Logic beyond formulas: a proof system on graphs

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Logic Beyond Formulas: A Proof System on Graphs

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Abstract

In this paper we present a proof system that operates on graphs instead of formulas. We begin our quest with the well-known correspondence between formulas and cographs, which are undirected graphs that do not have P_4 (the four-vertex path) as vertex-induced subgraph; and then we drop that condition and look at arbitrary (undirected) graphs. The consequence is that we lose the tree structure of the formulas corresponding to the cographs. Therefore we cannot use standard proof theoretical methods that depend on that tree structure. In order to overcome this difficulty, we use a modular decomposition of graphs and some techniques from deep inference where inference rules do not rely on the main connective of a formula. For our proof system we show the admissibility of cut and a generalization of the splitting property. Finally, we show that our system is a conservative extension of multiplicative linear logic (MLL) with mix, meaning that if a graph is a cograph and provable in our system, then it is also provable in MLL+mix.

Keywords: Proof theory, cographs, graph modules, prime graphs, cut elimination, deep inference, splitting, analycity

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1 Introduction

The notion of formula is central to all applications of logic and proof theory in computer science, ranging from the formal verification of software, where a formula describes a property that the program should satisfy, to logic programming, where a formula represents a program [27, 31], and functional programming, where a formula represents a type [25]. Proof theoretical methods are also employed in concurrency theory, where a formula can represent a process whose behaviours may be extracted from a proof of the formula [5, 22, 23, 30]. This *formulas-as-processes* paradigm is not as well-investigated as the *formulas-as-properties*, *formulas-as-programs* and *formulas-as-types* paradigms mentioned before. In our opinion, a reason for this is that the notion of formula reaches its limitations when it comes to describing processes as they are studied in concurrency theory.

For example, BV [17] and *pomset logic* [37] are proof systems which extend linear logic with a notion of sequential composition and can model series-parallel orders. However, series-parallel orders cannot express some ubiquitous patterns of *causal dependencies* such as producer-consumer queues [28], which are within the scope of pomsets [36], event structures [33], and Petri nets [34]. The essence of this problem is already visible when we consider *symmetric dependencies*, such as separation, which happens to be the dual concept to concurrency in the *formulas-as-processes* paradigm.

Let us use some simple examples to explain the problem. Suppose we are in a situation where two processes A and B can communicate with each other, written as $A \parr B$, or can be separated from each other, written as $A \otimes B$, such that no communication is possible. Now assume we have four atomic processes a, b, c, and d, from which we form the two processes $P = (a \otimes b) \parr (c \otimes d)$ and $Q = (a \parr c) \otimes (b \parr d)$. Both are perfectly fine formulas of multiplicative linear logic (MLL) [15]. In P, we have that a is separated from b but can communicate with c and d. Similarly, d can communicate with a and b but is separated from c, and so on. On the other hand, in Q, a can only communicate with c and is separated from the other two, and d can only communicate with b, and is separated from the other two. We can visualize this situation via graphs where a, b, c, and d are the vertices, and we draw an edge between two vertices if they are separated, and no edge if they can communicate. Then P and Q correspond to the two graphs shown below.

$$P = (a \otimes b) \parr (c \otimes d) \qquad Q = (a \parr c) \otimes (b \parr d)$$

(1)

It should also be possible to describe a situation where a is separated from b, and b is separated from c, and c is separated from d, but a can communicate with c and d, and b can

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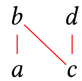
communicate with d, as indicated by the graph below.



(2)

However, this graph cannot be described by a formula in such a way that was possible for the two graphs in (1). Consequently, the tools of proof theory, that have been developed over the course of the last century, and that were very successful for the *formulas-as-properties*, *formulas-as-programs*, and *formulas-as-types* paradigms, can be used for the *formulas-as-processes* paradigm only if situations as in (2) above are forbidden. This seems to be a very strong and unnatural restriction, and the purpose of this paper is to propose a way to change this unsatisfactory situation.

We will present a proof system, called GS (for *graphical proof system*), whose objects of reason are not formulas but graphs, giving the example in (2) the same status as the examples in (1). In a less informal way, one could say that standard proof systems work on cographs (which are the class of graphs that correspond to formulas as in (1)), and our proof systems works on arbitrary graphs. In order for this to make sense, this proof system should obey the following basic properties:

1. *Consistency*: There are graphs that are not provable.
2. *Transitivity*: The proof system should come with an implication that is transitive, i.e., if we can prove that A implies B and that B implies C, then we should also be able to prove that A implies C.
3. *Analycity*: As we no longer have formulas, we cannot ask that every formula that occurs in a proof is a subformula of its conclusion. But we can ask that in a proof search situation, there is always only a finite number of ways to apply an inference rule.
4. *Conservativity*: There should be a well-known logic L based on formulas such that when we restrict our proof system to graphs corresponding to formulas, then we prove exactly the theorems of L.
5. *Minimality*: We want to make as few assumptions as possible, so that the theory we develop is as general as possible.

Properties 1-3 are standard for any proof system, and they are usually proved using cut elimination. In that respect our paper is no different. We introduce a notion of cut and show its admissibility for GS. Then Properties 1-3 are immediate consequences, and also Property 4 will follow from cut admissibility, where in our case the logic L is multiplicative linear logic (MLL) with mix [2, 14, 15].

Finally, Property 5 is of a more subjective nature. In our case, we only make the following two basic assumptions:

1. For any graph A, we should be able to prove that A implies A.
2. If a graph A is provable, then the graph $G = C[A]$ is also provable[1], provided that $C[\cdot]$ is a provable context. This can be compared to the *necessitation rule* of modal logic, which says that if A is provable then so is $\Box A$, except that in our case the $\Box$ is replaced by the provable graph context $C[\cdot]$.

All other properties of the system GS follow from the need to obtain admissibility of cut. This means that this paper does not present some random system, but follows the underlying principles of proof theory.

In Section 2, we give preliminaries on cographs, which form the class of graphs that correspond to formulas as in (1). Then, in Section 3 we give some preliminaries on modules and prime graphs, which are needed for our move away from cographs, so that in Section 4, we can present our proof system, which uses the notation of open deduction [18] and follows the principles of deep inference [4, 17, 19]. In Section 5 we show some properties of our system, and Sections 6, 7, and 8 are dedicated to cut elimination. Finally, in Section 9, we show that our system is a conservative extension of MLL+mix.

The contributions of this paper can thus be summarized as follows:

- We present (to our knowledge) the first proof system that is not tied to formulas/cographs but handles arbitrary (undirected) graphs instead.
- We prove a *Splitting Lemma* (in Section 6), which is often a crucial ingredient in a proof of cut elimination in a deep inference system. But in our case the statement and the proof of this lemma is different from standard deep inference systems, in particular, the general method proposed by Aler Tubella in her PhD [44] does not apply. But we still use the name *Splitting Lemma*, as it serves the same purpose.
- We propose a cut rule which corresponds to the standard cut rule in a deep inference system, and show its admissibility. But again, due to the different nature of our proof system, the standard methods must be adapted.

2 From Formulas to Graphs

Definition 2.1. A ***(simple, undirected) graph*** G is a pair $\langle V_G, E_G \rangle$ where V_G is a set of vertices and E_G is a set of two-element subsets of V_G. We omit the index G when it is clear from the context. For $v, w \in V_G$ we write vw as an abbreviation for $\{v, w\}$. A graph G is ***finite*** if its vertex set V_G is finite. Let L be a set and G be a graph. We say that G is ***L-labelled*** (or just ***labelled*** if L is clear from context) if every vertex in V_G is associated with an element of L, called its ***label***. We write $\ell_G(v)$ to denote the label of the vertex

[1] Formally, the notation $G = C[A]$ means that A is a module of G, and $C[\cdot]$ is the graph obtained from G by removing all vertices belonging to A. We give the formal definition in Section 3.

v in G. A graph G' is a ***subgraph*** of a graph G, denoted as $G' \subseteq G$ iff $V_{G'} \subseteq V_G$ and $E_{G'} \subseteq E_G$. We say that G' is an ***induced subgraph*** of G if G' is a subgraph of G and for all $v, w \in V_{G'}$, if $vw \in E_G$ then $vw \in E_{G'}$. The ***size*** of a graph G, denoted by $|G|$, is the number of its vertices, i.e., $|G| = |V_G|$.

In the following, we will just say *graph* to mean a finite, undirected, labelled graph, where the labels come from the set $\mathcal{A}$ of atoms which is the (disjoint) union of a countable set of propositional variables $\mathcal{V} = \{a, b, c, \ldots\}$ and their duals $\mathcal{V}^\perp = \{a^\perp, b^\perp, c^\perp, \ldots\}$.

Since we are mainly interested in how vertices are labelled, but not so much in the identity of the underlying vertex, we heavily rely on the notion of graph isomorphism.

Definition 2.2. Two graphs G and G' are ***isomorphic*** if there exists a bijection $f \colon V_G \to V_{G'}$ such that for all $v, u \in V_G$ we have $vu \in E_G$ iff $f(v)f(u) \in E_{G'}$ and $\ell_G(v) = \ell_{G'}(f(v))$. We denote this as $G \simeq_f G'$, or simply as $G \simeq G'$ if f is clear from context or not relevant.

In the following, we will, in diagrams, forget the identity of the underlying vertices, showing only the label, as in the examples in the introduction.

In the rest of this section we recall the characterization of those graphs that correspond to formulas. For simplicity, we restrict ourselves to only two connectives, and for reasons that will become clear later, we use the $\parr$ (***par***) and $\otimes$ (***tensor***) of linear logic [15]. More precisely, ***formulas*** are generated by the grammar

$$\phi, \psi \coloneqq \circ \mid a \mid a^\perp \mid \phi \parr \psi \mid \phi \otimes \psi \tag{3}$$

where $\circ$ is the ***unit***, and a can stand for any propositional variable in $\mathcal{V}$. As usual, we can define the negation of formulas inductively by letting $a^{\perp\perp} = a$ for all $a \in \mathcal{V}$, and by using the De Morgan duality between $\parr$ and $\otimes$: $(\phi \parr \psi)^\perp = \phi^\perp \otimes \psi^\perp$ and $(\phi \otimes \psi)^\perp = \phi^\perp \parr \psi^\perp$; the unit is self-dual: $\circ^\perp = \circ$.

On formulas we define the following structural equivalence relation:

$$\begin{array}{cc}
\phi \parr (\psi \parr \xi) \equiv (\phi \parr \psi) \parr \xi & \phi \otimes (\psi \otimes \xi) \equiv (\phi \otimes \psi) \otimes \xi \\
\phi \parr \psi \equiv \psi \parr \phi & \phi \otimes \psi \equiv \psi \otimes \phi \\
\phi \parr \circ \equiv \phi & \phi \otimes \circ \equiv \phi
\end{array}$$

In order to translate formulas to graphs, we define the following two operations on graphs:

Definition 2.3. Let $G = \langle V_G, E_G \rangle$ and $H = \langle V_H, E_H \rangle$ be graphs with $V_G \cap V_H = \emptyset$. We define the ***par*** and ***tensor*** operations between them as follows:

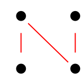
$$\begin{array}{rcl}
G \parr H & = & \langle V_G \cup V_H, E_G \cup E_H \rangle \\
G \otimes H & = & \langle V_G \cup V_H, E_G \cup E_H \cup \{vw \mid v \in V_G, w \in V_H\} \rangle
\end{array}$$

For a formula ϕ, we can now define its associated graph $[\![\phi]\!]$ inductively as follows: $[\![\circ]\!] = \emptyset$ the empty graph; $[\![a]\!] = a$ a single-vertex graph whose vertex is labelled by a (by a sight abuse of notation, we denote that graph also by a); similarly $[\![a^\perp]\!] = a^\perp$; finally we define $[\![\phi \parr \psi]\!] = [\![\phi]\!] \parr [\![\psi]\!]$ and $[\![\phi \otimes \psi]\!] = [\![\phi]\!] \otimes [\![\psi]\!]$.

Theorem 2.4. *For any two formulas, $\phi \equiv \psi$ iff $[\![\phi]\!] = [\![\psi]\!]$.*

Proof. By a straightforward induction. □

Definition 2.5. A graph is ***P_4-free*** (or ***N-free*** or ***Z-free***) iff it does not have an induced subgraph of the shape



(4)

Theorem 2.6. *Let G be a graph. Then there is a formula ϕ with $[\![\phi]\!] = G$ iff G is P_4-free.*

A proof of this can be found, e.g., in [32] or [17].

