



**HAL**  
open science

# Logic beyond formulas: a proof system on graphs

Matteo Acclavio, Ross Horne, Lutz Strassburger

► **To cite this version:**

Matteo Acclavio, Ross Horne, Lutz Strassburger. Logic beyond formulas: a proof system on graphs. LICS 2020 - 35th ACM/IEEE Symposium on Logic in Computer Science, Jul 2020, Saarbrücken, Germany. pp.38-52, 10.1145/3373718.3394763 . hal-02560105

**HAL Id: hal-02560105**

**<https://inria.hal.science/hal-02560105v1>**

Submitted on 1 May 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Logic Beyond Formulas: A Proof System on Graphs

Matteo Acclavio\*  
Télécom SudParis and CNRS  
SAMOVAR UMR 5157  
France

Ross Horne  
Computer Science  
University of Luxembourg  
Esch-sur-Alzette, Luxembourg  
ross.horne@uni.lu

Lutz Straßburger  
Inria, Equipe Partout  
Ecole Polytechnique  
LIX UMR 7161  
France

## Abstract

In this paper we present a proof system that operates on graphs instead of formulas. We begin our quest with the well-known correspondence between formulas and cographs, which are undirected graphs that do not have  $P_4$  (the four-vertex path) as vertex-induced subgraph; and then we drop that condition and look at arbitrary (undirected) graphs. The consequence is that we lose the tree structure of the formulas corresponding to the cographs. Therefore we cannot use standard proof theoretical methods that depend on that tree structure. In order to overcome this difficulty, we use a modular decomposition of graphs and some techniques from deep inference where inference rules do not rely on the main connective of a formula. For our proof system we show the admissibility of cut and a generalization of the splitting property. Finally, we show that our system is a conservative extension of multiplicative linear logic (MLL) with mix, meaning that if a graph is a cograph and provable in our system, then it is also provable in MLL+mix.

**Keywords:** Proof theory, cographs, graph modules, prime graphs, cut elimination, deep inference, splitting, analycity

## ACM Reference Format:

Matteo Acclavio, Ross Horne, and Lutz Straßburger. 2020. Logic Beyond Formulas: A Proof System on Graphs. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS '20)*, July 8–11, 2020, Saarbrücken, Germany. ACM, New York, NY, USA, 33 pages.

## 1 Introduction

The notion of formula is central to all applications of logic and proof theory in computer science, ranging from the formal verification of software, where a formula describes a property that the program should satisfy, to logic programming, where a formula represents a program [27, 31], and functional programming, where a formula represents a type [25]. Proof theoretical methods are also employed in concurrency theory, where a formula can represent a process whose behaviours may be extracted from a proof of

the formula [5, 22, 23, 30]. This *formulas-as-processes* paradigm is not as well-investigated as the *formulas-as-properties*, *formulas-as-programs* and *formulas-as-types* paradigms mentioned before. In our opinion, a reason for this is that the notion of formula reaches its limitations when it comes to describing processes as they are studied in concurrency theory.

For example, BV [17] and *pomset logic* [37] are proof systems which extend linear logic with a notion of sequential composition and can model series-parallel orders. However, series-parallel orders cannot express some ubiquitous patterns of *causal dependencies* such as producer-consumer queues [28], which are within the scope of pomsets [36], event structures [33], and Petri nets [34]. The essence of this problem is already visible when we consider *symmetric dependencies*, such as separation, which happens to be the dual concept to concurrency in the *formulas-as-processes* paradigm.

Let us use some simple examples to explain the problem. Suppose we are in a situation where two processes  $A$  and  $B$  can communicate with each other, written as  $A \bowtie B$ , or can be separated from each other, written as  $A \otimes B$ , such that no communication is possible. Now assume we have four atomic processes  $a, b, c$ , and  $d$ , from which we form the two processes  $P = (a \otimes b) \bowtie (c \otimes d)$  and  $Q = (a \bowtie c) \otimes (b \bowtie d)$ . Both are perfectly fine formulas of multiplicative linear logic (MLL) [15]. In  $P$ , we have that  $a$  is separated from  $b$  but can communicate with  $c$  and  $d$ . Similarly,  $d$  can communicate with  $a$  and  $b$  but is separated from  $c$ , and so on. On the other hand, in  $Q$ ,  $a$  can only communicate with  $c$  and is separated from the other two, and  $d$  can only communicate with  $b$ , and is separated from the other two. We can visualize this situation via graphs where  $a, b, c$ , and  $d$  are the vertices, and we draw an edge between two vertices if they are separated, and no edge if they can communicate. Then  $P$  and  $Q$  correspond to the two graphs shown below.

$$\begin{array}{ccc}
 P = (a \otimes b) \bowtie (c \otimes d) & Q = (a \bowtie c) \otimes (b \bowtie d) & (1) \\
 \begin{array}{cc}
 b & d \\
 | & | \\
 a & c
 \end{array} & \begin{array}{cc}
 b & d \\
 \diagdown & \diagup \\
 a & c
 \end{array} & 
 \end{array}$$

\*Also with Inria, Equipe Partout.

It should also be possible to describe a situation where  $a$  is separated from  $b$ , and  $b$  is separated from  $c$ , and  $c$  is separated from  $d$ , but  $a$  can communicate with  $c$  and  $d$ , and  $b$  can

communicate with  $d$ , as indicated by the graph below.



However, this graph cannot be described by a formula in such a way that was possible for the two graphs in (1). Consequently, the tools of proof theory, that have been developed over the course of the last century, and that were very successful for the *formulas-as-properties*, *formulas-as-programs*, and *formulas-as-types* paradigms, can be used for the *formulas-as-processes* paradigm only if situations as in (2) above are forbidden. This seems to be a very strong and unnatural restriction, and the purpose of this paper is to propose a way to change this unsatisfactory situation.

We will present a proof system, called GS (for *graphical proof system*), whose objects of reason are not formulas but graphs, giving the example in (2) the same status as the examples in (1). In a less informal way, one could say that standard proof systems work on cographs (which are the class of graphs that correspond to formulas as in (1)), and our proof system works on arbitrary graphs. In order for this to make sense, this proof system should obey the following basic properties:

1. *Consistency*: There are graphs that are not provable.
2. *Transitivity*: The proof system should come with an implication that is transitive, i.e., if we can prove that  $A$  implies  $B$  and that  $B$  implies  $C$ , then we should also be able to prove that  $A$  implies  $C$ .
3. *Analycity*: As we no longer have formulas, we cannot ask that every formula that occurs in a proof is a subformula of its conclusion. But we can ask that in a proof search situation, there is always only a finite number of ways to apply an inference rule.
4. *Conservativity*: There should be a well-known logic  $L$  based on formulas such that when we restrict our proof system to graphs corresponding to formulas, then we prove exactly the theorems of  $L$ .
5. *Minimality*: We want to make as few assumptions as possible, so that the theory we develop is as general as possible.

Properties 1-3 are standard for any proof system, and they are usually proved using cut elimination. In that respect our paper is no different. We introduce a notion of cut and show its admissibility for GS. Then Properties 1-3 are immediate consequences, and also Property 4 will follow from cut admissibility, where in our case the logic  $L$  is multiplicative linear logic (MLL) with mix [2, 14, 15].

Finally, Property 5 is of a more subjective nature. In our case, we only make the following two basic assumptions:

1. For any graph  $A$ , we should be able to prove that  $A$  implies  $A$ .

2. If a graph  $A$  is provable, then the graph  $G = C[A]$  is also provable<sup>1</sup>, provided that  $C[\cdot]$  is a provable context. This can be compared to the *necessitation rule* of modal logic, which says that if  $A$  is provable then so is  $\Box A$ , except that in our case the  $\Box$  is replaced by the provable graph context  $C[\cdot]$ .

All other properties of the system GS follow from the need to obtain admissibility of cut. This means that this paper does not present some random system, but follows the underlying principles of proof theory.

In Section 2, we give preliminaries on cographs, which form the class of graphs that correspond to formulas as in (1). Then, in Section 3 we give some preliminaries on modules and prime graphs, which are needed for our move away from cographs, so that in Section 4, we can present our proof system, which uses the notation of open deduction [18] and follows the principles of deep inference [4, 17, 19]. In Section 5 we show some properties of our system, and Sections 6, 7, and 8 are dedicated to cut elimination. Finally, in Section 9, we show that our system is a conservative extension of MLL+mix.

The contributions of this paper can thus be summarized as follows:

- We present (to our knowledge) the first proof system that is not tied to formulas/cographs but handles arbitrary (undirected) graphs instead.
- We prove a *Splitting Lemma* (in Section 6), which is often a crucial ingredient in a proof of cut elimination in a deep inference system. But in our case the statement and the proof of this lemma is different from standard deep inference systems, in particular, the general method proposed by Aler Tubella in her PhD [44] does not apply. But we still use the name *Splitting Lemma*, as it serves the same purpose.
- We propose a cut rule which corresponds to the standard cut rule in a deep inference system, and show its admissibility. But again, due to the different nature of our proof system, the standard methods must be adapted.

## 2 From Formulas to Graphs

**Definition 2.1.** A (*simple, undirected*) **graph**  $G$  is a pair  $\langle V_G, E_G \rangle$  where  $V_G$  is a set of vertices and  $E_G$  is a set of two-element subsets of  $V_G$ . We omit the index  $G$  when it is clear from the context. For  $v, w \in V_G$  we write  $vw$  as an abbreviation for  $\{v, w\}$ . A graph  $G$  is **finite** if its vertex set  $V_G$  is finite. Let  $L$  be a set and  $G$  be a graph. We say that  $G$  is  **$L$ -labelled** (or just **labelled** if  $L$  is clear from context) if every vertex in  $V_G$  is associated with an element of  $L$ , called its **label**. We write  $\ell_G(v)$  to denote the label of the vertex

<sup>1</sup>Formally, the notation  $G = C[A]$  means that  $A$  is a module of  $G$ , and  $C[\cdot]$  is the graph obtained from  $G$  by removing all vertices belonging to  $A$ . We give the formal definition in Section 3.

$v$  in  $G$ . A graph  $G'$  is a **subgraph** of a graph  $G$ , denoted as  $G' \subseteq G$  iff  $V_{G'} \subseteq V_G$  and  $E_{G'} \subseteq E_G$ . We say that  $G'$  is an **induced subgraph** of  $G$  if  $G'$  is a subgraph of  $G$  and for all  $v, w \in V_{G'}$ , if  $vw \in E_G$  then  $vw \in E_{G'}$ . The **size** of a graph  $G$ , denoted by  $|G|$ , is the number of its vertices, i.e.,  $|G| = |V_G|$ .

In the following, we will just say *graph* to mean a finite, undirected, labelled graph, where the labels come from the set  $\mathcal{A}$  of atoms which is the (disjoint) union of a countable set of propositional variables  $\mathcal{V} = \{a, b, c, \dots\}$  and their duals  $\mathcal{V}^\perp = \{a^\perp, b^\perp, c^\perp, \dots\}$ .

Since we are mainly interested in how vertices are labelled, but not so much in the identity of the underlying vertex, we heavily rely on the notion of graph isomorphism.

**Definition 2.2.** Two graphs  $G$  and  $G'$  are **isomorphic** if there exists a bijection  $f: V_G \rightarrow V_{G'}$  such that for all  $v, u \in V_G$  we have  $vu \in E_G$  iff  $f(v)f(u) \in E_{G'}$  and  $\ell_G(v) = \ell_{G'}(f(v))$ . We denote this as  $G \simeq_f G'$ , or simply as  $G \simeq G'$  if  $f$  is clear from context or not relevant.

In the following, we will, in diagrams, forget the identity of the underlying vertices, showing only the label, as in the examples in the introduction.

In the rest of this section we recall the characterization of those graphs that correspond to formulas. For simplicity, we restrict ourselves to only two connectives, and for reasons that will become clear later, we use the  $\wp$  (**par**) and  $\otimes$  (**tensor**) of linear logic [15]. More precisely, **formulas** are generated by the grammar

$$\phi, \psi := \circ \mid a \mid a^\perp \mid \phi \wp \psi \mid \phi \otimes \psi \quad (3)$$

where  $\circ$  is the **unit**, and  $a$  can stand for any propositional variable in  $\mathcal{V}$ . As usual, we can define the negation of formulas inductively by letting  $a^{\perp\perp} = a$  for all  $a \in \mathcal{V}$ , and by using the De Morgan duality between  $\wp$  and  $\otimes$ :  $(\phi \wp \psi)^\perp = \phi^\perp \otimes \psi^\perp$  and  $(\phi \otimes \psi)^\perp = \phi^\perp \wp \psi^\perp$ ; the unit is self-dual:  $\circ^\perp = \circ$ .

On formulas we define the following structural equivalence relation:

$$\begin{array}{ll} \phi \wp (\psi \wp \xi) \equiv (\phi \wp \psi) \wp \xi & \phi \otimes (\psi \otimes \xi) \equiv (\phi \otimes \psi) \otimes \xi \\ \phi \wp \psi \equiv \psi \wp \phi & \phi \otimes \psi \equiv \psi \otimes \phi \\ \phi \wp \circ \equiv \phi & \phi \otimes \circ \equiv \phi \end{array}$$

In order to translate formulas to graphs, we define the following two operations on graphs:

**Definition 2.3.** Let  $G = \langle V_G, E_G \rangle$  and  $H = \langle V_H, E_H \rangle$  be graphs with  $V_G \cap V_H = \emptyset$ . We define the **par** and **tensor** operations between them as follows:

$$\begin{aligned} G \wp H &= \langle V_G \cup V_H, E_G \cup E_H \rangle \\ G \otimes H &= \langle V_G \cup V_H, E_G \cup E_H \cup \{vw \mid v \in V_G, w \in V_H\} \rangle \end{aligned}$$

For a formula  $\phi$ , we can now define its associated graph  $\llbracket \phi \rrbracket$  inductively as follows:  $\llbracket \circ \rrbracket = \emptyset$  the empty graph;  $\llbracket a \rrbracket = a$  a single-vertex graph whose vertex is labelled by  $a$  (by a slight abuse of notation, we denote that graph also by  $a$ );

similarly  $\llbracket a^\perp \rrbracket = a^\perp$ ; finally we define  $\llbracket \phi \wp \psi \rrbracket = \llbracket \phi \rrbracket \wp \llbracket \psi \rrbracket$  and  $\llbracket \phi \otimes \psi \rrbracket = \llbracket \phi \rrbracket \otimes \llbracket \psi \rrbracket$ .

**Theorem 2.4.** For any two formulas,  $\phi \equiv \psi$  iff  $\llbracket \phi \rrbracket = \llbracket \psi \rrbracket$ .

*Proof.* By a straightforward induction.  $\square$

**Definition 2.5.** A graph is  **$P_4$ -free** (or  **$N$ -free** or  **$Z$ -free**) if it does not have an induced subgraph of the shape



**Theorem 2.6.** Let  $G$  be a graph. Then there is a formula  $\phi$  with  $\llbracket \phi \rrbracket = G$  iff  $G$  is  $P_4$ -free.

A proof of this can be found, e.g., in [32] or [17].

The graphs characterized by Theorem 2.6 are called **cographs**, because they are the smallest class of graphs containing all single-vertex graphs and being closed under complement and disjoint union.

Because of Theorem 2.6, one can think of standard proof system as *cograph proof systems*. Since in this paper we want to move from cographs to general graphs, we need to investigate, how much of the tree structure of formulas (which makes cographs so interesting for proof theory [26, 38, 42]) can be recovered for general graphs.

### 3 Modules and Prime Graphs

In this section we take some of the concepts that make working with formulas so convenient and lift them to graphs that are not  $P_4$ -free.

**Definition 3.1.** Let  $G$  be a graph. A **module** of  $G$  is an induced subgraph  $M = \langle V_M, E_M \rangle$  of  $G$  such that for all  $v \in V_G \setminus V_M$  and all  $x, y \in M$  we have  $vx \in E_G$  iff  $vy \in E_G$ .

Modules are used in this paper since they are for graphs what subformulas are for formulas.

**Notation 3.2.** Let  $G$  be a graph and  $M$  be a module of  $G$ . Let  $V_C = V_G \setminus V_M$  and let  $C$  be the graph obtained from  $G$  by removing all vertices in  $M$  (including incident edges). Let  $R \subseteq V_C$  be the set of vertices that are connected to a vertex in  $V_M$  (and hence to all vertices in  $M$ ). We denote this situation as  $G = C[M]_R$  and call  $C[\cdot]_R$  (or just  $C$ ) the **context** of  $M$  in  $G$ . Alternatively,  $C[M]_R$  can be defined as follows. If we write  $C[x]_R$  for a graph in which  $x$  is a distinct vertex and  $R$  is the set of neighbours  $x$ , then  $C[M]_R$  is the graph obtained from  $C[x]_R$  by substitution of  $x$  for  $M$ .

**Lemma 3.3.** Let  $G$  be a graph and  $M, N$  be modules of  $G$ . Then

1.  $M \cap N$  is a module of  $G$ ;
2. if  $M \cap N \neq \emptyset$ , then  $M \cup N$  is a module of  $G$ ; and
3. if  $N \not\subseteq M$  then  $M \setminus N$  is a module of  $G$ .



*Proof.* The first statement follows immediately from the definition. For the second one, let  $L = M \cap N \neq \emptyset$ , and let  $v \in G \setminus (V_M \cup V_N)$  and  $x, y \in V_M \cup V_N$ . If  $x, y$  are both in  $M$  or both in  $N$ , then we have immediately  $vx \in E_G$  iff  $vy \in E_G$ . So, let  $x \in V_M$  and  $y \in V_N$ , and let  $z \in L$ . We have  $vx \in E_G$  iff  $vz \in E_G$  iff  $vy \in E_G$ . Finally, for the last statement, let  $x, y \in V_M \setminus V_N$  and let  $v \in V_G \setminus (V_M \setminus V_N)$ . If  $v \notin V_M$ , we immediately have  $vx \in E_G$  iff  $vy \in E_G$ . So, let  $v \in V_M$ , and therefore  $v \in V_M \cap V_N$ . Let  $z \in V_N \setminus V_M$ . Then  $vx \in E_G$  iff  $zx \in E_G$  iff  $zy \in E_G$  iff  $vy \in E_G$ .  $\square$

**Definition 3.4.** Let  $G$  be a graph. A module  $M$  in  $G$  is **maximal** if for all modules  $M'$  of  $G$  such that  $M \neq G$  we have that  $M \subseteq M'$  implies  $M = M'$ .

**Definition 3.5.** A module  $M$  of a graph  $G$  is **trivial** iff either  $V_M = \emptyset$  or  $V_M$  is a singleton or  $V_M = V_G$ . A graph  $G$  is **prime** iff  $|V_G| \geq 2$  and all modules of  $G$  are trivial.

**Definition 3.6.** Let  $G$  be a graph with  $n$  vertices  $V_G = \{v_1, \dots, v_n\}$  and let  $H_1, \dots, H_n$  be  $n$  graphs. We define the **composition of  $H_1, \dots, H_n$  via  $G$** , denoted as  $G(H_1, \dots, H_n)$ , by replacing each vertex  $v_i$  of  $G$  by the graph  $H_i$ ; and there is an edge between two vertices  $x$  and  $y$  if either  $x$  and  $y$  are in the same  $H_i$  and  $xy \in E_{H_i}$  or  $x \in V_{H_i}$  and  $y \in V_{H_j}$  for  $i \neq j$  and  $v_i v_j \in E_G$ . Formally,  $G(H_1, \dots, H_n) = \langle V^*, E^* \rangle$  with

$$\begin{aligned} V^* &= \bigcup_{1 \leq i \leq n} V_{H_i} \\ E^* &= \bigcup_{1 \leq i \leq n} E_{H_i} \cup \{xy \mid x \in V_{H_i}, y \in V_{H_j}, v_i v_j \in E_G\} \end{aligned}$$

This concept allows us to decompose graphs into prime graphs (via Lemma 3.7 below) and recover a tree structure for an arbitrary graph, seeing prime graphs as generalized non-decomposable  $n$ -ary connectives. The two operations  $\wp$  and  $\otimes$ , defined in Definition 2.3 are then represented by the two prime graphs.

$$\wp: \bullet \quad \bullet \quad \text{and} \quad \otimes: \bullet \text{---} \bullet \quad (5)$$

If we name these graphs  $\wp$  and  $\otimes$ , respectively, then we can write  $\wp(G, H) = G \wp H$  and  $\otimes(G, H) = G \otimes H$ .

**Lemma 3.7.** *Let  $G$  be a nonempty graph. Then we have exactly one of the following four cases:*

- (i)  $G$  is a singleton graph.
- (ii)  $G = A \wp B$  for some  $A, B$  with  $A \neq \emptyset \neq B$ .
- (iii)  $G = A \otimes B$  for some  $A, B$  with  $A \neq \emptyset \neq B$ .
- (iv)  $G = P(A_1, \dots, A_n)$  for some prime graph  $P$  with  $n = |V_P| \geq 4$  and  $A_i \neq \emptyset$  for all  $0 \leq i \leq n$ .

*Proof.* Let  $G$  be given. If  $|G| = 1$ , we are in case (i). Now assume  $|G| > 1$ , and let  $M_1, \dots, M_n$  be the maximal modules of  $G$ . Now we have two cases:

- For all  $i, j \in \{1, \dots, n\}$  with  $i \neq j$  we have  $M_i \cap M_j = \emptyset$ . Since every vertex of  $G$  forms a module, every vertex must be part of a maximal module. Hence  $V_G = V_{M_1} \cup \dots \cup V_{M_n}$ . Therefore there is a graph  $P$  such that  $G = P(M_1, \dots, M_n)$ .

Since all  $M_i$  are maximal in  $G$ , we can conclude that  $P$  is prime. If  $|V_P| \geq 4$  we are in case (iv). If  $|V_P| < 4$  we are either in case (ii) or (iii), as the two graphs in (5) are only two prime graphs with  $|V_P| = 2$ , and there are no prime graphs with  $|V_P| = 3$ .

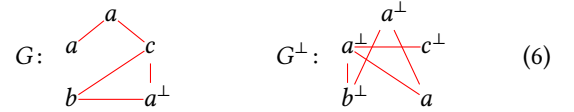
- We have some  $i \neq j$  with  $M_i \cap M_j \neq \emptyset$ . Let  $L = M_i \cap M_j$  and  $N = M_i \setminus M_j$  and  $K = M_j \setminus M_i$ . By Lemma 3.3,  $L, N, K$ , and  $M_i \cup M_j$  are all modules of  $G$ . Since  $M_i$  and  $M_j$  are maximal, it follows that  $G = M_i \cup M_j$ , and therefore  $G = N \otimes L \otimes K$  or  $G = N \wp L \wp K$ .  $\square$

## 4 The Proof System

To define a proof system, we need a notion of implication. To do so, we first introduce a notion of negation.

**Definition 4.1.** For a graph  $G = \langle V_G, E_G \rangle$ , we define its **dual**  $G^\perp = \langle V_G, E_{G^\perp} \rangle$  to have the same set of vertices, and an edge  $vw \in E_{G^\perp}$  iff  $vw \notin E_G$  (and  $v \neq w$ ). The label of a vertex  $v$  in  $G^\perp$  is the dual of the label of that vertex in  $G$ , i.e.,  $\ell_{G^\perp}(v) = \ell_G(v)^\perp$ . For any two graphs  $G$  and  $H$ , the **implication**  $G \multimap H$  is defined to be the graph  $G^\perp \wp H$ .

**Example 4.2.** To give an example, consider the graph  $G$  on the left below



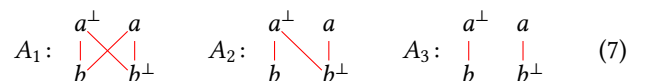
Its negation  $G^\perp$  is shown on the right above.

Observe that the dual graph construction defines the standard De Morgan dualities relating conjunction and disjunction, i.e., for every formula  $\phi$ , we have  $\llbracket \phi^\perp \rrbracket = \llbracket \phi \rrbracket^\perp$ . Furthermore, the De Morgan dualities extend to prime graphs, say  $P$ , as  $P(M_1, \dots, M_n)^\perp = P^\perp(M_1^\perp, \dots, M_n^\perp)$ , where  $P^\perp$  is the dual graph to  $P$ . Furthermore,  $P^\perp$  is prime if and only if  $P$  is prime. Thus each pair of prime graphs  $P$  and  $P^\perp$  defines a pair of connectives that are De Morgan duals to each other.

