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Unification of Drags and Confluence of Drag Rewriting

Jean-Pierre Jouannaud · Fernando Orejas

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Abstract Drags are a recent, natural generalization of terms which admit arbitrary cycles. A key aspect of drags is that they can be equipped with a composition operator so that rewriting amounts to replace a drag by another in a composition. In this paper, we develop a unification algorithm for drags that allows to check the local confluence property of a set of drag rewrite rules.

Keywords Drags · graphs · unification · confluence

PACS PACS code1 · PACS code2 · more

Mathematics Subject Classification (2010) MSC code1 · MSC code2 · more

1 Introduction

Rewriting with graphs has a long history in computer science, graphs being used to represent data structures, but also program structures, and even concurrent and distributed computational models. They therefore play a key rôle in program evaluation, transformation, and optimization, and more generally program analysis; see, for example, [3].

Our work is based on a recent, purely combinatorial, view of labeled graphs [8]. Drags are labeled graphs equipped with roots and *sprouts*, which are vertices without successors labeled by variables. Drags appear as a generalization of terms, they admit roots at arbitrary vertices, sharing, and cycles. Rewrite rules are then pairs of drags that preserve variables and number of roots, hence avoiding the creation of dangling edges when rewriting. A key aspect of drags is that they can be equipped with a composition operator so that matching a left-hand side of rule L w.r.t. an input drag D amounts to write D as the composition of a context graph C with L , and rewriting D with the rule $L \rightarrow R$ amounts to replace L with R in that composition. Composition plays indeed the rôle of both context grafting and substitution in the case of terms.

J.-P. Jouannaud

Université de Paris-Saclay, Projet INRIA Deducteam, Laboratoire de Méthodes Formelles, Saclay, France
E-mail: jeanpierre.jouannaud@gmail.com

F.Orejas

Universitat Politècnica de Catalunya, Department of Computer Science, Barcelona, Spain

To assess our claim that drags are a natural generalization of terms, we extend the most useful term rewriting techniques to drags: the recursive path ordering [7], unification (Section 3) and local confluence (Section 4).

Our first main result here is that unification is unitary and can be performed in quadratic time and space, a complexity bound which is not shown to be sharp and is possibly not. In the case of terms, unification is based on overlapping two terms at a non-variable subterm, from which a recursive propagation process takes place that identifies the labels of both term fragments as long as no label is a variable. The unification process for drags is similar, starting at a set of partner vertices at which the overlap takes place. Propagation operates on pairs of vertices which have not been propagated yet provided no vertex in a pair is a sprout. [Propagation may of course fail, for example at a pair of vertices labeled by different function symbols. When it succeeds, a most general unifier can be extracted from the propagation's result.](#)

Our second main result is that local confluence of a set of drag rewrite rules can be checked by the usual joinability test of their critical pairs. This is so because local confluence follows easily in the non-overlapping case since the rewritten drag is then the composition of two drags that are both rewritten independently of each other. The so-called disjoint and ancestor cases that pop up in the case of terms are therefore both captured here by the same case, hence showing the advantage of packaging context grafting and substitution within a single composition mechanism. As a result, confluence is decidable for terminating finite sets of drag rewrite rules, as is the case for term rewrite rules. Comparisons with the literature is addressed in Section 5. An interesting relationship between unification of drags and of rational dags is pointed out in conclusion.

2 The Drag Model [8]

To ameliorate notational burden, we use vertical bars $|\cdot|$ to denote various quantities, such as length of lists, size of sets or of expressions. We use \emptyset for an empty list, set, or multiset, \cup for set and multiset union, \cdot for list concatenation, and \setminus for set or multiset difference. We mix these, too, and denote by $K \setminus V$ the sublist of a list K obtained by filtering out those elements belonging to a set V . [\[1..n\] is the set \(or list\) of natural numbers from 1 to n.](#) We will also identify a singleton list, set, or multiset with its contents to avoid unnecessary clutter.

Drags are finite *directed rooted labeled multi-graphs*. We presuppose: a set of nullary variable symbols \mathcal{E} , used to label [some vertices without outgoing edges, called sprouts](#); and a set of function symbols Σ , disjoint from \mathcal{E} , whose elements, equipped with a fixed arity, are used as labels for all other vertices called *internal*.

Definition 1 (Drags) A *drag* is a tuple $\langle V, R, L, X, S \rangle$, where

1. V is a finite set of *vertices*;
2. $R : [1..|R|] \rightarrow V$ is a finite list of vertices, called *roots*;
3. $S \subseteq V$ is a set of *sprouts*, leaving $V \setminus S$ to be the *internal* vertices;
4. $L : V \rightarrow \Sigma \cup \mathcal{E}$ is the *labeling* function, mapping internal vertices $V \setminus S$ to labels from the vocabulary Σ and sprouts S to labels from the vocabulary \mathcal{E} , writing $v : f$ for $f = L(v)$;
5. $X : V \rightarrow V^*$ is the *successor* function, mapping each vertex $v \in V$ to a list of vertices in V whose length equals the arity of its label.

The pair (R, S) will be called the *interface* of the drag D .

We use R for both the function itself and its resulting list $[R(1) .. R(n)]$, where $n = |R|$. We will need sometimes the number of times a vertex u occurs in R , for which we will use the notation $R(u)$, as if R were a multiset.

The labeling function extends to lists, sets, and multisets of vertices as expected.

If $b \in X(a)$, then (a, b) is a directed *edge* with *source* a and *target* b . We also write aXb . We also say that b is a successor of a , and a a predecessor of b . The reflexive-transitive closure X^* of the relation X is called *accessibility*. A vertex v is said to be *accessible from* vertex u , and likewise that u *accesses* v , if uX^*v , otherwise it is *unreachable* from u . Vertex v is *accessible* if it is accessible from some root, and *unreachable* otherwise. A drag is *clean* if all its vertices are accessible, *linear* if no two sprouts have the same label.

Note that sprouts may be roots, this is essential for having a nice algebra of drags.

Terms as ordered trees, sequences of terms (forests), terms with shared subterms (dags) and sequences of dags (jungles [17]) are all particular kinds of clean rooted drags. The drag having no vertex, called the *empty drag* (which is also the empty tree), is clean too.

It will sometimes be convenient to [identify a sprout of a linear drag with the variable symbol that is its label](#).

Given a drag $D = \langle V, R, L, X, S \rangle$, we make use of the following notations: $\mathcal{V}er(D)$ for its set of vertices; $\mathcal{I}nt(D)$ for its set of internal vertices; $\mathcal{S}(D)$ for its set of sprouts; X_D for its successor function; $\mathcal{R}(D)$ for its roots (list or set, depending on context); \mathcal{L}_D for its labeling function; $\mathcal{V}ar(D)$ for the set of variables labeling its sprouts; $\mathcal{M}\mathcal{V}ar(D)$ for the multiset of those variables; $\mathcal{A}cc(D) = X^*(R)$ for its set of accessible vertices; $|D|$, its size, for the number of accessible internal vertices plus the size of R (using therefore the above interpretation of roots); and $in(v, D)$, the *in-degree* of vertex v in D , for the number of predecessors of v in D .

2.1 Equality of drags

Drags are particular graphs, the name of their vertices is not relevant. The order of roots in their list is not relevant either, as we shall see when defining rewriting. The only thing that matters is whether two sprouts are labeled by the same or by different variables.

Equality of drags will of course play a key role when it comes to unification.

We define a *vertex renaming* to be a bijection between two finite sets of vertices that restricts to internal vertices and sprouts, and a *variable renaming* to be a bijection between two finite sets of variables.

Definition 2 Two clean drags $D = \langle V, R, L, X, S \rangle$ and $D' = \langle V', R', L', X', S' \rangle$ in this order, are *equal modulo renaming*, namely a vertex renaming $\iota : V \rightarrow V'$ and a variable renaming $\alpha : \mathcal{V}ar(D) \rightarrow \mathcal{V}ar(D')$ (we also say that D' is a *renaming* of D), iff:

1. $\forall u, v \in V : uXv \text{ iff } \iota(u)X'\iota(v)$
2. $\forall u \in \mathcal{I}nt(D) : L(u) = L'(\iota(u))$
3. $\forall s \in S : L'(\iota(u)) = \alpha(L(u))$
4. there exists a bijection $\beta : R \rightarrow R'$ such that $\forall i \in [1 .. |R|] R'(\beta(i)) = R(i)$.

We then write $D =_{\alpha}^{\iota} D'$. The drags D and D' are said to be *equal* if ι is the identity, and *identical* if α is the identity as well.

The subscript α and superscript ι may be omitted when no ambiguity arises with definitional equality (which actually corresponds to identity with identical lists of roots). In

particular, in the absence of annotations, equality should always be interpreted as definitional in definitions.

Equality of drags modulo renaming is of course an equivalence relation, since the identity is a bijection, inverse of a bijection is a bijection, and bijections compose.

Two drags are *disjoint* if they have no vertex nor variable in common. By *renaming apart* two drags D, E , we mean defining two disjoint drags D', E' such that D' and E' are equal modulo renaming to D and E respectively.

The disjoint union of two drags D, E , written $D \oplus E$ is a drag obtained by first renaming D and E apart, and then forming the union of their labeled vertices and edges, and the concatenation of their roots, those of D coming first. In case D and E don't share vertices and/or variables, their vertices and/or variables will be kept identical so as to facilitate technicalities: $D \oplus E$ will then be the juxtaposition of D and E (in this order). Since juxtaposition is clearly associative, we denote by $\Sigma_i D_i$ the juxtaposition of several drags.

2.2 Composition of drags

In this section we introduce a main operation on drags that generalizes the notion of substitution for trees.

A variable in a drag should be understood as a potential connection to a root of another drag, as specified by a connection device called a *switchboard*. A switchboard ξ is a pair of partial injective functions, one for each drag, whose *domain* $\mathcal{D}om(\xi)$ and *image* $\mathcal{I}m(\xi)$ are a set of sprouts of one drag and a set of positions in the list of roots of the other, respectively.

Definition 3 (Switchboard) Let $D = \langle V, R, L, X, S \rangle$ and $D' = \langle V', R', L', X', S' \rangle$ be drags. A *switchboard* ξ for D, D' is a pair $(\xi_D : S \rightarrow \mathcal{D}om(R'); \xi_{D'} : S' \rightarrow \mathcal{D}om(R))$ of partial injective functions such that

1. $\forall s, t \in S$ s.t. $L(s) = L(t) : s \in \mathcal{D}om(\xi_D)$ iff $t \in \mathcal{D}om(\xi_D)$ and $R'(\xi_D(s)) = R'(\xi_D(t))$;
2. $\forall s, t \in S'$ s.t. $L'(s) = L'(t) : s \in \mathcal{D}om(\xi_{D'})$ iff $t \in \mathcal{D}om(\xi_{D'})$ and $R(\xi_{D'}(s)) = R(\xi_{D'}(t))$;
3. ξ is *well-behaved*: it does not induce any cycle among sprouts, using ξ, R, R' relationally:
 $\nexists n > 0, s_1, \dots, s_{n+1} \in S, t_1, \dots, t_n \in S' : s_1 = s_{n+1}, \forall i \in [1..n] s_i \xi_D R' t_i \xi_{D'} R s_{i+1}$

We also say that $\langle D', \xi \rangle$ is an *extension* of D , and that it is *clean* when D' is.

Sprouts labelled by the same variable should be connected by ξ to the same vertex – unless ξ is undefined for them all– which must then occur several times in the list of roots, as required by the first two conditions and the injectivity of ξ 's components. These two conditions are of course automatically satisfied by switchboards, called *linear*, defined for sprouts whose variables are all different.

Note that we could define ξ as a partial function from $\mathcal{D}om(\xi_D) \cup \mathcal{D}om(\xi_{D'})$ when these domains are disjoint sets, we have actually implicitly used this property in the above explanation. Defining its value would however require consider it as a pair of functions using R and R' respectively.

Given two drags D, D' , a partial map ξ from the vertices of D to the vertices of D' –in practice, a switchboard component–, $W \subseteq \mathcal{V}er(D)$ and $W' \subseteq \mathcal{V}er(D')$, we denote by $\xi_{W \rightarrow W'}$ the restriction of ξ to the vertices v of W whose image $R'(\xi(v))$ is in W' . We will use this notation to restrict a switchboard to particular subdrags.

We now move to the composition operation on drags induced by a switchboard. The essence of this operation is that the union of the two drags is formed, but with sprouts in the domain of the switchboards merged with the roots to which the switchboard images refer.

Merging sprouts with their images requires one to worry about the case where multiple sprouts are merged successively, when switchboards map sprout to rooted-sprout to rooted-sprout, until, eventually, an internal vertex of one of the two drags must be reached because a switchboard is well-behaved. That vertex is called *target*:

Definition 4 (Target) Let $D = \langle V, R, L, X, S \rangle$ and $D' = \langle V', R', L', X', S' \rangle$ be drags such that $V \cap V' = \emptyset$, and ξ be a switchboard for D, D' . The *target* $\xi^*(s)$ is a mapping from sprouts in $S \cup S'$ to vertices in $V \cup V'$ defined as follows:

Let $v = R'(n)$ if $s \in S$, and $v = R(n)$ if $s \in S'$, where $n = \xi(s)$.

