnosis of Time-Statistics

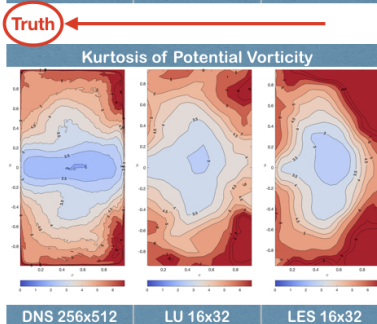
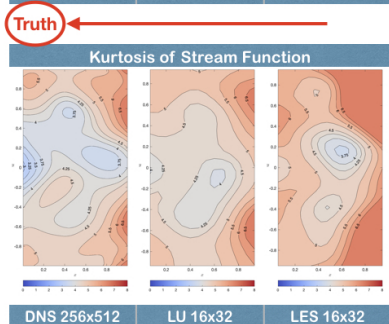
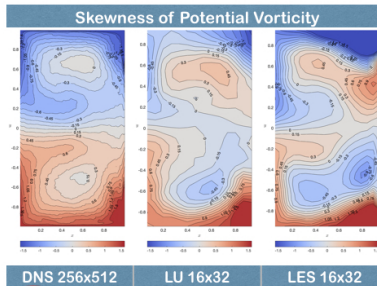
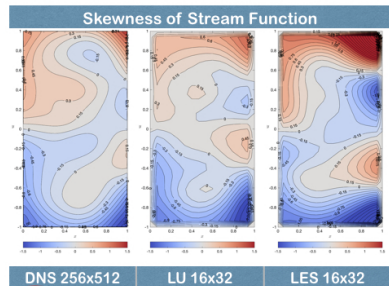