We will now develop our proof system based on the above notion of negation as graph duality. From the requirements mentioned in the introduction it follows that:

- (i) for any  $G$ , the graph  $G \multimap G$  should be provable;
- (ii) if  $G \neq \emptyset$  then  $G$  and  $G^\perp$  should not be both provable;
- (iii) the implication  $\multimap$  should be transitive, i.e., if  $G \multimap H$ , and  $H \multimap K$  are provable then so should be  $G \multimap K$ ;
- (iv) the implication  $\multimap$  should be closed under context, i.e., if  $G \multimap H$  is provable and  $C[\cdot]_R$  is an arbitrary context, then  $C[G]_R \multimap C[H]_R$  should be provable;
- (v) if  $A$  and  $C$  are provable graphs, and  $R \subseteq V_C$ , then the graph  $C[A]_R$  should also be provable.

**Example 4.3.** As an example, consider the following three graphs:



The graph  $A_1$  on the left should clearly be provable, as it corresponds to the formula  $(a^\perp \wp a) \otimes (b \wp b^\perp)$ , which is provable in MLL. The graph  $A_3$  on the right should not be provable, as it corresponds to the formula  $(a^\perp \otimes b) \wp (a \otimes b^\perp)$ , which is not provable in MLL. But what about the graph  $A_2$  in the middle? It does not correspond to a formula, and therefore we cannot resort to MLL. Nonetheless, we can make the following observations. If  $A_2$  were provable, then so would be the graph  $A_4$  shown below:

$$A_4: \begin{array}{c} a^\perp \quad a \\ | \quad | \\ \text{---} \quad \text{---} \\ | \quad | \\ a \quad a^\perp \end{array} \quad (8)$$

as it is obtained from  $A_2$  by a simple substitution. However,  $A_4^\perp = A_4$ , and therefore  $A_4^\perp$  and  $A_4$  would both be provable, which would be a contradiction and should be ruled out. Hence,  $A_2$  should not be provable.

We can make further observations without having presented the proof system yet: Notice that  $A_1 \multimap A_2$  cannot hold, as otherwise we would be able to use  $A_1$  and modus ponens to establish that  $A_2$  is provable, which cannot hold as we just observed. By applying a dual argument,  $A_2 \multimap A_3$  cannot hold. Hence, implication is not simply subset inclusion of edges.<sup>2</sup>

For presenting the inference system we use a *deep inference* formalism [17, 19], which allows rewriting inside an arbitrary context and admits a rather flexible composition of derivations. In our presentation we will follow the notation of *open deduction*, introduced in [18].

Let us start with the following two inference rules

$$\text{i}\downarrow \frac{\emptyset}{A^\perp \wp A} \quad \text{ss}\uparrow \frac{B \otimes A}{B[A]_S} \quad S \subseteq V_B, S \neq V_B \quad (9)$$

which are induced by the two Points (i) and (v) above, and which are called **identity down** and **super switch up**, respectively. The  $\text{i}\downarrow$  says that for arbitrary graphs  $C$  and  $A$  and any  $R \subseteq V_C$ , if  $C$  is provable, then so is the graph  $C[A \wp A^\perp]_R$ . Similarly, the rule  $\text{ss}\uparrow$  says that whenever  $C[B \otimes A]_R$  is provable, then so is  $C[B[A]_S]_R$  for any three graphs  $A, B, C$  and any  $R \subseteq V_C$  and  $S \subseteq V_B$ . The condition  $S \neq V_B$  is there to avoid a trivial rule instance, as  $B[A]_S = B \otimes A$  if  $S = V_B$ .

**Definition 4.4.** An **inference system**  $S$  is a set of inference rules. We define the set of **derivations** in  $S$  inductively below, and we denote a derivation  $\mathcal{D}$  in  $S$  with premise  $G$  and conclusion  $H$ , as follows:

$$\begin{array}{c} G \\ \mathcal{D} \parallel S \\ H \end{array}$$

<sup>2</sup>However, the converse holds in our particular case: We will see later that whenever we have  $G \multimap H$  and  $V_G = V_H$  then  $E_H \subseteq E_G$ . But this observation is not true in general for logics on graphs. For example in the extension of Boolean logic, defined in [8], it does not hold.

1. Every graph  $G$  is a derivation (also denoted by  $G$ ) with premise  $G$  and conclusion  $G$ .
2. If  $\mathcal{D}_1$  is a derivation with premise  $G_1$  and conclusion  $H_1$ , and  $\mathcal{D}_2$  is a derivation with premise  $G_2$  and conclusion  $H_2$ , then  $\mathcal{D}_1 \wp \mathcal{D}_2$  is a derivation with premise  $G_1 \wp G_2$  and conclusion  $H_1 \wp H_2$ , and similarly,  $\mathcal{D}_1 \otimes \mathcal{D}_2$  is a derivation with premise  $G_1 \otimes G_2$  and conclusion  $H_1 \otimes H_2$ , denoted as

$$\begin{array}{c} G_1 \\ \mathcal{D}_1 \parallel S \\ H_1 \end{array} \wp \begin{array}{c} G_2 \\ \mathcal{D}_2 \parallel S \\ H_2 \end{array} \quad \text{and} \quad \begin{array}{c} G_1 \\ \mathcal{D}_1 \parallel S \\ H_1 \end{array} \otimes \begin{array}{c} G_2 \\ \mathcal{D}_2 \parallel S \\ H_2 \end{array}$$

respectively.

3. If  $\mathcal{D}_1$  is a derivation with premise  $G_1$  and conclusion  $H_1$ , and  $\mathcal{D}_2$  is a derivation with premise  $G_2$  and conclusion  $H_2$ , and

$$\frac{H_1}{G_2} \quad r$$

is an instance of an inference rule  $r$ , then  $\mathcal{D}_2 \circ_r \mathcal{D}_1$  is a derivation with premise  $G_2$  and conclusion  $H_2$ , denoted as

$$\frac{\begin{array}{c} G_1 \\ \mathcal{D}_1 \parallel S \\ H_1 \end{array}}{G_2} \quad r \quad \text{or} \quad \frac{\begin{array}{c} G_1 \\ \mathcal{D}_1 \parallel S \\ H_1 \end{array}}{G_2} \quad r$$

If  $H_1 \simeq_f G_2$  we can compose  $\mathcal{D}_1$  and  $\mathcal{D}_2$  directly to  $\mathcal{D}_2 \circ \mathcal{D}_1$ , denoted as

$$\begin{array}{c} G_1 \\ \mathcal{D}_1 \parallel S \\ H_1 \end{array} \quad \text{or} \quad \begin{array}{c} G_1 \\ \mathcal{D}_1 \parallel S \\ H_1 \end{array} \quad \text{or} \quad \begin{array}{c} G_1 \\ \mathcal{D}_1 \parallel S \\ H_1 \end{array} \quad (10)$$

If  $f$  is the identity, i.e.,  $H_1 = G_2$ , we can write  $\mathcal{D}_2 \circ \mathcal{D}_1$  as

$$\begin{array}{c} G_1 \\ \mathcal{D}_1 \parallel S \\ H_1 \\ \mathcal{D}_2 \parallel S \\ H_2 \end{array} \quad \text{or} \quad \begin{array}{c} G_1 \\ \mathcal{D}_1 \parallel S \\ G_2 \\ \mathcal{D}_2 \parallel S \\ H_2 \end{array}$$

A **proof** in  $S$  is a derivation in  $S$  whose premise is  $\emptyset$ . A graph  $G$  is **provable** in  $S$  iff there is a proof in  $S$  with conclusion  $G$ . We denote this as  $\vdash_S G$  (or simply as  $\vdash G$  if  $S$  is clear from context). The **length** of a derivation  $\mathcal{D}$ , denoted by  $|\mathcal{D}|$ , is the number of inference rule instances in  $\mathcal{D}$ .

**Remark 4.5.** If we have a derivation  $\mathcal{D}$  from  $A$  to  $B$ , and a context  $G[\cdot]_R$ , then we also have a derivation from  $G[A]_R$  to  $G[B]_R$ . We can write this derivation as

$$\begin{array}{c} G[A]_R \\ G[\mathcal{D}]_R \parallel \\ G[B]_R \end{array} \quad \text{or} \quad \begin{array}{c} A \\ G[\mathcal{D} \parallel]_R \\ B \end{array}$$

**Example 4.6.** Let us emphasize that the conclusion of a proof in our system is not a formula but a graph. The following derivation is an example of a proof of length 2, using only  $i\downarrow$  and  $ss\uparrow$ :

$$\begin{array}{c} i\downarrow \\ \frac{\emptyset}{a^\perp \text{---} b^\perp \quad a \quad b} \\ c^\perp \quad d^\perp \quad c \text{---} d \\ ss\uparrow \\ \frac{a^\perp \text{---} b^\perp \quad a \quad b}{c^\perp \quad d^\perp \quad c \text{---} d} \end{array} \quad (11)$$

where the  $ss\uparrow$  instance moves the module  $d$  in the context consisting of vertices labelled  $a, b, c$ . The derivation in (11) establishes that the following implication is provable:

$$\begin{array}{c} a \quad b \\ | \quad | \\ c \text{---} d \end{array} \quad \multimap \quad \begin{array}{c} a \quad b \\ | \quad | \\ c \quad d \end{array} \quad (12)$$

which is a fact beyond the scope of formulas.

As in other deep inference systems, we can give for the rules in (9) their *duals*, or *corules*. In general, if

$$r \frac{G}{H}$$

is an instance of a rule, then

$$r^\perp \frac{H^\perp}{G^\perp}$$

is an instance of the dual rule. The corules of the two rules in (9) are the following:

$$i\uparrow \frac{A \otimes A^\perp}{\emptyset} \quad ss\downarrow \frac{B[A]_S}{B \wp A} \quad S \subseteq V_B, S \neq \emptyset \quad (13)$$

called **identity up** (or **cut**) and **super switch down**, respectively. We have the side condition  $S \neq \emptyset$  to avoid a triviality, as  $B[A]_S = B \wp A$  if  $S = \emptyset$ .

**Example 4.7.** The implication in (12) can also be proven using only  $ss\downarrow$  and  $i\downarrow$  instead of  $ss\uparrow$  and  $i\downarrow$ , as the following

proof of length 3 shows:

$$\begin{array}{c} i\downarrow \\ \frac{\emptyset}{a^\perp \text{---} b^\perp \quad a \quad b} \\ c^\perp \quad c \\ i\downarrow \\ \frac{a^\perp \text{---} b^\perp \quad a \quad b}{c^\perp \quad c \quad d \quad d^\perp} \\ ss\downarrow \\ \frac{a^\perp \text{---} b^\perp \quad a \quad b}{c^\perp \quad d^\perp \quad c \quad d} \end{array} \quad (14)$$

**Definition 4.8.** Let  $S$  be an inference system. We say that an inference rule  $r$  is **derivable** in  $S$  iff

$$\text{for every instance } r \frac{G}{H} \text{ there is a derivation } \frac{G}{H} \text{ in } S.$$

We say that  $r$  is **admissible** in  $S$  iff

$$\text{for every instance } r \frac{G}{H} \text{ we have that } \vdash_S G \text{ implies } \vdash_S H.$$

If  $r \in S$  then  $r$  is trivially derivable and admissible in  $S$ .

Most deep inference systems in the literature (e.g. [4, 17, 19, 20, 24, 40]) contain the **switch** rule:

$$s \frac{(A \wp B) \otimes C}{A \wp (B \otimes C)} \quad (15)$$

One can immediately see that it is its own dual and is a special case of both  $ss\downarrow$  and  $ss\uparrow$ . We therefore have the following:

**Lemma 4.9.** *If in an inference system  $S$  one of the rules  $ss\downarrow$  and  $ss\uparrow$  is derivable, then so is  $s$ .*

**Remark 4.10.** In a standard deep inference system for formulas we also have the converse of Lemma 4.9, i.e., if  $s$  is derivable, then so are  $ss\uparrow$  and  $ss\downarrow$  (see, e.g., [41]). However, in the case of arbitrary graphs this is no longer true, and the rules  $ss\uparrow$  and  $ss\downarrow$  are strictly more powerful than  $s$ .

**Lemma 4.11.** *Let  $S$  be an inference system. If the rules  $i\downarrow$  and  $i\uparrow$  and  $s$  are derivable in  $S$ , then for every rule  $r$  that is derivable in  $S$ , also its corule  $r^\perp$  is derivable in  $S$ .*

*Proof.* Suppose we have two graphs  $G$  and  $H$ , and a derivation from  $G$  to  $H$  in  $S$ . Then it suffices to show that we can

construct a derivation from  $H^\perp$  to  $G^\perp$  in  $S$ :

$$\frac{\frac{\text{i}\downarrow \frac{\emptyset}{G^\perp \wp G} \otimes H^\perp}{s}}{G^\perp \wp \frac{\frac{G}{H} \parallel s \otimes H^\perp}{\text{i}\uparrow \frac{\emptyset}{\emptyset}}}$$

Note that  $\emptyset \otimes H^\perp = H^\perp$  and  $G^\perp \wp \emptyset = G^\perp$ .  $\square$

**Lemma 4.12.** *If the rules  $\text{i}\uparrow$  and  $s$  are admissible for an inference system  $S$ , then  $\multimap$  is transitive, i.e., if  $\vdash_S G \multimap H$  and  $\vdash_S H \multimap K$  then  $\vdash_S G \multimap K$ .*

*Proof.* We can construct the following derivation

$$\frac{\frac{\frac{\emptyset}{G^\perp \wp H} \parallel s \otimes \frac{\emptyset}{H^\perp \wp K} \parallel s}{s}}{G^\perp \wp \frac{\frac{H \otimes (H^\perp \wp K)}{\text{i}\uparrow \frac{H \otimes H^\perp}{\emptyset} \wp K}}{s}}$$

from  $\emptyset$  to  $G^\perp \wp K$  in  $S$ .  $\square$

Lemma 4.12 is the reason why  $\text{i}\uparrow$  is also called *cut*. In a well-designed deep inference system for formulas, the two rules  $\text{i}\downarrow$  and  $\text{i}\uparrow$  can be restricted in a way that they are only applicable to atoms, i.e., replaced by the following two rules that we call **atomic identity down** and **atomic identity up**, respectively:

$$\text{ai}\downarrow \frac{\emptyset}{a^\perp \wp a} \quad \text{and} \quad \text{ai}\uparrow \frac{a \otimes a^\perp}{\emptyset} \quad (16)$$

We would like to achieve something similar for our proof system on graphs. For this it is necessary to be able to decompose prime graphs into atoms, but the two rules  $\text{ss}\downarrow$  and  $\text{ss}\uparrow$  cannot do this, as they are only able to move around modules in a graph. For this reason, we add the following two rules to our system:

$$\text{p}\downarrow \frac{(M_1 \wp N_1) \otimes \dots \otimes (M_n \wp N_n)}{P^\perp(M_1, \dots, M_n) \wp P(N_1, \dots, N_n)} \quad P \text{ prime, } |V_P| \geq 4 \quad (17)$$

called **prime down**, and

$$\text{p}\uparrow \frac{P(M_1, \dots, M_n) \otimes P^\perp(N_1, \dots, N_n)}{(M_1 \otimes N_1) \wp \dots \wp (M_n \otimes N_n)} \quad P \text{ prime, } |V_P| \geq 4 \quad (18)$$

called **prime up**. In both cases, the side condition is that  $P$  needs to be a prime graph and has at least 4 vertices. We also require that for all  $i \in \{1, \dots, n\}$  at least one of  $M_i$  and  $N_i$  is nonempty in an application of  $\text{p}\downarrow$  and  $\text{p}\uparrow$ . The reason for

these conditions is not that the rules would become unsound otherwise, but that the rules are derivable in the general case, as we will see in Lemma 5.2 in the next section.

**Example 4.13.** Below is a derivation of length 5 using the  $\text{p}\downarrow$ -rule, and proves that a prime graph implies itself.

$$\text{p}\downarrow \frac{\frac{\text{ai}\downarrow \frac{\emptyset}{a^\perp \wp a} \otimes \text{ai}\downarrow \frac{\emptyset}{b^\perp \wp b} \otimes \text{ai}\downarrow \frac{\emptyset}{c^\perp \wp c} \otimes \text{ai}\downarrow \frac{\emptyset}{d^\perp \wp d}}{a^\perp \wp b^\perp \quad a \wp b \quad c^\perp \wp d^\perp \quad c \wp d}}$$

This completes the presentation of our system, which is shown in Figure 1.

**Definition 4.14.** We define **system SGS** to be the set  $\{\text{ai}\downarrow, \text{ss}\downarrow, \text{p}\downarrow, \text{p}\uparrow, \text{ss}\uparrow, \text{ai}\uparrow\}$  of inference rules shown in Figure 1. The **down-fragment** (resp. **up-fragment**) of SGS consists of the rules  $\{\text{ai}\downarrow, \text{ss}\downarrow, \text{p}\downarrow\}$  (resp.  $\{\text{ai}\uparrow, \text{ss}\uparrow, \text{p}\uparrow\}$ ) and is denoted by  $\text{SGS}\downarrow$  (resp.  $\text{SGS}\uparrow$ ). The down-fragment  $\text{SGS}\downarrow$  is also called **system GS**.

## 5 Properties of the System

The first observation about SGS is that the general forms of the identity rules  $\text{i}\downarrow$  and  $\text{i}\uparrow$  are derivable, as we show in Lemma 5.1 below. Next, we have a similar result for the prime rules, for which also a general form is derivable, i.e., they can be applied to any graph instead of only prime graphs.

**Lemma 5.1.** *The rule  $\text{i}\downarrow$  is derivable in  $\text{SGS}\downarrow$ , and dually, the rule  $\text{i}\uparrow$  is derivable in  $\text{SGS}\uparrow$ .*

*Proof.* We show by induction on  $G$ , that  $G^\perp \wp G$  has a proof in  $\text{SGS}\downarrow$ , using Lemma 3.7.

- (i) If  $G$  is a singleton graph, we can apply  $\text{ai}\downarrow$ .
- (ii) If  $G = A \wp B$  then  $G^\perp = B^\perp \otimes A^\perp$ , and we can construct

$$\frac{\frac{\frac{\emptyset}{\mathcal{D}_1 \parallel \text{SGS}\downarrow} \otimes \frac{\emptyset}{\mathcal{D}_2 \parallel \text{SGS}\downarrow}}{\text{ss}\downarrow \frac{B^\perp \otimes A^\perp \wp A}{(B^\perp \otimes A^\perp) \wp A}} \wp B}{B^\perp \otimes A^\perp \wp A}$$

where  $\mathcal{D}_1$  and  $\mathcal{D}_2$  exist by induction hypothesis.

- (iii) If  $G = A \otimes B$ , we proceed similarly.
- (iv) If  $G = P(A_1, \dots, A_n)$  for  $P$  prime and  $|V_P| \geq 4$ , we get

$$\text{p}\downarrow \frac{\frac{\frac{\emptyset}{\mathcal{D}_1 \parallel \text{SGS}\downarrow} \otimes \dots \otimes \frac{\emptyset}{\mathcal{D}_n \parallel \text{SGS}\downarrow}}{P^\perp(A_1^\perp, \dots, A_n^\perp) \wp P(A_1, \dots, A_n)}}{P^\perp(A_1^\perp, \dots, A_n^\perp) \wp P(A_1, \dots, A_n)}$$



$$\begin{array}{c}
\text{ai}\downarrow \frac{\emptyset}{a^\perp \wp a} \quad \text{ss}\downarrow \frac{B[A]_S}{B \wp A} \quad S \subseteq V_B, S \neq \emptyset \\
\text{p}\downarrow \frac{(M_1 \wp N_1) \otimes \dots \otimes (M_n \wp N_n)}{P^\perp(M_1, \dots, M_n) \wp P(N_1, \dots, N_n)} \quad P \text{ prime}, |V_P| \geq 4
\end{array}
\quad \Bigg| \quad
\begin{array}{c}
\text{ai}\uparrow \frac{a \otimes a^\perp}{\emptyset} \quad \text{ss}\uparrow \frac{B \otimes A}{B[A]_S} \quad S \subseteq V_B, S \neq V_B \\
\text{p}\uparrow \frac{P(M_1, \dots, M_n) \otimes P^\perp(N_1, \dots, N_n)}{(M_1 \otimes N_1) \wp \dots \wp (M_n \otimes N_n)} \quad P \text{ prime}, |V_P| \geq 4
\end{array}$$

**Figure 1.** The inference rules for systems GS (rules ai↓, ss↓, p↓ on the left) and SGS (all rules in the figure).

where  $\mathcal{D}_1, \dots, \mathcal{D}_n$  exist by induction hypothesis.  $\square$

**Lemma 5.2.** *For any graph  $G$  with  $|V_G| = n$ , and graphs  $M_1, N_1, \dots, M_n, N_n$ , we have derivations*

$$\begin{array}{c}
(M_1 \wp N_1) \otimes \dots \otimes (M_n \wp N_n) \\
\parallel_{\text{SGS}\downarrow} \\
G^\perp(M_1, \dots, M_n) \wp G(N_1, \dots, N_n)
\end{array} \quad (19)$$

and dually

$$\begin{array}{c}
G(M_1, \dots, M_n) \otimes G^\perp(N_1, \dots, N_n) \\
\parallel_{\text{SGS}\uparrow} \\
(M_1 \otimes N_1) \wp \dots \wp (M_n \otimes N_n)
\end{array} \quad (20)$$

*Proof.* We only show (19), and proceed by induction on the size of  $G$ , using Lemma 3.7.

- (i) If  $G$  is a singleton graph, the statement holds trivially.
- (ii) If  $G = A \wp B$  then  $G(N_1, \dots, N_n) = A(N_1, \dots, N_k) \wp B(N_{k+1}, \dots, N_n)$  for some  $1 \leq k \leq n$ . We therefore have

$$\begin{array}{c}
\begin{array}{c} (M_1 \wp N_1) \otimes \dots \otimes (M_k \wp N_k) \\ \mathcal{D}_1 \parallel_{\text{SGS}\downarrow} \\ A^\perp(M_1, \dots, M_k) \wp A(N_1, \dots, N_k) \end{array} \otimes \begin{array}{c} (M_{k+1} \wp N_{k+1}) \otimes \dots \otimes (M_n \wp N_n) \\ \mathcal{D}_2 \parallel_{\text{SGS}\downarrow} \\ B^\perp(M_{k+1}, \dots, M_n) \wp B(N_{k+1}, \dots, N_n) \end{array} \\
\text{ss}\downarrow \frac{}{} \\
\begin{array}{c} (A^\perp(M_1, \dots, M_k) \wp A(N_1, \dots, N_k)) \otimes B^\perp(M_{k+1}, \dots, M_n) \\ \text{ss}\downarrow \frac{}{} \\ (A^\perp(M_1, \dots, M_k) \otimes B^\perp(M_{k+1}, \dots, M_n)) \wp A(N_1, \dots, N_k) \end{array} \wp B(N_{k+1}, \dots, N_n)
\end{array}$$

where  $\mathcal{D}_1$  and  $\mathcal{D}_2$  exist by induction hypothesis.

- (iii) If  $G = A \otimes B$ , we proceed similarly.
- (iv) If  $G = P(A_1, \dots, A_n)$  for  $P$  prime and  $|V_P| \geq 4$ , we have an instance of p↓.

The derivation in (20) can be constructed dually.  $\square$

Next, observe that Lemmas 4.11 and 4.12 hold for system SGS. In particular, we have that if  $\vdash_{\text{SGS}} A \multimap B$  and  $\vdash_{\text{SGS}} B \multimap C$  then  $\vdash_{\text{SGS}} A \multimap C$  because  $i\uparrow \in \text{SGS}$ . The main result of this paper is that Lemma 4.12 does also hold for GS. More precisely, we have the following theorem:

**Theorem 5.3** (Cut Admissibility). *The rule  $i\uparrow$  is admissible for GS.*

To prove this theorem, we will show that the whole up-fragment of SGS is admissible for GS.

**Theorem 5.4.** *The rules  $ai\uparrow, ss\uparrow, p\uparrow$  are admissible for GS.*

Then Theorem 5.3 follows immediately from Theorem 5.4 and the second statement in Lemma 5.1.

The following three sections are devoted to the proof of Theorem 5.4. But before, let us finish this section by exhibiting some immediate consequences of Theorem 5.3.