1. If $v \in (V \cup V') \setminus (S \cup S')$, then $\xi^*(s) = v$.
2. If $v \in (S \cup S') \setminus \mathcal{D}om(\xi)$, then $\xi^*(s) = v$.
3. If $v \in \mathcal{D}om(\xi)$, then $\xi^*(s) = \xi^*(v)$.

The target mapping $\xi^*(-)$ is extended to all vertices of D and D' by letting $\xi^*(v) = v$ when $v \in (V \setminus S) \cup (V' \setminus S')$.

Example 1 Consider the last of the three examples in Figure 1, in which a drag D , whose list of roots is $R = [f h x]$ (identifying vertices with their label) is composed with a second drag whose list of roots is $R' = [g y y]$, via the switchboard $\{x \mapsto 3, y \mapsto 2\}$. We calculate the target of the two sprouts: $x \xi 3 R' y \xi 2 R h$; hence $\xi^*(x) = \xi^*(y) = h$.

We are now ready for defining the composition of two drags. Its set of vertices will be the union of two components: the internal vertices of both drags, and their sprouts which are not in the domain of the switchboard. The labeling is inherited from that of the components.

Definition 5 (Composition) Let $D = \langle V, R, L, X, S \rangle$ and $D' = \langle V', R', L', X', S' \rangle$ be drags such that $V \cap V' = \emptyset$, and let ξ be a switchboard for D, D' . Their *composition* is the drag $D \otimes_{\xi} D' = \langle V'', R'', L'', X'', S'' \rangle$, with interface (R'', S'') denoted $(R, S) \otimes_{\xi} (R', S')$, where

1. $V'' = (V \cup V') \setminus \mathcal{D}om(\xi)$;
2. $S'' = (S \cup S') \setminus \mathcal{D}om(\xi)$;
3. $R'' = \xi^*(R([1..|R|] \setminus \mathcal{I}m(\xi_{D'})) \cdot \xi^*(R'([1..|R'|] \setminus \mathcal{I}m(\xi_D)))$;
4. $L''(v) = L(v)$ if $v \in V \cap V''$; and $L''(v) = L'(v)$ if $v \in V' \cap V''$;
5. $X''(v) = \xi^*(X(v))$ if $v \in V \setminus S$; and $X''(v) = \xi^*(X'(v))$ if $v \in V' \setminus S'$

If ξ_D is surjective and $\xi_{D'}$ total, a kind of switchboard that will play a key rôle for rewriting, then *all* sprouts of D' disappear in the composed drag, while all vertices of D' which are also roots become rootless vertices in the composed drag.

Example 2 We show in Figure 1 three examples of compositions. The first is a substitution of terms. The second uses a bi-directional switchboard, which induces a cycle. In that example, the remaining root is the first (red) root of the first drag which has two roots, the first red, the other black. The third example shows how sprouts that are also roots connect to roots in the composition (colors black and blue indicate roots' origin, while red indicates a root that disappears in the composition). Since x points at y and y at the second root of the first drag, a cycle is created on the vertex of the resulting drag which is labelled by h . Further, the third root of the first drag has become the second root of the result, while the first (resp., second) root of the second drag has become the third (resp., fourth) root of the result. This agrees of course with the definition, as shown by the following calculations (started in Example 1): $\xi^*([1, 2, 3] \setminus [2]) = \xi^*([1, 3]) = [f, h]$; and $\xi^*([1, 2, 3] \setminus [2]) = \xi^*([1, 2]) = [g, h]$, hence the list of roots of the resulting drag is $[f, h, g, h]$.

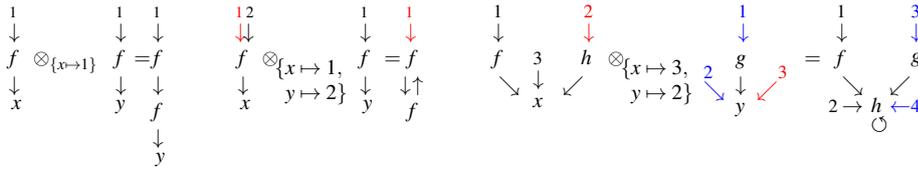


Fig. 1 Different forms of composition: substitution, formation of a cycle, and transfer of roots.

The definition of composition does not assume any property of the input drags. **Composing a single-rooted clean drag D having at least one internal vertex** with a non-clean drag C consisting of a single non-rooted sprout labelled x , has an observable effect on D : the result of the composition $D \otimes_{\{x \mapsto 1\}} C$ is the drag D' , which is D deprived of its root, hence is non-clean since D has internal vertices. In other words, any clean drag D can be sent by an appropriate composition to a drag whose set of accessible vertices is empty, hence is identified with the empty drag. This also implies that D can be sent to any drag U , once cleaned, by taking $C = x \oplus U$.

We end up this section by showing that equality is observational:

Lemma 1 *Let D, E be drags that are equal modulo, and $\langle C, \xi \rangle$ an extension of D . Then, there exists an extension $\langle C, \zeta \rangle$ of E such that $C \otimes_{\xi} D$ and $C \otimes_{\zeta} E$ are equal modulo.*

Proof Let $D =_{\alpha, \beta}^l E$. It suffices to define $\zeta_C = \alpha \xi_C$ and $\zeta_E = \xi_D \circ \iota^{-1}$, and to extend the bijections ι and β as the respective identities on the vertices of C which do not belong to $\mathcal{D}om(\xi_C)$, and on the roots of C which do not belong to $\mathcal{I}m(\xi_C)$.

The important observation is that β becomes the identity in case ξ_C is surjective, hence explaining why the order of roots in drags is irrelevant as far as rewriting is concerned.

2.3 Drag algebra

Composition has important algebraic properties, existence of identities and associativity [6]. We recall the second which will be needed later on, and describe a particular case for which composition is commutative.

Lemma 2 (Associativity) *Let U, V, W be three drags sharing no vertices nor variables, and ζ, ξ two switchboards for respectively (V, W) and $(U, V \otimes_{\xi} W)$. Then, there exist switchboards θ, γ for respectively (U, V) and $((U \otimes_{\theta} V), W)$ such that $(U \otimes_{\theta} V) \otimes_{\gamma} W = U \otimes_{\xi} (V \otimes_{\zeta} W)$.*

Lemma 2 is proved in a particular case in [8]. Here is the proof for the general case that is needed later in the proof of Lemma 7.

Proof Given a switchboard ξ for U, V , $s \in U' \subseteq \mathcal{S}(U)$, and $V' \subseteq \mathcal{V}er(V)$, we define $\xi_{U' \rightarrow V'}(s) = \xi_U(s)$ if $s \in U'$ and $\xi(s)$ reaches a rooted vertex of V' . V' may be omitted when equal to V .

We now define first θ by looking at all possibilities to go from U to V and back by composing ξ and ζ . For γ , we need to go from $U \otimes_{\theta} V$ to W and back. There are very few, because $(\mathcal{D}om(\zeta) \cup \mathcal{I}m(\zeta)) \cap \mathcal{D}om(\xi) = \emptyset$ since the sprouts in the domain of ξ are no more available for ζ . To help the reading, we abuse our notations by identifying a number in a list of roots with the associated rooted vertex. We get:

$$\begin{aligned}\theta &= \langle \zeta_{U \rightarrow V}, \zeta_{V \rightarrow U} \cup \xi_{V \rightarrow \mathcal{D}om(\zeta_{W \rightarrow U})} \circ \zeta_{W \rightarrow U} \rangle \\ \gamma &= \langle \xi_V \cup \zeta_{U \rightarrow W} \cup \xi_{V \rightarrow \mathcal{D}om(\zeta_{U \rightarrow W})} \circ \zeta_{U \rightarrow W}, \zeta_{W \rightarrow U} \cup \xi_W \rangle\end{aligned}$$

We show that θ is a switchboard, the proof for γ is similar.

Firstly, θ_V is a map, since $\mathcal{D}om(\zeta_V) \cap \mathcal{D}om(\xi_V) = \emptyset$. Secondly, θ is well-behaved since ζ is well-behaved, hence the composition $\zeta_{U \rightarrow V} \circ \zeta_{V \rightarrow U}$ cannot not cycle, and $\mathcal{D}om(\xi) \cap \mathcal{I}m(\zeta) = \emptyset$, hence the compositions $\zeta_{U \rightarrow V} \circ \xi_V$ and $\xi_{V \rightarrow \mathcal{D}om(\zeta_{W \rightarrow U})} \circ \zeta_{W \rightarrow U} \circ \xi_{U \rightarrow V}$ are empty.

We are left showing that both compositions are identical. To this end, we show that any sprout s belonging to U, V or W is sent to the same vertex v by $\xi^* \circ \zeta^*$ and $\theta^* \circ \gamma^*$ and that the same edges end up to v in both cases. Both follow from the fact that the path followed from s to v is the same for $\xi^* \circ \zeta^*$ and $\theta^* \circ \gamma^*$ by definition of θ, γ . \square

The composition of two drags D, D' is in general not commutative because its list of roots is the concatenation of the remaining roots of D followed by the remaining roots of D' . However, composition is commutative if all roots of either D or D' are connected by the switchboard, a property that will be useful later:

Lemma 3 (Commutativity) *Let D, D' be two drags sharing no vertices and ξ a switchboard for (D, D') such that either of $\xi_D, \xi_{D'}$ is surjective. Then $D \otimes_\xi D' = D' \otimes_\xi D$.*

2.4 Drag rewriting

Rewriting with drags is similar to rewriting with trees: we first select an instance of the left-hand side L of a rule in a drag D by exhibiting an extension $\langle W, \xi \rangle$ such that $D = W \otimes_\xi L$ – this is drag matching, then replace L by the corresponding right-hand side R in the composition. First, we define what kind of drags are allowed in rules:

Definition 6 A *pattern* is a clean drag containing no isolated sprout and all of whose vertices have at most one root.

Since a vertex of a pattern cannot have more than one root, it will be convenient to designate by v the root number of a rooted vertex v .

Definition 7 (Rule) A *drag rewrite rule* is a pair of clean drags, written $L \rightarrow R$, such that:

(i) L is a pattern, (ii) $|\mathcal{R}(R)| = |\mathcal{R}(L)|$, and (iii) $\mathcal{V}ar(R) \subseteq \mathcal{V}ar(L)$.

A set of drag rewrite rules is called a *drag rewriting system*.

A *renaming* of a rule $L \rightarrow R$ is a rule $L' \rightarrow R'$ such that $L' =^l_\alpha L$ and $R' =^{l'}_\alpha R$.

Condition (i) does not show up in [8]. Although it seems restrictive, it is not. Assume we need to match a drag $D = h(a)$ which has two roots at vertex h named 1 and 2, with a rule $h(x) \rightarrow x$ which has a single root at h named 1. Then, it would suffice to have an extension $\langle z \oplus a, \{z \mapsto 1, x \mapsto 3\} \rangle$, where z has two roots, 1 and 2, and a a single root numbered 3. Then, root transfer will ensure that the result is indeed D (up to drag equality). By exploiting the root transfer mechanism, condition (1) will slightly simplify unification of patterns.

Condition (ii) and (iii) ensure that L and R have a perfect fit with any extension of L , hence avoiding avoid the creation of ill-formed drags when rewriting.

Rewriting drags uses a specific kind of switchboard, which allows one to “encompass” a pattern L within drag D , so that all roots and sprouts of L disappear from the composition:

Definition 8 (Rewriting extension) The pair $\langle C, \xi \rangle$ is a *rewriting extension* of a clean drag L if $C \otimes_{\xi} L$ is a clean drag and ξ a *rewriting switchboard*, that is, ξ_C is linear and surjective and ξ_L is total.

The definition of rewriting extension in [8] assumes that C is clean, before to deduce that $C \otimes_{\xi} L$ is clean, our assumption here, hence making rewriting slightly more general and at the same time better behaved.

We now need an important observation absent from [8].

A (*drag*) *morphism* $o : D \rightarrow D'$ is a mapping from the vertices of D to those of D' that preserves labels, the successor function, and is root non-decreasing, that is, a vertex v of D' should have as many roots as all the u 's together such that $o(u) = v$. Further, sprouts labeled by the same variable must be mapped to the same vertex. A morphism o is *injective* if $o(u) = o(v)$ implies either $u = v$ or u, v are sprouts labeled by the same variable. An injective morphism is an *injection* if it is the identity on internal vertices.