The graphs characterized by Theorem 2.6 are called ***cographs***, because they are the smallest class of graphs containing all single-vertex graphs and being closed under complement and disjoint union.

Because of Theorem 2.6, one can think of standard proof system as *cograph proof systems*. Since in this paper we want to move from cographs to general graphs, we need to investigate, how much of the tree structure of formulas (which makes cographs so interesting for proof theory [26, 38, 42]) can be recovered for general graphs.

3 Modules and Prime Graphs

In this section we take some of the concepts that make working with formulas so convenient and lift them to graphs that are not P_4-free.

Definition 3.1. Let G be a graph. A ***module*** of G is an induced subgraph $M = \langle V_M, E_M \rangle$ of G such that for all $v \in V_G \setminus V_M$ and all $x, y \in M$ we have $vx \in E_G$ iff $vy \in E_G$.

Modules are used in this paper since they are for graphs what subformulas are for formulas.

Notation 3.2. Let G be a graph and M be a module of G. Let $V_C = V_G \setminus V_M$ and let C be the graph obtained from G by removing all vertices in M (including incident edges). Let $R \subseteq V_C$ be the set of vertices that are connected to a vertex in V_M (and hence to all vertices in M). We denote this situation as $G = C[M]_R$ and call $C[\cdot]_R$ (or just C) the ***context*** of M in G. Alternatively, $C[M]_R$ can be defined as follows. If we write $C[x]_R$ for a graph in which x is a distinct vertex and R is the set of neighbours x, then $C[M]_R$ is the graph obtained from $C[x]_R$ by substitution of x for M.

Lemma 3.3. *Let G be a graph and M, N be modules of G. Then*

1. *$M \cap N$ is a module of G;*
2. *if $M \cap N \neq \emptyset$, then $M \cup N$ is a module of G; and*
3. *if $N \not\subseteq M$ then $M \setminus N$ is a module of G.*

Proof. The first statement follows immediately from the definition. For the second one, let $L = M \cap N \neq \emptyset$, and let $v \in G \setminus (V_M \cup V_N)$ and $x, y \in V_M \cup V_N$. If x, y are both in M or both in N, then we have immediately $vx \in E_G$ iff $vy \in E_G$. So, let $x \in V_M$ and $y \in V_N$, and let $z \in L$. We have $vx \in E_G$ iff $vz \in E_G$ iff $vy \in E_G$. Finally, for the last statement, let $x, y \in V_M \setminus V_N$ and let $v \in V_G \setminus (V_M \setminus V_N)$. If $v \notin V_M$, we immediately have $vx \in E_G$ iff $vy \in E_G$. So, let $v \in V_M$, and therefore $v \in V_M \cap V_N$. Let $z \in V_N \setminus V_M$. Then $vx \in E_G$ iff $zx \in E_G$ iff $zy \in E_G$ iff $vy \in E_G$. □

Definition 3.4. Let G be a graph. A module M in G is ***maximal*** if for all modules M' of G such that $M \neq G$ we have that $M \subseteq M'$ implies $M = M'$.

Definition 3.5. A module M of a graph G is ***trivial*** iff either $V_M = \emptyset$ or V_M is a singleton or $V_M = V_G$. A graph G is ***prime*** iff $|V_G| \geq 2$ and all modules of G are trivial.

Definition 3.6. Let G be a graph with n vertices $V_G = \{v_1, \ldots, v_n\}$ and let $H_1, \ldots, H_n$ be n graphs. We define the ***composition of*** $\boldsymbol{H_1, \ldots, H_n}$ ***via*** $\boldsymbol{G}$, denoted as $G(\!|H_1, \ldots, H_n|\!)$, by replacing each vertex v_i of G by the graph H_i; and there is an edge between two vertices x and y if either x and y are in the same H_i and $xy \in E_{H_i}$ or $x \in V_{H_i}$ and $y \in V_{H_j}$ for $i \neq j$ and $v_i v_j \in E_G$. Formally, $G(\!|H_1, \ldots, H_n|\!) = \langle V^*, E^* \rangle$ with

$$V^* = \bigcup_{1 \leq i \leq n} V_{H_i}$$
$$E^* = \bigcup_{1 \leq i \leq n} E_{H_i} \cup \{xy \mid x \in V_{H_i}, y \in V_{H_j}, v_i v_j \in E_G\}$$

This concept allows us to decompose graphs into prime graphs (via Lemma 3.7 below) and recover a tree structure for an arbitrary graph, seeing prime graphs as generalized non-decomposable n-ary connectives. The two operations $\parr$ and $\otimes$, defined in Definition 2.3 are then represented by the two prime graphs.

$$\parr : \quad \bullet \qquad \bullet \qquad \text{and} \qquad \otimes : \quad \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \tag{5}$$

If we name these graphs $\parr$ and $\otimes$, respectively, then we can write $\parr(\!|G, H|\!) = G \parr H$ and $\otimes(\!|G, H|\!) = G \otimes H$.

Lemma 3.7. *Let G be a nonempty graph. Then we have exactly one of the following four cases:*

(i) G is a singleton graph.
(ii) $G = A \parr B$ for some A, B with $A \neq \emptyset \neq B$.
(iii) $G = A \otimes B$ for some A, B with $A \neq \emptyset \neq B$.
(iv) $G = P(\!|A_1, \ldots, A_n|\!)$ for some prime graph P with $n = |V_P| \geq 4$ and $A_i \neq \emptyset$ for all $0 \leq i \leq n$.

Proof. Let G be given. If $|G| = 1$, we are in case (i). Now assume $|G| > 1$, and let $M_1, \ldots, M_n$ be the maximal modules of G. Now we have two cases:

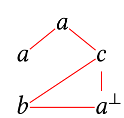
- For all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ we have $M_i \cap M_j = \emptyset$. Since every vertex of G forms a module, every vertex must be part of a maximal module. Hence $V_G = V_{M_1} \cup \cdots \cup V_{M_n}$. Therefore there is a graph P such that $G = P(\!|M_1, \ldots, M_n|\!)$. Since all M_i are maximal in G, we can conclude that P is prime. If $|V_P| \geq 4$ we are in case (iv). If $|V_P| < 4$ we are either in case (ii) or (iii), as the two graphs in (5) are only two prime graphs with $|V_P| = 2$, and there are no prime graphs with $|V_P| = 3$.
- We have some $i \neq j$ with $M_i \cap M_j \neq \emptyset$. Let $L = M_i \cap M_j$ and $N = M_i \setminus M_j$ and $K = M_j \setminus M_i$. By Lemma 3.3, L, N, K, and $M_i \cup M_j$ are all modules of G. Since M_i and M_j are maximal, it follows that $G = M_i \cup M_j$, and therefore $G = N \otimes L \otimes K$ or $G = N \parr L \parr K$. □

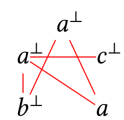
4 The Proof System

To define a proof system, we need a notion of implication. To do so, we first introduce a notion of negation.

Definition 4.1. For a graph $G = \langle V_G, E_G \rangle$, we define its ***dual*** $G^\perp = \langle V_G, E_{G^\perp} \rangle$ to have the same set of vertices, and an edge $vw \in E_{G^\perp}$ iff $vw \notin E_G$ (and $v \neq w$). The label of a vertex v in $G^\perp$ is the dual of the label of that vertex in G, i.e., $\ell_{G^\perp}(v) = \ell_G(v)^\perp$. For any two graphs G and H, the ***implication*** $G \multimap H$ is defined to be the graph $G^\perp \parr H$.

Example 4.2. To give an example, consider the graph G on the left below

G: 

$G^\perp$: 

(6)

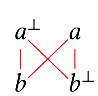
Its negation $G^\perp$ is shown on the right above.

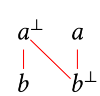
Observe that the dual graph construction defines the standard De Morgan dualities relating conjunction and disjunction, i.e., for every formula ϕ, we have $[\![\phi^\perp]\!] = [\![\phi]\!]^\perp$. Furthermore, the De Morgan dualities extend to prime graphs, say P, as $P(\!|M_1, \ldots, M_n|\!)^\perp = P^\perp(\!|M_1^\perp, \ldots, M_n^\perp|\!)$, where $P^\perp$ is the dual graph to P. Furthermore, $P^\perp$ is prime if and only if P is prime. Thus each pair of prime graphs P and $P^\perp$ defines a pair of connectives that are De Morgan duals to each other.

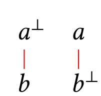
We will now develop our proof system based on the above notion of negation as graph duality. From the requirements mentioned in the introduction it follows that:

(i) for any G, the graph $G \multimap G$ should be provable;
(ii) if $G \neq \emptyset$ then G and $G^\perp$ should not be both provable;
(iii) the implication $\multimap$ should be transitive, i.e., if $G \multimap H$, and $H \multimap K$ are provable then so should be $G \multimap K$;
(iv) the implication $\multimap$ should be closed under context, i.e., if $G \multimap H$ is provable and $C[\cdot]_R$ is an arbitrary context, then $C[G]_R \multimap C[H]_R$ should be provable;
(v) if A and C are provable graphs, and $R \subseteq V_C$, then the graph $C[A]_R$ should also be provable.

Example 4.3. As an example, consider the following three graphs:

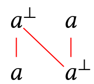
A_1: 

A_2: 

A_3: 

(7)

The graph A_1 on the left should clearly be provable, as it corresponds to the formula $(a^\perp \parr a) \otimes (b \parr b^\perp)$, which is provable in MLL. The graph A_3 on the right should not be provable, as it corresponds to the formula $(a^\perp \otimes b) \parr (a \otimes b^\perp)$, which is not provable in MLL. But what about the graph A_2 in the middle? It does not correspond to a formula, and therefore we cannot resort to MLL. Nonetheless, we can make the following observations. If A_2 were provable, then so would be the graph A_4 shown below:

A_4:  (8)

as it is obtained from A_2 by a simple substitution. However, $A_4^\perp = A_4$, and therefore $A_4^\perp$ and A_4 would both be provable, which would be a contradiction and should be ruled out. Hence, A_2 should not be provable.

We can make further observations without having presented the proof system yet: Notice that $A_1 \multimap A_2$ cannot hold, as otherwise we would be able to use A_1 and modus ponens to establish that A_2 is provable, which cannot hold as we just observed. By applying a dual argument, $A_2 \multimap A_3$ cannot hold. Hence, implication is not simply subset inclusion of edges.[2]

For presenting the inference system we use a *deep inference* formalism [17, 19], which allows rewriting inside an arbitrary context and admits a rather flexible composition of derivations. In our presentation we will follow the notation of *open deduction*, introduced in [18].

Let us start with the following two inference rules

$$\mathsf{i}{\downarrow}\,\frac{\varnothing}{A^\perp \parr A} \qquad\qquad \mathsf{ss}{\uparrow}\,\frac{B \otimes A}{B[A]_S}\;{\scriptstyle S\subseteq V_B,\ S\neq V_B} \tag{9}$$

which are induced by the two Points (i) and (v) above, and which are called ***identity down*** and ***super switch up***, respectively. The $\mathsf{i}{\downarrow}$ says that for arbitrary graphs C and A and any $R \subseteq V_C$, if C is provable, then so is the graph $C[A \parr A^\perp]_R$. Similarly, the rule $\mathsf{ss}{\uparrow}$ says that whenever $C[B \otimes A]_R$ is provable, then so is $C[B[A]_S]_R$ for any three graphs A, B, C and any $R \subseteq V_C$ and $S \subseteq V_B$. The condition $S \neq V_B$ is there to avoid a trivial rule instance, as $B[A]_S = B \otimes A$ if $S = V_B$.