**Corollary 5.5.** *For every graph  $G$ , we have  $\vdash_{\text{SGS}} A$  iff  $\vdash_{\text{GS}} A$ .*

**Corollary 5.6.** *For all graphs  $G$  and  $H$ , we have*

$$\vdash_{\text{GS}} G \multimap H \iff \frac{\emptyset}{G^\perp \wp H} \parallel_{\text{GS}} \iff \frac{\emptyset}{G^\perp \wp H} \parallel_{\text{SGS}} \iff \frac{G}{H} \parallel_{\text{SGS}}$$

*Proof.* The first equivalence is just the definition of  $\vdash$ . The second equivalence follows from Theorem 5.4, and the last equivalence follows from the two derivations

$$\begin{array}{c} \frac{\emptyset}{G^\perp \wp H} \parallel_{\text{GS}} \end{array} \quad \text{and} \quad \begin{array}{c} \frac{G \otimes \frac{\emptyset}{G^\perp \wp H}}{\frac{G \otimes G^\perp}{\emptyset} \wp H} \parallel_{\text{SGS}} \end{array}$$

together with Lemma 5.1.  $\square$

**Corollary 5.7.** *We have  $\vdash A \otimes B$  iff  $\vdash A$  and  $\vdash B$ .*

*Proof.* This follows immediately by inspecting the inference rules of GS.  $\square$

**Corollary 5.8.** *We have  $\vdash P(M_1, \dots, M_n)$  with  $P$  prime and  $n \geq 4$  and  $M_i \neq \emptyset$  for all  $i = \{1, \dots, n\}$ , if and only if there is at least one  $i = \{1, \dots, n\}$  such that  $\vdash M_i$  and  $\vdash P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n)$ .*

This can be seen as a generalization of the previous corollary, and it is proved similarly.

**Remark 5.9.** The system GS forms a proof system in the sense of Cook and Reckhow [10], as the time complexity of checking the correct application of inference rules is polynomial, since the modular decomposition of graphs can be obtained in linear time [29]. Also whenever graph isomorphism is used to compose derivations, as in (10), we assume that the isomorphism  $f$  is explicitly given.

**Theorem 5.10.** *Provability in GS is decidable and in NP.*

*Proof.* This follows immediately from the observation that to each graph only finitely many inference rules can be applied,

and that the length of a derivation in GS is  $O(n^3)$  where  $n$  is the number of vertices in the conclusion. This can be seen as follows: every inference rule application in GS, when seen bottom-up, removes either two vertices or at least one edge. No rule can introduce vertices or edges.<sup>3</sup>  $\square$

## 6 Splitting

The standard syntactic method for proving cut elimination in the sequent calculus is to permute the cut rule upwards in the proof and decomposing the cut formula along its main connective, and so inductively reduce the cut rank. However, in our proof system this method cannot be applied, as derivations can be constructed in a more flexible way than in the sequent calculus. For this reason, the *splitting* technique has been developed in the literature on deep inference [17, 21, 24, 41]. However, since we are working on general graphs instead of formulas, the generic method developed by Aler Tubella [44], cannot directly be applied in our case. For this reason, we needed to adapt the method and prove all lemmas from scratch. The central lemma is the following:

**Lemma 6.1** (Splitting). *Let  $G, A, B$  be graphs, let  $P$  be a prime graph with  $n = |V_P| \geq 4$ , let  $M_1, \dots, M_n$  be nonempty graphs, and let  $a$  be an atom.*

(1) *If  $\vdash_{\text{GS}} G \wp (A \otimes B)$  then there are a context  $C[\cdot]_R$  and graphs  $K_A$  and  $K_B$ , such that there are derivations*

$$\begin{array}{c} C[K_A \wp K_B]_R \\ \mathcal{D}_G \parallel_{\text{GS}} \\ G \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_C \parallel_{\text{GS}} \\ C \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_A \parallel_{\text{GS}} \\ K_A \wp A \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_B \parallel_{\text{GS}} \\ K_B \wp B \end{array}.$$

(2) *If  $\vdash_{\text{GS}} G \wp P(M_1, \dots, M_n)$ , then there are*

- *either a context  $C[\cdot]_R$  and graphs  $K_1, \dots, K_n$ , such that there are derivations*

$$\begin{array}{c} C[P^\perp(K_1, \dots, K_n)]_R \\ \mathcal{D}_G \parallel_{\text{GS}} \\ G \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_C \parallel_{\text{GS}} \\ C \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_i \parallel_{\text{GS}} \\ K_i \wp M_i \end{array}$$

for all  $i \in \{1, \dots, n\}$ ,

- *or a context  $C[\cdot]_R$  and graphs  $K_X$  and  $K_Y$  such that there are derivations*

$$\begin{array}{c} C[K_X \wp K_Y]_R \\ \mathcal{D}_G \parallel_{\text{GS}} \\ G \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_C \parallel_{\text{GS}} \\ C \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_X \parallel_{\text{GS}} \\ K_X \wp M_i \end{array},$$

$$\begin{array}{c} \emptyset \\ \mathcal{D}_Y \parallel_{\text{GS}} \\ K_Y \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n) \end{array}$$

for some  $i \in \{1, \dots, n\}$ .

<sup>3</sup>It can in fact be shown that the length is  $O(n^2)$  (see also [6]), but as the details are not needed for this paper, we leave them to the reader.

(3) *If  $\vdash_{\text{GS}} G \wp a$  then there is a context  $C[\cdot]_R$  such that there are derivations*

$$\begin{array}{c} C[a^\perp]_R \\ \mathcal{D}_G \parallel_{\text{GS}} \\ G \end{array} \quad \text{and} \quad \begin{array}{c} \emptyset \\ \mathcal{D}_C \parallel_{\text{GS}} \\ C \end{array}.$$

Note that in the statement of Lemma 6.1, the first case (1) is superfluous, as it is a special case of (2), when we see  $\otimes$  as a prime graph, as indicated in (5) in Section 3. In this case the two subcases of (2) collapse. We nonetheless decided for pedagogical reasons to list case (1) explicitly. It shows how our splitting lemma is related to the standard splitting lemmas in the deep inference literature [17, 19, 21, 24, 41, 44], and thus enables the reader to see where the two subcases in case (2) come from.

The idea of splitting is that, in a provable “sequent-like graph”, consisting of a number of disjoint connected components, we can select any of these components as the *principal component* and apply a derivation to the other components, such that eventually a rule breaking down the principal component can be applied. This allows us to approximate the effect of applying rules in the sequent calculus.

We will use the proof in (14) as an example to explain this idea. In the conclusion we have 3 connected components. We can select the  $N$ -shape component on the right as the principal component, and apply case (2) of Lemma 6.1 to reorganise the proof (14) such that an instance of  $\text{p}\downarrow$  involving the  $N$ -shape can be applied, as in Example 4.13. The bottom-most step of such a reorganised proof is shown below:

$$\begin{array}{c} \begin{array}{cc} a^\perp & b^\perp \\ \swarrow & \searrow \\ c^\perp & d^\perp \end{array} \quad \begin{array}{cc} a & b \\ | & | \\ c & d \end{array} \\ \text{ss}\downarrow \\ \begin{array}{cc} a^\perp & b^\perp \\ \swarrow & \searrow \\ c^\perp & d^\perp \end{array} \quad \begin{array}{cc} a & b \\ | & | \\ c & d \end{array} \end{array} \quad (21)$$

If, on the other hand we pick in (14) the  $d^\perp$  as principal component, and apply case (3) of Lemma 6.1, we get the following derivation

$$\begin{array}{c} d^\perp \wp \left( \text{ai}\downarrow \frac{\emptyset}{a^\perp \wp a} \otimes \text{ai}\downarrow \frac{\emptyset}{b^\perp \wp b} \otimes \text{ai}\downarrow \frac{\emptyset}{c^\perp \wp c} \otimes d \right) \\ \text{p}\downarrow \\ \begin{array}{cc} a^\perp & b^\perp \\ \swarrow & \searrow \\ d^\perp & c^\perp \end{array} \quad \begin{array}{cc} a & b \\ | & | \\ c & d \end{array} \end{array}$$

which we can complete to a proof with an application of the rule  $\text{ai}\downarrow$ .

A significant departure from established splitting lemmas in the literature, is the need for contexts in the premises of derivations. This is required to cope with graphs such as the

following:

$$\begin{array}{c}
 a^\perp \\
 \swarrow \quad \downarrow \quad \searrow \\
 a \quad b^\perp \quad c^\perp \quad c \text{---} b
 \end{array}
 \quad (22)$$

If we take  $a$  as the principal component, and apply case (3) of Lemma 6.1, we get a nonempty context  $C$ . Notice, furthermore, the above graph is provable only by applying rules deep inside the modular decomposition of the graph, as follows:

$$\begin{array}{c}
 \begin{array}{c} b^\perp \quad c^\perp \quad c \text{---} b \\ \hline \text{ai}\downarrow \\ \begin{array}{c} a \quad a^\perp \\ \swarrow \quad \downarrow \quad \searrow \\ b^\perp \quad c^\perp \quad c \text{---} b \end{array} \\ \hline \text{ss}\downarrow \\ \begin{array}{c} a^\perp \\ \swarrow \quad \downarrow \quad \searrow \\ a \quad b^\perp \quad c^\perp \quad c \text{---} b \end{array} \end{array}
 \end{array}
 \quad (23)$$

This shows that deep inference is necessary for this kind of proof theory on graphs.

The second subcase in case (2) of Lemma 6.1 is required for examples such as the following:

$$\begin{array}{c}
 b \quad a^\perp \quad b^\perp \\
 \downarrow \quad \downarrow \quad \downarrow \\
 a \quad c \quad c^\perp
 \end{array}
 \quad (24)$$

If we select the  $N$ -shape as the principal component and try to apply  $p\downarrow$ , then  $a$  and  $a^\perp$  can no longer communicate. Therefore, we must first move  $b^\perp$  or  $c^\perp$  into the structure and apply an  $ai\downarrow$ , in order to destroy the prime graph. For example, by using  $b^\perp$  to cancel out  $b$ , we obtain a provable graph of the form  $a \wp (c \otimes a^\perp) \wp c^\perp$ .

The proof of Lemma 6.1 proceeds by induction on the size of the derivation by exhaustively considering all ways in which the bottommost rule can interact with the principal component. It can be found in Appendix A.

## 7 Context Reduction

The Splitting Lemma 6.1 only applies in a shallow context, i.e., the outermost nodes in the modular tree construction of a graph (see Lemma 3.7). In order to use splitting for cut elimination, we need to apply it in arbitrary contexts. For this we need the context reduction lemma.

**Lemma 7.1** (Context reduction). *Let  $A$  be a graph and  $G[\cdot]_S$  be a context. If  $\vdash_{GS} G[A]_S$  then there is a graph  $K$  and a context  $C[\cdot]_R$  such that there are derivations*

$$\begin{array}{c} \emptyset \\ \mathcal{D}_C \parallel GS \\ C \end{array}
 \quad \text{and} \quad
 \begin{array}{c} \emptyset \\ \mathcal{D}_A \parallel GS \\ K \wp A \end{array}
 \quad \text{and} \quad
 \begin{array}{c} C[K \wp X]_R \\ \mathcal{D}_G \parallel GS \\ G[X]_S \end{array}$$

for any graph  $X$ .

The proof of this lemma proceeds by a case analysis on the structure of the context  $G[\cdot]_S$  employing splitting at each

step. It can be found in Appendix B. We show here only one case.

Assume  $G[A]_S = G'' \wp P(M_1[A]_{S'}, M_2, \dots, M_n)$  for some  $G''$ , prime graph  $P$  and  $M_1, \dots, M_n$ . Applying Lemma 6.1.(2) gives us three different cases, of which we show here only one: We get  $C'[\cdot]_{R'}$  and  $K_X$  and  $K_Y$ , such that

$$\begin{array}{c} C'[K_X \wp K_Y]_{R'} \\ \mathcal{D}_G'' \parallel \\ G'' \end{array}
 , \quad
 \begin{array}{c} \emptyset \\ \mathcal{D}_C \parallel \\ C' \end{array}
 , \quad
 \begin{array}{c} \emptyset \\ \mathcal{D}_X \parallel \\ K_X \wp M_1[A]_{S'} \end{array}
 ,$$

$$\begin{array}{c} \emptyset \\ \mathcal{D}_Y \parallel \\ K_Y \wp P(\emptyset, M_2, \dots, M_n) \end{array}
 .$$

We apply the induction hypothesis to  $\mathcal{D}_X$  and get  $K$  and  $C''[\cdot]_{R''}$ , such that

$$\begin{array}{c} \emptyset \\ \mathcal{D}_C'' \parallel \\ C'' \end{array}
 , \quad
 \begin{array}{c} \emptyset \\ \mathcal{D}_A \parallel \\ K \wp A \end{array}
 , \quad
 \begin{array}{c} C''[K \wp X]_{R''} \\ \mathcal{D}_G' \parallel GS \\ K_X \wp M_1[X]_{S'} \end{array}$$

for any  $X$ . We let  $C[\cdot]_R = C'[K_Y \wp P(C''[\cdot]_{R''}, M_2, \dots, M_n)]_{R'}$  and obtain  $\mathcal{D}_C$  via

$$\begin{array}{c} \emptyset \\ \mathcal{D}_C' \parallel GS \\ \begin{array}{c} \emptyset \\ \mathcal{D}_Y' \parallel GS \\ \begin{array}{c} \emptyset \\ \mathcal{D}_C'' \parallel GS \\ C'' \end{array} \end{array} \end{array}
 \wp Q(\dots)
 \end{array}
 ]_{R'}$$

and  $\mathcal{D}_G$  is as follows:

$$\begin{array}{c} C'[\dots]_{R'} \\ \text{ss}\downarrow \\ \begin{array}{c} C''[K \wp X]_{R''} \\ \mathcal{D}_G' \parallel \\ K_X \wp M_1[X]_{S'} \end{array} \wp P(M_2, \dots, M_n) \\ \hline \begin{array}{c} C'[K_X \wp K_Y]_{R'} \\ \mathcal{D}_G'' \parallel \\ G'' \end{array} \wp P(M_1[X]_{S'}, M_2, \dots, M_n) \end{array}
 .$$

The other cases follow by a similar reasoning.

## 8 Elimination of the Up-Fragment

In this section we discuss how we use splitting and context reduction to prove Theorem 5.4, i.e., the admissibility of the rules  $ai\uparrow$ ,  $ss\uparrow$ , and  $p\uparrow$ . For the rules  $ai\uparrow$  and  $ss\uparrow$ , this is similar to ordinary deep inference systems (see, e.g., [9, 21, 24, 41]). But for  $p\uparrow$ , there are surprising differences. In particular, we need to invoke an induction on the “size of the cut formula”. In other cut elimination proofs in deep inference, there is no need for such an induction, as it is outsourced to the splitting lemma.



$$\text{ax} \frac{}{a, a^\perp} \quad \otimes \frac{\Gamma, \phi \quad \Delta, \psi}{\Gamma, \phi \otimes \psi, \Delta} \quad \wp \frac{\Gamma, \phi, \psi}{\Gamma, \phi \wp \psi} \quad \text{mix} \frac{\Gamma \quad \Delta}{\Gamma, \Delta}$$

**Figure 3.** The inference rules of the system  $\text{MLL}_X$

$C_3[\cdot]_{R_3}$  and  $C_4[\cdot]_{R_4}$ , such that

$$\begin{array}{c} C_3[K_Z \wp K_W]_{R_3} \\ \mathcal{D}_8 \parallel \\ L_P \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_9 \parallel \\ C_3 \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_Z \parallel \\ K_Z \wp M_i \end{array}, \\ \\ \begin{array}{c} \emptyset \\ \mathcal{D}_W \parallel \\ K_W \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n) \end{array}, \\ \\ \begin{array}{c} C_4[H_X \wp H_Y]_{R_4} \\ \mathcal{D}_{10} \parallel \\ L_{P^\perp} \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_{11} \parallel \\ C_4 \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_X \parallel \\ H_X \wp N_j \end{array}, \\ \\ \begin{array}{c} \emptyset \\ \mathcal{D}_Y \parallel \\ H_Y \wp P^\perp(N_1, \dots, N_{j-1}, \emptyset, N_{j+1}, \dots, N_n) \end{array}$$

for some  $i, j \in \{1, \dots, n\}$ . In this case we use the derivation in Figure 2 to prove  $G[(M_1 \otimes N_1) \wp \dots \wp (M_n \otimes N_n)]_S$ . More precisely, Figure 2 shows the case  $i < j$ , the cases  $i = j$  and  $i > j$  are similar. The derivation  $\mathcal{D}^*$  exists by the second statement in Lemma 5.2. If  $i \neq j$ , this derivation consists of a single  $p\uparrow$  instance. If  $i = j$ , it can be a longer derivation containing all rules of  $\text{SGS}\uparrow$  (where the instances of  $\text{ai}\uparrow$  and  $\text{ss}\uparrow$  can be eliminated by the previous two theorems). The important observation to make is that all instances of  $p\uparrow$  occurring in  $\mathcal{D}^*$  have smaller size than the one we started with. Therefore we can invoke the induction hypothesis.

## 9 Conservativity

We are now able to show that GS is a conservative extension of unit-free multiplicative linear logic with mix ( $\text{MLL}_X$ ) [15]. The formulas of  $\text{MLL}_X$  are as in (3), but without the unit, and inference rules of  $\text{MLL}_X$  are shown in Figure 3 where  $\Gamma$  and  $\Delta$  are *sequents*, i.e. multisets of formulas, separated by commas. We write  $\vdash_{\text{MLL}_X} \Gamma$  if the sequent  $\Gamma$  is provable in  $\text{MLL}_X$ , i.e. if there is a derivation in  $\text{MLL}_X$  with conclusion  $\Gamma$ .

**Lemma 9.1.** *If  $\vdash_{\text{GS}} A$  and  $A$  is a cograph, then there is a derivation*

$$\begin{array}{c} \emptyset \\ \mathcal{D} \parallel \text{GS} \\ A \end{array} \quad (25)$$

such that every graph occurring in  $\mathcal{D}$  is a cograph.

The proof of this lemma proceeds by contradiction using Lemma 6.1. It can be found in Appendix D.

**Lemma 9.2.** *Let  $A$  and  $B$  be cographs. Then*

$$\text{ss}\downarrow \frac{A}{B} \quad \Longrightarrow \quad \begin{array}{c} A \\ \parallel \{s\} \\ B \end{array} \quad (26)$$

*Proof.* By Theorem 2.6, the graphs  $A$  and  $B$  are cographs iff there are formulas  $\phi$  and  $\psi$  with  $\llbracket \phi \rrbracket = A$  and  $\llbracket \psi \rrbracket = B$ . Now the statement follows from the corresponding statement for formulas (see e.g., Lemma 4.3.20 in [41]).  $\square$

**Theorem 9.3.** *Let  $A$  be a cograph. Then  $\vdash_{\text{GS}} A$  iff  $\vdash_{\{\text{ai}\downarrow, s\}} A$ .*

*Proof.* The implication from right to left follows immediately from the fact that  $s$  is a special case of  $\text{ss}\downarrow$  (see Lemma 4.9). For the implication from left to right, apply Lemma 9.1 to get a derivation  $\mathcal{D}$  that only uses cographs. Hence the rule  $p\downarrow$  is not used in  $\mathcal{D}$ . Therefore, by Lemma 9.2, we can get a derivation  $\mathcal{D}'$  that uses the rules  $\text{ai}\downarrow$  and  $s$ .  $\square$

**Corollary 9.4.** *For any unit-free formula  $\phi$ ,*

$$\vdash_{\text{MLL}_X} \phi \quad \Longleftrightarrow \quad \vdash_{\text{GS}} \llbracket \phi \rrbracket$$

*Proof.* It has been shown before (see, e.g., [19, 41] that a unit-free formula  $\phi$  is provable in  $\text{MLL}_X$  iff it is provable in  $\{\text{ai}\downarrow, s\}$  (note that in (15) we can have  $B = \emptyset$ ). Now the statement follows from Theorem 9.3 and Theorem 2.6.  $\square$

**Corollary 9.5.** *Provability in GS is NP-complete.*

*Proof.* Since  $\text{MLL}_X$  is NP-complete, we can conclude from Corollary 9.4 that GS is NP-hard. Containment in NP has been proved in Theorem 5.10.  $\square$

## 10 Discussion and related work

Here we draw attention to challenges surrounding GS. Using examples, such as (22) and (24), we have already explained why GS necessarily demands deep inference. Since no established deep inference system matches GS we have a fundamentally new proof system. Furthermore, we explain in this section that simply taking an established semantics for  $\text{MLL}_X$  based on graphs and dropping the restriction to cographs does not immediately yield a semantics for GS.

**Criteria for proof nets.** Graphical approaches to proof nets such as *R&B-graphs* [38] have valid definitions when we drop the restriction to cographs. However, we show that (at least without strengthening criteria), these definition do not yield a semantics for a logic over graphs, since logical principles laid out in the introduction are violated.

Consider again graph (8), which is not provable in GS. In an *R&B-graph* we draw blue edges representing the axiom links of proof nets, as shown below for graph (8).

$$\begin{array}{c} a^\perp \text{---} a \\ | \quad | \\ b \text{---} b^\perp \end{array} \quad (27)$$

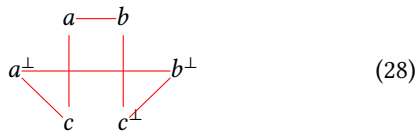
The established correctness criterion for *R&B-graphs* would wrongly accept the above graph. The reason is the cycle of 4



nodes alternating between red and blue edges has a chord. Notice this observation is independent of the rules of the system GS, since, in Sec. 4, we showed that graph (8) cannot be provable in a system subject to the logical principle of consistency.

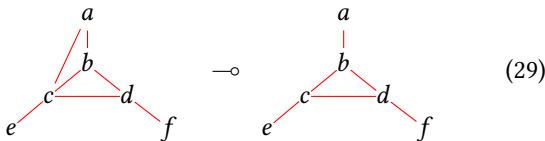
**What about cliques and stable sets?** The switch rule has the property that it reflects edges and maximal cliques. That is: if there is an edge in the conclusion it will also appear in the premise and every maximal clique in the premise is a superset of some maximal clique in the conclusion. Indeed, mappings reflecting maximal cliques and preserving stable sets (mutually independent nodes) have a long history in program semantics [3] which led to coherence spaces and the discovery of linear logic [15], see also [8, 12, 13]. Therefore it is a reasonable starting point to try generalising switch by using such maximal clique reflecting homomorphisms, instead of  $ss\downarrow$ . Indeed this is how we discovered  $ss\downarrow$ , which is sound with respect to such homomorphisms.

Unfortunately, replacing  $ss\downarrow$  with maximal clique reflecting homomorphisms yields a system distinct from our graphical system, for example the following would be provable, but is not provable in GS.



We may try replacing both  $ss\downarrow$  and  $ss\uparrow$  using a stronger symmetric notion of homomorphism where, in addition, every maximal stable set in the conclusion is a superset of some maximal stable set in the premise. Using such a homomorphism which is both maximal clique reflecting and stable set preserving as a rule, the above example is not provable. To see why, observe that at some point either  $a$  and  $\bar{a}$  or  $b$  and  $\bar{b}$  must be brought together into a module where they can interact, but this cannot be achieved while preserving the maximal stable set  $\{a, b^\perp\}$ .

Notice however, that if we replace  $ss\downarrow$  and  $ss\uparrow$  by the symmetric homomorphism described above, the implication below would be provable.



In contrast, the above is not provable in GS, since both sides are distinct prime graphs; and there is no suitable way to apply  $ss\downarrow$ . Thus, we would obtain a distinct system from GS by using such homomorphisms.

Studying logics coming out of reflecting maximal cliques and preserving maximal stable sets is currently a topic of active research and leads to possible extensions of Boolean logic to graphs [7, 8, 45].

**Generalised connectives.** In this paper, we use a modular decomposition of graphs based on prime graphs (see Lemma 3.7). The connectives  $\wp$  and  $\otimes$  are given by the prime graphs on two vertices. This choice is coherent with the graphs operations of union, join and composition, i.e.  $G \otimes H = \otimes(G, H)$  and  $G \wp H = \wp(G, H)$ . Pushing forward this idea, any graph can be interpreted as a (*multiplicative*) *generalized connective* [1, 11, 16]. In particular, in light of Lemma 3.7, every prime graph defines a non-decomposable connective. Furthermore, our Lemma 3.7 also provides a more refined notion of decomposition than the  $\wp$ - $\otimes$  decomposition known in the literature. However, the exact relation between the two constructions requires further investigation. Note, for example, that the number of pairs of orthogonal 6-ary non-decomposable connectives known at the time of writing is strictly smaller than the number of pairs of dual prime graphs on 6 vertices. Nonetheless, we conjecture that there is a correspondence between connectives defined by means of orthogonal sets of partitions and connectives defined by means of graphs.