Lemma 4 *Given a clean drag D , there exists a rewriting extension $\langle C, \xi \rangle$ of the pattern L such that $D = C \otimes_{\xi} L$ iff there exists an injection $o : L \rightarrow D$.*

Proof The only if direction follows directly from Definition 5. Conversely, the extension $\langle C, \xi \rangle$ is easily constructed from the map o as follows: each vertex $o(u)$ of D such that u is a root of L is replaced by a fresh sprout $s_u : x_u$ whose number of roots is that of $o(u)$ minus 1 (hence, can be 0), and the edges to its successors discarded, so that $\xi(s_u) = u$ (there is no ambiguity here since L is a pattern); each vertex $o(s)$, where s is a sprout of L is equipped with a new root r , so that $\xi(s) = r$. By construction, the obtained drag is D , hence is clean, giving a rewriting extension $\langle C, \xi \rangle$ of L that verifies the claim. \square

Note that the existence of an injection from a variable renaming of L to D is equivalent to the existence of an injective morphism from L to D :

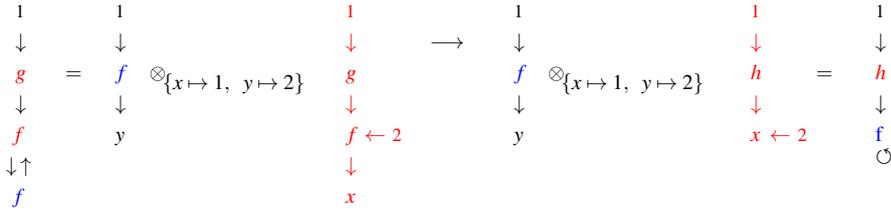
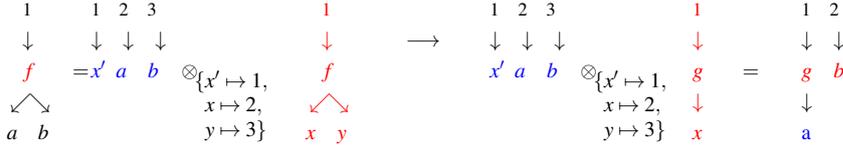
Lemma 5 *Let L' be a renaming of a pattern L and $\langle C, \xi \rangle$ be a rewriting extension of L' . Then, there exists an injective morphism $o : L \rightarrow C \otimes_{\xi} L'$.*

Proof Let $\mu : L \rightarrow L'$ the variable renaming. By Lemma 4, there exists an injection $v : L' \rightarrow C \otimes_{\xi} L'$. Then $o = \mu \circ v$ satisfies the claim. \square

The morphism o plays the rôle of the rewriting position for terms:

Definition 9 (Rewriting) Let \mathcal{R} be a drag rewrite system, $L' \rightarrow R'$ a renaming of a rewrite rule $L \rightarrow R \in \mathcal{R}$, $\langle C, \xi \rangle$ a rewriting extension of both L' and R' , $D = C \otimes_{\xi} L'$, and $D' = C \otimes_{\xi} R'$. We then say that L matches D and that D *rewrites* to D' at o , written $D \xrightarrow{o}_{L \rightarrow R} D'$ or $D \xrightarrow{o}_{\mathcal{R}} D'$, where o is the injective morphism from L to D provided by Lemma 5.

The assumption that $\langle C, \xi \rangle$ is a rewriting extension for R' does not follow from the assumption that it is one for L' : take $f(x) \rightarrow x$, with one root on each side. Take now for D a loop on a rooted vertex labelled by f , for C the context reduced to the sprout y with two roots, and $\xi = \{x \mapsto 1, y \mapsto 1\}$. Then, $\langle C, \xi \rangle$ is well-behaved for $f(x)$ but ill-behaved for x since x is mapped by ξ to y and y to x . It turns out that there exists no switchboard ξ well-behaved for both L and R that permits rewriting this cycle with this rule. This degenerated case went unnoticed in [8].


Fig. 2 Rewriting and cycles.

Fig. 3 Rewriting and connected components.

Because ξ is a rewriting switchboard, ξ_C must be linear, implying that the variables labeling the sprouts of C that are not already sprouts of D must all be different. Then, ξ_C must be surjective, implying that the roots of L —that is, the corresponding root vertices together with their associated edges—, hence those of R , disappear in the composition, a case where the composition is commutative—we shall mostly write the context on the left, though. Further, ξ_L must be total, implying that the sprouts of L (hence those of R) disappear in the composition. Finally, D and C being clean, it is easy to show that D' is clean as well, which is therefore a property rather than a requirement.

Example 3 In Fig. 2, the (red) rewrite rule $g(f(x)) \rightarrow h(x)$, whose roots are g and f on the left-hand side and h and x on the right-hand side, applies with a blue context, colours which are reflected in the input term (showing the rule applies across its cycle) and output term.

Example 4 This time, the rooted term $f(a, b)$ in Fig. 3 is rewritten to the drag made of two components, the rooted terms $g(a)$ and b . Note that allowing the non-clean right-hand side made of the rooted drag $g(x)$ and the non-rooted term y , as in [8], would result in the clean rooted term $g(a)$, the component b being then rootless and thrown away.

Lemma 4 is important, since it implies that the result of rewriting a drag at some position o is unique, as it is for trees.

Rewriting drags is of course monotonic with respect to composition, which subsumes monotonicity and stability of rewriting terms:

Lemma 6 (Monotonicity) *Assume that $D \rightarrow_{L \rightarrow R} D'$ and let $\langle C, \xi \rangle$ be an extension of D such that $C \otimes_{\xi} D$ is clean. Then $C \otimes_{\xi} D \rightarrow_{L \rightarrow R} C \otimes_{\xi} D'$.*

We are now finished with the material from [8] needed for the rest of this paper.

3 Unification

Unification of two terms s, t is somewhat simple: a substitution applied to both identifies them (makes them equal). Assuming s, t share no variable, this substitution is simply the

union of two substitutions, one for s and one for t . A substitution is just a particular case of composition as we have seen at Example 4, using a switchboard whose one component is empty, hence our definition of unification will be based on composition: two patterns U, V are unified by composing them with some rewriting extensions $\langle C, \xi \rangle$ and $\langle D, \zeta \rangle$, resulting in the same drag W , same referring here to drag equality modulo renaming.

We could be satisfied with that definition, but we also want to take care of our particular use of unification to characterize drags, called *overlaps*, that can be rewritten in two different ways with two rules $L \rightarrow R$ and $G \rightarrow D$. In the case of terms, one of L, G stands above the other in the overlap, that is, G is unified with the subterm of L at some position p , or vice versa. If σ is a unifier, the overlap is then either $L\sigma$ or $G\sigma$ (or both if p is the root position). The situation is different with graphical structures: none is above the other, they just share some common subdrag. This is the situation that we are going to define now.

While two terms s, t are unified *at their root*, the solution being a substitution σ such that $s\sigma = t\sigma$, two drags U, V are unified at partner vertices (\bar{u}, \bar{v}) , the solution being a pair of extensions $\langle C, \xi \rangle$ of U and $\langle D, \zeta \rangle$ of V that identifies $C \otimes_{\xi} U$ and $D \otimes_{\zeta} V$ at these partner vertices:

Definition 10 Given drags U, V , we call *partner vertices* two lists L_U, L_V of equal length of internal vertices of U and V , respectively, such that no two vertices $u, u' \in L_U$ (resp., $v, v' \in L_V$) are in relationship with X_U (resp., X_V).

The two lists of partner vertices \bar{u} and \bar{v} can also be organized as a set of unordered pairs $\{(u_i, v_i)\}_i$. The order between the elements of a pair is not important since one must be in U and the other in V , and U, V share no vertex, hence eliminating any potential ambiguity.

In the coming definition, U and V are patterns that have been renamed apart. We further assume for simplicity that these patterns are connected drags.

Definition 11 A drag *unification problem*, $U[\bar{u}] = V[\bar{v}]$, is a pair (U, V) of connected patterns that have been renamed apart, together with partner vertices $P = (\bar{u}, \bar{v})$. A *solution* (or *unifier*) of the drag unification problem $U[\bar{u}] = V[\bar{v}]$ is a pair of rewriting extensions $\langle C, \xi \rangle$ and $\langle D, \zeta \rangle$ of U and V respectively, such that $C \otimes_{\xi} U$ and $D \otimes_{\zeta} V$ are *identified at* (\bar{u}, \bar{v}) , that is: (i) $C \otimes_{\xi} U = {}^t D \otimes_{\zeta} V$, and (ii) $\bar{v} = \iota(\bar{u})$.

A drag equal modulo renaming to $C \otimes_{\xi} U$ (hence to $D \otimes_{\zeta} V$) is called *overlap* of U, V at (\bar{u}, \bar{v}) .

A unification problem $U[\bar{u}] = V[\bar{v}]$ is *solvable* if it has a solution. We denote by $Sol(U[\bar{u}] = V[\bar{v}])$ the (possibly empty) set of all its solutions.

The overlap drags $W = C \otimes_{\xi} U$ and $W' = D \otimes_{\zeta} V$ witness the property that U and V are embedded in W and W' respectively, and that these two embeddings, o and o' coincide at a list of *partner vertices* (condition (ii)) and recursively at their successors (condition (i)). Note that we could have allowed $W = C \otimes_{\xi} U$ and $W' = D \otimes_{\zeta} V$ to be equal modulo an arbitrary renaming including a variable renaming: these two definitions are actually equivalent.

Solutions of a unification problem are defined with the context drag coming first in the products $C \otimes_{\xi} U$ and $D \otimes_{\zeta} V$, which is of course consistent with our definition of rewriting. We will stick to this order throughout the paper even if it does not actually matter since composition is commutative for rewriting extensions.

Example 5 (Figure 4) Let $U = f(h(x))$ and $V = g(h(a))$, in which U has two roots, f and h in this order, and V has two roots g and h in this order (root numbers of U, V being written in bold face). Let the partner vertices be $\{(h, h)\}$. Consider now the rewriting extension

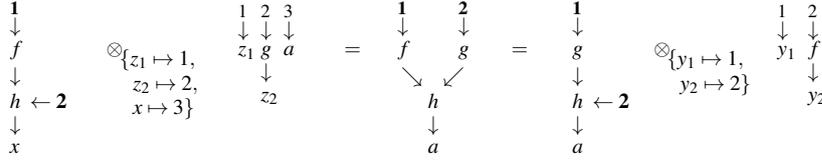


Fig. 4 Overlap drag

$\langle C, \xi \rangle$ such that $C = z_1 \oplus g(z_2) \oplus a$ which has three roots at z_1 , g and a in this order and $\xi = \{z_1 \mapsto 1, z_2 \mapsto 2, x \mapsto 3\}$. Then $C \otimes_{\xi} U$ is the drag with two roots at f and g in this order sharing the subdrag $h(a)$.

Consider now the rewriting extension $\langle D, \zeta \rangle$ such that $D = f(y_1) \oplus y_2$ which has two roots at f and y_1 in this order and $\zeta = \{y_1 \mapsto 1, y_2 \mapsto 2\}$. Then $D \otimes_{\zeta} V$ is equal to $C \otimes_{\xi} U$ (hence coincide on the figure), and the pair of rewriting extensions $\langle C, \xi \rangle, \langle D, \zeta \rangle$ is therefore a solution.

Note that flipping the two roots of V would give a different unification problem, whose solution would simply require to slightly change ζ : changing the order of roots in either U or V does not alter unifiability. \square

Indeed, we want unification to be minimal, that is, to capture all possible extensions that identify U and V . In a first subsection, we define the subsumption order on drags (and drag extensions) and show that it is well-founded. This order aims at defining precisely the notion of minimality of solutions. In a second subsection, we show that unification of drags is unitary, as for terms and dags.

3.1 Subsumption

Definition 12 We say that a clean drag U is an *instance* of a clean drag V , or that V *subsumes* U , and write $U \succeq V$, if there exists a rewriting extension $\langle C, \xi \rangle$ of V such that $U = C \otimes_{\xi} V$.

Note that V being clean, U must be clean as well by definition of a rewriting extension.

In the following, we assume for convenience that the sprouts of U, V are labelled by different sets of variables.

Lemma 7 \succeq is a quasi-order on clean drags, called subsumption, whose strict part is a well-founded order. Two clean drags are equivalent modulo subsumption iff they are equal modulo variable renaming and the number of roots of their rooted vertices.

Proof The relation \succeq being reflexive, we show transitivity. Let U, V, W be three clean drags whose sprouts are labelled by pairwise disjoint sets of variable, such that $U \succeq V \succeq W$. Then, $U = C \otimes_{\xi} V$ and $V = D \otimes_{\zeta} W$, for rewriting extensions $\langle C, \xi \rangle$ of V and $\langle D, \zeta \rangle$ of W . By Lemma 2, $U = E \otimes_{\theta} W$, for some rewriting extension $\langle E, \theta \rangle$ of W , hence $U \succeq W$.

Assume that $U \succeq V \succeq U$, hence $U = C \otimes_{\xi} V$ and $V = D \otimes_{\zeta} U$, using the same notations as above. It follows that C and D have no internal vertex, hence are a bunch of sprouts. Therefore, U and V have the same internal vertices, while their sprouts correspond bijectively.

Further, U and V must have the same (modulo renaming) rooted vertices, since a rootless vertex cannot become rooted by composition, but the number of vertices of a rooted vertex can be increased by composition, or decreased down to zero.