Definition 4.4. An ***inference system*** S is a set of inference rules. We define the set of ***derivations*** in S inductively below, and we denote a derivation $\mathcal{D}$ in S with premise G and conclusion H, as follows:

$$\begin{array}{c} G \\ \mathcal{D} \Vert \mathsf{S} \\ H \end{array}$$

1. Every graph G is a derivation (also denoted by G) with premise G and conclusion G.
2. If $\mathcal{D}_1$ is a derivation with premise G_1 and conclusion H_1, and $\mathcal{D}_2$ is a derivation with premise G_2 and conclusion H_2, then $\mathcal{D}_1 \parr \mathcal{D}_2$ is a derivation with premise $G_1 \parr G_2$ and conclusion $H_1 \parr H_2$, and similarly, $\mathcal{D}_1 \otimes \mathcal{D}_2$ is a derivation with premise $G_1 \otimes G_2$ and conclusion $H_1 \otimes H_2$, denoted as
$$\begin{array}{c} G_1 \\ \mathcal{D}_1 \Vert \mathsf{S} \\ H_1 \end{array} \parr \begin{array}{c} G_2 \\ \mathcal{D}_2 \Vert \mathsf{S} \\ H_2 \end{array} \qquad \text{and} \qquad \begin{array}{c} G_1 \\ \mathcal{D}_1 \Vert \mathsf{S} \\ H_1 \end{array} \otimes \begin{array}{c} G_2 \\ \mathcal{D}_2 \Vert \mathsf{S} \\ H_2 \end{array}$$
respectively.
3. If $\mathcal{D}_1$ is a derivation with premise G_1 and conclusion H_1, and $\mathcal{D}_2$ is a derivation with premise G_2 and conclusion H_2, and
$$\mathsf{r}\,\frac{H_1}{G_2}$$
is an instance of an inference rule r, then $\mathcal{D}_2 \circ_{\mathsf{r}} \mathcal{D}_1$ is a derivation with premise G_2 and conclusion H_2, denoted as
$$\mathsf{r}\,\frac{\begin{array}{c} G_1 \\ \mathcal{D}_1 \Vert \mathsf{S} \\ H_1 \end{array}}{\begin{array}{c} G_2 \\ \mathcal{D}_2 \Vert \mathsf{S} \\ H_2 \end{array}} \qquad \text{or} \qquad \begin{array}{c} G_1 \\ \mathcal{D}_1 \Vert \mathsf{S} \\ \mathsf{r}\,\dfrac{H_1}{G_2} \\ \mathcal{D}_2 \Vert \mathsf{S} \\ H_2 \end{array}$$
If $H_1 \simeq_f G_2$ we can compose $\mathcal{D}_1$ and $\mathcal{D}_2$ directly to $\mathcal{D}_2 \circ \mathcal{D}_1$, denoted as
$$f\,\begin{array}{c} G_1 \\ \mathcal{D}_1 \Vert \mathsf{S} \\ H_1 \\ \cdots\cdots \\ G_2 \\ \mathcal{D}_2 \Vert \mathsf{S} \\ H_2 \end{array} \qquad \text{or} \qquad \simeq\,\begin{array}{c} G_1 \\ \mathcal{D}_1 \Vert \mathsf{S} \\ H_1 \\ \cdots\cdots \\ G_2 \\ \mathcal{D}_2 \Vert \mathsf{S} \\ H_2 \end{array} \qquad \text{or} \qquad \begin{array}{c} G_1 \\ \mathcal{D}_1 \Vert \mathsf{S} \\ H_1 \\ \cdots\cdots \\ G_2 \\ \mathcal{D}_2 \Vert \mathsf{S} \\ H_2 \end{array} \tag{10}$$
If f is the identity, i.e., $H_1 = G_2$, we can write $\mathcal{D}_2 \circ \mathcal{D}_1$ as
$$\begin{array}{c} G_1 \\ \mathcal{D}_1 \Vert \mathsf{S} \\ H_1 \\ \mathcal{D}_2 \Vert \mathsf{S} \\ H_2 \end{array} \qquad \text{or} \qquad \begin{array}{c} G_1 \\ \mathcal{D}_1 \Vert \mathsf{S} \\ G_2 \\ \mathcal{D}_2 \Vert \mathsf{S} \\ H_2 \end{array}$$

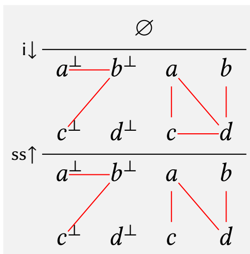
A ***proof*** in S is a derivation in S whose premise is $\varnothing$. A graph G is ***provable*** in S iff there is a proof in S with conclusion G. We denote this as $\vdash_{\mathsf{S}} G$ (or simply as $\vdash G$ if S is clear from context). The ***length*** of a derivation $\mathcal{D}$, denoted by $|\mathcal{D}|$, is the number of inference rule instances in $\mathcal{D}$.

[2] However, the converse holds in our particular case: We will see later that whenever we have $G \multimap H$ and $V_G = V_H$ then $E_H \subseteq E_G$. But this observation is not true in general for logics on graphs. For example in the extension of Boolean logic, defined in [8], it does not hold.

Remark 4.5. If we have a derivation $\mathcal{D}$ from A to B, and a context $G[\cdot]_R$, then we also have a derivation from $G[A]_R$ to $G[B]_R$. We can write this derivation as

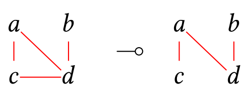
$$\begin{array}{c} G[A]_R \\ {\scriptstyle G[\mathcal{D}]_R}\,\Vert \\ G[B]_R \end{array} \quad \text{or} \quad G\left[\begin{array}{c} A \\ {\scriptstyle \mathcal{D}}\,\Vert \\ B \end{array}\right]_R$$

Example 4.6. Let us emphasize that the conclusion of a proof in our system is not a formula but a graph. The following derivation is an example of a proof of length 2, using only $\mathsf{i}\!\downarrow$ and $\mathsf{ss}\!\uparrow$:



(11)

where the $\mathsf{ss}\!\uparrow$ instance moves the module d in the context consisting of vertices labelled a, b, c. The derivation in (11) establishes that the following implication is provable:



(12)

which is a fact beyond the scope of formulas.

As in other deep inference systems, we can give for the rules in (9) their *duals*, or *corules*. In general, if

$$\mathsf{r}\,\frac{G}{H}$$

is an instance of a rule, then

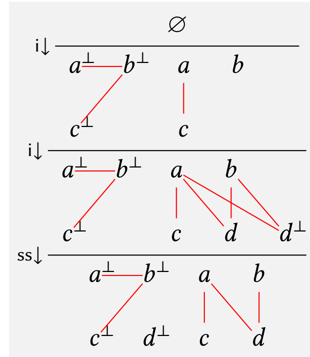
$$\mathsf{r}^\perp\,\frac{H^\perp}{G^\perp}$$

is an instance of the dual rule. The corules of the two rules in (9) are the following:

$$\mathsf{i}\!\uparrow\,\frac{A \otimes A^\perp}{\varnothing} \qquad\qquad \mathsf{ss}\!\downarrow\,\frac{B[A]_S}{B \mathbin{⅋} A}\; {\scriptstyle S\subseteq V_B,\; S\neq\varnothing} \tag{13}$$

called ***identity up*** (or ***cut***) and ***super switch down***, respectively. We have the side condition $S \neq \varnothing$ to avoid a triviality, as $B[A]_S = B ⅋ A$ if $S = \varnothing$.

Example 4.7. The implication in (12) can also be proven using only only $\mathsf{ss}\!\downarrow$ and $\mathsf{i}\!\downarrow$ instead of $\mathsf{ss}\!\uparrow$ and $\mathsf{i}\!\downarrow$, as the following proof of length 3 shows:



(14)

Definition 4.8. Let S be an inference system. We say that an inference rule r is ***derivable*** in S iff

$$\text{for every instance } \mathsf{r}\,\frac{G}{H} \text{ there is a derivation } \begin{array}{c} G \\ {\scriptstyle \mathcal{D}}\,\Vert\,{\scriptstyle \mathsf{S}} \\ H \end{array} .$$

We say that r is ***admissible*** in S iff

$$\text{for every instance } \mathsf{r}\,\frac{G}{H} \text{ we have that } \vdash_\mathsf{S} G \text{ implies } \vdash_\mathsf{S} H .$$

If $\mathsf{r} \in \mathsf{S}$ then r is trivially derivable and admissible in S.

Most deep inference systems in the literature (e.g. [4, 17, 19, 20, 24, 40]) contain the ***switch*** rule:

$$\mathsf{s}\,\frac{(A ⅋ B) \otimes C}{A ⅋ (B \otimes C)} \tag{15}$$

On can immediately see that it is its own dual and is a special case of both $\mathsf{ss}\!\downarrow$ and $\mathsf{ss}\!\uparrow$. We therefore have the following:

Lemma 4.9. *If in an inference system* S *one of the rules* $\mathsf{ss}\!\downarrow$ *and* $\mathsf{ss}\!\uparrow$ *is derivable, then so is* s.

Remark 4.10. In a standard deep inference system for formulas we also have the converse of Lemma 4.9, i.e., if s is derivable, then so are $\mathsf{ss}\!\uparrow$ and $\mathsf{ss}\!\downarrow$ (see, e.g., [41]). However, in the case of arbitrary graphs this is no longer true, and the rules $\mathsf{ss}\!\uparrow$ and $\mathsf{ss}\!\downarrow$ are strictly more powerful than s.

Lemma 4.11. *Let* S *be an inference system. If the rules* $\mathsf{i}\!\downarrow$ *and* $\mathsf{i}\!\uparrow$ *and* s *are derivable in* S*, then for every rule* r *that is derivable in* S*, also its corule* $\mathsf{r}^\perp$ *is derivable in* S.

Proof. Suppose we have two graphs G and H, and a derivation from G to H in S. Then it suffices to show that we can construct a derivation from $H^\perp$ to $G^\perp$ in S:

$$\mathsf{s}\,\dfrac{\left(\mathsf{i}{\downarrow}\,\dfrac{\varnothing}{G^\perp \parr G}\right) \otimes H^\perp}{G^\perp \parr \mathsf{i}{\uparrow}\,\dfrac{\begin{array}{c} G\\ \Vert\,\mathsf{S}\\ H\end{array} \otimes H^\perp}{\varnothing}}$$

Note that $\varnothing \otimes H^\perp = H^\perp$ and $G^\perp \parr \varnothing = G^\perp$. □

Lemma 4.12. *If the rules* $\mathsf{i}{\uparrow}$ *and* s *are admissible for an inference system* S*, then* $\multimap$ *is transitive, i.e., if* $\vdash_S G \multimap H$ *and* $\vdash_S H \multimap K$ *then* $\vdash_S G \multimap K$*.*

Proof. We can construct the following derivation

$$\mathsf{s}\,\dfrac{\begin{array}{c}\varnothing\\ \Vert\,\mathsf{S}\\ G^\perp \parr H\end{array} \otimes \begin{array}{c}\varnothing\\ \Vert\,\mathsf{S}\\ H^\perp \parr K\end{array}}{G^\perp \parr \mathsf{s}\,\dfrac{H \otimes (H^\perp \parr K)}{\left(\mathsf{i}{\uparrow}\,\dfrac{H \otimes H^\perp}{\varnothing}\right) \parr K}}$$

from $\varnothing$ to $G^\perp \parr K$ in S. □

Lemma 4.12 is the reason why $\mathsf{i}{\uparrow}$ is also called *cut*. In a well-designed deep inference system for formulas, the two rules $\mathsf{i}{\downarrow}$ and $\mathsf{i}{\uparrow}$ can be restricted in a way that they are only applicable to atoms, i.e., replaced by the following two rules that we call ***atomic identity down*** and ***atomic identity up***, respectively:

$$\mathsf{ai}{\downarrow}\,\frac{\varnothing}{a^\perp \parr a} \qquad \text{and} \qquad \mathsf{ai}{\uparrow}\,\frac{a \otimes a^\perp}{\varnothing} \tag{16}$$

We would like to achieve something similar for our proof system on graphs. For this it is necessary to be able to decompose prime graphs into atoms, but the two rules $\mathsf{ss}{\downarrow}$ and $\mathsf{ss}{\uparrow}$ cannot do this, as they are only able to move around modules in a graph. For this reason, we add the following two rules to our system:

$$\mathsf{p}{\downarrow}\,\frac{(M_1 \parr N_1) \otimes \cdots \otimes (M_n \parr N_n)}{P^\perp(\!|M_1, \ldots, M_n|\!) \parr P(\!|N_1, \ldots, N_n|\!)}\ {\scriptstyle P \text{ prime, } |V_P| \geq 4} \tag{17}$$

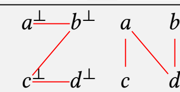
called ***prime down***, and

$$\mathsf{p}{\uparrow}\,\frac{P(\!|M_1, \ldots, M_n|\!) \otimes P^\perp(\!|N_1, \ldots, N_n|\!)}{(M_1 \otimes N_1) \parr \cdots \parr (M_n \otimes N_n)}\ {\scriptstyle P \text{ prime, } |V_P| \geq 4} \tag{18}$$

called ***prime up***. In both cases, the side condition is that P needs to be a prime graph and has at least 4 vertices. We also require that for all $i \in \{1, \ldots, n\}$ at least one of M_i and N_i is nonempty in an application of $\mathsf{p}{\downarrow}$ and $\mathsf{p}{\uparrow}$. The reason for these conditions is not that the rules would become unsound otherwise, but that the rules are derivable in the general case, as we will see in Lemma 5.2 in the next section.