## 11 Conclusion

Guided by logical principles, we have devised a minimal proof system (GS in Fig. 1) that operates directly over graphs, rather than formulas. Negation is generalised in terms of graph duality, while disjunction is disjoint union of graphs, allowing us to define implication “ $G$  implies  $H$ ” as the standard “not  $G$  or  $H$ ” (see Def. 4.1). All other design decisions are then fixed by our guiding logical principles. Most of these principles follow from cut elimination (Theorem 5.3), to which the majority of this paper is dedicated. We also confirm that GS conservatively extends  $MLL_\chi$  (Corollary 9.4) – a logic at the core of many proof systems.

Surprisingly, even for such a minimal generalisation of logic to graphs, deep inference is necessary. Proof systems for classical logic,  $MLL_\chi$  and many other logics *may* be expressed using deep inference, but deep inference is generally not necessary, since many standard logics have presentations in the sequent calculus where all inferences are applied at the root of some formula in a sequent. In contrast, for some logics (e.g., BV [17, 43] and modal logic S5 [35, 39]), deep inference is necessary in order to define a proof system satisfying cut elimination. System GS goes further than the aforementioned systems in that all intermediate lemmas such as splitting (Lemma 6.1) and context reduction (Lemma 7.1) also demand a deep formulation, requiring additional context awareness. As such we were required to generalise the basic mechanisms of deep inference itself in order to establish cut elimination (Theorem 5.3) for a logic over graphs. This is due to a property of general graphs that is forbidden in formulas – that the shortest path between any two connected nodes may be greater than two; and hence, when we apply

reasoning inside a module (i.e., a context), there may exist paths of dependencies that indirectly constrain the module.

**Acknowledgments.** We are very grateful for many insightful discussions with Anupam Das.

## References

- [1] Matteo Acclavio and Roberto Maieli. 2020. Generalized Connectives for Multiplicative Linear Logic. In *28th EACSL Annual Conference on Computer Science Logic (CSL 2020) (LIPIcs)*, Maribel Fernández and Anca Muscholl (Eds.), Vol. 152. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 6:1–6:16. <https://doi.org/10.4230/LIPIcs.CSL.2020.6>
- [2] Gianluigi Bellin. 1997. Subnets of proof-nets in multiplicative linear logic with MIX. *Mathematical Structures in Computer Science* 7, 6 (1997), 663–669. <https://doi.org/10.1017/S0960129597002326>
- [3] Gérard Berry. 1978. Stable models of typed  $\lambda$ -calculi. In *Automata, Languages and Programming*, Giorgio Ausiello and Corrado Böhm (Eds.). Springer, Berlin, Heidelberg, 72–89.
- [4] Kai Brännler and Alwen Fernanto Tiu. 2001. A Local System for Classical Logic. In *Logic for Programming, Artificial Intelligence, and Reasoning*, Robert Nieuwenhuis and Andrei Voronkov (Eds.). Springer, Berlin, Heidelberg, 347–361. [https://doi.org/10.1007/3-540-45653-8\\_24](https://doi.org/10.1007/3-540-45653-8_24)
- [5] Paola Bruscoli. 2002. A Purely Logical Account of Sequentiality in Proof Search. In *Logic Programming*, Peter J. Stuckey (Ed.). Springer, Berlin, Heidelberg, 302–316. [https://doi.org/10.1007/3-540-45619-8\\_21](https://doi.org/10.1007/3-540-45619-8_21)
- [6] Paola Bruscoli and Lutz Straßburger. 2017. On the Length of Medial-Switch-Mix Derivations. In *Logic, Language, Information, and Computation - 24th International Workshop, WoLLIC 2017, London, UK, July 18-21, 2017, Proceedings (Lecture Notes in Computer Science)*, Juliette Kennedy and Ruy J. G. B. de Queiroz (Eds.), Vol. 10388. Springer, 68–79. [https://doi.org/10.1007/978-3-662-55386-2\\_5](https://doi.org/10.1007/978-3-662-55386-2_5)
- [7] Cameron Calk. 2016. A graph theoretical extension of boolean logic. (2016). <http://www.anupamdas.com/graph-bool.pdf> Bachelor’s thesis.
- [8] Cameron Calk, Anupam Das, and Tim Waring. 2020. Beyond formulas-as-cographs: an extension of Boolean logic to arbitrary graphs. (2020). arXiv:cs.LO/2004.12941
- [9] Kaustuv Chaudhuri, Nicolas Guenot, and Lutz Straßburger. 2011. The Focused Calculus of Structures. In *CSL ’11 (LIPIcs)*, Marc Bezem (Ed.), Vol. 12. Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 159–173. <https://doi.org/10.4230/LIPIcs.CSL.2011.159>
- [10] Stephen A. Cook and Robert A. Reckhow. 1979. The Relative Efficiency of Propositional Proof Systems. *J. Symb. Log.* 44, 1 (1979), 36–50. <https://doi.org/10.2307/2273702>
- [11] Vincent Danos and Laurent Regnier. 1989. The structure of the multiplicatives. *Arch. Math. Log.* 28, 3 (1989), 181–203. <https://doi.org/10.1007/BF01622878>
- [12] Anupam Das and Lutz Straßburger. 2015. No complete linear term rewriting system for propositional logic. In *26th International Conference on Rewriting Techniques and Applications (RTA 2015) (LIPIcs)*, Maribel Fernández (Ed.), Vol. 36. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 127–142. <https://doi.org/10.4230/LIPIcs.RTA.2015.127>
- [13] Anupam Das and Lutz Straßburger. 2016. On linear rewriting systems for Boolean logic and some applications to proof theory. *Logical Methods in Computer Science* 12, 4 (2016), 1–27. [https://doi.org/10.2168/LMCS-12\(4:9\)2016](https://doi.org/10.2168/LMCS-12(4:9)2016)
- [14] Arnaud Fleury and Christian Retoré. 1994. The mix rule. *Mathematical Structures in Computer Science* 4, 2 (1994), 273–285. <https://doi.org/10.1017/S0960129500000451>
- [15] Jean-Yves Girard. 1987. Linear Logic. *Theoretical Computer Science* 50 (1987), 1–102. [https://doi.org/10.1016/0304-3975\(87\)90045-4](https://doi.org/10.1016/0304-3975(87)90045-4)
- [16] Jean-Yves Girard. 2000. On the meaning of logical rules II: multiplicatives and additives. *NATO ASI Series F: Computer and Systems Sciences* 175 (2000), 183–212.
- [17] Alessio Guglielmi. 2007. A System of Interaction and Structure. *ACM Transactions on Computational Logic* 8, 1 (2007), 1–64. <https://doi.org/10.1145/1182613.1182614>
- [18] Alessio Guglielmi, Tom Gundersen, and Michel Parigot. 2010. A Proof Calculus Which Reduces Syntactic Bureaucracy. In *Proceedings of the 21st International Conference on Rewriting Techniques and Applications (LIPIcs)*, Christopher Lynch (Ed.), Vol. 6. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 135–150. <https://doi.org/10.4230/LIPIcs.RTA.2010.135>
- [19] Alessio Guglielmi and Lutz Straßburger. 2001. Non-commutativity and MELL in the Calculus of Structures. In *Computer Science Logic*, Laurent Fribourg (Ed.). Springer, Berlin, Heidelberg, 54–68. [https://doi.org/10.1007/3-540-44802-0\\_5](https://doi.org/10.1007/3-540-44802-0_5)
- [20] Alessio Guglielmi and Lutz Straßburger. 2002. A Non-commutative Extension of MELL. In *Logic for Programming, Artificial Intelligence, and Reasoning*, Matthias Baaz and Andrei Voronkov (Eds.). Springer, Berlin, Heidelberg, 231–246. [https://doi.org/10.1007/3-540-36078-6\\_16](https://doi.org/10.1007/3-540-36078-6_16)
- [21] Alessio Guglielmi and Lutz Straßburger. 2011. A system of interaction and structure V: the exponentials and splitting. *Mathematical Structures in Computer Science* 21, 3 (2011), 563–584. <https://doi.org/10.1017/S096012951100003X>
- [22] Ross Horne. 2019. The Sub-Additives: A Proof Theory for Probabilistic Choice extending Linear Logic. In *4th International Conference on Formal Structures for Computation and Deduction, FSCD 2019, June 24-30, 2019, Dortmund, Germany (LIPIcs)*, Herman Geuvers (Ed.), Vol. 131. Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 23:1–23:16. <https://doi.org/10.4230/LIPIcs.FSCD.2019.23>
- [23] Ross Horne and Alwen Tiu. 2019. Constructing weak simulations from linear implications for processes with private names. *Mathematical Structures in Computer Science* 29, 8 (2019), 1275–1308. <https://doi.org/10.1017/S0960129518000452>
- [24] Ross Horne, Alwen Tiu, Bogdan Aman, and Gabriel Ciobanu. 2019. De Morgan Dual Nominal Quantifiers Modelling Private Names in Non-Commutative Logic. *ACM Trans. Comput. Log.* 20, 4 (2019), 22:1–22:44. <https://doi.org/10.1145/3325821>
- [25] William Alvin Howard. 1980. The Formulae-as-Types Notion of Construction. In *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, J. P. Seldin and J. R. Hindley (Eds.). Academic Press, London, 479–490.
- [26] Dominic Hughes. 2006. Proofs Without Syntax. *Annals of Mathematics* 164, 3 (2006), 1065–1076. <https://doi.org/10.4007/annals.2006.164.1065>
- [27] Naoki Kobayashi and Akinori Yonezawa. 1993. ACL – a Concurrent Linear Logic Programming Paradigm. In *Proceedings of the 1993 International Symposium on Logic Programming (ILPS ’93)*. MIT Press, Cambridge, MA, USA, 279–294.
- [28] Kamal Lodaya and Pascal Weil. 2000. Series-parallel languages and the bounded-width property. *Theoretical Computer Science* 237, 1 (2000), 347 – 380. [https://doi.org/10.1016/S0304-3975\(00\)00031-1](https://doi.org/10.1016/S0304-3975(00)00031-1)
- [29] Ross M. McConnell and Jeremy P. Spinrad. 1994. Linear-Time Modular Decomposition and Efficient Transitive Orientation of Comparability Graphs. In *Proceedings of the Fifth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA ’94)*. Society for Industrial and Applied Mathematics, USA, 536–545.
- [30] Dale Miller. 1993. The  $\pi$ -calculus as a theory in linear logic: Preliminary results. In *Extensions of Logic Programming*, E Lamma and P. Mello (Eds.). Springer, Berlin, Heidelberg, 242–264. [https://doi.org/10.1007/3-540-56454-3\\_13](https://doi.org/10.1007/3-540-56454-3_13)
- [31] Dale Miller, Gopalan Nadathur, Frank Pfenning, and Andre Scedrov. 1991. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic* 51, 1-2 (1991), 125–157. [https://doi.org/10.1016/0168-0072\(91\)90068-W](https://doi.org/10.1016/0168-0072(91)90068-W)

- [32] Rolf H. Möhring. 1989. Computationally Tractable Classes of Ordered Sets. In *Algorithms and Order*, Ivan Rival (Ed.). Springer Netherlands, Dordrecht, 105–193. [https://doi.org/10.1007/978-94-009-2639-4\\_4](https://doi.org/10.1007/978-94-009-2639-4_4)
- [33] Mogens Nielsen, Gordon Plotkin, and Glynn Winskel. 1981. Petri nets, event structures and domains, part I. *Theoretical Computer Science* 13, 1 (1981), 85 – 108. [https://doi.org/10.1016/0304-3975\(81\)90112-2](https://doi.org/10.1016/0304-3975(81)90112-2)
- [34] Carl Adam Petri. 1977. Interpretations of net theory. *Comput. Surveys* 9, 3 (1977), 223–252.
- [35] Francesca Poggiolesi. 2008. A cut-free simple sequent calculus for modal logic S5. *The Review of Symbolic Logic* 1, 1 (2008), 3–15. <https://doi.org/10.1017/S1755020308080040>
- [36] Vaughan Pratt. 1986. Modeling concurrency with partial orders. *International Journal of Parallel Programming* 15, 1 (1986), 33–71. <https://doi.org/10.1007/BF01379149>
- [37] Christian Retoré. 1997. Pomset logic: A non-commutative extension of classical linear logic. In *Typed Lambda Calculi and Applications*, Philippe de Groote and J. Roger Hindley (Eds.). Springer, Berlin, Heidelberg, 300–318. [https://doi.org/10.1007/3-540-62688-3\\_43](https://doi.org/10.1007/3-540-62688-3_43)
- [38] Christian Retoré. 2003. Handsome proof-nets: perfect matchings and cographs. *Theoretical Computer Science* 294, 3 (2003), 473–488. [https://doi.org/10.1016/S0304-3975\(01\)00175-X](https://doi.org/10.1016/S0304-3975(01)00175-X)
- [39] Phiniki Stouppa. 2007. A deep inference system for the modal logic S5. *Studia Logica* 85, 2 (2007), 199–214. <https://doi.org/10.1007/s11225-007-9028-y>
- [40] Lutz Straßburger. 2002. A Local System for Linear Logic. In *Logic for Programming, Artificial Intelligence, and Reasoning*, Matthias Baaz and Andrei Voronkov (Eds.). Springer, Berlin, Heidelberg, 388–402. [https://doi.org/10.1007/3-540-36078-6\\_26](https://doi.org/10.1007/3-540-36078-6_26)
- [41] Lutz Straßburger. 2003. *Linear Logic and Noncommutativity in the Calculus of Structures*. Ph.D. Dissertation. Technische Universität Dresden.
- [42] Lutz Straßburger. 2017. Combinatorial Flows and Their Normalisation. In *2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017) (LIPIcs)*, Dale Miller (Ed.), Vol. 84. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 31:1–31:17. <https://doi.org/10.4230/LIPIcs.FSCD.2017.31>
- [43] Alwen Fernanto Tiu. 2006. A System of Interaction and Structure II: The Need for Deep Inference. *Logical Methods in Computer Science* 2, 2 (2006), 1–24. [https://doi.org/10.2168/LMCS-2\(2:4\)2006](https://doi.org/10.2168/LMCS-2(2:4)2006)
- [44] Andrea Aler Tubella. 2017. *A study of normalisation through subatomic logic*. Ph.D. Dissertation. University of Bath.
- [45] Timothy Waring. 2019. A Graph theoretic extension of Boolean logic. (2019). [http://anupamdas.com/thesis\\_tim-waring.pdf](http://anupamdas.com/thesis_tim-waring.pdf) Master’s thesis.

## A Proof of Splitting (Lemma 6.1)

**Observation A.1.** Whenever we have a derivation from  $A$  to  $B$  in GS, then  $|A| \leq |B|$ , as the rules  $\text{ss}\downarrow$  and  $\text{p}\downarrow$  do not change the size of a graph, and the rule  $\text{ai}\downarrow$  deletes two vertices when going up in a derivation.

**Lemma A.2.** Let  $C_1[\cdot]_{R_1}, \dots, C_n[\cdot]_{R_n}$  be contexts. If  $\vdash_{\text{GS}} C_i$  for all  $i \in \{1, \dots, n\}$ , then  $\vdash_{\text{GS}} C_n[\dots C_2[C_1]_{R_2}]_{R_n}$ .

*Proof.* We proceed by induction on  $n$ . The base case for  $n = 1$  is trivial, and the inductive case for  $n > 1$  is this derivation:

$$\frac{\begin{array}{c} \emptyset \\ \mathcal{D}_n \parallel \\ C_n[ \begin{array}{c} \emptyset \\ \mathcal{D}' \parallel \\ C_{n-1}[\dots C_2[C_1]_{R_2}]_{R_{n-1}} \end{array} ]_{R_1} \end{array}}{\quad},$$

where  $\mathcal{D}_n$  is the derivation for  $C_n$  and  $\mathcal{D}'$  exists by induction hypothesis.  $\square$

*Proof of Lemma 6.1.* We prove all three statements simultaneously, by induction on the pair  $\langle |F|, |\mathcal{D}| \rangle$ , ordered lexicographically, where  $F$  is the graph provable in the premise of each statement (i.e.,  $F = G \wp (A \otimes B)$  in (1), and  $F = G \wp P(M_1, \dots, M_n)$  in (2), and  $F = G \wp a$  in (3)) and  $\mathcal{D}$  is the proof of  $F$ .

(1) Assume we have a proof  $\mathcal{D}$  of  $G \wp (A \otimes B)$ . We have to find graphs  $K_A$  and  $K_B$ , and a context  $C[\cdot]_R$ , and derivations  $\mathcal{D}_G, \mathcal{D}_C, \mathcal{D}_A, \mathcal{D}_B$ , as in the statement of Lemma 6.1.(1). Note that if one of  $G, A, B$  is empty, then the statement holds trivially (with  $C = \emptyset$ , see Corollary 5.7). We now assume that  $G, A, B$  are all non-empty and make a case analysis on the bottommost rule instance  $r$  in  $\mathcal{D}$ .

(a) The rule  $r$  acts inside one of  $G, A$ , or  $B$ . I.e., the derivation  $\mathcal{D}$  is of shape

$$\frac{\begin{array}{c} \emptyset \\ \mathcal{D}' \parallel \\ \frac{G'}{r} \frac{G}{G} \wp (A \otimes B) \end{array}}{\quad} \quad \text{or} \quad \frac{\begin{array}{c} \emptyset \\ \mathcal{D}' \parallel \\ G \wp \left( \frac{A'}{r} \frac{A}{A} \otimes B \right) \end{array}}{\quad} \quad \text{or} \quad \frac{\begin{array}{c} \emptyset \\ \mathcal{D}' \parallel \\ G \wp (A \otimes \frac{B'}{r} \frac{B}{B}) \end{array}}{\quad}$$

for some  $\mathcal{D}'$ . In each case we can apply the induction hypothesis to  $\mathcal{D}'$  as  $|\mathcal{D}'| < |\mathcal{D}|$  and conclude immediately by adding the corresponding application of  $r$  to  $\mathcal{D}_G$  or  $\mathcal{D}_A$  or  $\mathcal{D}_B$ , respectively.

(b)  $G = G'' \wp G'$  and  $\mathcal{D}$  is of shape

$$\frac{\begin{array}{c} \emptyset \\ \mathcal{D}' \parallel \\ G'' \wp (A \otimes B)[G']_S \end{array}}{\text{ss}\downarrow \frac{G'' \wp G' \wp (A \otimes B)}}.$$

We make a case analysis on the structure of the graph  $(A \otimes B)[G']_S$ , using Lemma 3.7:

- (i)  $(A \otimes B)[G']_S$  cannot be the singleton graph as neither  $A$  nor  $B$  are empty.
- (ii)  $(A \otimes B)[G']_S$  cannot be a par as this would imply  $S = \emptyset$ , contradicting the side condition of  $\text{ss}\downarrow$ .
- (iii)  $(A \otimes B)[G']_S$  is a tensor. We have the following possibilities:

- (I)  $(A \otimes B)[G']_S = A[G']_{S'} \otimes B$  for some  $S' \subseteq S$ . We can apply the induction hypothesis to  $\mathcal{D}'$ , and get  $C[\cdot]_R$  and  $K'_A$  and  $K_B$  such that

$$\frac{C[K'_A \wp K_B]_R}{\mathcal{D}'_G \parallel \text{GS}} \quad , \quad \frac{\emptyset}{\mathcal{D}_C \parallel \text{GS}} \quad , \quad \frac{\emptyset}{K'_A \wp A[G']_S} \quad , \quad \frac{\emptyset}{K_B \wp B}.$$

We can let  $K_A = G' \wp K'_A$  and obtain  $\mathcal{D}_G$  and  $\mathcal{D}_A$  via

$$\text{ss}\downarrow \frac{C[G' \wp K'_A \wp K_B]_R}{G' \wp \frac{C[K'_A \wp K_B]_R}{\mathcal{D}'_G \parallel \text{GS}}} \quad \text{and} \quad \frac{\begin{array}{c} \emptyset \\ \mathcal{D}'_A \parallel \text{GS} \\ K'_A \wp A[G']_S \end{array}}{\text{ss}\downarrow \frac{G' \wp K'_A \wp A}}.$$

respectively.

- (II)  $(A \otimes B)[G']_S = A \otimes B[G']_{S'}$  for some  $S' \subseteq S$ . This case is similar to the previous one.
- (III)  $(A \otimes B)[G']_S = A'' \otimes (A' \otimes B)[G']_{S'}$  for some  $S' \subseteq S$ , where  $A = A'' \otimes A'$  and  $A'' \neq \emptyset \neq A'$ . We can apply the induction hypothesis to  $\mathcal{D}'$  and get  $K''_A$  and  $L$  and  $C'[\cdot]_{R'}$  such that

$$\frac{C'[K''_A \wp L]_{R_1}}{\mathcal{D}'_G \parallel \text{GS}} \quad , \quad \frac{\emptyset}{\mathcal{D}'_C \parallel \text{GS}} \quad , \quad \frac{\emptyset}{C'}$$

$$\frac{\emptyset}{\mathcal{D}'_A \parallel \text{GS}} \quad , \quad \frac{\emptyset}{L \wp (A' \otimes B)[G']_{S'}}.$$

From  $\mathcal{D}_L$  we get that  $\vdash_{\text{GS}} L \wp G' \wp (A \otimes B)$  (via the rule  $\text{ss}\downarrow$ ). To this we can apply the induction hypothesis to get a context  $C''[\cdot]_{R''}$  and  $K'_A$  and  $K_B$  such that

$$\frac{C''[K'_A \wp K_B]_{R''}}{\mathcal{D}'_G \parallel \text{GS}} \quad , \quad \frac{\emptyset}{\mathcal{D}'_C \parallel \text{GS}} \quad , \quad \frac{\emptyset}{K'_A \wp A'} \quad , \quad \frac{\emptyset}{K_B \wp B}.$$

We can let  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$  (and hence  $C = C'[C'']_{R'}$ ), and  $K_A = K''_A \wp K'_A$ . We obtain  $\mathcal{D}_G$  via

$$\text{ss}\downarrow \frac{C' \left[ \begin{array}{c} \text{ss}\downarrow \frac{C''[K''_A \wp K'_A \wp K_B]_{R''}}{K''_A \wp \left( \begin{array}{c} C''[K'_A \wp K_B]_{R''} \\ \mathcal{D}'_G \parallel \\ L \wp G' \end{array} \right)} \end{array} \right]_{R'}}{C' \left[ \begin{array}{c} C''[K''_A \wp L]_{R'} \\ \mathcal{D}''_G \parallel \\ G'' \end{array} \right]_{R'} \wp G'}$$

and  $\mathcal{D}_A$  via

$$\text{ss}\downarrow \frac{\left( \begin{array}{c} \emptyset \\ \mathcal{D}'_{A'} \parallel \\ K''_A \wp (A'' \otimes \left( \begin{array}{c} \emptyset \\ \mathcal{D}'_{A'} \parallel \\ K'_A \wp A' \end{array} \right)) \end{array} \right)}{K''_A \wp K'_A \wp (A'' \otimes A')}$$

Finally,  $\mathcal{D}_C$  is obtained via Lemma A.2 from  $\mathcal{D}'_C$  and  $\mathcal{D}''_C$ .

(IV)  $(A \otimes B)[G']_S = (A \otimes B')[G']_{S'} \otimes B''$  for some  $S' \subseteq S$ , where  $B = B' \otimes B''$  and  $B' \neq \emptyset \neq B''$ .

This case is similar to the previous one.