Assume now that $U > V$. Then, either $|U| > |V|$, or $|U| = |V|$. In the latter case, they cannot have the same number of variables labelling their sprouts, since otherwise U and V would be identical up to variable names, hence contradicting our assumption. Since $U > V$, then $U = C \otimes_{\xi} V$ where C cannot have internal vertices since $|U| = |V|$, and ξ cannot be bijective, otherwise $U \simeq V$. Hence ξ maps at least two variables of V to a same (rooted) variable of C , which becomes a variable of U in the composition. Well-foundedness follows, since the number of variables labelling the sprouts of a drag of a given size is bounded. \square

The subsumption quasi-order for drags, despite its name, does not generalize the subsumption quasi-order for terms, which does not take the context into account, but only the substitution. The existence of cycles in drags makes it however impossible, in general, to separate the substitution from the context. Our subsumption quasi-order corresponds therefore to what is called *encompassment* of terms, that is, a subterm of one is an instance of the other. On the other hand, its equivalence generalizes the case of terms, since encompassment and subsumption for terms have the same equivalence.

Given a clean drag D whose one vertex u is rooted, it is always possible to add new roots at u by composition with a rewriting extension, namely the rewriting extension $\langle z, \{z \mapsto u\} \rangle$, where z is a fresh many-rooted sprout. It is possible as well to remove a root (if u is accessible from some other vertex in D in case it has a single root) by composition with the rewriting extension $\langle z, \{z \mapsto u\} \rangle$ where z is a fresh rootless sprout. On the other hand, a single-rooted vertex u that is not accessible from any other vertex cannot lose its only root by composition with a rewriting extension. So, there may be many equivalent rewriting extensions of a pattern L , whose compositions with L will only differ in the number of roots of the vertices of the resulting drags. This will be used to ease the construction of most general unifiers, by choosing the number of roots that makes unification easiest.

3.2 Unification algorithm

Since subsumption is well-founded, the set of solutions of a unification problem $U[\bar{u}] = V[\bar{v}]$ has minimal elements when non-empty. What is yet unclear is how to compute them, and whether there are several or one as for terms. This is the problem we address now.

Identifying $C \otimes_{\xi} U$ and $D \otimes_{\zeta} V$ at a pair of vertices (u, v) requires that u and v have the same label, and that the property can be recursively propagated to their corresponding pairs of successors. Since C, D are yet unknown, this propagation takes place on vertices of U and V , hence on the drag $U \oplus V$. To organize the propagation, we shall mark the pair (u, v) with a fresh red natural number before the propagation has taken place (the initial partner vertices will hold marks $\mathbf{1}, \dots, \mathbf{|\bar{u}|}$), and turn this mark into blue once the propagation has taken place. In case one of u, v is a sprout, no propagation occurs, it is enough to turn the red mark into blue (in practice, we can mark it in blue from the beginning). To ensure freshness, we shall memorize the number c of pairs of vertices that have been marked so far, and increment it by one at each use of a mark. Our syntax for marking a vertex is as in $u : f \cdot \mathbf{i}_1 \cdots \mathbf{i}_n$ if u has label f and (blue or red) marks $\mathbf{i}_1, \dots, \mathbf{i}_n$. Vertex u , label f or marks may be omitted when convenient. The drag $U \oplus V$ in which some pairs of vertices hold a same mark is called a *marked unification problem*. Two vertices u, v of a marked unification problem $U \oplus V[u][v]$, are on the *same side* if they both belong to either U or V , and on *opposite sides* otherwise.

Propagation computes therefore a succession of marked unification problems, denoted by $U \oplus_k V$, starting with the marked unification problem $U \oplus_0 V$ whose marked vertices are exactly the partner vertices. Propagation will stop when there are no more pairs of internal vertices holding a red mark, unless one of the following two situations occurs: two sprouts hold the same variable; some vertex marked \mathbf{i} and \mathbf{j} provides a link between two other different vertices marked \mathbf{i} and \mathbf{j} respectively; In both cases, a new pair of vertices must now be marked if not marked already, giving then possibly rise to new propagations.

When no rule is applicable, the procedure stops at some step k with vertices, some of whose are marked and the other unmarked. An internal vertex u is said to be *singular* if it doesn't share a mark with another vertex. Vertices that are unreachable from the partner vertices are particular singular vertices, but some reachable internal vertices may end up unmarked as well, in which case they too are singular.

The procedure described at Figure 5 computes therefore an equivalence relation between the vertices of two drags U, V to be unified from which their most general unifier is extracted in Section 3.6 when no failure occurs. It consists of a set of transformation rules operating on marked unification problems. The rules construct the equivalence relation by marking pairs of vertices of $U \oplus V$ that figure out new edges of a specific kind between vertices of $U \oplus V$, in the style of Patterson and Wegman unification algorithm [29]. A related idea appears also in [21]. Failure rules return the special expression \perp .

Some rules are reminiscent from the unification rules for terms [6], although we don't use the same rule names except for *Merge*. For example, we use *Propagate* rather than *Decompose* to stress the fact that drags cannot be treated as terms. The failure rules also depend on the roots present in an equivalence class, since drag equality checks their number at all pairs of corresponding vertices.

We write $U \oplus_k V[w]$, k being possibly omitted, to single out a vertex w in a marked unification problem $U \oplus_k V$, a notation that extends to several vertices as expected. We assume that vertices singled out in a left-hand side of rule are pairwise different, and that a pair of vertices sharing a mark is never marked again. It follows that the configurations $U \oplus_k V[u \cdot i \cdot j][v \cdot i \cdot j]$ and $U \oplus_k V[u \cdot i \cdot i]$ cannot occur.

Unification of finite terms differs from unification of infinite rational terms by only one rule called occur-check. Since terms and rational terms are two particular cases of drags, one might expect that the occur-check rule applies in case the occur-check cannot be solved by forming a cycle. This is actually a particular case of a more general principle: identifying a sprout with an internal vertex of the same drag requires mapping that sprout to the internal vertex which should therefore be rooted. This is taken care of by *Lack of root*, of which an occur check is the particular case where the internal vertex u is accessible from sprout s . *Lack of root* applies to a completely different situation, when no root is available to mimic singular vertices on the side where they are missing.

To exemplify our conventions, note that applying *Merge* requires that $U \oplus V$ has at least two different vertices, s and t , which are sprouts holding the same label. These sprouts may be unmarked, or marked, either blue or red, in which case these marks must be different since marked pairs are never marked again.

Example 6 In our example of Figure 6, unification of the initial two drags proceeds in eleven steps and succeeds. *Propagation* steps are labelled by the red mark processed while *Transitivity* steps are labelled by the generated mark. This explains why some steps have the same label. When a red mark labels a sprout, as in step 4, the mark is simply turned blue, as explained early on in this paragraph.

$$\begin{array}{l}
\text{Propagate} \quad U \oplus V \left[\begin{array}{c} u : f \cdot \mathbf{i} \\ \swarrow \quad \searrow \\ s_1 \quad \dots \quad s_n \end{array} \right] \left[\begin{array}{c} v : f \cdot \mathbf{i} \\ \swarrow \quad \searrow \\ t_1 \quad \dots \quad t_n \end{array} \right] \\
\Rightarrow \\
U \oplus V \left[\begin{array}{c} f \cdot \mathbf{i} \\ \swarrow \quad \searrow \\ s_1 \cdot \mathbf{c} + \mathbf{1} \quad \dots \quad s_n \cdot \mathbf{c} + \mathbf{n} \end{array} \right] \left[\begin{array}{c} f \cdot \mathbf{i} \\ \swarrow \quad \searrow \\ t_1 \cdot \mathbf{c} + \mathbf{1} \quad \dots \quad t_n \cdot \mathbf{c} + \mathbf{n} \end{array} \right] \\
\\
\text{Variable case} \quad U \oplus V[s : x \cdot \mathbf{c}][u : f \cdot \mathbf{c}] \Rightarrow U \oplus V[s : x \cdot \mathbf{c}][u : f \cdot \mathbf{c}] \\
\\
\text{Merge} \quad U \oplus V[s : x][t : x] \Rightarrow U \oplus V[s \cdot \mathbf{c} + \mathbf{1}][t \cdot \mathbf{c} + \mathbf{1}] \\
\\
\text{Transitivity} \quad U \oplus V[u \cdot \mathbf{i} \cdot \mathbf{j}][v \cdot \mathbf{i}][w \cdot \mathbf{j}] \Rightarrow U \oplus V[v \cdot \mathbf{c} + \mathbf{1}][w \cdot \mathbf{c} + \mathbf{1}] \\
\\
\text{Symbol conflict} \quad U \oplus V[u : f \cdot \mathbf{i}][v : g \cdot \mathbf{i}] \Rightarrow \perp \quad \text{if } f \neq g \\
\\
\text{Internal conflict} \quad U \oplus V[u \cdot \mathbf{i}][v \cdot \mathbf{i}] \Rightarrow \perp \\
\text{if internal vertices } u, v \text{ are on the same side} \\
\\
\text{Lack of root} \quad U \oplus V[u \cdot \mathbf{i}][v \cdot \mathbf{i}][w] \Rightarrow \perp \\
\text{if } \left\{ \begin{array}{l} \text{case 1: the rootless internal vertex } u \text{ and the sprout } v \text{ are on the same side} \\ \text{case 2: the rootless internal vertex } u \text{ and vertex } v \text{ are on opposite sides,} \\ \text{and } w \text{ is a singular vertex ancestor of } v \end{array} \right.
\end{array}$$

Fig. 5 Drag unification rules

Example 7 An example of failure is given at Figure 7. The first 5 steps are all *Propagation* or *Variable case* steps. Step 6 is a *Transitivity* step and step 7 an *Internal conflict*.

An important, immediate property of the unification rules is termination:

Lemma 8 *Unification rules terminate in quadratic time and space w.r.t. the size of the input.*

Proof Since a pair of identical vertices is never marked, a pair of different vertices is never marked twice, and added sprouts take place of unreachable vertices that are never marked, the number of marked vertices of a unification problem $U[u] = V[v]$ is at most equal to $(|U| + |V|) \times (|U| + |V| - 1)$. \square

3.3 Unification congruences

Correctness of a set of unification rules is the property that the solutions of a unification problem are preserved by application of the rules, until some normal form is obtained which contains them all. Defining precisely what preservation means is the problem we tackle now.

As a general fact, congruences are at the heart of unification and of unification algorithms. In our case, solutions define congruences, and markings define congruences as well. Preservation then relates both kinds of congruences, those defined by markings being coarser than those defined by solutions.

The notion of congruence on terms applies to drags directly:

Definition 13 An equivalence relation \equiv on the set of vertices of a drag $U \oplus V$ is a *congruence* if it satisfies the following properties:

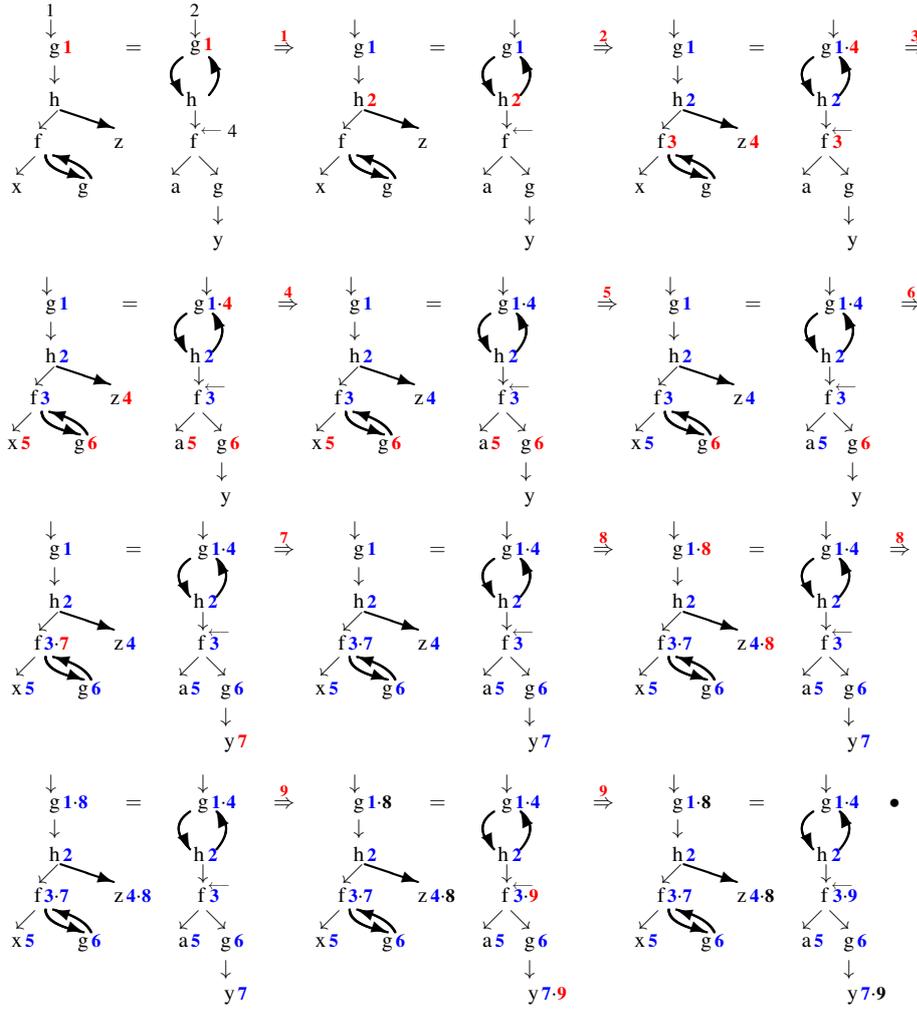


Fig. 6 Successful unification of two patterns.