Example 4.13. Below is a derivation of length 5 using the $\mathsf{p}{\downarrow}$-rule, and proves that a prime graph implies itself.

$$\mathsf{p}{\downarrow}\,\dfrac{\left(\mathsf{ai}{\downarrow}\,\dfrac{\varnothing}{a^\perp \parr a}\right) \otimes \left(\mathsf{ai}{\downarrow}\,\dfrac{\varnothing}{b^\perp \parr b}\right) \otimes \left(\mathsf{ai}{\downarrow}\,\dfrac{\varnothing}{c^\perp \parr c}\right) \otimes \left(\mathsf{ai}{\downarrow}\,\dfrac{\varnothing}{d^\perp \parr d}\right)}{}$$



This completes the presentation of our system, which is shown in Figure 1.

Definition 4.14. We define ***system SGS*** to be the set $\{\mathsf{ai}{\downarrow}, \mathsf{ss}{\downarrow}, \mathsf{p}{\downarrow}, \mathsf{p}{\uparrow}, \mathsf{ss}{\uparrow}, \mathsf{ai}{\uparrow}\}$ of inference rules shown in Figure 1. The ***down-fragment*** (resp. ***up-fragment***) of SGS consists of the rules $\{\mathsf{ai}{\downarrow}, \mathsf{ss}{\downarrow}, \mathsf{p}{\downarrow}\}$ (resp. $\{\mathsf{ai}{\uparrow}, \mathsf{ss}{\uparrow}, \mathsf{p}{\uparrow}\}$) and is denoted by SGS↓ (resp. SGS↑). The down-fragment SGS↓ is also called ***system* GS**.

5 Properties of the System

The first observation about SGS is that the general forms of the identity rules $\mathsf{i}{\downarrow}$ and $\mathsf{i}{\uparrow}$ are derivable, as we show in Lemma 5.1 below. Next, we have a similar result for the prime rules, for which also a general form is derivable, i.e., they can be applied to any graph instead of only prime graphs.

Lemma 5.1. *The rule* $\mathsf{i}{\downarrow}$ *is derivable in* SGS↓*, and dually, the rule* $\mathsf{i}{\uparrow}$ *is derivable in* SGS↑*.*

Proof. We show by induction on G, that $G^\perp \parr G$ has a proof in SGS↓, using Lemma 3.7.

(i) If G is a singleton graph, we can apply $\mathsf{ai}{\downarrow}$.
(ii) If $G = A \parr B$ then $G^\perp = B^\perp \otimes A^\perp$, and we can construct

$$\dfrac{\begin{array}{c}\varnothing\\ \mathcal{D}_1\,\Vert\,\mathsf{SGS}{\downarrow}\\ B^\perp \parr B\end{array}}{\left(\mathsf{ss}{\downarrow}\,\dfrac{B^\perp \otimes \begin{array}{c}\varnothing\\ \mathcal{D}_2\,\Vert\,\mathsf{SGS}{\downarrow}\\ A^\perp \parr A\end{array}}{(B^\perp \otimes A^\perp) \parr A}\right) \parr B}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ exist by induction hypothesis.

(iii) If $G = A \otimes B$, we proceed similarly.
(iv) If $G = P(\!|A_1, \ldots, A_n|\!)$ for P prime and $|V_P| \geq 4$, we get

$$\mathsf{p}{\downarrow}\,\dfrac{\begin{array}{c}\varnothing\\ \mathcal{D}_1\,\Vert\,\mathsf{SGS}{\downarrow}\\ A_1^\perp \parr A_1\end{array} \otimes \cdots \otimes \begin{array}{c}\varnothing\\ \mathcal{D}_n\,\Vert\,\mathsf{SGS}{\downarrow}\\ A_n^\perp \parr A_n\end{array}}{P^\perp(\!|A_1^\perp, \ldots, A_n^\perp|\!) \parr P(\!|A_1, \ldots, A_n|\!)}$$

$$
\mathsf{ai}{\downarrow}\,\frac{\varnothing}{a^\perp \parr a}
\qquad
\mathsf{ss}{\downarrow}\,\frac{B[A]_S}{B \parr A}\;{\scriptstyle S\subseteq V_B,\ S\neq\varnothing}
\qquad\Bigg|\qquad
\mathsf{ai}{\uparrow}\,\frac{a \otimes a^\perp}{\varnothing}
\qquad
\mathsf{ss}{\uparrow}\,\frac{B \otimes A}{B[A]_S}\;{\scriptstyle S\subseteq V_B,\ S\neq V_B}
$$

$$
\mathsf{p}{\downarrow}\,\frac{(M_1 \parr N_1)\otimes\cdots\otimes(M_n \parr N_n)}{P^\perp\llparenthesis M_1,\ldots,M_n\rrparenthesis \parr P\llparenthesis N_1,\ldots,N_n\rrparenthesis}\;{\scriptstyle P \text{ prime}, |V_P|\geq 4}
\qquad\Bigg|\qquad
\mathsf{p}{\uparrow}\,\frac{P\llparenthesis M_1,\ldots,M_n\rrparenthesis \otimes P^\perp\llparenthesis N_1,\ldots,N_n\rrparenthesis}{(M_1 \otimes N_1)\parr\cdots\parr(M_n \otimes N_n)}\;{\scriptstyle P \text{ prime}, |V_P|\geq 4}
$$

Figure 1. The inference rules for systems GS (rules ai↓, ss↓, p↓ on the left) and SGS (all rules in the figure).

where $\mathcal{D}_1,\ldots,\mathcal{D}_n$ exist by induction hypothesis. □

Lemma 5.2. *For any graph G with $|V_G| = n$, and graphs $M_1, N_1, \ldots, M_n, N_n$, we have derivations*

$$
\begin{array}{c}
(M_1 \parr N_1)\otimes\cdots\otimes(M_n \parr N_n)\\
\Vert\,\mathsf{SGS}{\downarrow}\\
G^\perp\llparenthesis M_1,\ldots,M_n\rrparenthesis \parr G\llparenthesis N_1,\ldots,N_n\rrparenthesis
\end{array}
\tag{19}
$$

and dually

$$
\begin{array}{c}
G\llparenthesis M_1,\ldots,M_n\rrparenthesis \otimes G^\perp\llparenthesis N_1,\ldots,N_n\rrparenthesis\\
\Vert\,\mathsf{SGS}{\uparrow}\\
(M_1 \otimes N_1)\parr\cdots\parr(M_n \otimes N_n)
\end{array}
\tag{20}
$$

Proof. We only show (19), and proceed by induction on the size of G, using Lemma 3.7.

(i) If G is a singleton graph, the statement holds trivially.

(ii) If $G = A \parr B$ then $G\llparenthesis N_1,\ldots,N_n\rrparenthesis = A\llparenthesis N_1,\ldots,N_k\rrparenthesis \parr B\llparenthesis N_{k+1},\ldots,N_n\rrparenthesis$ for some $1 \leq k \leq n$. We therefore have

$$
\mathsf{ss}{\downarrow}\,\dfrac{
\begin{array}{c}
(M_1 \parr N_1)\otimes..\otimes(M_k \parr N_k)\\
\mathcal{D}_1\,\Vert\,\mathsf{SGS}{\downarrow}\\
A^\perp\llparenthesis M_1,..,M_k\rrparenthesis \parr A\llparenthesis N_1,..,N_k\rrparenthesis
\end{array}
\otimes
\begin{array}{c}
(M_{k+1} \parr N_{k+1})\otimes..\otimes(M_n \parr N_n)\\
\mathcal{D}_2\,\Vert\,\mathsf{SGS}{\downarrow}\\
B^\perp\llparenthesis M_{k+1},..,M_n\rrparenthesis \parr B\llparenthesis N_{k+1},..,N_n\rrparenthesis
\end{array}
}{
\mathsf{ss}{\downarrow}\,\dfrac{(A^\perp\llparenthesis M_1,..,M_k\rrparenthesis \parr A\llparenthesis N_1,..,N_k\rrparenthesis)\otimes B^\perp\llparenthesis M_{k+1},..,M_n\rrparenthesis}{(A^\perp\llparenthesis M_1,..,M_k\rrparenthesis \otimes B^\perp\llparenthesis M_{k+1},..,M_n\rrparenthesis)\parr A\llparenthesis N_1,..,N_k\rrparenthesis}
\parr B\llparenthesis N_{k+1},..,N_n\rrparenthesis
}
$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ exist by induction hypothesis.

(iii) If $G = A \otimes B$, we proceed similarly.

(iv) If $G = P\llparenthesis A_1,\ldots,A_n\rrparenthesis$ for P prime and $|V_P| \geq 4$, we have an instance of p↓.

The derivation in (20) can be constructed dually. □

Next, observe that Lemmas 4.11 and 4.12 hold for system SGS. In particular, we have that if $\vdash_{\mathsf{SGS}} A \multimap B$ and $\vdash_{\mathsf{SGS}} B \multimap C$ then $\vdash_{\mathsf{SGS}} A \multimap C$ because $\mathsf{i}{\uparrow} \in \mathsf{SGS}$. The main result of this paper is that Lemma 4.12 does also hold for GS. More precisely, we have the following theorem:

Theorem 5.3 (Cut Admissibility)**.** *The rule* i↑ *is admissible for* GS.

To prove this theorem, we will show that the whole up-fragment of SGS is admissible for GS.

Theorem 5.4. *The rules* ai↑*,* ss↑*,* p↑ *are admissible for* GS.

Then Theorem 5.3 follows immediately from Theorem 5.4 and the second statement in Lemma 5.1.

The following three sections are devoted to the proof of Theorem 5.4. But before, let us finish this section by exhibiting some immediate consequences of Theorem 5.3.

Corollary 5.5. *For every graph G, we have $\vdash_{\mathsf{SGS}} A$ iff $\vdash_{\mathsf{GS}} A$.*

Corollary 5.6. *For all graphs G and H, we have*

$$
\vdash_{\mathsf{GS}} G \multimap H \iff
\begin{array}{c}\varnothing\\ \Vert\,\mathsf{GS}\\ G^\perp \parr H\end{array}
\iff
\begin{array}{c}\varnothing\\ \Vert\,\mathsf{SGS}\\ G^\perp \parr H\end{array}
\iff
\begin{array}{c}G\\ \Vert\,\mathsf{SGS}\\ H\end{array}
$$

Proof. The first equivalence is just the definition of $\vdash$. The second equivalence follows from Theorem 5.4, and the last equivalence follows from the two derivations

$$
\mathsf{i}{\downarrow}\,\dfrac{\varnothing}{G^\perp \parr \begin{array}{c}G\\ \Vert\\ H\end{array}}
\qquad\text{and}\qquad
\mathsf{ss}{\downarrow}\,\dfrac{G \otimes \begin{array}{c}\varnothing\\ \Vert\\ G^\perp \parr H\end{array}}{\mathsf{i}{\uparrow}\,\dfrac{G\otimes G^\perp}{\varnothing} \parr H}
$$

together with Lemma 5.1. □

Corollary 5.7. *We have $\vdash A \otimes B$ iff $\vdash A$ and $\vdash B$.*

Proof. This follows immediately by inspecting the inference rules of GS. □

Corollary 5.8. *We have $\vdash P\llparenthesis M_1,\ldots,M_n\rrparenthesis$ with P prime and $n \geq 4$ and $M_i \neq \varnothing$ for all $i = \{1,\ldots,n\}$, if and only if there is at least one $i = \{1,\ldots,n\}$ such that $\vdash M_i$ and $\vdash P\llparenthesis M_1,\ldots,M_{i-1},\varnothing,M_{i+1},\ldots,M_n\rrparenthesis$.*

This can be seen as a generalization of the previous corollary, and it is proved similarly.

Remark 5.9. The system GS forms a proof system in the sense of Cook and Reckhow [10], as the time complexity of checking the correct application of inference rules is polynomial, since the modular decomposition of graphs can be obtained in linear time [29]. Also whenever graph isomorphism is used to compose derivations, as in (10), we assume that the isomorphism f is explicitly given.

Theorem 5.10. *Provability in* GS *is decidable and in* **NP**.