(iv)  $(A \otimes B)[G']_S$  is composed via a prime graph  $Q$ , i.e., it is of shape  $Q(A_1 \otimes B_1, \dots, A_h \otimes B_h, G')$  with  $|V_Q| = h + 1 \geq 4$  and  $A_1, \dots, A_h$  being modules of  $A$  and  $B_1, \dots, B_h$  being modules of  $B$ , such that at least one of  $A_i, B_i$  is non-empty for every  $i \in \{1, \dots, h\}$ .<sup>5</sup> Note that we also have

$$Q(A_1 \otimes B_1, \dots, A_h \otimes B_h, \emptyset) = A \otimes B \quad (30)$$

We apply the induction hypothesis to  $\mathcal{D}'$  and get one of the following three cases

( $\alpha$ ) We get  $K_1, \dots, K_h, K_{G'}$  and a context  $C'[\cdot]_{R'}$ , such that

$$C'[Q^\perp(K_1, \dots, K_h, L)]_R \quad \mathcal{D}''_G \parallel \quad G''$$

$$\mathcal{D}'_C \parallel \quad \mathcal{D}_i \parallel \quad \mathcal{D}_L \parallel$$

$$C' \quad K_i \wp (A_i \otimes B_i) \quad L \wp G'$$

<sup>5</sup>Note that at least one  $A_i$  or  $B_i$  has to be empty for some  $i$ , in order to make this case possible.

for all  $1 \leq i \leq h$ . Now observe that because of (30) we have the following proof

$$\text{ss}\downarrow \frac{\left( \begin{array}{c} \emptyset \\ \mathcal{D}'_C \parallel \\ C' \left[ \begin{array}{c} \emptyset \\ \mathcal{D}_1 \parallel \\ K_1 \wp (A_1 \otimes B_1) \end{array} \right] \otimes \dots \otimes \left[ \begin{array}{c} \emptyset \\ \mathcal{D}_h \parallel \\ K_h \wp (A_h \otimes B_h) \end{array} \right]_{R'} \end{array} \right)}{\mathcal{D} \parallel \quad Q^\perp(K_1, \dots, K_h, \emptyset) \wp (A \otimes B)} \\ C'[Q^\perp(K_1, \dots, K_h, \emptyset)]_{R'} \wp (A \otimes B)$$

where  $\tilde{\mathcal{D}}$  is given by Lemma 5.2. To this proof we can apply the induction hypothesis (since  $G'$  is non-empty), and get  $K_A, K_B$  and a context  $C[\cdot]_R$ , such that

$$C[K_A \wp K_B]_R \quad \mathcal{D}'_G \parallel \quad C'[Q^\perp(K_1, \dots, K_h, \emptyset)]_R$$

$$\mathcal{D}_C \parallel \quad \mathcal{D}_A \parallel \quad \mathcal{D}_B \parallel$$

$$C \quad K_A \wp A \quad K_B \wp B$$

Now  $\mathcal{D}_G$  is the derivation

$$\text{ss}\downarrow \frac{\left( \begin{array}{c} C[K_A \wp K_B]_R \\ \mathcal{D}'_G \parallel \\ C'[Q^\perp(K_1, \dots, K_h, \left( \begin{array}{c} \emptyset \\ \mathcal{D}_L \parallel \\ L \wp G' \end{array} \right))]_{R'} \end{array} \right)}{C'[Q^\perp(K_1, \dots, K_h, L)]_R \quad \mathcal{D}''_G \parallel \quad G''}$$

( $\beta$ ) We have  $L_X$  and  $L_Y$  and  $C'[\cdot]_{R'}$  such that

$$C'[L_X \wp L_Y]_{R'} \quad \mathcal{D}''_G \parallel \quad \mathcal{D}'_C \parallel \quad \mathcal{D}_X \parallel$$

$$G'' \quad C' \quad L_X \wp (A_1 \otimes B_1)$$

$$\mathcal{D}_Y \parallel \quad L_Y \wp Q(\emptyset, A_2 \otimes B_2, \dots, A_h \otimes B_h, G')$$

(Note that for easier typesetting we pick without loss of generality  $A_1 \otimes B_1$  instead of an arbitrary  $A_i \otimes B_i$  for some  $i \in \{1, \dots, h\}$ .) From  $\mathcal{D}_Y$ , we get (via the rule  $\text{ss}\downarrow$ ) that

$$\vdash_{\text{GS}} L_Y \wp G' \wp Q(\emptyset, A_2 \otimes B_2, \dots, A_h \otimes B_h, \emptyset)$$

to which we now apply the induction hypothesis. Observe that by (30) we have  $Q(\emptyset, A_2 \otimes B_2, \dots, A_h \otimes B_h, \emptyset) = A' \otimes B'$ , for some  $A'$  and  $B'$  where  $A = A'[A_1]_S$  and  $B = B'[B_1]_T$



for some  $S, T$ . We therefore have  $K'_A$  and  $K'_B$  and  $C''[\cdot]_{R''}$  such that

$$\begin{array}{c} C''[K'_A \wp K_B]_{R''} \\ \mathcal{D}_G \parallel \\ L_Y \wp G' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_C \parallel \\ C'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_A \parallel \\ K'_A \wp A' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_B \parallel \\ K'_B \wp B' \end{array}.$$

Similarly, we can apply the induction hypothesis to  $\mathcal{D}_X$  and get have  $K''_A$  and  $K''_B$  and  $C'''[\cdot]_{R'''}$  such that

$$\begin{array}{c} C'''[K''_A \wp K''_B]_{R'''} \\ \mathcal{D}''_G \parallel \\ L_X \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}''_C \parallel \\ C''' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}''_A \parallel \\ K''_A \wp A_1 \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}''_B \parallel \\ K''_B \wp B_1 \end{array}.$$

We let  $K_A = K''_A \wp K'_A$ ,  $K_B = K''_B \wp K'_B$  and  $C[\cdot]_R = C'[C'''[C''[\cdot]_{R''}]_{R'}]_{R'}$ . Then  $\mathcal{D}_G$  is

$$\begin{array}{c} C''[\begin{array}{c} C'''[K''_A \wp K''_B \wp K'_A \wp K'_B]_{R'''} \\ \text{ss}\downarrow \\ \frac{C'''[K''_A \wp K''_B \wp C''[K'_A \wp K'_B]_{R''}}{K''_A \wp K''_B \wp C''[K'_A \wp K'_B]_{R''}} \end{array}]_{R'''} \\ C'[\text{ss}\downarrow \begin{array}{c} C'''[K''_A \wp K''_B]_{R'''} \\ \mathcal{D}''_G \parallel \\ L_X \end{array} \wp \begin{array}{c} C''[K'_A \wp K'_B]_{R''} \\ \mathcal{D}'_G \parallel \\ L_Y \wp G' \end{array}]_{R'} \\ \text{ss}\downarrow \\ \begin{array}{c} C'[L_X \wp L_Y]_{R'} \\ \mathcal{D}''_G \parallel \\ G'' \end{array} \wp G' \end{array},$$

Derivation  $\mathcal{D}_A$  is as follows:

$$\begin{array}{c} \begin{array}{c} \emptyset \\ \mathcal{D}'_A \parallel \\ K'_A \wp A' \end{array} \begin{array}{c} \emptyset \\ \mathcal{D}'_A \parallel \\ K''_A \wp A_1 \end{array} ]_S \\ \text{ss}\downarrow \\ \frac{K''_A \wp K'_A \wp A'[A_1]_S}{K''_A \wp K'_A \wp A'[A_1]_S} \end{array},$$

and  $\mathcal{D}_B$  is similar. Finally,  $\mathcal{D}_C$  exists by Lemma A.2.

(y) We have  $L_X$  and  $L_Y$  and  $C'[\cdot]_{R'}$  such that

$$\begin{array}{c} C'[L_X \wp L_Y]_{R'} \\ \mathcal{D}''_G \parallel \\ G'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_C \parallel \\ C' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_X \parallel \\ L_X \wp G' \end{array},$$

$$\begin{array}{c} \emptyset \\ \mathcal{D}_Y \parallel \\ L_Y \wp Q(A_1 \otimes B_1, \dots, A_h \otimes B_h, \emptyset) \end{array}.$$

We apply the induction hypothesis to  $\mathcal{D}_Y$ , using (30). This gives us  $K_A$  and  $K_B$  and  $C''[\cdot]_{R''}$  such that

$$\begin{array}{c} C''[K_A \wp K_B]_{R''} \\ \mathcal{D}''_G \parallel \\ L_Y \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_C \parallel \\ C'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_A \parallel \\ K_A \wp A \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_B \parallel \\ K_B \wp B \end{array}.$$

We let  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$  and get  $\mathcal{D}_C$  from Lemma A.2. Finally,  $\mathcal{D}_G$  is

$$\begin{array}{c} \begin{array}{c} \emptyset \\ \mathcal{D}_X \parallel \\ L_X \wp G' \end{array} \wp \begin{array}{c} C''[K_A \wp K_B]_{R''} \\ \mathcal{D}'_G \parallel \\ L_Y \end{array} ]_{R'} \\ \text{ss}\downarrow \\ \begin{array}{c} C'[L_X \wp L_Y]_{R'} \\ \mathcal{D}''_G \parallel \\ G'' \end{array} \wp G' \end{array}.$$

(c) In the next case to consider we also have  $G = G'' \wp G'$ , and  $\mathcal{D}$  is (without loss of generality) of shape

$$\begin{array}{c} \emptyset \\ \mathcal{D}' \parallel \\ G'' \wp G'[A \otimes B]_S \\ \text{ss}\downarrow \\ \frac{G'' \wp G' \wp (A \otimes B)}{G'' \wp G' \wp (A \otimes B)} \end{array}$$

We proceed by a case analysis on  $G'[A \otimes B]_S$ , using Lemma 3.7. Observe that  $G'[A \otimes B]_S$  cannot be a single-vertex graph, and without loss of generality, we can assume it is not of shape  $E \wp D$  for non-empty  $E, D$ . Hence, there are only two cases to consider:

(iii)  $G'[A \otimes B]_S$  is a tensor of two graphs. Without loss of generality, we can assume  $G'[A \otimes B]_S = E \otimes D[A \otimes B]_{S'}$ , with  $E \neq \emptyset$ . We apply the induction hypothesis to  $\mathcal{D}'$  and get  $C'[\cdot]_{R'}$  and  $K_E$  and  $K_D$  such that

$$\begin{array}{c} C'[K_E \wp K_D]_{R'} \\ \mathcal{D}''_G \parallel_{GS} \\ G'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_C \parallel_{GS} \\ C' \end{array},$$

$$\begin{array}{c} \emptyset \\ \mathcal{D}_E \parallel_{GS} \\ K_E \wp E \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_D \parallel_{GS} \\ K_D \wp D[A \otimes B]_{S'} \end{array}.$$

From  $\mathcal{D}_D$ , we get  $\vdash_{GS} K_D \wp D \wp (A \otimes B)$  via the  $\text{ss}\downarrow$ -rule. Since  $|E| > 0$  we can apply the induction hypothesis again, and get  $C''[\cdot]_{R''}$  and  $K_A$  and  $K_B$  such that

$$\begin{array}{c} C''[K_A \wp K_B]_{R''} \\ \mathcal{D}''_G \parallel_{GS} \\ K_D \wp D \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}''_C \parallel_{GS} \\ C'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_A \parallel_{GS} \\ K_A \wp A \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_B \parallel_{GS} \\ K_B \wp B \end{array}.$$

Note that  $G = G'' \wp (E \otimes D)$ . We therefore can let  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$  and obtain  $\mathcal{D}_C$  from

Lemma A.2. We obtain  $\mathcal{D}_G$  via

$$\begin{array}{c}
 \begin{array}{c}
 C''[\hat{K}]_{R''} \\
 \mathcal{D}'_G \parallel \\
 \emptyset \\
 K_D \wp (\mathcal{D}_E \parallel \otimes D) \\
 K_E \wp E
 \end{array} \\
 \text{ss}\downarrow \\
 \begin{array}{c}
 C'[K_E \wp K_D]_{R'} \\
 \mathcal{D}'_G \parallel \\
 G''
 \end{array} \wp (E \otimes D) \\
 \text{ss}\downarrow \\
 \begin{array}{c}
 C'[K_E \wp K_D]_{R'} \\
 \mathcal{D}'_G \parallel \\
 G''
 \end{array} \wp (E \otimes D)
 \end{array}$$

where  $\hat{K}$  abbreviates  $K_A \wp K_B$ .

- (iv)  $G'[A \otimes B]_S$  is composed via a prime graph  $Q$ . Without loss of generality, we can assume that  $G'[A \otimes B]_S = Q(N_1[A \otimes B]_{S'}, N_2, \dots, N_m)$  with  $|V_Q| = m \geq 4$ . We apply the induction hypothesis to  $\mathcal{D}'$  and get one of the following three cases:

- (iv.α) We have  $C'[\cdot]_{R'}$  and  $L_1, \dots, L_m$  such that

$$\begin{array}{c}
 C'[Q^+(L_1, \dots, L_m)]_{R'} \\
 \mathcal{D}'_G \parallel \text{GS} \\
 G''
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_C \parallel \text{GS} \\
 C'
 \end{array}
 ,$$

$$\begin{array}{c}
 \emptyset \\
 \mathcal{D}'_i \parallel \text{GS} \\
 L_i \wp N_i[A \otimes B]_{S'}
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_i \parallel \text{GS} \\
 L_i \wp N_i
 \end{array}$$

for  $2 \leq i \leq m$ . From  $\mathcal{D}'_i$ , we get (via the  $\text{ss}\downarrow$ -rule) that  $\vdash_{\text{GS}} L_i \wp N_i \wp (A \otimes B)$ , to which we can apply the induction hypothesis again and get  $C''[\cdot]_{R''}$  and  $K_A$  and  $K_B$  such that

$$\begin{array}{c}
 C''[K_A \wp K_B]_{R''} \\
 \mathcal{D}'_G \parallel \text{GS} \\
 L_1 \wp N_1
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_C \parallel \text{GS} \\
 C''
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}_A \parallel \text{GS} \\
 K_A \wp A
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}_B \parallel \text{GS} \\
 K_B \wp B
 \end{array}
 .$$

As  $G = G'' \wp Q(N_1, \dots, N_m)$ , we can (as before) let  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$  and obtain  $\mathcal{D}_G$  as

$$\begin{array}{c}
 \begin{array}{c}
 C''[\hat{K}]_{R''} \\
 \mathcal{D}'_G \parallel \\
 L_1 \wp N_1
 \end{array} \otimes \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_2 \parallel \\
 L_2 \wp N_2
 \end{array} \otimes \dots \otimes \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_m \parallel \\
 L_m \wp N_m
 \end{array} \\
 \text{pl}\downarrow \\
 Q^+(L_1, \dots, L_m) \wp Q(N_1, \dots, N_m) \\
 \text{ss}\downarrow \\
 \begin{array}{c}
 C'[Q^+(L_1, \dots, L_m)]_{R'} \\
 \mathcal{D}'_G \parallel \\
 G''
 \end{array} \wp Q(N_1, \dots, N_m)
 \end{array}$$

where  $\hat{K}$  abbreviates  $K_A \wp K_B$ , and  $\mathcal{D}_C$  is obtained via Lemma A.2.

- (iv.β) We have  $C'[\cdot]_{R'}$  and  $L_X$  and  $L_Y$  such that

$$\begin{array}{c}
 C'[L_X \wp L_Y]_{R'} \\
 \mathcal{D}'_G \parallel \text{GS} \\
 G''
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_C \parallel \text{GS} \\
 C'
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_X \parallel \text{GS} \\
 L_X \wp N_1[A \otimes B]_{S'}
 \end{array}
 ,$$

$$\begin{array}{c}
 \emptyset \\
 \mathcal{D}'_Y \parallel \text{GS} \\
 L_Y \wp Q(\emptyset, N_2, \dots, N_m)
 \end{array}
 ,$$

i.e., we assume  $i = 1$  in the second case of (2) in Lemma 6.1. From  $\mathcal{D}'_X$  we get (via the  $\text{ss}\downarrow$ ) that  $\vdash_{\text{GS}} L_X \wp N_1 \wp (A \otimes B)$ , to which we can apply the induction hypothesis again to get  $C''[\cdot]_{R''}$  and  $K_A$  and  $K_B$  such that

$$\begin{array}{c}
 C''[K_A \wp K_B]_{R''} \\
 \mathcal{D}'_G \parallel \text{GS} \\
 L_X \wp N_1
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_C \parallel \text{GS} \\
 C''
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}_A \parallel \text{GS} \\
 K_A \wp A
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}_B \parallel \text{GS} \\
 K_B \wp B
 \end{array}
 .$$

As before, we have  $G = G'' \wp Q(N_1, \dots, N_m)$ . We let  $C[\cdot]_R = C'[L_Y \wp Q(C''[\cdot]_{R''}, N_2, \dots, N_m)]_{R'}$ , and obtain  $\mathcal{D}_G$  as

$$\begin{array}{c}
 \begin{array}{c}
 C''[\hat{K}]_{R''} \\
 \mathcal{D}'_G \parallel \\
 L_X \wp N_1
 \end{array} \\
 \text{ss}\downarrow \\
 L_X \wp L_Y \wp Q(N_1, N_2, \dots, N_m) \\
 \text{ss}\downarrow \\
 \begin{array}{c}
 C'[L_X \wp L_Y]_{R'} \\
 \mathcal{D}'_G \parallel \text{GS} \\
 G''
 \end{array} \wp Q(N_1, N_2, \dots, N_m)
 \end{array}
 ,$$

where  $\hat{K} = K_A \wp K_B$ , and  $\mathcal{D}_C$  is obtained via Lemma A.2 from  $\mathcal{D}'_C$ ,  $\mathcal{D}'_Y$ , and  $\mathcal{D}'_C$ .

- (iv.γ) We have  $C'[\cdot]_{R'}$  and  $L_X$  and  $L_Y$  such that

$$\begin{array}{c}
 C'[L_X \wp L_Y]_{R'} \\
 \mathcal{D}'_G \parallel \text{GS} \\
 G
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_C \parallel \text{GS} \\
 C'
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_X \parallel \text{GS} \\
 L_X \wp N_2
 \end{array}
 ,$$

$$\begin{array}{c}
 \emptyset \\
 \mathcal{D}'_Y \parallel \text{GS} \\
 L_Y \wp Q(N_1[A \otimes B]_{S'}, \emptyset, N_3, \dots, N_m)
 \end{array}
 ,$$

i.e., we assume  $i = 2$  in the second case of (2) in Lemma 6.1 (the cases  $i \geq 3$  are similar). From  $\mathcal{D}'_Y$  we get  $\vdash_{\text{GS}} L_Y \wp Q(N_1, \emptyset, N_3, \dots, N_m) \wp (A \otimes B)$ , to which we can apply the induction hypothesis again, and get  $C''[\cdot]_{R''}$  and  $K_A$  and  $K_B$  such that

$$\begin{array}{c}
 C''[K_A \wp K_B]_{R''} \\
 \mathcal{D}'_G \parallel \text{GS} \\
 L_Y \wp Q(N_1, \emptyset, N_3, \dots, N_m)
 \end{array}
 , \quad
 \begin{array}{c}
 \emptyset \\
 \mathcal{D}'_C \parallel \text{GS} \\
 C''
 \end{array}
 ,$$

$$\begin{array}{c} \emptyset \\ \mathcal{D}_A \parallel \text{GS} \\ K_A \wp A \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_B \parallel \text{GS} \\ K_B \wp B \end{array}.$$

As before, we have  $G = G' \wp Q(N_1, \dots, N_m)$  and let  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ . Thus, we can obtain  $\mathcal{D}_G$  as follows:

$$\begin{array}{c} \begin{array}{c} C''[\hat{K}]_{R''} \\ \mathcal{D}'_G \parallel \text{GS} \\ \emptyset \\ L_Y \wp Q(N_1, \mathcal{D}_3 \parallel \text{GS}, N_3, \dots, N_m) \\ L_X \wp N_2 \end{array} \\ \text{ss}\downarrow \\ \begin{array}{c} L_X \wp L_Y \wp Q(N_1, N_2, \dots, N_m) \end{array} \\ \text{ss}\downarrow \\ \begin{array}{c} C'[L_X \wp L_Y]_{R'} \\ \mathcal{D}'_G \parallel \text{GS} \\ G \end{array} \wp Q(N_1, N_2, \dots, N_m) \end{array}$$

where as before  $\hat{K} = K_A \wp K_B$  and  $\mathcal{D}_C$  is obtained via Lemma A.2.

- (d) In the final case to consider we have that  $G = G' \wp Q(N_1, \dots, N_m)$  where  $N_i$  non-empty for all  $i \in 1, \dots, m$ , and  $Q$  is prime with  $|V_Q| \geq 4$ , and  $\mathcal{D}$  is of shape

$$\begin{array}{c} \emptyset \\ \mathcal{D}' \parallel \text{GS} \\ G' \wp (N_1 \otimes (N_2 \wp (A_2 \otimes B_2)) \otimes \dots \otimes (N_m \wp (A_m \otimes B_m))) \\ \text{p}\downarrow \\ G' \wp Q(N_1, \dots, N_m) \wp (A \otimes B) \end{array}$$

This is only possible if

$$A \otimes B = Q^\perp(\emptyset, A_2 \otimes B_2, \dots, A_m \otimes B_m) \quad (31)$$

(observe at least one component of the prime connective  $Q^\perp$  must be empty for this equality to hold and we take without loss of generality the first). We apply the induction hypothesis to  $\mathcal{D}'$  and get  $C'[\cdot]_{R'}$  and  $K_1, \dots, K_m$ , such that

$$\begin{array}{c} C'[K_1 \wp \dots \wp K_m]_{R'} \\ \mathcal{D}'_G \parallel \text{GS} \\ G' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_C \parallel \text{GS} \\ C' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_1 \parallel \text{GS} \\ K_1 \wp N_1 \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_i \parallel \text{GS} \\ K_i \wp N_i \wp (A_i \otimes B_i) \end{array}$$

for all  $i \in \{2, \dots, m\}$ . Then we can construct the following proof, by using (31):

$$\begin{array}{c} \emptyset \\ \mathcal{D}'_C \parallel \text{GS} \\ \begin{array}{c} \emptyset \\ \mathcal{D}'_2 \parallel \text{GS} \\ K_2 \wp N_2 \wp (A_2 \otimes B_2) \end{array} \otimes \dots \otimes \begin{array}{c} \emptyset \\ \mathcal{D}'_m \parallel \text{GS} \\ K_m \wp N_m \wp (A_m \otimes B_m) \end{array} \\ \text{ss}\downarrow \\ \begin{array}{c} \mathcal{D}' \parallel \text{GS} \\ Q(\emptyset, K_2 \wp N_2, \dots, K_m \wp N_m) \wp (A \otimes B) \end{array} \\ \text{ss}\downarrow \\ C'[Q(\emptyset, K_2 \wp N_2, \dots, K_m \wp N_m)]_{R'} \wp (A \otimes B) \end{array}$$

where  $\tilde{\mathcal{D}}$  is given by Lemma 5.2. To this proof we can apply the induction hypothesis again (since  $N_1$  is non empty), and get  $C[\cdot]_R$  and  $K_A$  and  $K_B$ , such that

$$\begin{array}{c} C[K_A \wp K_B]_R \\ \mathcal{D}'_G \parallel \text{GS} \\ C'[P(\emptyset, K_2 \wp N_2, \dots, K_m \wp N_m)]_{R'} \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_C \parallel \text{GS} \\ C \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_A \parallel \text{GS} \\ K_A \wp A \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_B \parallel \text{GS} \\ K_B \wp B \end{array}$$

It remains to give  $\mathcal{D}_G$  which is as follows:

$$\begin{array}{c} C[K_A \wp K_B]_R \\ \mathcal{D}'_G \parallel \text{GS} \\ \begin{array}{c} \emptyset \\ \mathcal{D}'_1 \parallel \text{GS} \\ Q(K_1 \wp N_1, K_2 \wp N_2, \dots, K_m \wp N_m) \end{array} \\ \text{ss}\downarrow \\ \begin{array}{c} \mathcal{D}_s \parallel \text{GS} \\ K_1 \wp \dots \wp K_m \wp Q(M_1, \dots, M_m) \end{array} \\ \text{ss}\downarrow \\ \begin{array}{c} C'[K_1 \wp \dots \wp K_m]_{R'} \\ \mathcal{D}'_G \parallel \text{GS} \\ G' \end{array} \wp Q(N_1, \dots, N_m) \end{array}$$

where  $\mathcal{D}_s$  consist of  $m$  applications of the rule  $\text{ss}\downarrow$ .