1. any two equivalent internal vertices u, v have identical labels;
2. the successors of equivalent internal vertices are pairwise equivalent;
3. sprouts with identical label are equivalent.

The main difference between terms and drags is that the latter may have cycles, hence a sprout can be equivalent to any other vertex in a given drag while it cannot in a term.

We now define the congruence associated with the solutions of a given unification problem:

Definition 14 Let $U[\bar{u}] = V[\bar{v}]$ be a unification problem. To any solution $S = (\langle C, \xi \rangle, \langle D, \zeta \rangle)$ such that $C \otimes_{\xi} U[\bar{u}] =^l D \otimes_{\zeta} V[\bar{v}]$, we associate the least equivalence $=_S$ on the vertices of $U \oplus V$ such that:

1. $w \in \mathcal{V}er(U), w' \in \mathcal{V}er(V)$: then $w =_S w'$ iff $\zeta^*(w') = \iota(\xi^*(w))$;

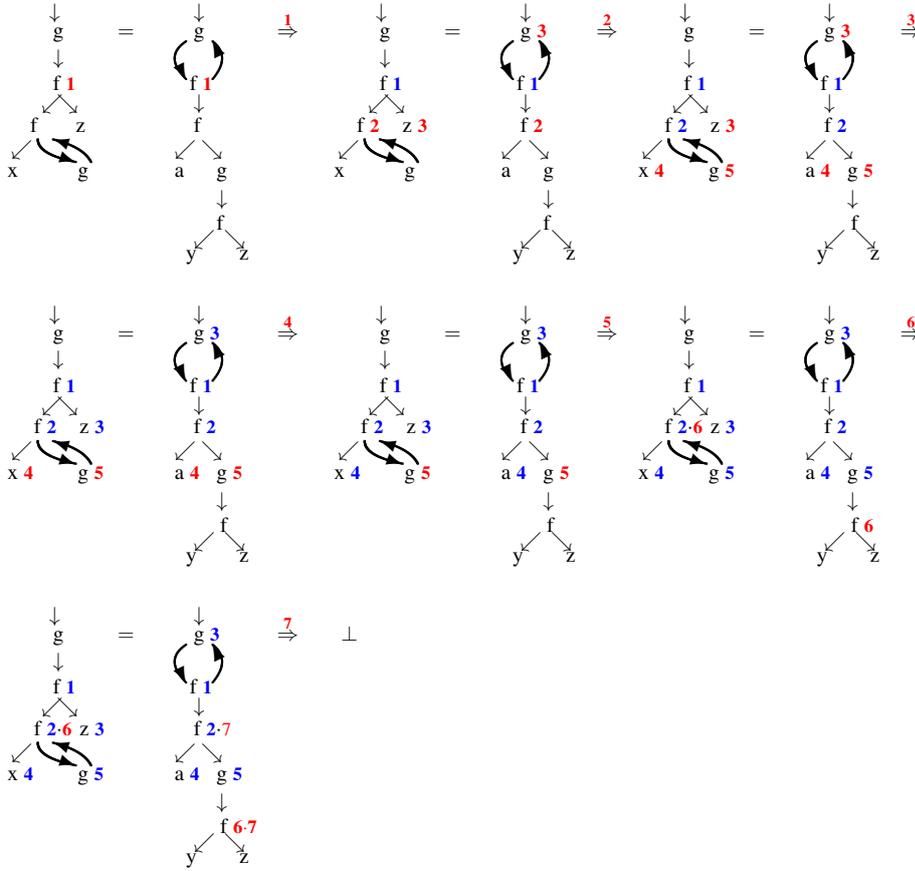


Fig. 7 Unification of two patterns (failure case).

2. $w \in \mathcal{Ver}(U), w' \in \mathcal{S}(U)$: then $w =_S w'$ iff $\xi^*(w') = \xi^*(w)$;
3. $w \in \mathcal{Ver}(V), w' \in \mathcal{S}(V)$: then $w =_S w'$ iff $\zeta^*(w') = \zeta^*(w)$.

Unification equivalence $=_{unif}$ is now defined as the intersection $\bigcap_{S \in \text{Sol}(U[\bar{u}] = V[\bar{v}])} =_S$.

Note that, if w, w' are both sprouts of U (or of V) holding the same variable, then they must be sent to the same vertex by ξ (or by ζ), hence they are equivalent.

Lemma 9 (Unification congruence) *Given a solution S of a unification problem $U[\bar{u}] = V[\bar{v}]$, $=_S$ is a congruence on the vertices of $U \uplus V$ such that $\forall i \in [1..|\bar{u}|]: u_i =_S v_i$, and $=_{unif}$ is a congruence on those vertices satisfying the same property.*

Proof First, $=_S$ is an equivalence generated by a solution, it is therefore a congruence.

Further, $\xi^*(u) = u$ and $\zeta^*(v) = v$ since u, v are internal, and $v_i = t(u_i)$ since S is a solution. It follows from its definition that \equiv_{unif} is a congruence satisfying the same property. \square

We now define the congruence computed by the unification rules:

Definition 15 Given a marked unification problem $U \oplus_k V$, we denote by \equiv_k the binary relation on the vertices of the drag $U \oplus V$ generated by all pairs of vertices that share a common mark. When unification succeeds, we define *marking equivalence* \equiv_{unif} as $\bigcup_k \equiv_k$.

The rules show that vertices of $U[\bar{u}] \oplus_k V[\bar{v}]$ are marked by natural numbers in $[1..k]$. Since the unification rules never remove markings, \equiv_k is monotonically increasing with k :

Lemma 10 $\equiv_k \subseteq \equiv_l$ for all $l \geq k$ such that \equiv_l is defined.

It follows that \equiv_{unif} coincides with \equiv_n defined by $U \oplus_n V$, the obtained normal form of $U \oplus_0 V$ at step n . We believe that this normal form is unique, a property not needed here.

Lemma 11 (Marking congruence) \equiv_{unif} is a congruence on the vertices of $U \oplus V$ generated by (\bar{u}, \bar{v}) .

Proof By definition, \equiv_k is symmetric, hence is an equivalence thanks to *Transitivity*. Hence \equiv is an equivalence. Since unification has terminated with success, *Propagation* and Lemma 10 ensures the first two properties of a congruence, and *Merge* and Lemma 10 the third. Finally, *Initialization* and Lemma 10 ensure the last required property. \square

It should by now be clear that, although they are defined quite differently, \equiv_{unif} and \equiv_{unif} are nevertheless the same congruence on the vertices of $U \oplus V$. The proof of this key property is the matter of the next two sections.

3.4 Soundness of the unification rules

Soundness is the property that the marking algorithm succeeds iff the unification problem is solvable. We split the soundness lemma into two, depending whether the unification problem is solvable or unsolvable.

Lemma 12 Let $U[\bar{u}] = V[\bar{v}]$ be a solvable unification problem, and $S = (\langle C, \xi \rangle, \langle D, \zeta \rangle)$ a solution such that $C \otimes_\xi U[\bar{u}] =^t D \otimes_\zeta V[\bar{v}]$ for some map t . Then, $\forall k : \equiv_k \subseteq =_S$.

Proof By induction on k .

- $u \equiv_0 v$. Then, u, v are two partner vertices, hence $u =_S v$ by Lemma 11.
- $u \equiv_{k+1} v$. If $u \equiv_k v$, we conclude by induction on k . Otherwise, $U \oplus_{k+1} V[u \cdot c + 1][v \cdot c + 1]$. If $u = v$, the result holds since $=_S$ is an equivalence. Otherwise, there are three cases (*Variable case* implies $u \equiv_k v$), depending on the rule used to generate the marked pair:
 - *Propagate*: there exist internal vertices u', v' and $i \leq k$ such that $U \oplus_k V[u' : f \cdot i][v' : f \cdot i]$, and u, v are corresponding successors of u', v' in $U \oplus V$. By induction hypothesis, $u' =_S v'$. Hence $u =_S v$ since $=_S$ is a congruence by Lemma 11.
 - *Merge*: u, v are different sprouts such that $U \oplus_k V[u : x][t : x]$. Then, U, V don't share vertices nor variables, u, v must belong both to either U or V . Then $u =_S v$ by definition of a switchboard.
 - *Transitivity*: u, v are different vertices such that $U \oplus_k V[w \cdot i \cdot j][u \cdot i][v \cdot j]$. By induction hypothesis, $w =_S u$ and $w =_S v$. Hence $u =_S v$ since $=_S$ is an equivalence. \square

Lemma 13 Assume that $\equiv_{k+1} = \perp$ for some k . Then $U[\bar{u}] = V[\bar{v}]$ is an unsolvable unification problem.

Proof Assume by contradiction that the problem is solvable. By Lemma 12, any solution of the unification problem is a solution at step k . We discuss the three possible cases of failure in turn, showing that each one leads to a contradiction, that is, there is no solution at step k :

1. *Symbol conflict*: because $=_S$ is a congruence.
2. *Internal conflict*: because composition cannot identify different internal vertices.
3. *Lack of root*: assume there exists C, D, ξ, ζ such that $C \otimes_\xi U$ and $D \otimes_\xi V$ are identified at step k . Wlog, we assume that u is a vertex of U . There are two cases, one for each failure condition.

Assume there is a sprout s in U sharing a mark with u , hence $\xi^*(u) = u$ and $\xi^*(s)$ are the same vertex in $C \otimes_\xi U$. Then $\xi^*(s) = u$, requiring that u is rooted, a contradiction.

Assume that there are vertices v, w in V such that v shares a mark with u and w is a predecessor of v which is singular. Then, $\xi^*(v) = \xi^*(u)$, hence there would be an edge in $C \otimes_\xi U$ mimicking the edge from $\zeta^*(w) = w$ to $\zeta^*(u)$. Since w is singular in $U \oplus V$, the vertex corresponding to w in $C \otimes_\xi U$ must be a vertex from C . Therefore, u has one more incoming edge in $C \otimes_\xi U$ than in U , which is impossible since u is rootless. \square

3.5 Solved forms

Completeness is the converse property. Instead of proving it directly, we show first here that the unification rules return an equivalence in *solved form* from which most general unifying extensions can be constructed in Section 3.6

Definition 16 (Solved form) Given a solvable unification problem $U[\bar{u}] = V[\bar{v}]$, a congruence \equiv on $U \oplus V$ containing the partner vertices (\bar{u}, \bar{v}) is in *solved form* iff

1. no two internal vertices of U or of V are equivalent
2. An internal vertex u must be rooted in the following cases:
 - (a) there exists a sprout s equivalent to u in the same drag;
 - (b) u is equivalent to a vertex v of the other drag whose one predecessor is singular.

Note that the conditions for a congruence containing the partner vertices to be in solved form are directly inherited from the application conditions of the failure rules.

Non-trivial equivalence classes of a congruence in solved form have a specific structure:

Lemma 14 *Given two patterns U, V , let \equiv be a congruence in solved form on the vertices of $U \oplus V$ that contains the partner vertices (\bar{u}, \bar{v}) . Then a non-trivial equivalence class is a set of vertices $S = \{u_i\}_{i=1}^{i=m} \cup \{v_j\}_{j=1}^{j=n}$ such that $m, n \geq 1$ and $\forall i < m \forall j < n$, u_i and v_j are sprouts.*

Proof Follows directly from Definition 16. \square

The equivalence classes of a congruence in solved form can therefore contain any number of variables, but at most one internal vertex from each drag U, V .

We now show that the unification rules deliver solved forms:

Lemma 15 *Assuming the unification problem $U[\bar{u}] = V[\bar{v}]$ does not fail, the equivalence \equiv_{unif} defined on the vertices of $U \oplus V$ by a marked unification problem in normal form is the least congruence in solved form generated by (\bar{u}, \bar{v}) .*

Proof By Lemma 8, \equiv_{unif} is well defined, and by lemma 11, it is a congruence generated by (\bar{u}, \bar{v}) . It is the least such one generated by these partner vertices, since it contains them and any congruence is closed under *Propagate*, *Transitivity*, and *Merge*.

We are left showing that a failure rule applies to unification problems which are not in solved form, contradicting the assumption of a successful unification.