Proof. This follows immediately from the observation that to each graph only finitely many inference rules can be applied, and that the length of a derivation in GS is $O(n^3)$ where n is the number of vertices in the conclusion. This can be seen as follows: every inference rule application in GS, when seen bottom-up, removes either two vertices or at least one edge. No rule can introduce vertices or edges.[3] $\square$

6 Splitting

The standard syntactic method for proving cut elimination in the sequent calculus is to permute the cut rule upwards in the proof and decomposing the cut formula along its main connective, and so inductively reduce the cut rank. However, in our proof system this method cannot be applied, as derivations can be constructed in a more flexible way than in the sequent calculus. For this reason, the *splitting* technique has been developed in the literature on deep inference [17, 21, 24, 41]. However, since we are working on general graphs instead of formulas, the generic method developed by Aler Tubella [44], cannot directly be applied in our case. For this reason, we needed to adapt the method and prove all lemmas from scratch. The central lemma is the following:

Lemma 6.1 (Splitting). *Let G, A, B be graphs, let P be a prime graph with $n = |V_P| \geq 4$, let $M_1, \ldots, M_n$ be nonempty graphs, and let a be an atom.*

(1) *If $\vdash_{GS} G \parr (A \otimes B)$ then there are a context $C[\cdot]_R$ and graphs K_A and K_B, such that there are derivations*

$$\begin{array}{c} C[K_A \parr K_B]_R \\ \mathcal{D}_G \Vert \text{GS} \\ G \end{array}\ ,\quad \begin{array}{c} \varnothing \\ \mathcal{D}_C \Vert \text{GS} \\ C \end{array}\ ,\quad \begin{array}{c} \varnothing \\ \mathcal{D}_A \Vert \text{GS} \\ K_A \parr A \end{array}\ ,\quad \begin{array}{c} \varnothing \\ \mathcal{D}_B \Vert \text{GS} \\ K_B \parr B \end{array}\ .$$

(2) *If $\vdash_{GS} G \parr P(\!| M_1, \ldots, M_n |\!)$, then there are*

- *either a context $C[\cdot]_R$ and graphs $K_1, \ldots, K_n$, such that there are derivations*

$$\begin{array}{c} C[P^\perp(\!| K_1, \ldots, K_n |\!)]_R \\ \mathcal{D}_G \Vert \text{GS} \\ G \end{array}\ ,\quad \begin{array}{c} \varnothing \\ \mathcal{D}_C \Vert \text{GS} \\ C \end{array}\ ,\quad \begin{array}{c} \varnothing \\ \mathcal{D}_i \Vert \text{GS} \\ K_i \parr M_i \end{array}$$

  for all $i \in \{1, \ldots, n\}$,

- *or a context $C[\cdot]_R$ and graphs K_X and K_Y such that there are derivations*

$$\begin{array}{c} C[K_X \parr K_Y]_R \\ \mathcal{D}_G \Vert \text{GS} \\ G \end{array}\ ,\quad \begin{array}{c} \varnothing \\ \mathcal{D}_C \Vert \text{GS} \\ C \end{array}\ ,\quad \begin{array}{c} \varnothing \\ \mathcal{D}_X \Vert \text{GS} \\ K_X \parr M_i \end{array}\ ,$$

$$\begin{array}{c} \varnothing \\ \mathcal{D}_Y \Vert \text{GS} \\ K_Y \parr P(\!| M_1, \ldots, M_{i-1}, \varnothing, M_{i+1}, \ldots, M_n |\!) \end{array}$$

  for some $i \in \{1, \ldots, n\}$.

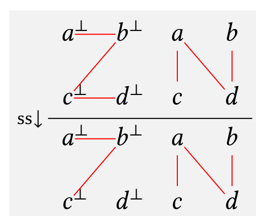
(3) *If $\vdash_{GS} G \parr a$ then there is a context $C[\cdot]_R$ such that there are derivations*

$$\begin{array}{c} C[a^\perp]_R \\ \mathcal{D}_G \Vert \text{GS} \\ G \end{array} \quad \textit{and} \quad \begin{array}{c} \varnothing \\ \mathcal{D}_C \Vert \text{GS} \\ C \end{array}\ .$$

Note that in the statement of Lemma 6.1, the first case **(1)** is superfluous, as it is a special case of **(2)**, when we see $\otimes$ as a prime graph, as indicated in (5) in Section 3. In this case the two subcases of **(2)** collapse. We nonetheless decided for pedagogical reasons to list case **(1)** explicitly. It shows how our splitting lemma is related to the standard splitting lemmas in the deep inference literature [17, 19, 21, 24, 41, 44], and thus enables the reader to see where the two subcases in case **(2)** come from.

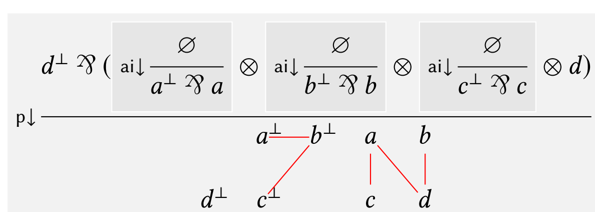
The idea of splitting is that, in a provable "sequent-like graph", consisting of a number of disjoint connected components, we can select any of these components as the *principal component* and apply a derivation to the other components, such that eventually a rule breaking down the principal component can be applied. This allows us to approximate the effect of applying rules in the sequent calculus.

We will use the proof in (14) as an example to explain this idea. In the conclusion we have 3 connected components. We can select the N-shape component on the right as the principal component, and apply case **(2)** of Lemma 6.1 to reorganise the proof (14) such that an instance of p↓ involving the N-shape can be applied, as in Example 4.13. The bottommost step of such a reorganised proof is shown below:



(21)

If, on the other hand we pick in (14) the $d^\perp$ as principal component, and apply case **(3)** of Lemma 6.1, we get the following derivation

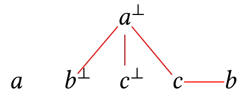


which we can complete to a proof with an an application of the rule ai↓.

A significant departure from established splitting lemmas in the literature, is the need for contexts in the premises of derivations. This is required to cope with graphs such as the

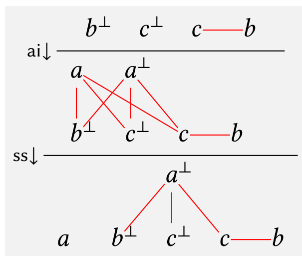
[3] It can in fact be shown that the length is $O(n^2)$ (see also [6]), but as the details are not needed for this paper, we leave them to the reader.

following:



(22)

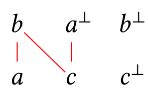
If we take a as the principal component, and apply case **(3)** of Lemma 6.1, we get a nonempty context C. Notice, furthermore, the above graph is provable only by applying rules deep inside the modular decomposition of the graph, as follows:



(23)

This shows that deep inference is necessary for this kind of proof theory on graphs.

The second subcase in case **(2)** of Lemma 6.1 is required for examples such as the following:



(24)

If we select the N-shape as the principal component and try to apply p↓, then a and $a^\perp$ can no longer communicate. Therefore, we must first move $b^\perp$ or $c^\perp$ into the structure and apply an ai↓, in order to destroy the prime graph. For example, by using $b^\perp$ to cancel out b, we obtain a provable graph of the form $a \parr (c \otimes a^\perp) \parr c^\perp$.

The proof of Lemma 6.1 proceeds by induction on the size of the derivation by exhaustively considering all ways in which the bottommost rule can interact with the principal component. It can be found in Appendix A.

7 Context Reduction

The Splitting Lemma 6.1 only applies in a shallow context, i.e., the outermost nodes in the modular tree construction of a graph (see Lemma 3.7). In order to use splitting for cut elimination, we need to apply it in arbitrary contexts. For this we need the context reduction lemma.

Lemma 7.1 (Context reduction)**.** *Let A be a graph and $G[\cdot]_S$ be a context. If $\vdash_{\mathsf{GS}} G[A]_S$ then there is a graph K and a context $C[\cdot]_R$ such that there are derivations*

$$\begin{array}{c}\varnothing\\ \mathcal{D}_C \Vert \mathsf{GS}\\ C\end{array} \quad\textit{and}\quad \begin{array}{c}\varnothing\\ \mathcal{D}_A \Vert \mathsf{GS}\\ K \parr A\end{array} \quad\textit{and}\quad \begin{array}{c}C[K \parr X]_R\\ \mathcal{D}_G \Vert \mathsf{GS}\\ G[X]_S\end{array}$$

for any graph X.

The proof of this lemma proceeds by a case analysis on the structure of the context $G[\cdot]_S$ employing splitting at each step. It can be found in Appendix B. We show here only one case.

Assume $G[A]_S = G'' \parr P(\!|M_1[A]_{S'}, M_2, \ldots, M_n|\!)$ for some G'', prime graph P and $M_1, \ldots, M_n$. Applying Lemma 6.1.**(2)** gives us three different cases, of which we show here only one: We get $C'[\cdot]_{R'}$ and K_X and K_Y, such that

$$\begin{array}{c}C'[K_X \parr K_Y]_{R'}\\ \mathcal{D}''_G \Vert\\ G''\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}'_C \Vert\\ C'\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_X \Vert\\ K_X \parr M_1[A]_{S'}\end{array}\ ,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_Y \Vert\\ K_Y \parr P(\!|\varnothing, M_2, \ldots, M_n|\!)\end{array}\ .$$

We apply the induction hypothesis to $\mathcal{D}_X$ and get K and $C''[\cdot]_{R''}$, such that

$$\begin{array}{c}\varnothing\\ \mathcal{D}''_C \Vert\\ C''\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_A \Vert\\ K \parr A\end{array}\ ,\quad \begin{array}{c}C''[K \parr X]_{R''}\\ \mathcal{D}'_G \Vert \mathsf{GS}\\ K_X \parr M_1[X]_{S'}\end{array}$$

for any X. We let $C[\cdot]_R = C'[K_Y \parr P(\!|C''[\cdot]_{R''}, M_2, \ldots, M_n|\!)]_{R'}$ and obtain $\mathcal{D}_C$ via

$$\begin{array}{c}\varnothing\\ \mathcal{D}_{C'} \Vert \mathsf{GS}\\ C'\Big[\begin{array}{c}\varnothing\\ \mathcal{D}'_Y \Vert \mathsf{GS}\\ L_Y \parr Q(\!|\begin{array}{c}\varnothing\\ \mathcal{D}'_{C'} \Vert \mathsf{GS}\\ C''\end{array}, N_2, \ldots, N_m|\!)\end{array}\Big]_{R'}\end{array}\ ,$$

and $\mathcal{D}_G$ is as follows:

$$\mathsf{ss}{\downarrow}\dfrac{C'\left[\mathsf{ss}{\downarrow}\dfrac{K_Y \parr P(\!|\begin{array}{c}C''[K \parr X]_{R''}\\ \mathcal{D}'_G \Vert\\ K_X \parr M_1[X]_{S'}\end{array}, M_2, \ldots, M_n|\!)}{K_X \parr K_Y \parr P(\!|M_1[X]_{S'}, M_2, \ldots, M_n|\!)}\right]_{R'}}{\begin{array}{c}C'[K_X \parr K_Y]_{R'}\\ \mathcal{D}''_G \Vert\\ G''\end{array} \parr P(\!|M_1[X]_{S'}, M_2, \ldots, M_n|\!)}\ .$$

The other cases follow by a similar reasoning.

8 Elimination of the Up-Fragment

In this section we discuss how we use splitting and context reduction to prove Theorem 5.4, i.e., the admissibility of the rules ai↑, ss↑, and p↑. For the rules ai↑ and ss↑, this is similar to ordinary deep inference systems (see, e.g., [9, 21, 24, 41]. But for p↑, there are surprising differences. In particular, we need to invoke an induction on the "size of the cut formula". In other cut elimination proofs in deep inference, there is no need for such an induction, as it is outsourced to the splitting lemma.