- (2) In this case, we assume  $\vdash_{\text{GS}} G \wp P(M_1, \dots, M_n)$  with  $P$  prime and  $|V_P| = n \geq 4$  and  $M_i$  nonempty for  $1 \leq i \leq n$ ; and aim to construct  $C[\cdot]_R$ ,  $\mathcal{D}_G$ ,  $\mathcal{D}_C$ , and either  $K_i$ ,  $\mathcal{D}_i$  or  $K_X$ ,  $K_Y$ ,  $\mathcal{D}_X$ ,  $\mathcal{D}_Y$ , as in the statement of the splitting lemma. As before, we make a case analysis on the bottommost rule instance  $r$  in  $\mathcal{D}$ .

- (a) If the rule  $r$  acts inside  $G$  then we can conclude immediately by using the induction hypothesis. Similarly, if the rule  $r$  acts inside one of the  $M_i$ , i.e.,  $\mathcal{D}$  is of shape

$$\begin{array}{c} \emptyset \\ \mathcal{D}' \parallel \text{GS} \\ G \wp P(M_1, \dots, M_{i-1}, \begin{array}{c} M'_i \\ r \\ M_i \end{array}, M_{i+1}, \dots, M_n) \end{array}$$

for some  $1 \leq i \leq n$ , we can apply the induction hypothesis, unless  $r$  is  $\text{ai}\downarrow$  and  $M'_i = \emptyset$ . Then we have

$$G \wp P(M_1, \dots, M_{i-1}, \text{ai}\downarrow \frac{\emptyset}{a^\perp \wp a}, M_{i+1}, \dots, M_n)$$

and can conclude immediately by letting  $C = K_X = \emptyset$ , and  $K_Y = G$ .

(b)  $G = G' \wp G''$  and  $\mathcal{D}$  is of shape

$$\text{ss}\downarrow \frac{\frac{\emptyset}{\mathcal{D}'' \parallel} \quad G'' \wp P(M_1, \dots, M_n)[G']_{R_P}}{G'' \wp G' \wp P(M_1, \dots, M_n)}$$

Now consider the possible forms of  $P(M_1, \dots, M_n)[G']_{R_P}$ , according to Lemma 3.7:

(i) It cannot be an atom since  $G'$  and  $M_i$  are non-empty.

(ii) It cannot be a par, due to conditions on  $\text{ss}\downarrow$ .

(iii) It can only be of the form  $P(M_1, \dots, M_n) \otimes G'$ .

In this case, we can apply the induction hypothesis to obtain  $C'[\cdot]_{R'}$ ,  $K_P$ ,  $K'_G$  such that

$$C'[K_P \wp L'_G]_{R'} \quad , \quad \frac{\emptyset}{\mathcal{D}'_C \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_{K_G} \parallel} \quad , \quad \frac{\emptyset}{K_P \wp P(M_1, \dots, M_n)}$$

By the induction hypothesis we have context  $C''[\cdot]_{R''}$  and graph  $\hat{K}$  such that

$$\frac{C''[\hat{K}]_{R''}}{\mathcal{D}'' \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}'_C \parallel} \quad , \quad C''$$

and either  $\hat{K} = P^\perp(K_1, \dots, K_n)$ , for some  $K_1, \dots, K_n$ , where for all  $i \in \{1, \dots, n\}$

$$\frac{\emptyset}{\mathcal{D}_i \parallel} \quad , \quad K_i \wp M_i$$

or  $\hat{K} = K_X \wp K_Y$  for some  $K_X$  and  $K_Y$  such that

$$\frac{\emptyset}{\mathcal{D}_X \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_Y \parallel} \quad , \quad K_Y \wp P(M_1, \dots, M_{j-1}, \emptyset, M_{j+1}, \dots, M_n)$$

for some  $j \in \{1, \dots, n\}$ .

Using the above, we can construct the following derivation  $\mathcal{D}_G$

$$\text{ss}\downarrow \frac{C'[\hat{K}]_{R''} \quad \frac{\emptyset}{\mathcal{D}_{K_G} \parallel} \quad \wp \quad \frac{\emptyset}{K'_G \wp G''}}{C'[K_P \wp L'_G]_{R'} \quad \wp \quad G''}$$

Let  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$  and obtain  $\mathcal{D}_C$  by using Lemma A.2.

(iv) Otherwise,  $P(M_1, \dots, M_n)[G']_{R_P}$  is a prime graph in which case we have the following possibilities.

(I)  $\mathcal{D}$  is of the shape

$$\text{ss}\downarrow \frac{\frac{\emptyset}{\mathcal{D}' \parallel} \quad G'' \wp P(M_1, \dots, M_{j-1}, M_j[G']_S, M_{j+1}, \dots, M_n)}{G'' \wp G' \wp P(M_1, \dots, M_n)}$$

for some  $1 \leq j \leq n$ . We apply the induction hypothesis to  $\mathcal{D}'$  and get one of the following three sub-cases:

(b.(iv).(I). $\alpha$ ) We have  $C[\cdot]_R$  and  $L_j$  and  $K_1, \dots, K_{j-1}, K_{j+1}, \dots, K_n$  such that

$$C[P^\perp(K_1, \dots, K_{j-1}, L_j, K_{j+1}, \dots, K_n)]_R \quad , \quad \frac{\emptyset}{\mathcal{D}'_G \parallel} \quad , \quad G''$$

$$\frac{\emptyset}{\mathcal{D}_C \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}'_j \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_i \parallel} \quad , \quad L_j \wp M_j[G']_S \quad , \quad K_i \wp M_i$$

for  $i \in \{1, \dots, n\}$  with  $i \neq j$ . We let  $K_j = L_j \wp G'$ . Then  $\mathcal{D}_G$  is the derivation

$$\text{ss}\downarrow \frac{C[P^\perp(K_1, \dots, K_{j-1}, L_j \wp G', K_{j+1}, \dots, K_n)]_R}{C[P^\perp(K_1, \dots, K_{j-1}, L_j, K_{j+1}, \dots, K_n)]_R \quad \wp \quad G''} \quad \wp \quad G'$$

and  $\mathcal{D}_j$  is

$$\text{ss}\downarrow \frac{\frac{\emptyset}{\mathcal{D}'_j \parallel} \quad L_j \wp M_j[G']_S}{L_j \wp G' \wp M_j}$$

(b.iv).(I). $\beta$ ) We have  $C[\cdot]_R$  and  $L_X$  and  $K_Y$  such that

$$\begin{array}{c} C[L_X \wp K_Y]_R \\ \mathcal{D}'_G \parallel \\ G'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_C \parallel \\ C \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ L_X \wp M_j[G']_S \end{array},$$

$$\begin{array}{c} \emptyset \\ \mathcal{D}_Y \parallel \\ K_Y \wp P(M_1, \dots, M_{j-1}, \emptyset, M_{j+1}, \dots, M_n) \end{array}$$

We let  $K_X = L_X \wp G'$  and let  $\mathcal{D}_G$  and  $\mathcal{D}_X$  be the derivations

$$\text{ss}\downarrow \frac{C[L_X \wp G' \wp K_Y]_R}{\begin{array}{c} C[L_X \wp K_Y]_R \\ \mathcal{D}'_G \parallel \\ G'' \end{array} \wp G'} \quad \text{and} \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ L_X \wp M_j[G']_S \\ \text{ss}\downarrow \\ L_X \wp G' \wp M_j \end{array}$$

(b.iv).(I). $\gamma$ ) We have  $C[\cdot]_R$  and  $K_X$  and  $L_Y$  such that

$$\begin{array}{c} C[K_X \wp L_Y]_R \\ \mathcal{D}'_G \parallel \\ G'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_C \parallel \\ C \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_X \parallel \\ K_X \wp M_i \end{array},$$

$$\begin{array}{c} \emptyset \\ \mathcal{D}'_Y \parallel \\ L_Y \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_j[G'], \dots, M_n) \end{array}$$

for some  $i \neq j$ . We let  $K_Y = G' \wp L_Y$ , we obtain  $\mathcal{D}_G$  as in the previous case, and  $\mathcal{D}_Y$  is

$$\text{ss}\downarrow \frac{\begin{array}{c} \emptyset \\ \mathcal{D}'_Y \parallel \\ L_Y \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_j[G'], \dots, M_n) \end{array}}{G' \wp L_Y \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n)}$$

(II)  $\mathcal{D}$  is of the shape

$$\text{ss}\downarrow \frac{\begin{array}{c} \emptyset \\ \mathcal{D}' \parallel \\ G'' \wp Q(G', N_2, \dots, N_k) \end{array}}{G'' \wp G' \wp P(M_1, \dots, M_n)}$$

where  $Q$  is a prime graph such that  $|Q| = k > n$  such that

$$Q(\emptyset, N_2, \dots, N_k) = P(M_1, \dots, M_n) \quad (32)$$

and each  $N_i$  for  $i \in \{1, \dots, k\}$  is a module of some  $M_j$  where  $j \in \{1, \dots, n\}$ .

By the induction hypothesis we have  $\hat{K}$ ,  $C'[\cdot]_{R'}$  such that

$$\begin{array}{c} C'[\hat{K}]_{R'} \\ \mathcal{D}''_G \parallel \\ G'' \end{array}$$

And one of ( $\alpha$ ), ( $\beta$ ) or ( $\gamma$ ) holds as follows.

( $\alpha$ )  $\hat{K} = Q^\perp(\emptyset, K_2, \dots, K_k)$  and

$$\begin{array}{c} \emptyset \\ \mathcal{D}_C \parallel \\ C \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_L \parallel \\ L \wp G' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_i \parallel \\ K_i \wp N_i \end{array}$$

for  $i \in \{1, \dots, k\}$ .

Hence we have the following derivation, named  $\mathcal{D}''$

$$\text{ss}\downarrow \frac{C'[Q^\perp(\begin{array}{c} \emptyset \\ \mathcal{D}_L \parallel \\ L \wp G' \end{array}, K_2, \dots, K_k)]_{R'}}{C'[Q^\perp(\emptyset, K_2, \dots, K_k)]_{R'} \wp G'}$$

Now, we have the following proof, where  $\mathcal{D}_Q$  is given by Lemma 5.2 and by appealing to (32),

$$\mathcal{D}_Q \frac{\begin{array}{c} \emptyset \\ \mathcal{D}_2 \parallel \\ K_2 \wp N_2 \end{array} \otimes \dots \otimes \begin{array}{c} \emptyset \\ \mathcal{D}_k \parallel \\ K_k \wp N_k \end{array}}{Q^\perp(\emptyset, K_2, \dots, K_k) \wp P(M_1, \dots, M_n)}$$

Since  $N_1$  is nonempty, the size of  $Q^\perp(\emptyset, K_2, \dots, K_k) \wp Q(G', N_2, \dots, N_k)$  is less than the size of  $G \wp P(M_1, \dots, M_n)$ . Hence we can apply the induction hypothesis to obtain  $\hat{\mathcal{D}}$  such that

$$\begin{array}{c} C''[\hat{K}']_{R''} \\ \hat{\mathcal{D}} \parallel \\ Q^\perp(\emptyset, K_2, \dots, K_k) \end{array}$$

where  $\hat{\mathcal{D}}$  satisfies the conditions of the splitting (providing  $K_i$ ,  $K_X$ ,  $K_Y$ , etc.).  $\mathcal{D}_G = C'[\hat{\mathcal{D}}]_{R'} \circ \mathcal{D}''$ ,  $C[\cdot]_R = C'[C''[\cdot]_{R'']_{R'}}$  and  $\mathcal{D}_C$  is given by Lemma A.2.

( $\beta$ )  $\hat{K} = K'_X \wp K'_Y$  such that (appealing to (32))

$$\begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ K'_X \wp G' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_Y \parallel \\ K'_Y \wp P(M_1, \dots, M_n) \end{array}$$

In this case we have derivation  $\mathcal{D}''$ , defined as follows

$$\text{ss}\downarrow \frac{C'[\begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ gK'_X \wp G' \end{array} \wp K'_Y]_{R'}}{C'[K'_X \wp K'_Y]_{R'} \wp G'}$$



Now, since  $G'$  is nonempty the size of  $K'_Y \wp P(M_1, \dots, M_n)$  is strictly less than the size of  $G \wp P(M_1, \dots, M_n)$ . Hence we can apply the induction hypothesis to obtain  $\hat{\mathcal{D}}$  such that

$$\begin{array}{c} C''[\hat{K}']_{R''} \\ \parallel \\ \hat{\mathcal{D}} \\ Q^\perp(\emptyset, K_2, \dots, K_k) \end{array}$$

where  $\hat{\mathcal{D}}$  satisfies most of the conditions of the splitting (providing  $K_i, K_X, K_Y$ , etc.),  $\mathcal{D}_G = C'[\hat{\mathcal{D}}]_{R'} \circ \mathcal{D}''$ ,  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$  and  $\mathcal{D}_C$  is given by Lemma A.2.

(Y)  $\hat{K} = K'_X \wp K'_Y$  such that for some  $\ell \in \{2, \dots, k\}$

$$\begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ K'_X \wp N_\ell \end{array}$$

$$\begin{array}{c} \emptyset \\ \mathcal{D}'_Y \parallel \\ K'_Y \wp Q(G', N_2, \dots, N_{\ell-1}, \emptyset, N_{\ell+1}, \dots, N_k) \end{array}$$

In the case that  $N_\ell = M_m$  for some  $m \in \{1..k\}$ , proof  $\mathcal{D}_Y$  is given by (using (32))

$$\begin{array}{c} \emptyset \\ \mathcal{D}'_Y \parallel \\ K'_Y \wp Q(G', N_2, \dots, N_{\ell-1}, \emptyset, N_{\ell+1}, \dots, N_k) \\ \hline \text{ss}\downarrow \\ K'_Y \wp G' \wp P(M_1, \dots, M_{m-1}, \emptyset, M_{m+1}, \dots, M_n) \end{array}$$

Hence we can conclude immediately by taking  $K_Y = K'_Y \wp G'$ ,  $K_X = K'_X$ ,  $\mathcal{D}_X = \mathcal{D}'_X$ ,  $\mathcal{D}_G = \mathcal{D}''$  and  $C[\cdot]_R = C'[\cdot]_{R'}$ .

Otherwise we have to consider the scenario when we have  $M_m = M'[N_j]_{R_m}$  for some non-empty  $M'$  (recall that  $N_j$  must be a module of some such  $M_m$ ). In this scenario, the following are equivalent (by using (32))

$$Q(\emptyset, N_2, \dots, N_{\ell-1}, \emptyset, N_{\ell+1}, \dots, N_k)$$

$$P(M_1, \dots, M_{m-1}, M', M_{m+1}, \dots, M_n)$$

$\vdash K'_Y \wp G' \wp P(M_1, \dots, M_{m-1}, \emptyset, M_{m+1}, \dots, M_n)$  holds, using  $\mathcal{D}'_Y$  and  $\text{ss}\downarrow$ , and is strictly smaller than  $G \wp P(M_1, \dots, M_n)$ , since  $N_j$  is non-empty. Hence we can apply the induction hypothesis to obtain  $\hat{K}'$  and  $C'[\cdot]_{R'}$  such that

$$\begin{array}{c} C'[\hat{K}']_{R'} \\ \parallel \\ [K'_Y; G'] \end{array}$$

There are then three cases to check (A), (B) and (C) as follows.

(A) Let  $\hat{K}' = P^\perp(K_1, \dots, L, K_{m+1}, \dots, K_n)$  and

$$\begin{array}{c} \emptyset \\ \mathcal{D}'_m \parallel \\ L \wp M' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_i \parallel \\ K_i \wp M_i \end{array}$$

for  $i \in \{1 \dots m-1\} \cup \{m+1 \dots n\}$ .

From the above we can construct the following derivation  $\mathcal{D}_G$

$$\begin{array}{c} C'[P^\perp(K_1, \dots, K'_X \wp L, K_{m+1}, \dots, K_n)]_{R'} \\ \text{ss}\downarrow \\ C'[\begin{array}{c} C'[P^\perp(K_1, \dots, L, K_{m+1}, \dots, K_n)]_{R'} \\ \parallel \\ K'_Y \wp G' \end{array}]_{R'} \\ \hline \text{ss}\downarrow \\ C'[K'_X \wp K'_Y]_{R'} \\ \parallel \\ G'' \wp G' \end{array}$$

Also we can construct proof  $\mathcal{D}_m$  as follows

$$\begin{array}{c} \emptyset \\ \mathcal{D}'_m \parallel \\ L \wp M'[\begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ K'_X \wp N_j \end{array}]_{R_m} \\ \hline \text{ss}\downarrow \\ K'_X \wp L \wp M'[N_j]_{R_m} \end{array}$$

We conclude by setting  $K_m = K'_X; L$  and  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ , where  $\mathcal{D}_C$  is obtained using Lemma A.2.

(B) Let  $\hat{K}' = K_X \wp \hat{K}_Y$  where w.l.o.g.

$$\begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ K_X \wp M_1 \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}'_Y \parallel \\ K_Y \wp P(\emptyset, \dots, M_{m-1}, M', M_{m+1}, \dots, M_n) \end{array}$$

In this case we have derivation  $\mathcal{D}_G$ , defined as follows

$$\begin{array}{c} K_X \wp K'_X \wp \hat{K}_Y \\ \text{ss}\downarrow \\ C'[\begin{array}{c} K_X \wp \hat{K}_Y \\ \parallel \\ K'_Y \wp G' \end{array}]_{R'} \\ \hline \text{ss}\downarrow \\ C'[K'_X \wp K'_Y]_{R'} \\ \parallel \\ G'' \wp G' \end{array}$$

We also have the following proof, named  $\mathcal{D}_Y$

$$\text{ss}\downarrow \frac{\begin{array}{c} \emptyset \\ \mathcal{D}'_Y \parallel \\ \hat{K}_Y \approx P(\emptyset, \dots, M_{m-1}, M'_j \parallel \begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ K'_X \approx N_j \end{array} ]_{R_m, M_{m+1}, \dots, M_n} \end{array}}{K'_X \approx \hat{K}_Y \approx P(\emptyset, \dots, M'_j[N_j]_{R_m}, M_{m+1}, \dots, M_n)}$$

We conclude by setting  $K_Y = K'_X \approx \hat{K}_Y$  and  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ , where  $\mathcal{D}_C$  is obtained using Lemma A.2.

(C) Let  $\hat{K}' = \hat{K}_X \approx K_Y$  where w.l.o.g.

$$\begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ \hat{K}_X \approx M' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_Y \parallel \\ K_Y \approx P(\emptyset, \dots, M_{m-1}, \emptyset, M_{m+1}, \dots, M_n) \end{array}$$

In this case we have derivation  $\mathcal{D}_G$ , defined as follows

$$\text{ss}\downarrow \frac{\begin{array}{c} \hat{K}_X \approx K'_X \approx \hat{K}_Y \\ \text{ss}\downarrow \parallel \\ C'[\begin{array}{c} \hat{K}_X \approx K_Y \\ \parallel \\ K'_X \approx G' \end{array}]_{R'} \end{array}}{C'[K'_X \approx K'_Y]_{R'} \parallel G'' \approx G'}$$

We also have the following proof, named  $\mathcal{D}_X$

$$\text{ss}\downarrow \frac{\begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ \hat{K}_X \approx M'[\begin{array}{c} \emptyset \\ \mathcal{D}'_X \parallel \\ K'_X \approx N_j \end{array}]_{R_m} \end{array}}{\hat{K}_X \approx K'_X \approx M'[N_j]_{R_m}}$$

We conclude by setting  $K_X = K'_X \approx \hat{K}_X$  and  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ , where  $\mathcal{D}_C$  is obtained using Lemma A.2.

(c) If  $G = G'' \approx G'$  (with  $G' \neq \emptyset$ ) and  $\mathcal{D}$  is of shape

$$\text{ss}\downarrow \frac{\begin{array}{c} \emptyset \\ \mathcal{D}' \parallel \\ G'' \approx G'[P(M_1, \dots, M_n)]_S \end{array}}{G'' \approx G' \approx P(M_1, \dots, M_n)}$$

we proceed as in case (1.c) by a case analysis on the shape of  $G'[P(M_1, \dots, M_n)]_S$  via Lemma 3.7, and for the same reasons as above, there are two cases.

(iii)  $G'[P(M_1, \dots, M_n)]_S$  is a tensor of two graphs. Without loss of generality, we can assume  $G'[P(M_1, \dots, M_n)]_S = E \otimes D[P(M_1, \dots, M_n)]_{S'}$ , with  $E \neq \emptyset$ . We apply the induction hypothesis to  $\mathcal{D}'$  and get  $C'[\cdot]_{R'}$  and  $K_E$  and  $K_D$  such that

$$\begin{array}{c} C'[K_E \approx K_D]_{R'} \\ \mathcal{D}'_G \parallel \text{GS} \\ G'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_{C'} \parallel \text{GS} \\ C' \end{array},$$

$$\begin{array}{c} \emptyset \\ \mathcal{D}_E \parallel \text{GS} \\ K_E \approx E \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_D \parallel \text{GS} \\ K_D \approx D[P(M_1, \dots, M_n)]_{S'} \end{array}$$

From  $\mathcal{D}_D$ , we get  $\vdash_{\text{GS}} K_D \approx D \approx P(M_1, \dots, M_n)$  via the  $\text{ss}\downarrow$ -rule. Since  $|E| > 0$  we can apply the induction hypothesis again, giving us one of the following two cases:

• either a context  $C''[\cdot]_{R''}$  and graphs  $K_1, \dots, K_n$ , such that

$$\begin{array}{c} C''[P^\perp(K_1, \dots, K_n)]_{R''} \\ \mathcal{D}'_G \parallel \text{GS} \\ K_D \approx D \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_{C''} \parallel \text{GS} \\ C'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_i \parallel \text{GS} \\ K_i \approx M_i \end{array}$$

for all  $i \in \{1, \dots, n\}$ ,

• or a context  $C''[\cdot]_{R''}$  and graphs  $K_X$  and  $K_Y$  such that

$$\begin{array}{c} C''[K_X \approx K_Y]_{R''} \\ \mathcal{D}'_G \parallel \text{GS} \\ K_D \approx D \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_{C''} \parallel \text{GS} \\ C'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_X \parallel \text{GS} \\ K_X \approx M_i \end{array},$$

$$\begin{array}{c} \emptyset \\ \mathcal{D}_Y \parallel \text{GS} \\ K_Y \approx P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n) \end{array}$$

for some  $i \in \{1, \dots, n\}$ .

In the first case we let  $\hat{K} = P^\perp(K_1, \dots, K_n)$  and in the second  $\hat{K} = K_X \approx K_Y$ . In both cases we have  $G = G'' \approx (E \otimes D)$  and we can let  $C[\cdot]_R$  and  $\mathcal{D}_G$  and  $\mathcal{D}_C$  as in case (1.c.iii) above.<sup>6</sup>

(iv)  $G'[P(M_1, \dots, M_n)]_S$  is composed via a prime graph  $Q$ . This case is similar to case (1.c.iv) above. Without loss of generality, we can assume that  $G'[P(M_1, \dots, M_n)]_S = Q(N_1[P(M_1, \dots, M_n)]_{S'}, N_2, \dots, N_m)$  for some prime graph  $Q$  and  $m \geq 4$ , and  $N_i \neq \emptyset$  for all  $1 \leq i \leq m$ . Applying the induction hypothesis to  $\mathcal{D}'$  gives us one of the following three cases:

(iv.α) We have  $C'[\cdot]_{R'}$  and  $L_1, \dots, L_m$  such that

$$\begin{array}{c} C'[Q^\perp(L_1, \dots, L_m)]_{R'} \\ \mathcal{D}'_G \parallel \\ G'' \end{array}, \quad \begin{array}{c} \emptyset \\ \mathcal{D}_{C'} \parallel \\ C' \end{array},$$

<sup>6</sup>This is the reason for using the abbreviation  $\hat{K}$  in that case.

$$\frac{\emptyset}{\mathcal{D}'_1 \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}'_i \parallel} \quad , \quad \frac{\emptyset}{L_1 \wp N_1 [P(M_1, \dots, M_n)]_{S'}} \quad , \quad \frac{\emptyset}{L_i \wp N_i}$$

for  $2 \leq i \leq m$ . From  $\mathcal{D}'_1$ , we get (via the rule  $ss\downarrow$ )  $\vdash_{GS} L_1 \wp N_1 \wp P(M_1, \dots, M_n)$  to which we can apply the induction hypothesis again, to get one of the following two cases:

- either a context  $C''[\cdot]_{R''}$  and graphs  $K_1, \dots, K_n$ , such that

$$\frac{C''[P^\perp(K_1, \dots, K_n)]_{R''}}{\mathcal{D}''_G \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}'_{C'} \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}_i \parallel GS} \quad , \quad \frac{\emptyset}{L_1 \wp N_1} \quad , \quad \frac{\emptyset}{C''} \quad , \quad \frac{\emptyset}{K_i \wp M_i}$$

for all  $i \in \{1, \dots, n\}$ ,

- or a context  $C''[\cdot]_{R''}$  and graphs  $K_X$  and  $K_Y$  such that

$$\frac{C''[K_X \wp K_Y]_{R''}}{\mathcal{D}''_G \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}_{C'} \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}_X \parallel GS} \quad , \quad \frac{\emptyset}{L_1 \wp N_1} \quad , \quad \frac{\emptyset}{C''} \quad , \quad \frac{\emptyset}{K_X \wp M_i}$$

$$\frac{\emptyset}{\mathcal{D}_Y \parallel GS} \quad , \quad \frac{\emptyset}{K_Y \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n)}$$

for some  $i \in \{1, \dots, n\}$ .