1. Let $U \oplus V[u][v]$ such that $u \equiv_{unif} v$, $u, v \in \mathcal{Snt}(U)$ and $u \neq v$. By definition of \equiv_{unif} , there exists some $k > 0$ such that $U \oplus_k V[u \cdot i][v \cdot i]$ for some i , hence *Internal Conflict* applies at all steps from $k + 1$.
2. Assume the equivalence class of u contains a single other vertex $v \in \mathcal{Snt}(V)$ which is rooted. By definition of \equiv_{unif} , there exists $k > 0$ such that $U \oplus_k V[u \cdot i][v \cdot i]$ for some i . By assumption, the class of v contains the same two elements at all steps $k' \geq k$. Hence *Root conflict* applies to the result of unification.
3. Let $u \in \mathcal{Snt}(U)$ be rootless. We proceed by contradiction, showing in both sub-cases that a failure rule applies to the result of unification, contradicting the assumption of success. Assume there exists a vertex v equivalent to u which is either a sprout of U or the successor in V of a vertex w which is single. By definition of \equiv_{unif} , there exists some $k > 0$ such that $U \oplus_k V[u \cdot i][v \cdot i]$ for some i , hence *Lack of root* applies at step k . Since $U \oplus V$ remains unchanged during the monotone marking process, *Lack of root* applies at all steps $k' \geq k$, hence to the result of unification. \square

3.6 Construction of the most general unifying extensions

We now show that a solved form is always solvable, hence their name. Here, the input is a congruence in solved form, which can be seen as a specific unification problem. We therefore construct a most general unifying extension for that solved form.

Definition 17 [mgu] Given a unification problem $U[\bar{u}] = V[\bar{v}]$ and an equivalence \equiv on the vertices of $U \oplus V$ which is in solved form, we define the unifying extensions $\langle C, \xi \rangle$ of U and $\langle D, \zeta \rangle$ of V , as well as the renaming $\iota : C \otimes_{\xi} U \rightarrow D \otimes_{\zeta} V$ such that $C \otimes_{\xi} C =^t D \otimes_{\zeta} V$.

Let $S = \{u, s_1, \dots, s_{m_S}; t_1, \dots, t_{n_S}, v\}$ be an equivalence class containing internal vertices u from U and v from V that are possibly absent, $m_S \geq 0$ sprouts $\{s_i\}_i$ originating from U and $n_S \geq 0$ sprouts $\{t_j\}_j$ originating from V . The construction is by case on the form of S .

At step 1, we set up an infrastructure of fresh sprouts in C and D that will serve connecting C, D with U, V and ensuring that each vertex in the composition has zero or one root. At step 2, we define the mapping ι . At step 3, we define the successor functions for C and D .

1. For each rooted internal vertex u in U (respectively, $v \in V$) belonging to some class S , we include a fresh sprout $s_S : x_S$ in C , (resp. $t_S : y_S$ in D), with $i + m_S$ roots (resp., $i + n_S$ roots), where i is the number of roots of the internal vertex v from V (resp., u from U) belonging to S if there is one, otherwise 1.
For each rooted sprout r in U (resp., in V), we include a fresh rootless sprout s_r in C (resp., t_r in D).
Define $\xi_C(s_S) = u$, $\xi_C(s_r) = r$, and for each sprout s_i in S , $\xi_U(s_i) = s_S$ (resp., $\xi_D(t_S) = v$, $\xi_D(t_r) = r$, and for each sprout t_j in S , $\xi_V(t_j) = t_S$).
% the root of s will disappear in the composition, while the edges of C ending in s_S will then end up in u .
2. - For each class S containing internal vertices u, v from U, V respectively, define $\iota(u) = v$.
- For each class S containing a single internal vertex u from U (resp., v from V), include in D (resp., C) a fresh internal vertex u_S equipped with $n_S + r$ roots, where r is the number

- of roots, zero or one, of u , with $\mathcal{L}_D(u_S) = \mathcal{L}_U(u)$ (resp., v_S equipped with $m_S + r'$ roots, r' is the number of roots of v , and $\mathcal{L}_C(v_S) = \mathcal{L}_V(v)$); define $\iota(u) = u_S$ (resp., $\iota(v_S) = v$), and for each sprout t_j in S , $\zeta_V(t_j) = u_S$ (resp., for each sprout s_i in S , $\xi_U(s_i) = v_S$);
- For each class S containing no internal vertex, include two sprouts s_S in C and t_S in D both labelled x_S , equipped with $1 + m_S$ and $1 + n_S$ roots respectively. Define $\iota(s_S) = t_S$, and for each sprout s_i in S , $\xi_U(s_i) = s_S$ (for each sprout t_j in S , $\zeta_V(t_j) = t_S$).
3. For each internal vertex v_S in C (resp., u_S in D) associated with the class S , let $X_V(\iota(v_S)) = \langle v_1 \dots v_k \rangle$ (resp., $X_U(\iota(u_S)) = \langle u_1 \dots u_k \rangle$). We define $X_C(v_S)$ (resp. $X_D(u_S)$) as the tuple $\langle w_1 \dots w_k \rangle$, where, denoting by S_i the class of w_i :
- $w_i = \iota^{-1}(v_i)$ if $\iota^{-1}(v_i)$ is in C , (resp., $w_i = \iota(u_i)$ if $\iota(u_i)$ is in D),
 - $w_i = s_{S_i}$ (resp., $w_i = t_{S_i}$), if $\iota^{-1}(v_i)$ is in U (resp. $\iota(u_i)$ is in V).

Before to show that we have defined a solution, we develop two examples. In both cases, the given congruence in solved form is obtained from the marking congruence resulting from applying the unification algorithm. The solution obtained is therefore the most general one for the starting unification problem, not only for the solved form.

Example 8 Figure 8 shows the two marked drags obtained by our unification algorithm at Figure 6, as well as the context drag, switchboard, and overlapping drags obtained by composition with the two marked drags.

The equivalence in solved form on vertices has 5 classes, whose elements are given by their position in the drags (using for that purpose words over the natural numbers with Λ for the empty word and "." for concatenation, a notation named after Dewey) for both the left-hand and right-hand side trees:

$$(\Lambda, 1 \cdot 2; \Lambda), (1; 1), (1 \cdot 1; 1 \cdot 1 \cdot 2 \cdot 1, 1 \cdot 1), (1 \cdot 1 \cdot 1; 1 \cdot 1 \cdot 1), (1 \cdot 1 \cdot 2; 1 \cdot 1 \cdot 2)$$

For example, the first class contains the vertices sharing the marks **1, 4, 8**, that is the internal vertices labeled by g heading both drags, both at position Λ , and the sprout labeled by z , at position $1 \cdot 2$ of the left-hand side drag, it therefore appears before the semi-column. We use for class names their position in the above list of classes.

Steps one and two define the extension drags C, D , containing fresh sprouts (named here by their variable label x or y , lower-indexed by their class name and upper-indexed by their number of roots) and one internal vertex v_5 mirror of the internal vertex labeled a of its class. The fresh sprout y_1 associated with vertex Λ of the right-hand side drag has one root, like the other side's vertex. But x_1 has two roots, since its class has one variable on its side, and one rooted internal vertex on the other side. Likewise, y_5 has one root since there is one sprout y on its side, and one rootless internal vertex on the other side. The mirror vertex v_5 of a is rooted since there is one sprout in the class on its side:

$$\begin{aligned} C &= x_1^2 \oplus v_5 = a^1 & \xi &= \{x_1 \mapsto \Lambda, x \mapsto v_5, z \mapsto x_1\} \\ D &= y_1^1 & \zeta &= \{y_1 \mapsto \Lambda\} \end{aligned}$$

□

Example 9 Figure 9 shows how cycles may result from unifying non-linear drags. The congruence obtained by unifying the input drags at the pair of roots labelled by f has 3 classes:

$$(\Lambda; \Lambda), (1, 2 \cdot 1; 1), (2; 1 \cdot 1, 2)$$

At step one, four new sprouts are added corresponding to the rooted vertices, two in the first class (we name them here by their variable), x_1^1 and y_1^1 , one in the second class, x_2^3

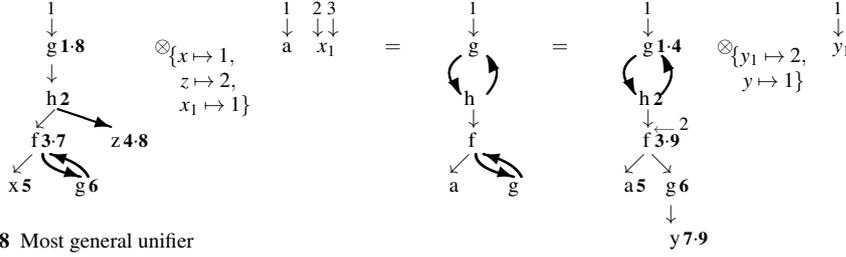
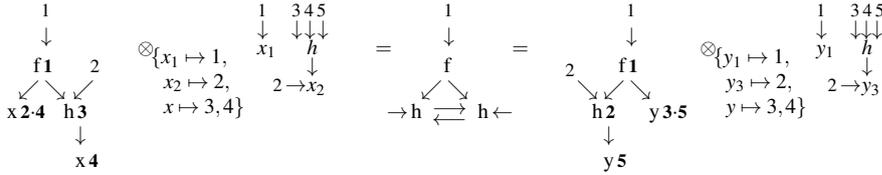


Fig. 8 Most general unifier



Note that the lefthand side h vertex of the unified drag is the mirror of the h vertex of the righthand side input drag in the left overlap, while the righthand side h vertex is the mirror vertex of the h vertex of the lefthand side input drag in the right overlap. So, both overlaps are not really identical as drags, also their drawing is the same.

Fig. 9 Building a most general unifier from a solved congruence

and one in the third class, y_3^3 , having 3 roots each. Accordingly, $\xi_C(x_1) = 1$, $\xi_C(x_2) = 2$ and $\zeta_D(y_1) = 1$, $\zeta_D(y_3) = 2$, map these sprouts to the corresponding internal vertex of the class, the root absorbed by the associated composition being recreated by transfer. Further, both x 's are sent to two roots of x_2 by ξ while both y 's are sent to two roots of y_3 by ζ . Each vertex labelled h in the drags will therefore keep its root after composition.

At step two, two mirror internal vertices are created, v_2 for C and u_3 for D that mimic the internal vertices of classes 2 and 3, that belong to the second and first drag respectively, and are both labelled by h . Each of these mirror vertices is equipped with one root, as is the internal vertex it mimics.

At step 3, edges are created in C and D for the mirror vertices v_2 and u_3 , that end up in the created sprouts of their respective classes, x_2 and y_3 . That explains the form of the subdrags headed by h in C and D that have both 4 roots.

$$\begin{aligned} C &= x_1^1 \oplus x_2^1 \oplus v_2^3 & \xi &= \{x_1 \mapsto 1, x_2 \mapsto 2, x \mapsto v_2\} \\ D &= y_1^1 \oplus y_3^1 \oplus u_3^3 & \zeta &= \{y_1 \mapsto 1, y_3 \mapsto 2, y \mapsto u_3\} \end{aligned}$$

□

We now prove that $\langle C, \xi \rangle$ and $\langle D, \zeta \rangle$ are rewriting extensions and a solution of the given unification problem.

Lemma 16 *Let \equiv be an equivalence in solved form for the unification problem $U[\bar{u}] = V[\bar{v}]$. Then, the most general unifying extensions $\langle C, \xi \rangle$ and $\langle D, \zeta \rangle$ is a solution of the unification problem.*

Proof We show first that $\langle C, \xi \rangle$ and $\langle D, \zeta \rangle$ are rewriting extensions. We carry our the proof for $\langle C, \xi \rangle$, the other being similar.

- First, the switchboard ξ is clearly well-defined.
- Totality of ξ_U : each sprout s of U belongs to some class S , hence is mapped by ξ to s_S .
- Surjectivity and linearity of ξ_C : by construction, for every internal rooted vertex, say u , belonging to a class S , there exists s_S in C such that $\xi(s_S) = u$. And for every rooted sprout r in U , there is a sprout s_r in C such that $\xi(s_r) = r$. Finally, C is linear by construction.
- Cleanliness: let u be a vertex in $C \otimes_\xi U$. By totality, it can't be a sprout of U .
If u is an internal vertex of C , then none of its ancestors can be vertices of U , hence all are mirror vertices of vertices in V with the same number of roots by construction, which ensures that u is accessible in $C \otimes_\xi U$.
If u is an internal vertex of U , then u is accessible in U , hence in $C \otimes U$ from some vertex u' which is rooted in U . If u' is rooted in $\otimes_\xi U$, we are done. Otherwise, u' must have a predecessor in C , which is accessible, as we have already proved.
If u is a sprout of C , then u is a fresh sprout s_S , the equivalence class S being a set of sprouts of U, V . In that case, depending whether all sprouts are on the V side or not, u has an ancestor in either U or C , which must be accessible by the two previous cases, hence u is accessible.