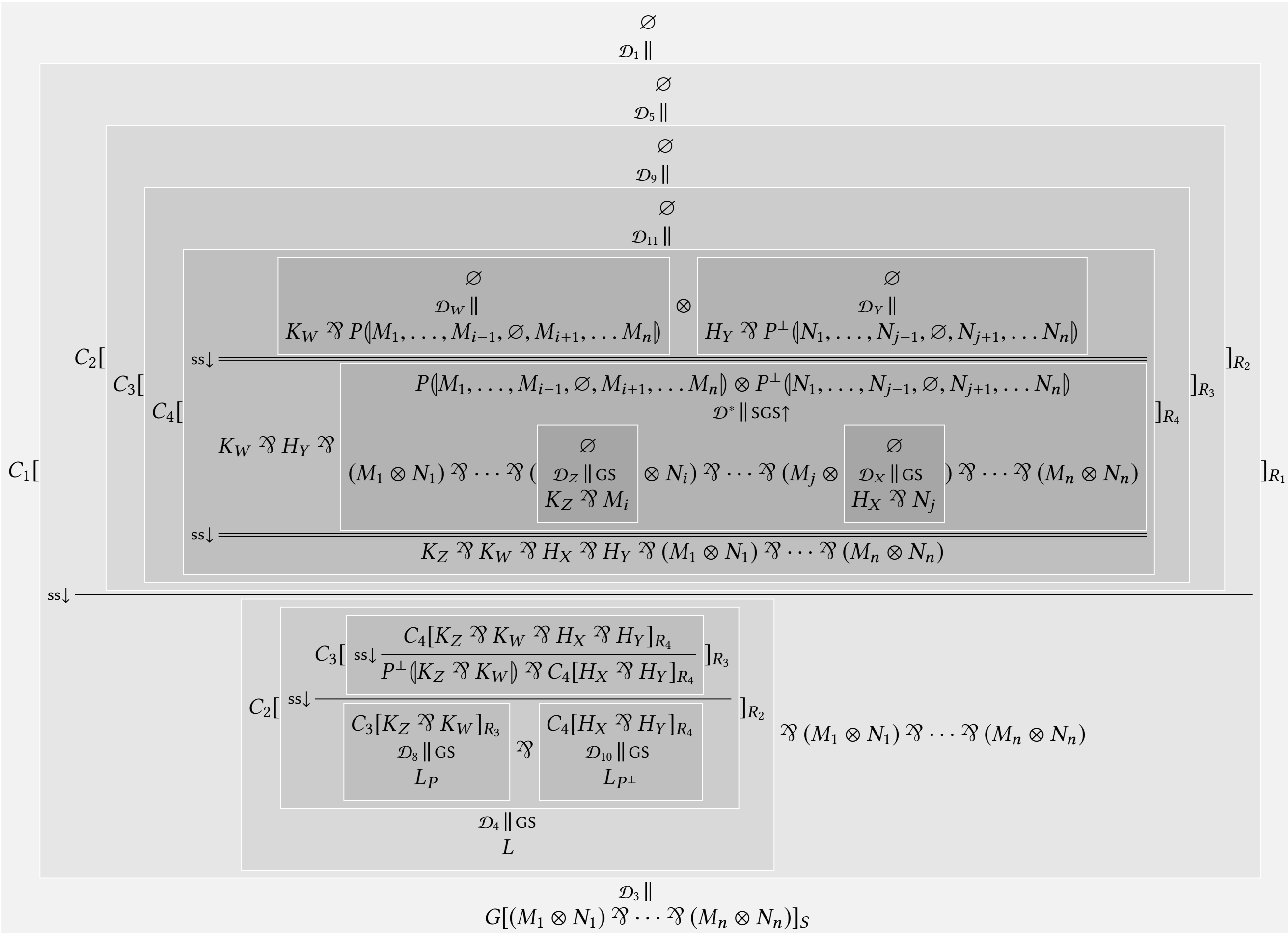


Figure 2. Derivation for the elimination of p↑.

Consider an instance of p↑, as follows.

$$\mathsf{p}\!\uparrow \frac{G[P(\!|M_1,\ldots,M_n|\!) \otimes P^{\perp}(\!|N_1,\ldots,N_n|\!)]_S}{G[(M_1 \otimes N_1) \parr \cdots \parr (M_n \otimes N_n)]_S}\ P \text{ prime, } |V_P| \geq 4$$

Here, we define the size of such an instance of p↑ as

$$\sum_{1 \leq i \leq n} (|M_i| + |N_i|)$$

i.e., the number of vertices in the subgraph that is modified by the rule. To prove admissibility of p↑, assume we have a proof of $G[P(\!|M_1,\ldots,M_n|\!) \otimes P^{\perp}(\!|N_1,\ldots,N_n|\!)]_S$. We apply Lemma 7.1 and get a graph L and a context $C_1[\cdot]_{R_1}$, such that there are derivations

$$\begin{array}{c}\varnothing\\ \mathcal{D}_1 \parallel\\ C_1\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_2 \parallel\\ L \parr (P(\!|M_1,\ldots,M_n|\!) \otimes P^{\perp}(\!|N_1,\ldots,N_n|\!))\end{array}\ ,\quad \begin{array}{c}C_1[L \parr X]_{R_1}\\ \mathcal{D}_3 \parallel\\ G[X]_S\end{array}$$

for any graph X. We apply Lemma 6.1.**(1)** to $\mathcal{D}_2$ and get graphs L_P and $L_{P^\perp}$ and a context $C_2[\cdot]_{R_2}$ such that

$$\begin{array}{c}C_2[L_P \parr L_{P^\perp}]_{R_2}\\ \mathcal{D}_4 \parallel\\ L\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_5 \parallel\\ C_2\end{array}\ ,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_6 \parallel\\ L_P \parr P(\!|M_1,\ldots,M_n|\!)\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_7 \parallel\\ L_{P^\perp} \parr P^{\perp}(\!|N_1,\ldots,N_n|\!)\end{array}\ .$$

Applying Lemma 6.1.**(2)** to $\mathcal{D}_6$ and $\mathcal{D}_7$ gives us four different cases, according to the two possible outcomes of case **(2)** in Lemma 6.1. We show here only the most complicated one[4], in which we get K_Z and K_W and H_X and H_Y and contexts

[4]The complete proof, together with the proofs for ai↑ and ss↑ can be found in Appendix C.

$$\mathsf{ax}\,\frac{}{a, a^\perp} \qquad \otimes\,\frac{\Gamma, \phi \quad \Delta, \psi}{\Gamma, \phi \otimes \psi, \Delta} \qquad \parr\,\frac{\Gamma, \phi, \psi}{\Gamma, \phi \parr \psi} \qquad \mathsf{mix}\,\frac{\Gamma \quad \Delta}{\Gamma, \Delta}$$

Figure 3. The inference rules of the system $\mathsf{MLL_X}$

$C_3[\cdot]_{R_3}$ and $C_4[\cdot]_{R_4}$, such that

$$\begin{array}{c} C_3[K_Z \parr K_W]_{R_3} \\ \mathcal{D}_8 \Vert \\ L_P \end{array}, \quad \begin{array}{c} \varnothing \\ \mathcal{D}_9 \Vert \\ C_3 \end{array}, \quad \begin{array}{c} \varnothing \\ \mathcal{D}_Z \Vert \\ K_Z \parr M_i \end{array},$$

$$\begin{array}{c} \varnothing \\ \mathcal{D}_W \Vert \\ K_W \parr P(\!|M_1, \ldots, M_{i-1}, \varnothing, M_{i+1}, \ldots M_n|\!) \end{array},$$

$$\begin{array}{c} C_4[H_X \parr H_Y]_{R_4} \\ \mathcal{D}_{10} \Vert \\ L_{P^\perp} \end{array}, \quad \begin{array}{c} \varnothing \\ \mathcal{D}_{11} \Vert \\ C_4 \end{array}, \quad \begin{array}{c} \varnothing \\ \mathcal{D}_X \Vert \\ H_X \parr N_j \end{array},$$

$$\begin{array}{c} \varnothing \\ \mathcal{D}_Y \Vert \\ H_Y \parr P^\perp(\!|N_1, \ldots, N_{j-1}, \varnothing, N_{j+1}, \ldots N_n|\!) \end{array}$$

for some $i, j \in \{1, \ldots, n\}$. In this case we use the derivation in Figure 2 to prove $G[(M_1 \otimes N_1) \parr \cdots \parr (M_n \otimes N_n)]_S$. More precisely, Figure 2 shows the case $i < j$, the cases $i = j$ and $i > j$ are similar. The derivation $\mathcal{D}^*$ exists by the second statement in Lemma 5.2. If $i \neq j$, this derivation consists of a single $\mathsf{p}\!\uparrow$ instance. If $i = j$, it can be a longer derivation containing all rules of $\mathsf{SGS}\!\uparrow$ (where the instances of $\mathsf{ai}\!\uparrow$ and $\mathsf{ss}\!\uparrow$ can be eliminated by the previous two theorems). The important observation to make is that all instances of $\mathsf{p}\!\uparrow$ occurring in $\mathcal{D}^*$ have smaller size than the one we started with. Therefore we can invoke the induction hypothesis.

9 Conservativity

We are now able to show that GS is a conservative extension of unit-free multiplicative linear logic with mix ($\mathsf{MLL_X}$) [15]. The formulas of $\mathsf{MLL_X}$ are as in (3), but without the unit, and inference rules of $\mathsf{MLL_X}$ are shown in Figure 3 where Γ and Δ are ***sequents***, i.e. multisets of formulas, separated by commas. We write $\vdash_{\mathsf{MLL_X}} \Gamma$ if the sequent Γ is provable in $\mathsf{MLL_X}$, i.e. if there is a derivation in $\mathsf{MLL_X}$ with conclusion Γ.

Lemma 9.1. *If $\vdash_{\mathsf{GS}} A$ and A is a cograph, then there is a derivation*

$$\begin{array}{c} \varnothing \\ \mathcal{D} \Vert \mathsf{GS} \\ A \end{array} \tag{25}$$

such that every graph occurring in $\mathcal{D}$ is a cograph.

The proof of this lemma proceeds by contradiction using Lemma 6.1. It can be found in Appendix D.

Lemma 9.2. *Let A and B be cographs. Then*

$$\mathsf{ss}\!\downarrow\frac{A}{B} \quad \Longrightarrow \quad \begin{array}{c} A \\ \Vert \{\mathsf{s}\} \\ B \end{array} \tag{26}$$

Proof. By Theorem 2.6, the graphs A and B are cographs iff there are formulas ϕ and ψ with $[\![\phi]\!] = A$ and $[\![\psi]\!] = B$. Now the statement follows from the corresponding statement for formulas (see e.g., Lemma 4.3.20 in [41]). ◻

Theorem 9.3. *Let A be a cograph. Then $\vdash_{\mathsf{GS}} A$ iff $\vdash_{\{\mathsf{ai}\downarrow, \mathsf{s}\}} A$.*

Proof. The implication from right to left follows immediately from the fact that s is a special case of $\mathsf{ss}\!\downarrow$ (see Lemma 4.9). For the implication from left to right, apply Lemma 9.1 to get a derivation $\mathcal{D}$ that only uses cographs. Hence the rule $\mathsf{p}\!\downarrow$ is not used in $\mathcal{D}$. Therefore, by Lemma 9.2, we can get a derivation $\mathcal{D}'$ that uses the rules $\mathsf{ai}\!\downarrow$ and s. ◻

Corollary 9.4. *For any unit-free formula ϕ,*

$$\vdash_{\mathsf{MLL_X}} \phi \quad \Longleftrightarrow \quad \vdash_{\mathsf{GS}} [\![\phi]\!]$$

Proof. It has been shown before (see, e.g., [19, 41] that a unit-free formula ϕ is provable in $\mathsf{MLL_X}$ iff it is provable in $\{\mathsf{ai}\!\downarrow, \mathsf{s}\}$ (note that in (15) we can have $B = \varnothing$). Now the statement follows from Theorem 9.3 and Theorem 2.6. ◻

Corollary 9.5. *Provability in* GS *is* **NP***-complete.*

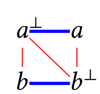
Proof. Since $\mathsf{MLL_X}$ is **NP**-complete, we can conclude from Corollary 9.4 that GS is **NP**-hard. Containment in **NP** has been proved in Theorem 5.10. ◻

10 Discussion and related work

Here we draw attention to challenges surrounding GS. Using examples, such as (22) and (24), we have already explained why GS necessarily demands deep inference. Since no established deep inference system matches GS we have a fundamentally new proof system. Furthermore, we explain in this section that simply taking an established semantics for $\mathsf{MLL_X}$ based on graphs and dropping the restriction to cographs does not immediately yield a semantics for GS.

Criteria for proof nets. Graphical approaches to proof nets such as *R&B*-graphs [38] have valid definitions when we drop the restriction to cographs. However, we show that (at least without strengthening criteria), these definition do not yield a semantics for a logic over graphs, since logical principles laid out in the introduction are violated.