In the first case we let  $\hat{K} = P^\perp(K_1, \dots, K_n)$  and in the second  $\hat{K} = K_X \wp K_Y$ . In both cases we have  $G = G'' \wp Q(N_1, \dots, N_m)$  and can let  $C[\cdot]_R$  and  $\mathcal{D}_G$  and  $\mathcal{D}_C$  as in case (1.c.iv. $\alpha$ ) above.<sup>7</sup>

(iv. $\beta$ ) We have  $C'[\cdot]_{R'}$  and  $L_X$  and  $L_Y$  such that

$$\frac{C'[L_X \wp L_Y]_{R'}}{\mathcal{D}'_G \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_X \parallel} \quad , \quad \frac{\emptyset}{L_X \wp N_1 [P(M_1, \dots, M_n)]_{S'}} \quad , \quad \frac{\emptyset}{\mathcal{D}_Y \parallel GS}$$

$$\frac{\emptyset}{\mathcal{D}_{C'} \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_Y \parallel} \quad , \quad \frac{\emptyset}{L_Y \wp Q(\emptyset, N_2, \dots, N_m)}$$

From  $\mathcal{D}_X$ , we get  $\vdash_{GS} L_X \wp N_1 \wp P(M_1, \dots, M_n)$ , to which we apply the induction hypothesis again and get one of the following two cases:

- either a context  $C''[\cdot]_{R''}$  and graphs  $K_1, \dots, K_n$ , such that

$$\frac{C''[P^\perp(K_1, \dots, K_n)]_{R''}}{\mathcal{D}''_G \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}'_{C'} \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}_i \parallel GS} \quad , \quad \frac{\emptyset}{L_X \wp N_1} \quad , \quad \frac{\emptyset}{C''} \quad , \quad \frac{\emptyset}{K_i \wp M_i}$$

for all  $i \in \{1, \dots, n\}$ ,

- or a context  $C''[\cdot]_{R''}$  and graphs  $K_X$  and  $K_Y$  such that

$$\frac{C''[K_X \wp K_Y]_{R''}}{\mathcal{D}''_G \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}'_{C'} \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}_X \parallel GS} \quad , \quad \frac{\emptyset}{L_X \wp N_1} \quad , \quad \frac{\emptyset}{C''} \quad , \quad \frac{\emptyset}{K_X \wp M_i}$$

$$\frac{\emptyset}{\mathcal{D}_Y \parallel GS} \quad , \quad \frac{\emptyset}{K_Y \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n)}$$

for some  $i \in \{1, \dots, n\}$ .

As above, we let in the first case  $\hat{K} = P^\perp(K_1, \dots, K_n)$  and in the second  $\hat{K} = K_X \wp K_Y$ . In both cases we have  $G = G'' \wp Q(N_1, \dots, N_m)$  and can let  $C[\cdot]_R$  and  $\mathcal{D}_G$  and  $\mathcal{D}_C$  as in case (1.c.iv. $\beta$ ) above.

(iv. $\gamma$ ) We have  $C'[\cdot]_{R'}$  and  $L_X$  and  $L_Y$  such that

$$\frac{C'[L_X \wp L_Y]_{R'}}{\mathcal{D}'_G \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_{C'} \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_X \parallel} \quad , \quad \frac{\emptyset}{L_X \wp N_2} \quad , \quad \frac{\emptyset}{G''}$$

$$\frac{\emptyset}{\mathcal{D}_Y \parallel} \quad , \quad \frac{\emptyset}{L_Y \wp Q(N_1 [P(M_1, \dots, M_n)]_{S'}, \emptyset, N_3, \dots, N_m)}$$

From  $\mathcal{D}_Y$  we get (via  $ss\downarrow$ )

$$\vdash_{GS} L_Y \wp Q(N_1, \emptyset, N_3, \dots, N_m) \wp P(M_1, \dots, M_n)$$

to which we apply the induction hypothesis again, and get one of the following two cases:

- either a context  $C''[\cdot]_{R''}$  and graphs  $K_1, \dots, K_n$ , such that

$$\frac{C''[P^\perp(K_1, \dots, K_n)]_{R''}}{\mathcal{D}''_G \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}'_{C'} \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}_i \parallel GS} \quad , \quad \frac{\emptyset}{L_Y \wp Q(N_1, \emptyset, N_3, \dots, N_m)} \quad , \quad \frac{\emptyset}{C''} \quad , \quad \frac{\emptyset}{K_i \wp M_i}$$

for all  $i \in \{1, \dots, n\}$ ,

- or a context  $C''[\cdot]_{R''}$  and graphs  $K_X$  and  $K_Y$  such that

$$\frac{C''[K_X \wp K_Y]_{R''}}{\mathcal{D}''_G \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}'_{C'} \parallel GS} \quad , \quad \frac{\emptyset}{\mathcal{D}_X \parallel GS} \quad , \quad \frac{\emptyset}{L_Y \wp Q(N_1, \emptyset, N_3, \dots, N_m)} \quad , \quad \frac{\emptyset}{C''} \quad , \quad \frac{\emptyset}{K_X \wp M_i}$$

$$\frac{\emptyset}{\mathcal{D}_Y \parallel GS} \quad , \quad \frac{\emptyset}{K_Y \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n)}$$

for some  $i \in \{1, \dots, n\}$ .

As above, we let in the first case  $\hat{K} = P^\perp(K_1, \dots, K_n)$  and in the second  $\hat{K} = K_X \wp K_Y$ . In both cases we have  $G = G'' \wp Q(N_1, \dots, N_m)$  and can let  $C[\cdot]_R$  and  $\mathcal{D}_G$  and  $\mathcal{D}_C$  as in case (1.c.iv. $\gamma$ ) above.

<sup>7</sup>This is the reason for using the abbreviation  $\hat{K}$  in that case.

- (d) Consider when we have  $G = G' \wp P^\perp(N_1, \dots, N_n)$  and  $\mathcal{D}$  is of shape

$$\frac{\frac{\emptyset}{\mathcal{D}' \parallel} \quad G' \wp ((N_1 \wp M_1) \otimes \dots \otimes (N_n \wp M_n))}{G' \wp P^\perp(N_1, \dots, N_n) \wp P(M_1, \dots, M_n)} \text{ p}\downarrow$$

By applying the induction hypothesis  $n - 1$  times to  $\mathcal{D}'$  and some uses of the  $\text{ss}\downarrow$  rule, we obtain a context  $C[\cdot]_R$  and graphs  $L_1, \dots, L_n$  such that

$$C[L_1 \wp \dots \wp L_n]_R, \quad \frac{\emptyset}{\mathcal{D}_C \parallel} \quad C, \quad \frac{\emptyset}{\mathcal{D}_i \parallel} \quad L_i \wp N_i \wp M_i$$

for  $i \in \{1, \dots, n\}$ . We let  $K_i = L_i \wp N_i$ , and construct  $\mathcal{D}_G$  as

$$\frac{\frac{C[\frac{P^\perp(L_1 \wp N_1, \dots, L_n \wp N_n)}{L_1 \wp \dots \wp L_n \wp P^\perp(N_1, \dots, N_n)} \parallel \{\text{ss}\downarrow\}]_R}{L_1 \wp \dots \wp L_n \wp P^\perp(N_1, \dots, N_n)} \text{ ss}\downarrow}{\frac{\mathcal{D}'_G \parallel}{G'} \wp P^\perp(N_1, \dots, N_n)} \text{ ss}\downarrow$$

- (e) Finally, consider when  $G = G'' \wp Q(N_1, \dots, N_k)$  where  $N_i$  non-empty for all  $j$ ,  $Q$  is prime,  $|V_Q| > |V_P|$ ,  $P(M_1, \dots, M_n) = Q^\perp(\emptyset, L_2, \dots, L_n)$  (observe at least one module of the prime connective  $Q^\perp$  must be empty for this equality to hold and we set that w.l.o.g. to be the first module, otherwise  $P^\perp$  and  $Q$  are isomorphic, contradicting  $|V_Q| > |V_P|$ ), and  $\mathcal{D}$  is of shape

$$\frac{\frac{\emptyset}{\mathcal{D}' \parallel} \quad G'' \wp (N_1 \otimes (N_2 \wp L_2) \otimes \dots \otimes (N_n \wp L_n))}{G'' \wp Q(N_1, \dots, N_k) \wp P(M_1, \dots, M_n)} \text{ p}\downarrow$$

By applying the induction hypothesis to  $\mathcal{D}'$  we can obtain  $C'[\cdot]_{R'}$  and  $K_i$  such that

$$C'[[K_1; K_2; \dots; K_n]]_{R'}, \quad \frac{\emptyset}{\mathcal{D}'_1 \parallel} \quad N_1 \wp K_1, \quad \frac{\emptyset}{\mathcal{D}'_i \parallel} \quad N_1 \wp K_i \wp M_i$$

Now observe we can construct the following proof, where  $\mathcal{D}_Q$  is given by Lemma 5.2,

$$\frac{\left( \frac{\emptyset}{\mathcal{D}'_2 \parallel} \quad [K_2; N_2; M_2] \quad ; \dots \quad ; \quad \frac{\emptyset}{\mathcal{D}'_k \parallel} \quad [K_k; N_k; M_k] \right)}{\mathcal{D}_Q \parallel} \quad Q(\emptyset, K_2 \wp N_2, \dots, K_k \wp N_k); P(M_1, \dots, M_n)]$$

Since  $G''$  is nonempty and does not contribute to the conclusion of the above proof the size of the conclusion is strictly less than the size of  $G \wp P(M_1, \dots, M_n)$ . Hence we can apply the induction proof to obtain  $C''[\cdot]_{R''}$  and  $\hat{K}$  such that

$$\frac{C''[\hat{K}]_{R''}}{\hat{\mathcal{D}} \parallel} \quad Q(\emptyset, K_2 \wp N_2, \dots, K_k \wp N_k)$$

where  $\hat{K}$  satisfies gives us most conditions of splitting such as  $K_i$  or  $K_X$  and  $K_Y$ . What remains is to set  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ , obtain  $\mathcal{D}_C$  using Lemma A.2, and construct derivation  $\mathcal{D}_G$  as follows

$$\frac{\frac{C'[\frac{C''[\hat{K}]_{R''}}{\hat{\mathcal{D}} \parallel} \quad Q(\frac{\emptyset}{\mathcal{D}'_1 \parallel} \quad K_1 \wp N_1, K_2 \wp N_2, \dots, K_k \wp N_k)]_{R'}}{K_1 \wp K_2 \wp \dots \wp K_n \wp Q(N_1, \dots, N_k)} \text{ ss}\downarrow}{\frac{C'[K_1 \wp K_2 \wp \dots \wp K_n]_{R'}}{\mathcal{D}'' \parallel} \quad G'' \wp Q(N_1, \dots, N_k)} \text{ ss}\downarrow$$

- (3) In this case, we assume  $\vdash_{\text{GS}} G \wp a$  and aim to construct  $C[\cdot]_R$ ,  $\mathcal{D}_G$ ,  $\mathcal{D}_C$  as in the statement of the splitting lemma. Observe that  $G \neq \emptyset$ , otherwise  $G \wp a$  would not be probable in GS. As in the other two cases, we make a case analysis on the bottommost rule instance  $r$  in  $\mathcal{D}$ .
- If  $r$  acts completely inside  $G$  then we proceed immediately by induction hypothesis, as in case (1.a) above.
  - As  $a$  is an atom, there is no case that corresponds to (1.b) or (2.b), and we leave that out.
  - In this case we have  $G = G'' \wp G'$  for  $G' \neq \emptyset$  and  $\mathcal{D}$  is of shape

$$\frac{\frac{\emptyset}{\mathcal{D}' \parallel} \quad G'' \wp G'[a]_S}{G'' \wp G' \wp a} \text{ ss}\downarrow$$

We proceed by a case analysis on  $G'[a]_S$ , using Lemma 3.7. As before, it is not an atom and not a par. Consequently, it is either a tensor or is composed via a prime graph  $Q$  with  $|V_Q| \geq 4$ . The two cases (iii) and (iv) are now almost literally the same as for case (1.c) above, and therefore not repeated here. The only difference is that we replace everywhere  $A \otimes B$  by  $a$  and  $K_A \wp K_B$  by  $a^\perp$ , and omit  $\mathcal{D}_A$  and  $\mathcal{D}_B$ .

- (d) Assume  $P$  is prime,  $|V_P| \geq 4$ , and  $M_i$  are nonempty in the derivation

$$\frac{\frac{\emptyset}{\mathcal{D}' \parallel} \quad G'' \wp ((a \wp M_1) \otimes M_2 \otimes \dots \otimes M_n)}{\rho \downarrow \quad G'' \wp P(M_1, \dots, M_n) \wp a}$$

Notice, w.l.o.g.,  $a$  is ready to interact with the first module of the  $P$  connective. By the induction hypothesis, we have  $C'[\cdot]_{R'}$ ,  $K_i$  such that

$$C'[K_1 \wp \dots \wp K_n]_{R'} \quad , \quad \frac{\emptyset}{\mathcal{D}_{C_1} \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_1 \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_i \parallel}$$

$$\frac{\mathcal{D}_G'' \parallel}{G''} \quad , \quad C' \quad , \quad K_1 \wp M_1 \wp a \quad , \quad K_i \wp M_i$$

for  $2 \leq i \leq n$ . Since  $\vdash K_1 \wp M_1 \wp a$ , by the induction hypothesis, we have  $C''[\cdot]_{R''}$  such that

$$\frac{\emptyset}{\mathcal{D}_L \parallel} \quad , \quad C''[a^\perp]_{R''} \quad , \quad K_1 \wp M_1$$

From the above, we can construct the following derivation,  $\mathcal{D}_G$ , as required, where  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ , and  $\mathcal{D}_C$  is given by Lemma A.2

$$\frac{\frac{\frac{\frac{\frac{\emptyset}{\mathcal{D}_L \parallel} \quad C''[a^\perp]_{R''}}{K_1 \wp M_1} \quad , \quad \frac{\emptyset}{\mathcal{D}_2 \parallel} \quad , \quad \dots \quad , \quad \frac{\emptyset}{\mathcal{D}_n \parallel} \quad }{C'[P(K_1 \wp \dots \wp K_n)]_{R'}}}{n \times \text{ss} \downarrow} \quad C'[K_1 \wp \dots \wp K_n \wp P(M_1, \dots, M_n)]_{R'}}$$

$$\text{ss} \downarrow \quad \frac{\frac{\emptyset}{\mathcal{D}_G'' \parallel} \quad C'[K_1 \wp \dots \wp K_n]_{R'} \quad \wp \quad P(M_1, \dots, M_n)}{G''}$$

- (e) In the final case we have  $G = G' \wp a^\perp$  and  $\mathcal{D}$  is of shape

$$\frac{\emptyset}{\mathcal{D}_G \parallel} \quad , \quad G' \wp \frac{\emptyset}{\text{ai} \downarrow \quad a^\perp \wp a}$$

We can immediately conclude by letting  $C = \emptyset$  (hence  $\mathcal{D}_C$  is trivial) and letting  $\mathcal{D}_G$  be

$$\frac{\emptyset}{\mathcal{D}_G'' \parallel} \quad , \quad G' \wp a^\perp$$

This completes the proof of the splitting lemma.  $\square$

## B Proof of Context Reduction (Lemma 7.1)

*Proof of Lemma 7.1.* We proceed by induction on the size of  $G[A]_S$ , making a case analysis based on Lemma 3.7.

- (i)  $G[A]_S$  has only one vertex. This is impossible as it is provable.  
 (ii)  $G[A]_S = G'' \wp G'[A]_S$ : In that case we make the same case analysis on  $G'[A]_S$ . But without loss of generality, we can assume that  $G'[A]_S$  is not a par.  
 (ii.i)  $G'[A]_S$  has only one vertex. Then  $G'[A]_S = A = a$  for some atom and  $G[A]_S = G'' \wp a$ . We apply Lemma 6.1.(3) and obtain  $C[\cdot]_R$  such that

$$\frac{\emptyset}{\mathcal{D}_G'' \parallel} \quad , \quad C[a^\perp]_R \quad , \quad G'' \quad , \quad \frac{\emptyset}{\mathcal{D}_C \parallel} \quad , \quad C$$

We let  $K = a^\perp$  and for any  $X$ , we can construct

$$\text{ss} \downarrow \quad \frac{\frac{\emptyset}{\mathcal{D}_G'' \parallel} \quad C[a^\perp \wp X]_R}{G'' \wp X}$$

as  $G[X]_S = G'' \wp X$ .

- (ii.iii)  $G'[A]_S$  is a tensor. Then we can assume without loss of generality that  $G[A]_S = G'' \wp (G'_1[A]_{S'} \otimes G_2)$ . By Lemma 6.1.(1), we get  $C'[\cdot]_{R'}$  and  $K_1$  and  $K_2$ , such that

$$C'[K_1 \wp K_2]_{R'} \quad , \quad \frac{\emptyset}{\mathcal{D}_C \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_1 \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_2 \parallel}$$

$$\frac{\mathcal{D}_G'' \parallel}{G''} \quad , \quad C' \quad , \quad K_1 \wp G'_1[A]_{S'} \quad , \quad K_2 \wp G_2$$

We apply the induction hypothesis to  $\mathcal{D}_1$  and get  $K$  and  $C''[\cdot]_{R''}$ , such that

$$\frac{\emptyset}{\mathcal{D}_C'' \parallel} \quad , \quad \frac{\emptyset}{\mathcal{D}_A \parallel} \quad , \quad \text{and} \quad \frac{\emptyset}{\mathcal{D}_G'' \parallel \text{GS}} \quad , \quad C''[K \wp X]_{R''} \quad , \quad K \wp A \quad , \quad K_1 \wp G'_1[X]_{S'}$$

for any  $X$ . We now let  $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ . We obtain  $\mathcal{D}_C$  from Lemma A.2 and construct  $\mathcal{D}_G$  as

$$\text{ss} \downarrow \quad \frac{\frac{\frac{\frac{\frac{\emptyset}{\mathcal{D}_G'' \parallel} \quad C''[K \wp X]_{R''}}{K_1 \wp (G'_1[X]_{S'} \otimes \frac{\emptyset}{\mathcal{D}_2 \parallel} \wp G_2)}{K_1 \wp K_2 \wp (G'_1[X]_{S'} \otimes G_2)} \quad , \quad C''[K_1 \wp K_2]_{R'} \quad \wp \quad (G'_1[X]_{S'} \otimes G_2)}{\text{ss} \downarrow} \quad C'[K_1 \wp K_2 \wp (G'_1[X]_{S'} \otimes G_2)]_{R'}}$$

since  $G[X]_S = G'' \wp (G'_1[X]_{S'} \otimes G_2)$ .



(ii.iv)  $G'[A]_S$  is composed via a prime graph  $P$  with  $|V_P| \geq 4$ . Then, without loss of generality we can assume that  $G'[A]_S = G'' \wp P(M_1[A]_{S'}, M_2, \dots, M_n)$ . Applying Lemma 6.1(2) gives us one of the following three cases:

( $\alpha$ ) We have  $C'[\cdot]_{R'}$  and  $K_1, \dots, K_n$ , such that

$$C'[P^\perp(K_1, \dots, K_n)]_{R'} \quad \emptyset \\ \mathcal{D}_G'' \parallel \quad , \quad \mathcal{D}_C'' \parallel \\ G'' \quad C'$$

$$\emptyset \quad \emptyset \\ \mathcal{D}_i \parallel \quad , \quad \mathcal{D}_i \parallel \\ K_1 \wp M_1[A]_{S'} \quad , \quad K_i \wp M_i$$

for  $2 \leq i \leq n$ . We can apply the induction hypothesis to  $\mathcal{D}_i$  and obtain  $K$  and  $C''[\cdot]_{R''}$ , such that

$$\emptyset \quad \emptyset \quad C''[K \wp X]_{R''} \\ \mathcal{D}_C'' \parallel \quad , \quad \mathcal{D}_A \parallel \quad , \quad \mathcal{D}_G'' \parallel \text{GS} \\ C'' \quad K \wp A \quad K_1 \wp M_1[X]_{S'}$$

for any  $X$ . We let  $C[\cdot]_R$  and  $\mathcal{D}_C$  as in case (ii.iii) above and obtain  $\mathcal{D}_G$  as

$$C' \left[ \begin{array}{c} C''[K \wp X]_{R''} \\ \mathcal{D}_G'' \parallel \\ K_1 \wp M_1[X]_{S'} \end{array} \otimes \begin{array}{c} \emptyset \\ \mathcal{D}_2 \parallel \\ K_2 \wp M_2 \end{array} \otimes \dots \otimes \begin{array}{c} \emptyset \\ \mathcal{D}_n \parallel \\ K_n \wp M_n \end{array} \right]_{R'} \\ \text{pl} \frac{P^\perp(K_1, \dots, K_n) \wp K_1 \wp P(M_1[X]_{S'}, M_2, \dots, M_n)}{C'[P^\perp(K_1, \dots, K_n)]_{R'} \wp P(M_1[X]_{S'}, M_2, \dots, M_n)} \\ \text{ss} \downarrow \\ \begin{array}{c} C'[P^\perp(K_1, \dots, K_n)]_{R'} \\ \mathcal{D}_G'' \parallel \\ G'' \end{array} \wp P(M_1[X]_{S'}, M_2, \dots, M_n)$$

where  $G[X]_S = G'' \wp P(M_1[X]_{S'}, M_2, \dots, M_n)$ .