Let us show that $U' = C \otimes_\xi U = {}^t D \otimes_\zeta V = V'$:

- By definition, $\mathcal{I}nt(C \otimes_\xi U) = \mathcal{I}nt(U) \cup \mathcal{I}nt(C) = \{u \in \mathcal{I}nt(U) \mid u \text{ is not singular}\} \cup \{u \in \mathcal{I}nt(U) \mid u \text{ is singular}\} \cup \{v_S \mid v \in \mathcal{I}nt(V) \text{ and } v \text{ is singular}\}$. Then, $\iota(\mathcal{I}nt(C \otimes_\xi U)) = \iota(\{u \in \mathcal{I}nt(U) \mid u \text{ is not singular}\}) \cup \iota(\{u \in \mathcal{I}nt(U) \mid u \text{ is singular}\}) \cup \iota(\{v_S \mid v \in \mathcal{I}nt(V) \text{ and } v \text{ is singular}\}) = \mathcal{I}nt(D \otimes_\zeta V)$;
- By definition, $\mathcal{S}(C \otimes_\xi U) = \{s_S \mid S \cap (\mathcal{I}nt(U) \cup \mathcal{I}nt(V)) = \emptyset\}$. Then, $\iota(\mathcal{S}(C \otimes_\xi U)) = \iota(\{s_S \mid S \cap (\mathcal{I}nt(U) \cup \mathcal{I}nt(V)) = \emptyset\}) = \mathcal{S}(D \otimes_\zeta V)$
- By definition the labels of internal vertices in $C \otimes_\xi U$ and $D \otimes_\zeta V$ coincide. Moreover, for each pair of sprouts s_S and t_S , their label is x_S in both drags (having x_S and $y_S \neq x_S$ instead would require an additional variable renaming to show that both drags are equal).
- For each $w \in \mathcal{I}nt(C \otimes_\xi U)$, if $X_{C \otimes_\xi U}(w) = \langle w_1 \dots w_k \rangle$, then $X_{D \otimes_\zeta V}(\iota(w)) = \langle \iota(w_1) \dots \iota(w_k) \rangle$, since \equiv is a congruence.
- Finally, the number of roots at each vertex u in $C \otimes_\xi U$ is equal to the number of roots at $\iota(u)$ in $D \otimes_\zeta V$: If s_S is a sprout in $C \otimes_\xi U$, then $\iota(s_S) = t_S$ and both sprouts have one root in $C \otimes_\xi U$ and $D \otimes_\zeta V$, by definition. If u is an internal vertex in U , whose equivalence class S does not include any internal vertex, by definition, u and $\iota(u)$ will have the same number of roots in $C \otimes_\xi U$ and $D \otimes_\zeta V$. Finally, if S includes internal vertices u, v , with $\iota(u) = v$, then, if both u and v are rooted in U and in V then, by definition, u and v will be rooted in $C \otimes_\xi U$ and $D \otimes_\zeta V$. But if one of the vertices u or v is unrooted in U or V , respectively then, both vertices will be unrooted in $C \otimes_\xi U$ and $D \otimes_\zeta V$. \square

As an important consequence, we have

Corollary 1 *Unification congruence \equiv_{unif} and marking congruence \equiv_{unif} coincide.*

Proof Let $U[\bar{u}] = V[\bar{v}]$ be a unification problem. The result is clear if it is unsolvable. Otherwise, let \equiv be a congruence in solved form for that problem. By Lemma 16, \equiv is the equivalence associated with the unifying extensions introduced at Definition 17, hence is coarser than \equiv_{unif} which is the intersection of all equivalences associated with the solutions of a given unification problem. Now, it is easy to see that \equiv_{unif} is itself a congruence in solved form, hence is coarser than \equiv by Lemma 15. \square

3.7 Completeness of the unification algorithm

Theorem 1 *Let \equiv be the equivalence returned by the unification algorithm for the problem $U[\bar{u}] = V[\bar{v}]$ when no failure occurs. Then, $\text{mgu}(U \oplus V, \equiv)$ is a most general unifier.*

Proof By Lemma 16, $(\langle C, \xi \rangle, \langle D, \zeta \rangle) = \text{mgu}(U[\bar{u}] \oplus V[\bar{v}], \equiv)$ is a unifier of the equation $U[\bar{u}] = V[\bar{v}]$. We are therefore left proving it is most general.

Let now S be a solution of $U[\bar{u}] = V[\bar{v}]$. By Corollary 1, S defines a unification equivalence whose classes must be (appropriate) unions of classes of \equiv . It therefore suffices to show that merging classes of a congruence in solved form increases (wrt subsumption) the associated solution constructed at Lemma 16.

We describe the situation when merging two mergeable classes S and S' . There are four cases up to symmetry depending upon the existence of internal vertices in these classes, since there can only be at most two internal vertices on opposite sides in a merged class. We carry out one case, with two internal vertices in S and none in S' , the others being similar.

Let therefore $S = \{u, s_1, \dots, s_m, t_1, \dots, t_n, v\}$ and $S' = \{s'_1, \dots, s'_m, t'_1, \dots, t'_n\}$. Lemma 16 creates the sprouts $s_S, t_S, s_{S'}, t_{S'}$, as well as additional rootless sprouts to annihilate the roots of the rooted sprouts of S and S' . After merging the two classes, Lemma 16 uses new sprouts s_m and t_m , whose number of roots is just the sum of those for s_S and $s_{S'}$, while the additional sprouts can remain the same. The latter solution is therefore an instance of the previous one via the following extension: two new sprouts s, s' equipped with two roots each. Then the extension's switchboard can map s_S and $s_{S'}$ to s_m and t_S and $t_{S'}$ to t_m , this will give a solution for the merged class.

Merging more than two classes obeys the same mechanism. Further, Lemma 2 ensures that merges can be done one by one. \square

3.8 Most general unifiers

We can therefore end up this section with our first main result:

Theorem 2 *Unification is unitary, and has quadratic worst case time and space complexity.*

Proof The first part of the claim follows from Lemma 12 and Theorem 1. For the second part, Lemma 8 shows that a solved form is obtained in quadratic time and space. Further, the solved form itself has a linear size in terms of the input unification problem, since the algorithm never generates new terms. As the construction of the most general unifier from the solved form is clearly done in linear time in terms of its size, hence in terms of the size of the input problem, the whole process takes at most quadratic time and space. \square

The question therefore arises whether these complexity bounds are sharp. A long way was necessary to go from the early unification algorithm by Robinson, which was exponential, to the linear time unification algorithm of Patterson and Wegman. It is indeed easy to go from quadratic to linear space by changing our marking technique, since a single number per vertex is enough for memorizing an equivalence class, instead of a list of numbers of quadratic length. Going from quadratic to sub-quadratic time complexity should be harder, and might require a smart strategy for applying the unification rules, as is the case for terms [23]. Whether the ultimate algorithm should be linear or not is an open problem for which an early guess would be premature.

4 Confluence

Local confluence of a term rewriting system follows from the joinability of its critical pairs, obtained by unifying overlapping left-hand sides of rules [22]. Our goal is to generalize this result to the drag framework.

To simplify the development, we assume throughout this section that left-hand sides of rules are connected drags that have been renamed apart. The general case of non-connected drags can be carried out to the price of slightly more involved statements and technicalities. We leave it as an exercise for the reader.

Definition 18 (Local peaks) A *local peak* is a pair of rewrites $S \leftarrow_{L \rightarrow R}^{\mu} U \rightarrow_{G \rightarrow D}^{\nu} T$ originating from a given drag U . A local peak is *critical* if there exist internal vertices u of L and v of G such that $\mu(u) = \nu(v) = w$, in which case we say that L and G *share* the vertex w . A local peak is *disjoint* otherwise.

In the case of terms, non-critical peaks are either disjoint or ancestors, depending on their relative positions, and both join. Here, there is no distinction between non-critical peaks, all are disjoint, a definition which covers both kinds of peaks for terms. This is the benefit of our definition of composition of drags, which subsumes both operations of context application and substitution for terms. Further, the coming commutation proof happens to be simpler, since non-linearities of variables are taken care of by the composition mechanism. It is just like the case of disjoint redexes for terms, the reason why we retained that name.

Lemma 17 (Commutation) Let $S \leftarrow_{L \rightarrow R}^{\mu} U \rightarrow_{G \rightarrow D}^{\nu} T$ be a disjoint local peak. Then, $S \rightarrow_{G \rightarrow D} V \leftarrow_{L \rightarrow R} T$ for some drag V .

Proof Without loss of generality, we assume that By definition of rewriting, there exist renamings $L' \rightarrow R'$ and $G' \rightarrow D'$ of respectively $L \rightarrow R$ and $G \rightarrow D$, and rewriting extensions $\langle C, \xi \rangle$ and $\langle D, \zeta \rangle$ of L' and G' respectively such that $C \otimes_{\xi} L' = U = D \otimes_{\zeta} G'$.

By assumption on the local peak, there exist no pair of vertices $u \in \mathcal{I}nt(L), v \in \mathcal{I}nt(G)$ such that $\mu(u) = w = \nu(v)$. It follows that $\mathcal{V}er(G') \subseteq \mathcal{V}er(C)$, and therefore, there exists an injection $\nu' : G' \rightarrow C$. By Lemma 4, there exists an extension $\langle E, \theta \rangle$ of G' such that $C = E \otimes_{\theta} G'$, hence $U = (E \otimes_{\theta} G') \otimes_{\xi} L'$. By Lemma 2, $U = E \otimes_{\gamma} (G' \otimes_{\eta} L')$, hence $S = E \otimes_{\gamma} (G' \otimes_{\eta} R')$ and $T = E \otimes_{\gamma} (D' \otimes_{\eta} L')$, since the result of rewriting at some position is unique, as a consequence of Lemma 4.

Then $V = E \otimes_{\gamma} (D' \otimes_{\eta} R')$ satisfies the claim thanks to Lemma 6. \square

Lemma 18 (Critical Peak) Let $S \leftarrow_{L \rightarrow R}^{\mu} U \rightarrow_{G \rightarrow D}^{\nu} T$ be a critical peak. Then, there exist $\bar{u} \in \mathcal{V}er(L)$ and $\bar{v} \in \mathcal{V}er(G)$, and two extensions $\langle C, \xi \rangle$ of L and $\langle D, \zeta \rangle$ of G such that $C \otimes_{\xi} L = \iota D \otimes_{\zeta} G$ and $\bar{u} = \iota(\bar{v})$.

In words, the extensions $\langle C, \xi \rangle$ and $\langle D, \zeta \rangle$ form a solution of the unification problem $U[\bar{u}] = V[\bar{v}]$.

Proof Let A be the subset of internal vertices of U belonging to both $o(\mathcal{I}nt(L))$ and $o'(\mathcal{I}nt(G))$. By assumption, $A \neq \emptyset$, implying that L and G overlap. The core of the proof is the definition of two lists \bar{v}, \bar{w} of partner vertices of L, G which generate A , that is, all vertices of A are accessible from both \bar{v} in L and \bar{w} in G . As partner vertices, v_i and w_i must coincide, that is, be mapped by o and o' to the same internal vertices in A .

Let us denote vertices of A by u, v and w according to their origin (via o and o'), in U, L and G respectively.

Because left-hand sides of rules are clean drags, all vertices $u \in A$ are accessible from rooted vertices of both L and G via o and o' , let r such a vertex in L . For each u consider the first vertex $v' \in A$ on a path from $o(r)$ to u , and let $v = o^{-1}(v')$. Since $v' \in A$, then $v' = o'(w)$ for some internal vertex W of G . The pair (v, w) is therefore a pair of partner vertices generating A via o and o' . Eliminating redundancies (with respect to accessibility) from all such pairs yields two lists of vertices that satisfy the conditions for being partner vertices generating A .

We now show that U is an overlap of the equation $L[\bar{v}] = G[\bar{w}]$. Since L and G match U , there exist renamings L' and G' of L and G respectively such that $U = L' \otimes_{\gamma} C = G' \otimes_{\delta} D$ for some rewriting extensions $\langle C, \gamma \rangle$ of L' and $\langle D, \delta \rangle$ of G' . Therefore $L \otimes_{\gamma} C = \iota G \otimes_{\delta} D$ for some renaming ι . We are left showing that $w = \iota(v)$ for each pair of partner vertices, which follows from the fact that both are sent to the same vertex of U by o and o' .

Hence L and G are unifiable at partner vertices (\bar{v}, \bar{w}) . \square

Definition 19 (Critical pair) Let $L \rightarrow R$ and $G \rightarrow D$ be two rules whose left-hand sides L and G are unifiable at partner vertices \bar{v}, \bar{w} , with most general overlap $C \otimes_{\xi} L' = C' \otimes_{\xi'} G'$, for some renamings $L' \rightarrow R'$ of $L \rightarrow R$ and $G' \rightarrow D'$ of $G \rightarrow D$. Then, $\langle C \otimes_{\xi} R', C' \otimes_{\xi'} D' \rangle$ is called a *critical pair* of $L \rightarrow R$ with $G \rightarrow D$ at partner vertices \bar{v}, \bar{w} .

Given a drag rewriting system, how many critical pairs can be generated? Their number is indeed bounded by the potential choices for partner vertices, which must satisfy the constraints stated in the proof of Lemma 18, and resist the inequality test in *Symbol conflict*. In practice, this number should remain small, as is the case for terms.