**Truth** ←

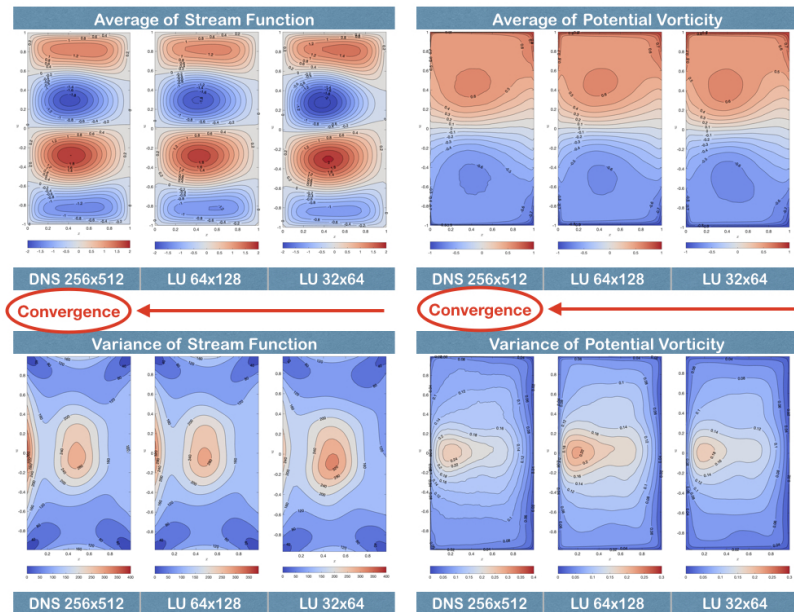
**Truth** ←



# Long-term Diagnosis of Time-Statistics

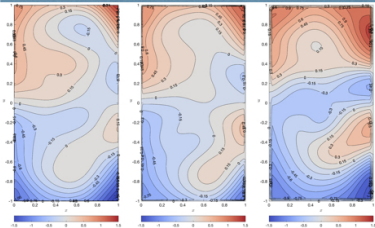


# Long-term Diagnosis of Time-Statistics



# Long-term Diagnosis of Time-Statistics

### Skewness of Stream Function



DNS 256x512

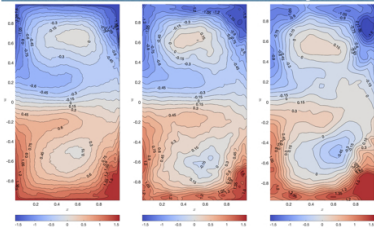
LU 64x128

LU 32x64

**Convergence**



### Skewness of Potential Vorticity



DNS 256x512

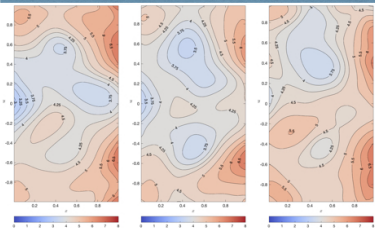
LU 64x128

LU 32x64

**Convergence**



### Kurtosis of Stream Function

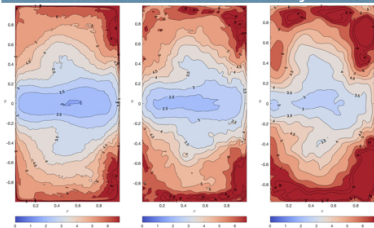


DNS 256x512

LU 64x128

LU 32x64

### Kurtosis of Potential Vorticity



DNS 256x512

LU 64x128

LU 32x64

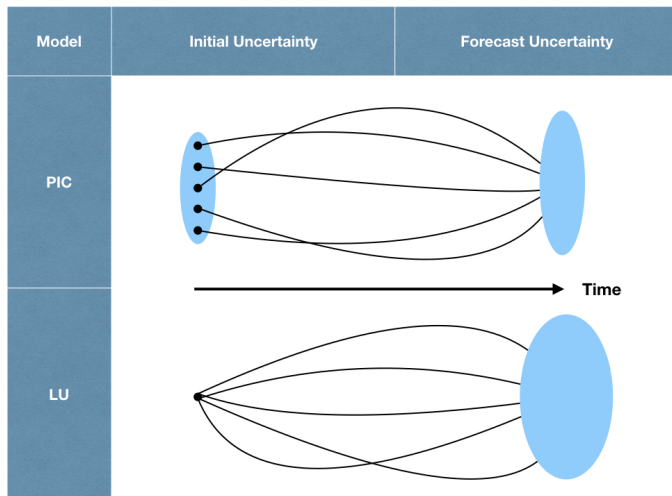


# Outline

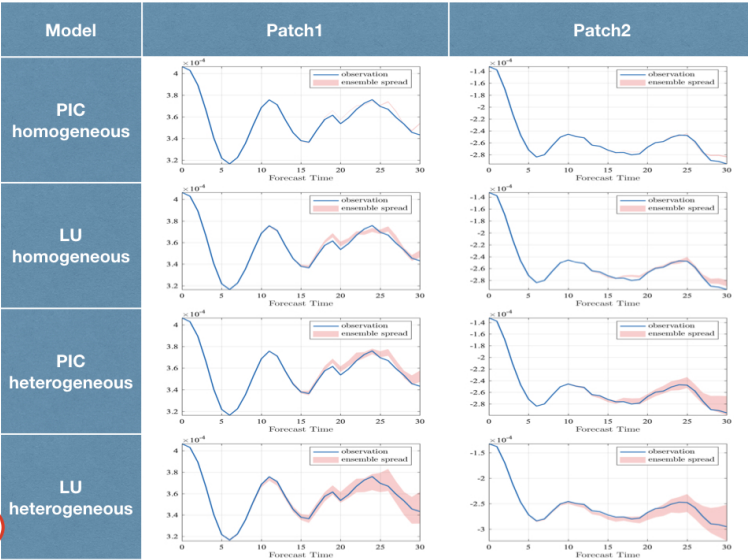
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# Short-term Verification of Ensemble Forecasts

Observed samples are the filtered DNS PV at each grid point; Ensemble simulations are performed with 30 particles from  $t = 0$  to  $t = 30$ , starting from the (perturbed) filtered DNS.



# Ensemble Spreads





- Rank histogram

( How well does the ensemble spread of the forecast represent the true uncertainty of the observations )

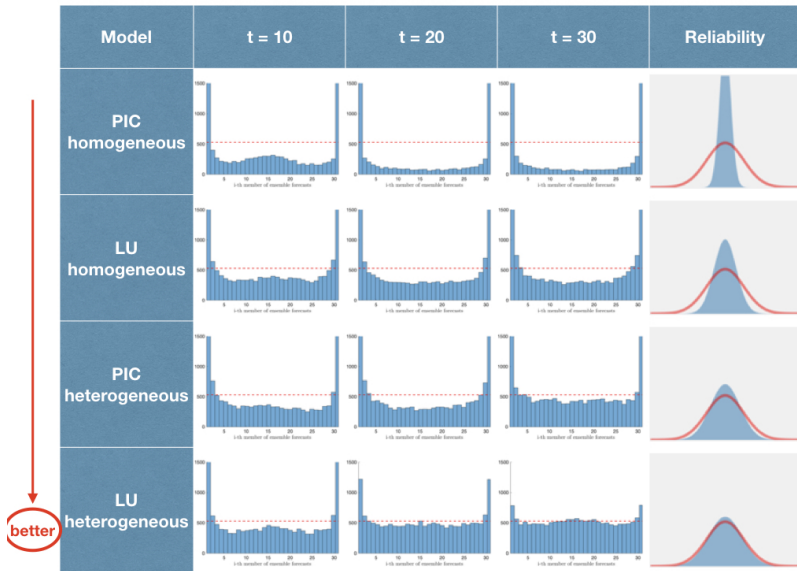
**Flat** : ensemble represent **well** the observed probability distribution;

**U-shaped** : ensemble spread too **small**, many observations falling outside the extremes of the ensemble;

**Dome-shaped** : ensemble spread too **large**, most observations falling near the center of the ensemble;

**Asymmetric** : ensemble contains **bias**.

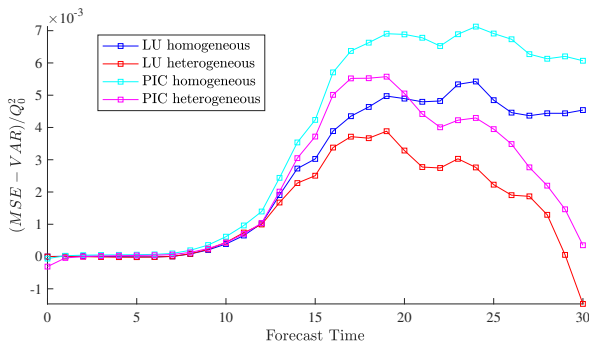
# Reliability of Ensemble Forecasts



# Reliability of Ensemble Forecasts

- The mean squared error (MSE) of the ensemble mean forecasts is identical to the average intra-ensemble sample variance (VAR), multiplied by an ensemble-size  $N$  - dependent inflation factor :

$$\overline{(q^o - \hat{\mathbb{E}}[q])^2}^x = \frac{N + 1}{N} \overline{\text{Var}[q]}^x$$



- Other UQ metrics [Resseguier, Li, Jouan, Derian, Mémén & Chapron, 2019] : Continuous ranked probability score (CRPS), Energy score, Variogram score.

- Better eddy representation based on stochastic transport;
- Capture better on a coarse mesh the correct long-term time-statistics;
- Spread is more accurate compared to PIC.

Work on ...

Strong interest in ensemble data assimilation with particle filter coupled with LU.

Thanks for Your Attention!