( $\beta$ ) We have  $C'[\cdot]_{R'}$  and  $K_X$  and  $K_Y$ , such that

$$C'[K_X \wp K_Y]_{R'} \quad \emptyset \quad \emptyset \\ \mathcal{D}_G'' \parallel \quad , \quad \mathcal{D}_C'' \parallel \quad , \quad \mathcal{D}_X \parallel \\ G'' \quad C' \quad K_X \wp M_1[A]_{S'}$$

$$\emptyset \\ \mathcal{D}_Y \parallel \\ K_Y \wp P(\emptyset, M_2, \dots, M_n)$$

We apply the induction hypothesis to  $\mathcal{D}_X$  and get  $K$  and  $C''[\cdot]_{R''}$ , such that

$$\emptyset \quad \emptyset \quad C''[K \wp X]_{R''} \\ \mathcal{D}_C'' \parallel \quad , \quad \mathcal{D}_A \parallel \quad , \quad \mathcal{D}_G'' \parallel \text{GS} \\ C'' \quad K \wp A \quad K_X \wp M_1[X]_{S'}$$

for any  $X$ . We let  $C[\cdot]_R = C'[K_Y \wp P(C''[\cdot]_{R''}, M_2, \dots, M_n)]_{R'}$  and obtain  $\mathcal{D}_C$  from

Lemma A.2, and  $\mathcal{D}_G$  is as follows:

$$C' \left[ \begin{array}{c} K_Y \wp P \left( \begin{array}{c} C''[K \wp X]_{R''} \\ \mathcal{D}_G'' \parallel \\ K_X \wp M_1[X]_{S'} \end{array} , M_2, \dots, M_n \right) \end{array} \right]_{R'} \\ \text{ss} \downarrow \\ \frac{K_X \wp K_Y \wp P(M_1[X]_{S'}, M_2, \dots, M_n)}{C'[K_X \wp K_Y]_{R'} \wp P(M_1[X]_{S'}, M_2, \dots, M_n)} \\ \text{ss} \downarrow \\ \begin{array}{c} C'[K_X \wp K_Y]_{R'} \\ \mathcal{D}_G'' \parallel \\ G'' \end{array} \wp P(M_1[X]_{S'}, M_2, \dots, M_n)$$

( $\gamma$ ) We have  $C'[\cdot]_{R'}$  and  $K_X$  and  $K_Y$ , such that

$$C'[K_X \wp K_Y]_{R'} \quad \emptyset \quad \emptyset \\ \mathcal{D}_G'' \parallel \quad , \quad \mathcal{D}_C'' \parallel \quad , \quad \mathcal{D}_X \parallel \\ G'' \quad C' \quad K_X \wp M_2$$

$$\emptyset \\ \mathcal{D}_Y \parallel \\ K_Y \wp P(M_1[X]_{S'}, \emptyset, M_3, \dots, M_n)$$

We apply the induction hypothesis to  $\mathcal{D}_Y$  and get  $K$  and  $C''[\cdot]_{R''}$ , such that

$$\emptyset \quad \emptyset \quad C''[K \wp X]_{R''} \\ \mathcal{D}_C'' \parallel \quad , \quad \mathcal{D}_A \parallel \quad , \quad \mathcal{D}_G'' \parallel \\ C'' \quad K \wp A \quad K_Y \wp P(M_1[X]_{S'}, \emptyset, M_3, \dots, M_n)$$

for any  $X$ . We let  $C[\cdot]_R$  and  $\mathcal{D}_C$  as in case (ii.iii) above and obtain  $\mathcal{D}_G$  as

$$C' \left[ \begin{array}{c} C''[K \wp X]_{R''} \\ \mathcal{D}_G'' \parallel \\ K_Y \wp P \left( \begin{array}{c} \emptyset \\ \mathcal{D}_X \parallel \\ K_X \wp M_2 \end{array} , M_3, \dots, M_n \right) \end{array} \right]_{R'} \\ \text{ss} \downarrow \\ \frac{K_X \wp K_Y \wp P(M_1[X]_{S'}, M_2, M_3, \dots, M_n)}{C'[K_X \wp K_Y]_{R'} \wp P(M_1[X]_{S'}, M_2, M_3, \dots, M_n)} \\ \text{ss} \downarrow \\ \begin{array}{c} C'[K_X \wp K_Y]_{R'} \\ \mathcal{D}_G'' \parallel \\ G'' \end{array} \wp P(M_1[X]_{S'}, M_2, M_3, \dots, M_n)$$

(iii)  $G[A]_S$  is a tensor. This case is as case (ii.iii) above, with  $G'' = \emptyset$ , and consequently  $C' = K_1 = K_2 = \emptyset$ .

(iv)  $G[A]_S$  is composed via a prime graph  $P$  with  $|V_P| \geq 4$ . This case is as case (ii.iv) above, with  $G'' = \emptyset$ . Consequently,  $C'$  and  $K_1, \dots, K_n$  (resp.  $K_X$  and  $K_Y$ ) are empty as well.  $\square$

## C Proof of admissibility of the up-rules (Section 8)

In this section we show how splitting and context reduction are used to show the admissibility of all rules in the up-fragment of SGS.

**Theorem C.1.** *The rule ai $\uparrow$  is admissible for GS.*



$$\frac{\emptyset}{\mathcal{D}_6 \parallel} \frac{\emptyset}{L_P \wp P(M_1, \dots, M_n)}, \quad \frac{\emptyset}{\mathcal{D}_7 \parallel} \frac{\emptyset}{L_{P^\perp} \wp P^\perp(N_1, \dots, N_n)}.$$

Applying Lemma 6.1.(2) to  $\mathcal{D}_6$  and  $\mathcal{D}_7$  gives us four different cases.

(a) We get  $K_1, \dots, K_n$  and  $H_1, \dots, H_n$  and contexts  $C_3[\cdot]_{R_3}$  and  $C_4[\cdot]_{R_4}$ , such that

$$\frac{C_3[P^\perp(K_1, \dots, K_n)]_{R_3}}{\mathcal{D}_8 \parallel} \frac{\emptyset}{L_P}, \quad \frac{\emptyset}{\mathcal{D}_9 \parallel} \frac{\emptyset}{C_3}, \quad \frac{\emptyset}{\mathcal{D}'_i \parallel} \frac{\emptyset}{K_i \wp M_i},$$

$$\frac{C_4[P(H_1, \dots, H_n)]_{R_4}}{\mathcal{D}_{10} \parallel} \frac{\emptyset}{L_{P^\perp}}, \quad \frac{\emptyset}{\mathcal{D}_{11} \parallel} \frac{\emptyset}{C_4}, \quad \frac{\emptyset}{\mathcal{D}''_j \parallel} \frac{\emptyset}{H_j \wp N_j},$$

for all  $i, j \in \{1, \dots, n\}$ . Then, our proof of  $G[(M_1 \otimes N_1) \wp \dots \wp (M_n \otimes N_n)]_S$  is shown in Figure 4, where  $\mathcal{D}^*$  exist by Lemma 5.2.

(b) We get  $K_1, \dots, K_n$  and  $H_X$  and  $H_Y$  and contexts  $C_3[\cdot]_{R_3}$  and  $C_4[\cdot]_{R_4}$ , such that

$$\frac{C_3[P^\perp(K_1, \dots, K_n)]_{R_3}}{\mathcal{D}_8 \parallel} \frac{\emptyset}{L_P}, \quad \frac{\emptyset}{\mathcal{D}_9 \parallel} \frac{\emptyset}{C_3}, \quad \frac{\emptyset}{\mathcal{D}'_i \parallel} \frac{\emptyset}{K_i \wp M_i},$$

$$\frac{C_4[H_X \wp H_Y]_{R_4}}{\mathcal{D}_{10} \parallel} \frac{\emptyset}{L_{P^\perp}}, \quad \frac{\emptyset}{\mathcal{D}_{11} \parallel} \frac{\emptyset}{C_4}, \quad \frac{\emptyset}{\mathcal{D}_X \parallel} \frac{\emptyset}{H_X \wp N_j},$$

$$\frac{\emptyset}{\mathcal{D}_Y \parallel} \frac{\emptyset}{H_Y \wp P^\perp(N_1, \dots, N_{j-1}, \emptyset, N_{j+1}, \dots, N_n)}$$

for all  $i \in \{1, \dots, n\}$  and some  $j \in \{1, \dots, n\}$ . The derivation for  $G[(M_1 \otimes N_1) \wp \dots \wp (M_n \otimes N_n)]_S$  is shown in Figure 5, where  $\mathcal{D}^*$  consists of  $n+1$  instances of  $\text{ss}\downarrow$  and the double-line in the center indicates two instances of  $\text{ss}\downarrow$ .

(c) We get  $K_Z$  and  $K_W$  and  $H_1, \dots, H_n$  and contexts  $C_3[\cdot]_{R_3}$  and  $C_4[\cdot]_{R_4}$ , such that

$$\frac{C_3[K_Z \wp K_W]_{R_3}}{\mathcal{D}_8 \parallel} \frac{\emptyset}{L_P}, \quad \frac{\emptyset}{\mathcal{D}_9 \parallel} \frac{\emptyset}{C_3}, \quad \frac{\emptyset}{\mathcal{D}_Z \parallel} \frac{\emptyset}{K_Z \wp M_i},$$

$$\frac{\emptyset}{\mathcal{D}_W \parallel} \frac{\emptyset}{K_W \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n)},$$

$$\frac{C_4[P(H_1, \dots, H_n)]_{R_4}}{\mathcal{D}_{10} \parallel} \frac{\emptyset}{L_{P^\perp}}, \quad \frac{\emptyset}{\mathcal{D}_{11} \parallel} \frac{\emptyset}{C_4}, \quad \frac{\emptyset}{\mathcal{D}''_j \parallel} \frac{\emptyset}{H_j \wp N_j},$$

for some  $i \in \{1, \dots, n\}$  and all  $j \in \{1, \dots, n\}$ . This case is similar to the previous one.

(d) We get  $K_Z$  and  $K_W$  and  $H_X$  and  $H_Y$  and contexts  $C_3[\cdot]_{R_3}$  and  $C_4[\cdot]_{R_4}$ , such that

$$\frac{C_3[K_Z \wp K_W]_{R_3}}{\mathcal{D}_8 \parallel} \frac{\emptyset}{L_P}, \quad \frac{\emptyset}{\mathcal{D}_9 \parallel} \frac{\emptyset}{C_3}, \quad \frac{\emptyset}{\mathcal{D}_Z \parallel} \frac{\emptyset}{K_Z \wp M_i},$$

$$\frac{\emptyset}{\mathcal{D}_W \parallel} \frac{\emptyset}{K_W \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n)},$$

$$\frac{C_4[H_X \wp H_Y]_{R_4}}{\mathcal{D}_{10} \parallel} \frac{\emptyset}{L_{P^\perp}}, \quad \frac{\emptyset}{\mathcal{D}_{11} \parallel} \frac{\emptyset}{C_4}, \quad \frac{\emptyset}{\mathcal{D}_X \parallel} \frac{\emptyset}{H_X \wp N_j},$$

$$\frac{\emptyset}{\mathcal{D}_Y \parallel} \frac{\emptyset}{H_Y \wp P^\perp(N_1, \dots, N_{j-1}, \emptyset, N_{j+1}, \dots, N_n)}$$

for some  $i, j \in \{1, \dots, n\}$ . In this case we use the derivation in Figure 6 to prove  $G[(M_1 \otimes N_1) \wp \dots \wp (M_n \otimes N_n)]_S$ . More precisely, Figure 6 shows the case  $i < j$ , the cases  $i = j$  and  $i > j$  are similar. The derivation  $\mathcal{D}^*$  exist by the second statement in Lemma 5.2. If  $i \neq j$ , this derivation consists of a single  $\text{p}\uparrow$  instance. If  $i = j$ , it can be a longer derivation containing all rules of  $\text{SGS}\uparrow$  (where the instances of  $\text{ai}\uparrow$  and  $\text{ss}\uparrow$  can be eliminated by the previous two theorems). The important observation to make is that all instances of  $\text{p}\uparrow$  occurring in  $\mathcal{D}^*$  have smaller size than the one we started with. Therefore we can invoke the induction hypothesis.  $\square$

## D Proof of Conservativity (Lemma 9.1)

*Proof of Lemma 9.1.* By way of contradiction, assume there is a cograph that is not provable with passing through a non-cograph. Let  $A$  be a minimal such graph, where we define the size of  $A$  as the lexicographic pair  $\langle |V_A|, |E_{A^+}| \rangle$ . The only way to create a non-cograph from a cograph while going up in a derivation is via the  $\text{ss}\downarrow$  as in

$$\text{ss}\downarrow \frac{P(M_1, \dots, M_{i-1}, M_i, M_{i+1}, \dots, M_n)}{M_i \wp P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n)}$$

where  $M_i$  and  $P(M_1, \dots, M_{i-1}, \emptyset, M_{i+1}, \dots, M_n)$  are cographs and  $P(M_1, \dots, M_n)$  is not. Without loss of generality, we assume  $i = 1$ . By minimality of  $A$ , we can assume that this  $\text{ss}\downarrow$  occurs as bottommost rule instance in  $\mathcal{D}$ , and we can also assume that it occurs in a shallow context, i.e., we have

$$A = G \wp M_1 \wp P(\emptyset, M_2, \dots, M_n)$$

for some  $G$ . Otherwise  $A = G \wp C[M_1 \wp P(\emptyset, M_2, \dots, M_n)]_{R_3}$  for some nontrivial context  $C[\cdot]_{R_3}$ , and we could apply context reduction to get a  $K$  with  $\vdash K \wp M_1 \wp P(\emptyset, M_2, \dots, M_n)$

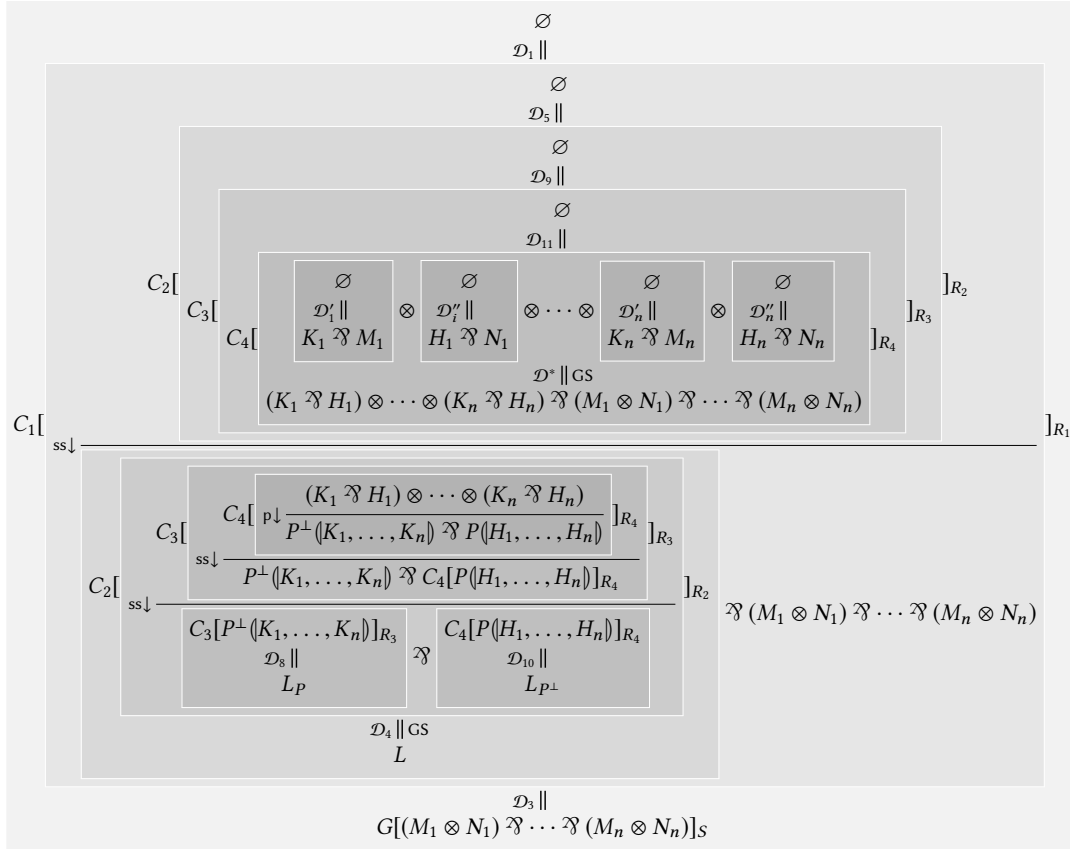


Figure 4. Derivation for case (a) in the proof of Theorem C.3

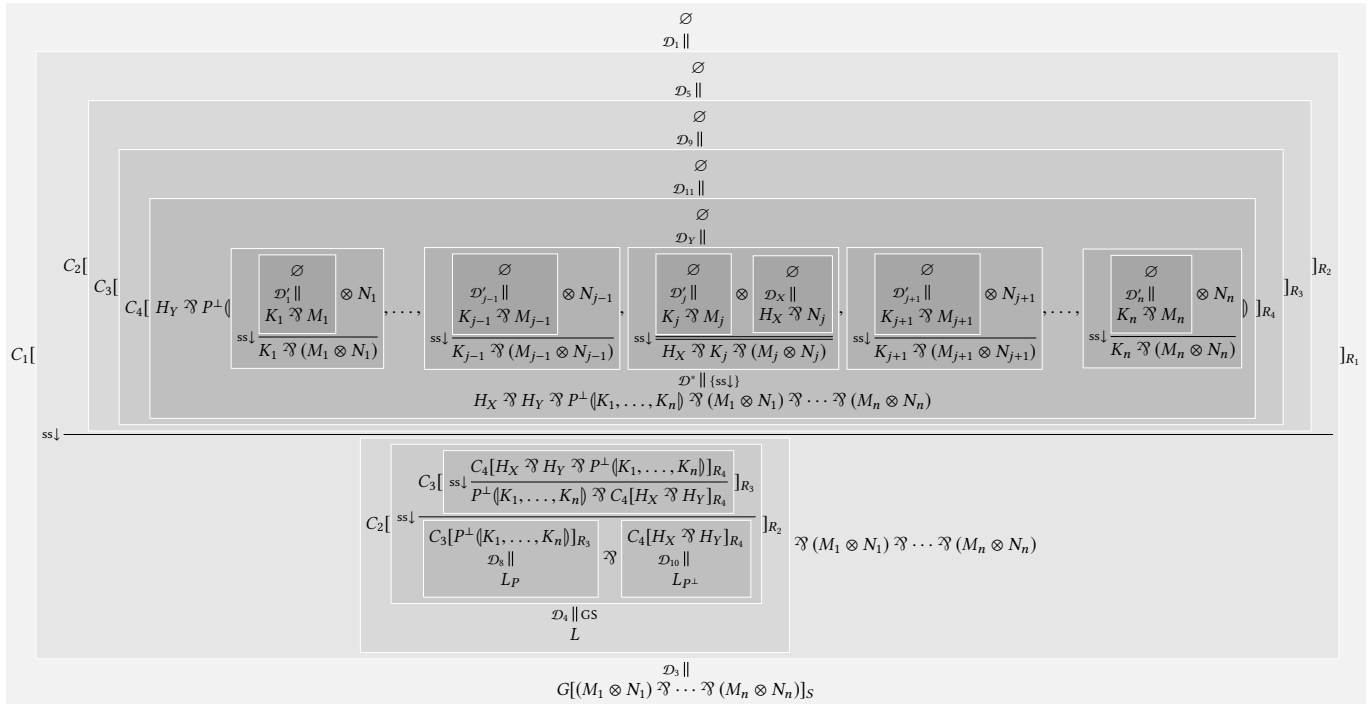


Figure 5. Derivation for case (b) in the proof of Theorem C.3

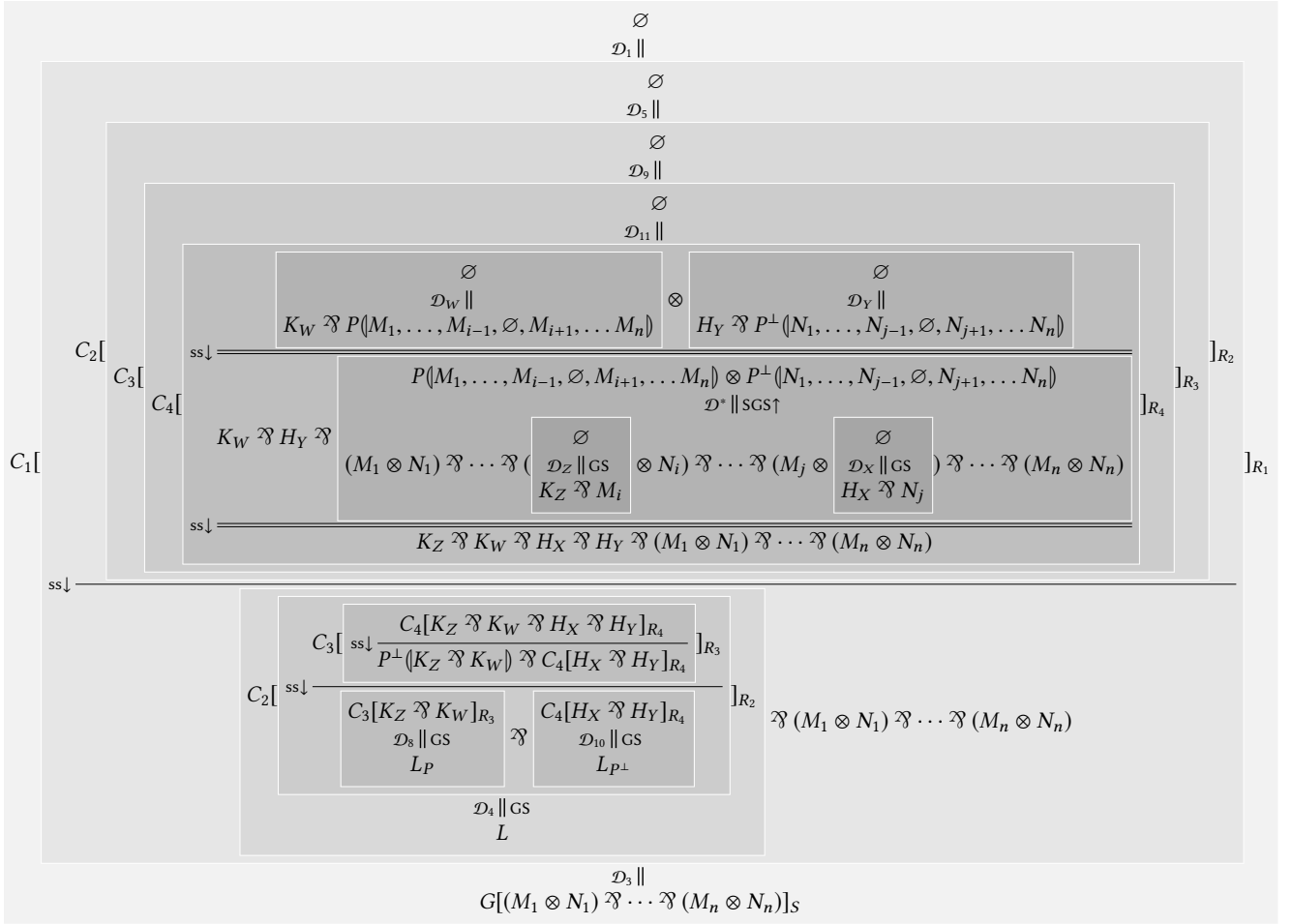


Figure 6. Derivation for case (d) in the proof of Theorem C.3.

contradicting the minimality of  $A$ . Hence,  $\mathcal{D}$  is of shape

$$\frac{\emptyset \quad \mathcal{D}' \parallel \text{GS}}{G \wp P(M_1, M_2, \dots, M_n)} \quad \text{ss}\downarrow \frac{G \wp M_1 \wp P(\emptyset, M_2, \dots, M_n)}{G \wp M_1 \wp P(\emptyset, M_2, \dots, M_n)}$$

and we apply splitting (Lemma 6.1.(2)) to  $\mathcal{D}'$ , yielding 2 possibilities, of which we show here only the first, the second one being simpler.

- there is a context  $C[\cdot]_R$  and graphs  $K_1, \dots, K_n$ , such that

$$\frac{C[P^\perp(K_1, \dots, K_n)]_R \quad \mathcal{D}_G \parallel \text{GS}}{G} \quad , \quad \frac{\emptyset}{C} \quad \text{GS} \quad , \quad \frac{\emptyset \quad \mathcal{D}' \parallel \text{GS}}{K_i \wp M_i}$$

for all  $i \in \{1, \dots, n\}$ . If  $|\mathcal{D}_G| = 0$  then  $G = C[P^\perp(K_1, \dots, K_n)]_R$  and we can have a derivation

$$\frac{C[P^\perp(K_1, \dots, K_n)]_R \quad \frac{\emptyset \quad \mathcal{D}_1 \parallel \text{GS}}{K_1 \wp M_1} \quad K_2, \dots, K_n}{C[P^\perp(K_1, \dots, K_n)]_R \wp M_1} \quad \wp \quad P(\emptyset, M_2, \dots, M_n) \quad (33)$$

contradicting the minimality of  $A$ . If  $|\mathcal{D}_G| \neq 0$  and there is a cograph  $G'$  occurring in  $\mathcal{D}_G$ , then  $G'$  has smaller size than  $G$ , contradicting the minimality of  $G \wp M_1 \wp P(\emptyset, M_2, \dots, M_n)$ . If there is no smaller cograph  $G'$  occurring in  $\mathcal{D}_G$  in, then the bottom-most rule instance in  $\mathcal{D}_G$  is a  $ss\downarrow$  creating a non-cograph. By the same argument as above, we can conclude that it must be in a shallow context. Then  $G = K_i \wp C[P^\perp(K_1, \dots, K_{i-1}, \emptyset, K_{i+1}, \dots, K_n)]_R$ . If  $i = 1$  we apply  $ss\downarrow$  to move  $M_1$  and  $K_1$  inside  $C[\cdot]_R$  and conclude by a similar reasoning as with (33). If  $i \neq 1$  we have  $P^\perp(K_1, \dots, K_{i-1}, \emptyset, K_{i+1}, \dots, K_n)$  and



$P(\emptyset, M_2, \dots, M_n)$  are cographs and again we get a contradiction to the minimality of  $A$ .  $\square$