Consider again graph (8), which is not provable in GS. In an *R&B*-graph we draw blue edges representing the axiom links of proof nets, as shown below for graph (8).

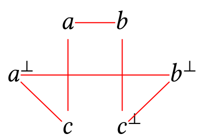


(27)

The established correctness criterion for *R&B*-graphs would wrongly accept the above graph. The reason is the cycle of 4 nodes alternating between red and blue edges has a chord. Notice this observation is independent of the rules of the system GS, since, in Sec. 4, we showed that graph (8) cannot be provable in a system subject to the logical principle of consistency.

What about cliques and stable sets? The switch rule has the property that it reflects edges and maximal cliques. That is: if there is an edge in the conclusion it will also appear in the premise and every maximal clique in the premise is a superset of some maximal clique in the conclusion. Indeed, mappings reflecting maximal cliques and preserving stable sets (mutually independent nodes) have a long history in program semantics [3] which led to coherence spaces and the discovery of linear logic [15], see also [8, 12, 13]. Therefore it is a reasonable starting point to try generalising switch by using such maximal clique reflecting homomorphisms, instead of ss↓. Indeed this is how we discovered ss↓, which is sound with respect to such homomorphisms.

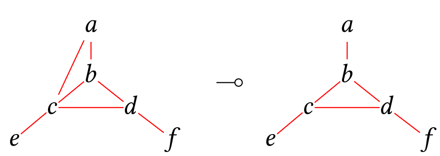
Unfortunately, replacing ss↓ with maximal clique reflecting homomorphisms yields a system distinct from our graphical system, for example the following would be provable, but is not provable in GS.



(28)

We may try replacing both ss↓ and ss↑ using a stronger symmetric notion of homomorphism where, in addition, every maximal stable set in the conclusion is a superset of some maximal stable set in the premise. Using such a homomorphism which is both maximal clique reflecting and stable set preserving as a rule, the above example is not provable. To see why, observe that at some point either a and $\overline{a}$ or b and $\overline{b}$ must be brought together into a module where they can interact, but this cannot be achieved while preserving the maximal stable set $\{a, b^\perp\}$.

Notice however, that if we replace ss↓ and ss↑ by the symmetric homomorphism described above, the implication below would be provable.



(29)

In contrast, the above is not provable in GS, since both sides are distinct prime graphs; and there is no suitable way to apply ss↓. Thus, we would obtain a distinct system from GS by using such homomorphisms.

Studying logics coming out of reflecting maximal cliques and preserving maximal stable sets is currently a topic of active research and leads to possible extensions of Boolean logic to graphs [7, 8, 45].

Generalised connectives. In this paper, we use a modular decomposition of graphs based on prime graphs (see Lemma 3.7). The connectives ⅋ and ⊗ are given by the prime graphs on two vertices. This choice is coherent with the graphs operations of union, join and composition, i.e. $G \otimes H = \otimes(\!|G, H|\!)$ and $G ⅋ H = ⅋(\!|G, H|\!)$. Pushing forward this idea, any graph can be interpreted as a *(multiplicative) generalized connective* [1, 11, 16]. In particular, in light of Lemma 3.7, every prime graph defines a non-decomposable connective. Furthermore, our Lemma 3.7 also provides a more refined notion of decomposition than the ⅋-⊗ decomposition known in the literature. However, the exact relation between the two constructions requires further investigation. Note, for example, that the number of pairs of orthogonal 6-ary non-decomposable connectives known at the time of writing is strictly smaller than the number of pairs of dual prime graphs on 6 vertices. Nonetheless, we conjecture that there is a correspondence between connectives defined by means of orthogonal sets of partitions and connectives defined by means of graphs.

11 Conclusion

Guided by logical principles, we have devised a minimal proof system (GS in Fig. 1) that operates directly over graphs, rather than formulas. Negation is generalised in terms of graph duality, while disjunction is disjoint union of graphs, allowing us to define implication "G implies H" as the standard "not G or H" (see Def. 4.1). All other design decisions are then fixed by our guiding logical principles. Most of these principles follow from cut elimination (Theorem 5.3), to which the majority of this paper is dedicated. We also confirm that GS conservatively extends MLL_X (Corollary 9.4) — a logic at the core of many proof systems.

Surprisingly, even for such a minimal generalisation of logic to graphs, deep inference is necessary. Proof systems for classical logic, MLL_X and many other logics *may* be expressed using deep inference, but deep inference is generally not necessary, since many standard logics have presentations in the sequent calculus where all inferences are applied at the root of some formula in a sequent. In contrast, for some logics (e.g., BV [17, 43] and modal logic $S5$ [35, 39]), deep inference is necessary in order to define a proof system satisfying cut elimination. System GS goes further than the aforementioned systems in that all intermediate lemmas such as splitting (Lemma 6.1) and context reduction (Lemma 7.1) also demand a deep formulation, requiring additional context awareness. As such we were required to generalise the basic mechanisms of deep inference itself in order to establish cut elimination (Theorem 5.3) for a logic over graphs. This is due to a property of general graphs that is forbidden in formulas — that the shortest path between any two connected nodes may be greater than two; and hence, when we apply reasoning inside a module (i.e., a context), there may exist paths of dependencies that indirectly constrain the module.

Acknowledgments. We are very grateful for many insightful discussions with Anupam Das.

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A Proof of Splitting (Lemma 6.1)

Observation A.1. Whenever we have a derivation from A to B in GS, then $|A| \leq |B|$, as the rules ss$\downarrow$ and p$\downarrow$ do not change the size of a graph, and the rule ai$\downarrow$ deletes two vertices when going up in a derivation.

Lemma A.2. *Let $C_1[\cdot]_{R_1}, \dots, C_n[\cdot]_{R_n}$ be contexts. If $\vdash_{\mathsf{GS}} C_i$ for all $i \in \{1, \dots, n\}$, then $\vdash_{\mathsf{GS}} C_n[\dots C_2[C_1]_{R_2}]_{R_n}$.*

Proof. We proceed by induction on n. The base case for $n = 1$ is trivial, and the inductive case for $n > 1$ is this derivation:

$$\begin{array}{c}\varnothing\\ \mathcal{D}_n \,\|\\ C_n\Big[\begin{array}{c}\varnothing\\ \mathcal{D}' \,\|\\ C_{n-1}[\dots C_2[C_1]_{R_2}]_{R_{n-1}}\end{array}\Big]_{R_1}\end{array},$$

where $\mathcal{D}_n$ is the derivation for C_n and $\mathcal{D}'$ exists by induction hypothesis. ◻

Proof of Lemma 6.1. We prove all three statements simultaneously, by induction on the pair $\langle |F|, |\mathcal{D}| \rangle$, ordered lexicographically, where F is the graph provable in the premise of each statement (i.e., $F = G \parr (A \otimes B)$ in (1), and $F = G \parr P(\!|M_1, \dots, M_n|\!)$ in (2), and $F = G \parr a$ in (3)) and $\mathcal{D}$ is the proof of F.

(1) Assume we have a proof $\mathcal{D}$ of $G \parr (A \otimes B)$. We have to find graphs K_A and K_B, and a context $C[\cdot]_R$, and derivations $\mathcal{D}_G$, $\mathcal{D}_C$, $\mathcal{D}_A$, $\mathcal{D}_B$, as in the statement of Lemma 6.1.(1). Note that if one of G, A, B is empty, then the statement holds trivially (with $C = \varnothing$, see Corollary 5.7). We now assume that G, A, B are all non-empty and make a case analysis on the bottommost rule instance r in $\mathcal{D}$.

(a) The rule r acts inside one of G, A, or B. I.e., the derivation $\mathcal{D}$ is of shape

$$\begin{array}{c}\varnothing\\ \mathcal{D}' \,\|\\ \mathsf{r}\dfrac{G'}{G} \parr (A \otimes B)\end{array} \quad\text{or}\quad \begin{array}{c}\varnothing\\ \mathcal{D}' \,\|\\ G \parr (\mathsf{r}\dfrac{A'}{A} \otimes B)\end{array} \quad\text{or}\quad \begin{array}{c}\varnothing\\ \mathcal{D}' \,\|\\ G \parr (A \otimes \mathsf{r}\dfrac{B'}{B})\end{array}$$

for some $\mathcal{D}'$. In each case we can apply the induction hypothesis to $\mathcal{D}'$ as $|\mathcal{D}'| < |\mathcal{D}|$ and conclude immediately by adding the corresponding application of r to $\mathcal{D}_G$ or $\mathcal{D}_A$ or $\mathcal{D}_B$, respectively.

(b) $G = G'' \parr G'$ and $\mathcal{D}$ is of shape

$$\mathsf{ss}\!\downarrow \dfrac{\begin{array}{c}\varnothing\\ \mathcal{D}' \,\|\\ G'' \parr (A \otimes B)[G']_S\end{array}}{G'' \parr G' \parr (A \otimes B)}\;.$$

We make a case analysis on the structure of the graph $(A \otimes B)[G']_S$, using Lemma 3.7:

(i) $(A \otimes B)[G']_S$ cannot be the singleton graph as neither A nor B are empty.

(ii) $(A \otimes B)[G']_S$ cannot be a par as this would imply $S = \varnothing$, contradicting the side condition of ss$\downarrow$.

(iii) $(A \otimes B)[G']_S$ is a tensor. We have the following possibilities:

(I) $(A \otimes B)[G']_S = A[G']_{S'} \otimes B$ for some $S' \subseteq S$. We can apply the induction hypothesis to $\mathcal{D}'$, and get $C[\cdot]_R$ and K'_A and K_B such that

$$\begin{array}{c}C[K'_A \parr K_B]_R\\ \mathcal{D}''_G \,\|\, \mathsf{GS}\\ G''\end{array},\quad \begin{array}{c}\varnothing\\ \mathcal{D}_C \,\|\, \mathsf{GS}\\ C\end{array},\quad \begin{array}{c}\varnothing\\ \mathcal{D}'_A \,\|\, \mathsf{GS}\\ K'_A \parr A[G']_S\end{array},\quad \begin{array}{c}\varnothing\\ \mathcal{D}_B \,\|\, \mathsf{GS}\\ K_B \parr B\end{array}.$$

We can let $K_A = G' \parr K'_A$ and obtain $\mathcal{D}_G$ and $\mathcal{D}_A$ via

$$\mathsf{ss}\!\downarrow \dfrac{C[G' \parr K'_A \parr K_B]_R}{G' \parr \begin{array}{c}C[K'_A \parr K_B]_R\\ \mathcal{D}''_G \,\|\, \mathsf{GS}\\ G''\end{array}} \quad\text{and}\quad \mathsf{ss}\!\downarrow \dfrac{\begin{array}{c}\varnothing\\ \mathcal{D}'_A \,\|\, \mathsf{GS}\\ K'_A \parr A[G']_S\end{array}}{G' \parr K'_A \parr A}\;.$$

respectively.

(II) $(A \otimes B)[G']_S = A \otimes B[G']_{S'}$ for some $S' \subseteq S$. This case is similar to the previous one.