We can now end up with our second main result:

Theorem 3 Let \mathcal{R} be a rewrite system on drags. Then, \mathcal{R} is locally confluent iff all its critical pairs are joinable.

Proof Let $S \xleftarrow{L \rightarrow R} U \xrightarrow{G \rightarrow D} T$, L and G being renamed apart. There are two cases:

1. G has no vertex in common with L . We then conclude by Lemma 17.
2. G has a vertex in common with L . By Lemma 18, there is a critical pair. By property of most general unifiers and Lemma 6, the joinability of the critical pair can be lifted to S and T . \square

As a direct consequence, a terminating drag rewrite system is confluent iff its critical pairs are joinable. If the rewrite system is non-terminating, confluence can still be achieved by using Van Oostrom's decreasing diagrams technique. Developing confluence criteria along these lines goes beyond the objectives of this paper. The case of terms has been thoroughly investigated, see [14, 15, 26]. We believe that similar investigations can be carried out for drags.

A particular case worth mentioning, as suggested to us by Jan-Willem Klop, is orthogonality. Orthogonal term rewriting systems are confluent, whether terminating or not. It so happens with drag rewriting systems, with the exact same definition of orthogonality:

Definition 20 A drag rewriting system is said to be *orthogonal* if it is critical pair free.

Note that left-linearity is not needed here: a non-left linear rule and the linear rule obtained by sharing all sprouts labeled by the same variable define the same rewriting relation on terms. Hence our definition of drag rewriting is inherently linear, as we have remarked already.

Theorem 4 *Orthogonal drag rewriting systems are confluent.*

Proof Lemma 17 shows that rewriting has the so-called diamond under the assumption of orthogonality, hence can be shown confluent by the standard pasting technique.

We are currently developing a new version of drags for which linearity is not built in the definition of composition as it is here. This new model would require the assumption of left-linearity for orthogonal systems.

5 Related work

The first, dominant model for graph rewriting was introduced in the mid-seventies by Hartmut Ehrig and his collaborators [13]. Referred to as DPO (Double Push-Out), this purely categorical model was then extended in various ways, but also specialized to specific classes of graphs, namely those that do not admit cycles [33]. In particular, termination and confluence techniques have been elaborated for various generalization of trees, such as rational trees, directed acyclic graphs, jungles, term-graphs, lambda-graphs, as well as for graphs in general. See [18] for a survey of various forms of graph rewriting and of available analysis techniques.

DPO applies to any category of graphs that has pushouts and unique pushout complements [11]. A rule is a span $L \leftarrow I \rightarrow R$, where I is the *interface* specifying which elements (vertices and edges) from the input graph G matching the left-hand side L by an injective morphism m are preserved by the transformation, the elements in $m(L \setminus I)$ being removed from G while the elements in $R \setminus I$ are added to G . The term DPO refers to the two pushouts generated by the span that define the result of a rewrite step. DPO suffers two drawbacks: applying a rewriting rule fails in case it results in dangling edges, and rules do not have variables.

Categorical approaches are very general, they do apply to many different kinds of graph structures. Besides DPO, the most popular one, they include many variations: matching by a non-injective morphism [11], arbitrary adhesive graph categories [11], single pushout transformation (SPO [12, 34]), or Sesqui-Pushout transformation (SqPO [5]), AGREE [4], and *Hyperedge Replacement Systems* [10]. A detailed comparison of the approach based on drags with all these approaches is not obvious and will be carried out in another paper [9].

DRAG rewriting aims instead at providing a faithful generalization of term rewriting techniques to a certain class of graphs named drags by generalizing to drags all constructions underlying term rewriting, i.e., subterm, substitution, matching, replacement and unification. This is done *constructively* by providing a composition operator for drags which does not pop up in the other, purely descriptive approaches, which aim at describing abstractly subgraph replacement. As a consequence, for a long time neither graphs nor rules included variables that can be substituted in the transformation process. A recent approach that has some similarities with DRAG transformation is *port graph rewriting* [16], where graphs include *ports* and *roles*, which, in a way, play a similar rôle as roots and variables in drags. However, the transformation process remains similar to DPO graph rewriting with interfaces [2].

Since most of these general approaches lack variables, most works that study graph unification concentrate in the specific case of *directed acyclic graphs* (dags) that are used to represent terms with shared subterms (see, e.g., [29]). A preliminary attempt to handling variables in graph unification is [28], where variables are used to represent labels equipping

the vertices or edges. A quite more general approach is [33], where variables represent hyperedges that may be substituted by *pointed* hypergraphs, but unification is solved there for a very restrictive case only. More recently, Hristakiev and Plump consider graph unification for their graph programming language GP2 [20]. Graphs in GP2 are symbolic graphs whose attributes's values are given by variables satisfying some set of constraints [27]. Variables are not substituted by graphs, but by constrained values.

In contrast, drag variables are real variables as in terms, and drag unification is shown here unitary, and decidable in quadratic time and space, a bound which we believe is not tight. It does not only subsume unification of trees, dags and jungles, but also of rational trees, dags or jungles, as we shall discover in the concluding remarks. The complexity analysis exploits the fact that the successors of a vertex are ordered and their number is fixed by the symbol labelling that vertex. Relaxing these constraints would blow up the number of most general unifiers resulting in a non-polynomial complexity of matching and unification.

Confluence of graph transformation systems was first studied by Plump, who defined the notion of graph critical pairs and proved their completeness, but also showed that local confluence is undecidable already for terminating systems [30–32]. He also considered the case of symbolic graphs [19]. A main problem with Plump's notion of critical pair is that, to compute the set of all critical pairs, we need a procedure that is exponential on the size of the lefthand sides of the rules. More precisely, according to Plump's definition, the set of critical pairs of two rules r_1, r_2 consists of all pairs of transformations $H_1 \leftarrow_{r_1} G \rightarrow_{r_2} H_2$ that are *parallel independent* (e.g., see [11]) and such that G is an overlap of L_1 and L_2 . This means that, in principle, to compute all possible critical pairs, we need to compute all possible overlaps of L_1 and L_2 and to check if they are parallel independent. Moreover, even if it is difficult to estimate what is the exact number of critical pairs, since it is difficult to estimate how many of these pairs of transformations will be parallel independent, we know that many of them are useless. Consequently, less prolific notions of critical pairs are introduced in [1, 24, 25]. For instance, [24] includes an example to show how large may be the different number of critical pairs depending on the approach considered. The example considers the definition of finite automata in terms of graph transformation. More precisely, a finite automaton is represented by a graph including: a) the state/transition diagram of the automaton b) a cursor (represented by a loop) on the vertex denoting the current state of the automaton, and c) a queue of symbols representing the word to be recognized. Then, the transformation rules describe how the given automaton works, i.e., when the first symbol in the queue is recognized by the automaton, the movement of the cursor and the deletion of the symbol. In this example, computing the critical pairs of that rule with itself gave the following results: the number of all the overlaps of the left-hand side of the rule with itself was 51,602; the number of critical pairs according to Plumps definition was 21,478; the number of critical pairs computed using the method presented in [25] was 49; finally, the number of critical pairs computed using the method presented in [1, 24] was 7.

Recently, local confluence was shown decidable for terminating graph rewriting with interfaces [2], where an interface is a subset of the indices of the given graph that are used to define an operation of graph composition by connecting the interfaces of the given graphs. Then, rewriting a graph with an interface, according to [2], means rewriting the graph but leaving the interface invariant. For instance, it would not be allowed to apply a rule if, as a consequence, a vertex in the interface would be deleted or if two vertices in the interface would be merged. With respect to confluence, a main difference between standard DPO rewriting and this variation is that, in DPO rewriting, two graphs G_1, G_2 are considered joinable if they can be rewritten into isomorphic graphs H_1, H_2 , respectively, but when the graphs have an interface I it would be required the existence of an isomorphism $h : H_1 \rightarrow$

H_2 such that $h(v) = v$, for every $v \in I$. This difference is the reason why joinability of critical pairs in standard DPO graph transformation does not imply local confluence, while that implication holds for graphs with interfaces, implying the decidability of confluence of terminating DPO rewriting of graphs with interfaces. Let us see an example from [2]:

Consider the following rewrite rules on undirected graphs with labelled edges, where $\leftarrow [a] \rightarrow$ represents an edge labelled by a and vertices are subindexed with numbers 1 and 2 to identify them and make the morphisms explicit:

$$r_1: [\bullet_1 \xrightarrow{a} \bullet_2] \longleftarrow [\bullet_1 \quad \bullet_2] \longrightarrow [\bullet_1 \curvearrowright \quad \bullet_2]$$

$$r_2: [\bullet_1 \xrightarrow{a} \bullet_2] \longleftarrow [\bullet_1 \quad \bullet_2] \longrightarrow [\bullet_1 \quad \bullet_2 \curvearrowright]$$

Then, among the possible critical pairs only the following two have non-trivial overlap:

$$[\bullet_1 \curvearrowright \quad \bullet_2] \longleftarrow [\bullet_1 \xrightarrow{a} \bullet_2] \longrightarrow [\bullet_1 \quad \bullet_2 \curvearrowright] \quad \text{and} \quad [\bullet_1 \curvearrowright] \longleftarrow [\bullet_1 \curvearrowright] \longrightarrow [\bullet_1 \curvearrowright]$$

which are trivially joinable. However, the two rules are not confluent, as we can see below:

$$[\bullet_1 \curvearrowright \quad \bullet_2] \xrightarrow{b} [\bullet_1 \quad \bullet_2] \longleftarrow [\bullet_1 \xrightarrow{a} \bullet_2] \longrightarrow [\bullet_1 \quad \bullet_2 \curvearrowright]$$

Let us see what happens when we work with graphs with interfaces. If we associate an interface, consisting of the two nodes 1 and 2, to the first graph that gave rise to the first critical pair above, we have:

$$\begin{array}{ccccc} & & [\bullet_1 \curvearrowright \quad \bullet_2] & \longleftarrow & [\bullet_1 \xrightarrow{a} \bullet_1] & \longrightarrow & [\bullet_1 \quad \bullet_2 \curvearrowright] \\ & \nearrow & & \nearrow & & \nearrow & \\ [\bullet_1 \quad \bullet_2] & & [\bullet_1 \quad \bullet_2] & & [\bullet_1 \quad \bullet_2] & & \end{array}$$

but $([\bullet_1 \quad \bullet_2] \longrightarrow [\bullet_1 \curvearrowright \quad \bullet_2])$ and $([\bullet_1 \quad \bullet_2] \longrightarrow [\bullet_1 \quad \bullet_2 \curvearrowright])$ are not isomorphic anymore, which means that this critical pair is not joinable, witnessing that the two rules are not confluent when applied to graphs with interfaces.

The authors point out that the situation is similar to the term rewriting case, for which local confluence of terminating systems is undecidable for ground rewriting rules, but decidable for open ones. This is only partially true, because in this case the interfaces do not play the rôle of variables, since the interfaces are not in the rules, but in the graphs that we rewrite, while in term rewriting the variables are in the rules (even if the terms that are rewritten may also have variables). When rewriting graphs with interfaces, the interfaces restrict the application of rules, since the interfaces must be preserved, while this is not the case when rewriting terms with rules including variables.

In the case of drags, the analogy to term rewriting is more faithful: confluence of graph/drag rewriting is not decidable for ground rules, but we have shown that it is decidable for rules with variables. Furthermore, as with term rewriting, drag critical pairs are computed via most general unifiers: all of them are useful in the absence of critical pair criteria. The reason why their number is limited, at most quadratic in the size of the input left-hand sides, is that, in our model, two vertices labeled by the same function symbol have the same number of successors, and these successors are ordered. Allowing for a variable number of successors (via associative function symbols) and disregarding their order (via commutative function symbols), as in Plump's model, would blow up the number of most

general unifiers, hence of critical pairs. However, we would only need checking the most general ones, as is the case with term rewriting.

6 Conclusion

Drags appear to be an extremely handy generalization of terms, dags and jungles: the intuitions behind them all are very similar, as well as the most important algorithms for implementing rewriting and testing its termination and confluence, despite the possibility of having arbitrary cycles in drags. This is made possible by a powerful composition operator.

Drags do not exactly generalize terms, though, as is pointed out in [8]. This is because our definition of composition *forces* sharing, as does term rewriting in practice. Capturing the term case requires using drag isomorphism instead of drag equality in presence of non-linear variables in rules. This is of course possible, and is currently being investigated.

So drags generalize dags (and jungles), by allowing for cycles. Unwinding these cycles yields infinite dags with finitely many subdags, that is, rational dags. The difference is that the ability to share a subdag requires that a context can always add an incoming edge to that subdag, which is not possible with drags, for which this subdag must be rooted beforehand. So, dags are drags equipped with roots at all vertices that need to be shared, making it then possible to build cycles by composition with a rewrite extension. Therefore drag unification must be at least as complex as rational dags unification, and therefore as rational tree unification, shown almost linear by Huet, whose algorithm actually solves without saying the rational dags unification problem [21]. It may well be that the exact complexity of dag and drag unification are the same, but we have not investigated it yet.

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