(III) $(A \otimes B)[G']_S = A'' \otimes (A' \otimes B)[G']_{S'}$ for some $S' \subseteq S$, where $A = A'' \otimes A'$ and $A'' \neq \varnothing \neq A'$. We can apply the induction hypothesis to $\mathcal{D}'$ and get K''_A and L and $C'[\cdot]_{R'}$ such that

$$\begin{array}{c}C'[K''_A \parr L]_{R_1}\\ \mathcal{D}''_G \,\|\\ G''\end{array},\quad \begin{array}{c}\varnothing\\ \mathcal{D}'_C \,\|\\ C'\end{array},$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}''_A \,\|\\ K''_A \parr A''\end{array},\quad \begin{array}{c}\varnothing\\ \mathcal{D}_L \,\|\\ L \parr (A' \otimes B)[G']_{S'}\end{array}.$$

From $\mathcal{D}_L$ we get that $\vdash_{\mathsf{GS}} L \parr G' \parr (A \otimes B)$ (via the rule ss$\downarrow$). To this we can apply the induction hypothesis to get a context $C''[\cdot]_{R''}$ and K'_A and K_B such that

$$\begin{array}{c}C''[K'_A \parr K_B]_{R''}\\ \mathcal{D}'_G \,\|\\ L \parr G'\end{array},\quad \begin{array}{c}\varnothing\\ \mathcal{D}''_C \,\|\\ C''\end{array},\quad \begin{array}{c}\varnothing\\ \mathcal{D}'_A \,\|\\ K'_A \parr A'\end{array},\quad \begin{array}{c}\varnothing\\ \mathcal{D}_B \,\|\\ K_B \parr B\end{array}.$$

We can let $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ (and hence $C = C'[C'']_{R'}$), and $K_A = K''_A \parr K'_A$. We obtain $\mathcal{D}_G$ via

$$\mathsf{ss}{\downarrow}\,\dfrac{C'\left[\mathsf{ss}{\downarrow}\,\dfrac{C''[K''_A \parr K'_A \parr K_B]_{R''}}{K''_A \parr \begin{array}{c} C''[K'_A \parr K_B]_{R''}\\ \mathcal{D}'_G \Vert \\ L \parr G'\end{array}}\right]_{R'}}{\begin{array}{c}C'[K''_A \parr L]_{R'}\\ \mathcal{D}''_G \Vert\\ G''\end{array} \parr G'} ,$$

and $\mathcal{D}_A$ via

$$\mathsf{ss}{\downarrow}\,\dfrac{\begin{array}{c}\varnothing\\ \mathcal{D}'_{A'} \Vert \\ K''_A \parr (A'' \otimes \begin{array}{c}\varnothing\\ \mathcal{D}_{A'} \Vert\\ K_{A'} \parr A'\end{array})\end{array}}{K''_A \parr K'_A \parr (A'' \otimes A')} .$$

Finally, $\mathcal{D}_C$ is obtained via Lemma A.2 from $\mathcal{D}'_C$ and $\mathcal{D}''_C$.

(IV) $(A \otimes B)[G']_S = (A \otimes B')[G']_{S'} \otimes B''$ for some $S' \subseteq S$, where $B = B' \otimes B''$ and $B' \neq \varnothing \neq B''$. This case is similar to the previous one.

(iv) $(A \otimes B)[G']_S$ is composed via a prime graph Q, i.e., it is of shape $Q(\!|A_1 \otimes B_1, \ldots, A_h \otimes B_h, G'|\!)$ with $|V_Q| = h + 1 \geq 4$ and $A_1, \ldots, A_h$ being modules of A and $B_1, \ldots, B_h$ being modules of B, such that at least one of A_i, B_i is non-empty for every $i \in \{1, \ldots, h\}$.[5] Note that we also have

$$Q(\!|A_1 \otimes B_1, \ldots, A_h \otimes B_h, \varnothing|\!) = A \otimes B \tag{30}$$

We apply the induction hypothesis to $\mathcal{D}'$ and get one of the following three cases

(α) We get $K_1, \ldots, K_h, K_{G'}$ and a context $C'[\cdot]_{R'}$, such that

$$\begin{array}{c}C'[Q^\perp(\!|K_1, \ldots, K_h, L|\!)]_R\\ \mathcal{D}''_G \Vert\\ G''\end{array} ,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}'_C \Vert\\ C'\end{array} ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_i \Vert\\ K_i \parr (A_i \otimes B_i)\end{array} ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_L \Vert\\ L \parr G'\end{array} .$$

for all $1 \leq i \leq h$. Now observe that because of (30) we have the following proof

$$\mathsf{ss}{\downarrow}\,\dfrac{\begin{array}{c}\varnothing\\ \mathcal{D}'_C \Vert\\ C'\left[\begin{array}{c}\begin{array}{c}\varnothing\\ \mathcal{D}_1 \Vert\\ K_1 \parr (A_1 \otimes B_1)\end{array} \otimes \cdots \otimes \begin{array}{c}\varnothing\\ \mathcal{D}_h \Vert\\ K_h \parr (A_h \otimes B_h)\end{array}]\\ \tilde{\mathcal{D}} \Vert\\ Q^\perp(\!|K_1, \ldots, K_h, \varnothing|\!) \parr (A \otimes B)\end{array}\right]_R\end{array}}{C'[Q^\perp(\!|K_1, \ldots, K_h, \varnothing|\!)]_{R'} \parr (A \otimes B)} ,$$

where $\tilde{\mathcal{D}}$ is given by Lemma 5.2. To this proof we can apply the induction hypothesis (since G' is non-empty), and get K_A, K_B and a context $C[\cdot]_R$, such that

$$\begin{array}{c}C[K_A \parr K_B]_R\\ \mathcal{D}'_G \Vert\\ C'[Q^\perp(\!|K_1, \ldots, K_h, \varnothing|\!)]_R\end{array} ,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_C \Vert\\ C\end{array} ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_A \Vert\\ K_A \parr A\end{array} ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_B \Vert\\ K_B \parr B\end{array} .$$

Now $\mathcal{D}_G$ is the derivation

$$\mathsf{ss}{\downarrow}\,\dfrac{\begin{array}{c}C[K_A \parr K_B]_R\\ \mathcal{D}'_G \Vert\\ C'[Q^\perp(\!|K_1, \ldots, K_h, \begin{array}{c}\varnothing\\ \mathcal{D}_L \Vert\\ L \parr G'\end{array}|\!)]_{R'}\end{array}}{\begin{array}{c}C'[Q^\perp(\!|K_1, \ldots, K_h, L|\!)]_R\\ \mathcal{D}''_G \Vert\\ G''\end{array} \parr G'} .$$

(β) We have L_X and L_Y and $C'[\cdot]_{R'}$ such that

$$\begin{array}{c}C'[L_X \parr L_Y]_{R'}\\ \mathcal{D}''_G \Vert\\ G''\end{array} ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}'_C \Vert\\ C'\end{array} ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_X \Vert\\ L_X \parr (A_1 \otimes B_1)\end{array} ,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_Y \Vert\\ L_Y \parr Q(\!|\varnothing, A_2 \otimes B_2, \ldots, A_h \otimes B_h, G'|\!)\end{array} .$$

(Note that for easier typesetting we pick without loss of generality $A_1 \otimes B_1$ instead of an arbitrary $A_i \otimes B_i$ for some $i \in \{1, \ldots, h\}$.) From $\mathcal{D}_Y$, we get (via the rule $\mathsf{ss}{\downarrow}$) that

$$\vdash_{\mathsf{GS}} L_Y \parr G' \parr Q(\!|\varnothing, A_2 \otimes B_2, \ldots, A_h \otimes B_h, \varnothing|\!)$$

to which we now apply the induction hypothesis. Observe that by (30) we have $Q(\!|\varnothing, A_2 \otimes B_2, \ldots, A_h \otimes B_h, \varnothing|\!) = A' \otimes B'$, for some A' and B' where $A = A'[A_1]_S$ and $B = B'[B_1]_T$

[5] Note that at least one A_i or B_i has to be empty for some i, in order to make this case possible.

for some S, T. We therefore have K'_A and K'_B and $C''[\cdot]_{R''}$ such that

$$\begin{array}{c} C''[K'_A \parr K_B]_{R''} \\ \mathcal{D}'_G \,\| \\ L_Y \parr G' \end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}''_C \,\| \\ C''\end{array}\ ,\quad \begin{array}{c}\emptyset \\ \mathcal{D}'_A\,\| \\ K'_A \parr A'\end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_B \,\| \\ K'_B \parr B'\end{array}\ .$$

Similarly, we can apply the induction hypothesis to $\mathcal{D}_X$ and get have K''_A and K''_B and $C'''[\cdot]_{R'''}$ such that

$$\begin{array}{c} C'''[K''_A \parr K''_B]_{R'''} \\ \mathcal{D}'''_G \,\| \\ L_X \end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'''_C \,\| \\ C'''\end{array}\ ,\quad \begin{array}{c}\emptyset \\ \mathcal{D}''_A\,\| \\ K''_A \parr A_1\end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}''_B \,\| \\ K''_B \parr B_1\end{array}\ .$$

We let $K_A = K''_A \parr K'_A$, $K_B = K''_B \parr K'_B$ and $C[\cdot]_R = C'[C'''[C''[\cdot]_{R''}]_{R'''}]_{R'}$. Then $\mathcal{D}_G$ is

$$\mathsf{ss}{\downarrow}\frac{C'\left[\ \mathsf{ss}{\downarrow}\dfrac{C'''\left[\ \mathsf{ss}{\downarrow}\dfrac{C''[K''_A \parr K''_B \parr K'_A \parr K'_B]_{R''}}{K''_A \parr K''_B \parr C''[K'_A \parr K'_B]_{R''}}\ \right]_{R'''}}{\begin{array}{c} C'''[K''_A \parr K''_B]_{R'''} \\ \mathcal{D}'''_G\,\| \\ L_X\end{array} \parr \begin{array}{c} C''[K'_A \parr K'_B]_{R''} \\ \mathcal{D}'_G\,\| \\ L_Y \parr G'\end{array}}\ \right]_{R'}}{\begin{array}{c} C'[L_X \parr L_Y]_{R'} \\ \mathcal{D}''_G\,\| \\ G''\end{array} \parr G'}\ ,$$

Derivation $\mathcal{D}_A$ is as follows:

$$\mathsf{ss}{\downarrow}\frac{\begin{array}{c}\emptyset \\ \mathcal{D}'_A\,\| \\ K'_A \parr A'\left[\ \begin{array}{c}\emptyset\\ \mathcal{D}''_A\,\|\\ K''_A \parr A_1\end{array}\ \right]_S\end{array}}{K''_A \parr K'_A \parr A'[A_1]_S}\ ,$$

and $\mathcal{D}_B$ is similar. Finally, $\mathcal{D}_C$ exists by Lemma A.2.

(γ) We have L_X and L_Y and $C'[\cdot]_{R'}$ such that

$$\begin{array}{c} C'[L_X \parr L_Y]_{R'} \\ \mathcal{D}''_G \,\| \\ G'' \end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_C \,\| \\ C'\end{array}\ ,\quad \begin{array}{c}\emptyset \\ \mathcal{D}_X\,\| \\ L_X \parr G'\end{array}\ ,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}_Y\,\| \\ L_Y \parr Q(\!|A_1 \otimes B_1, \ldots, A_h \otimes B_h, \emptyset|\!)\end{array}\ .$$

We apply the induction hypothesis to $\mathcal{D}_Y$, using (30). This gives us K_A and K_B and $C''[\cdot]_{R''}$ such that

$$\begin{array}{c} C''[K_A \parr K_B]_{R''} \\ \mathcal{D}'_G \,\| \\ L_Y \end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_{C'} \,\| \\ C''\end{array}\ ,\quad \begin{array}{c}\emptyset \\ \mathcal{D}_A\,\| \\ K_A \parr A\end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_B \,\| \\ K_B \parr B\end{array}\ .$$

We let $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ and get $\mathcal{D}_C$ from Lemma A.2. Finally, $\mathcal{D}_G$ is

$$\mathsf{ss}{\downarrow}\frac{C'\left[\ \begin{array}{c}\emptyset\\ \mathcal{D}_X\,\|\\ L_X \parr G'\end{array} \parr \begin{array}{c} C''[K_A \parr K_B]_{R''}\\ \mathcal{D}'_G\,\|\\ L_Y\end{array}\ \right]_{R'}}{\begin{array}{c} C'[L_X \parr L_Y]_{R'} \\ \mathcal{D}''_G\,\| \\ G''\end{array} \parr G'}\ .$$

(c) In the next case to consider we also have $G = G'' \parr G'$, and $\mathcal{D}$ is (without loss of generality) of shape

$$\mathsf{ss}{\downarrow}\frac{\begin{array}{c}\emptyset\\ \mathcal{D}'\,\|\\ G'' \parr G'[A \otimes B]_S\end{array}}{G'' \parr G' \parr (A \otimes B)}$$

We proceed by a case analysis on $G'[A \otimes B]_S$, using Lemma 3.7. Observe that $G'[A \otimes B]_S$ cannot be a single-vertex graph, and without loss of generality, we can assume it is not of shape $E \parr D$ for non-empty E, D. Hence, there are only two cases to consider:

(iii) $G'[A \otimes B]_S$ is a tensor of two graphs. Without loss of generality, we can assume $G'[A \otimes B]_S = E \otimes D[A \otimes B]_{S'}$, with $E \neq \emptyset$. We apply the induction hypothesis to $\mathcal{D}'$ and get $C'[\cdot]_{R'}$ and K_E and K_D such that

$$\begin{array}{c} C'[K_E \parr K_D]_{R'} \\ \mathcal{D}''_G \,\|\,\mathsf{GS} \\ G'' \end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_C \,\|\,\mathsf{GS} \\ C'\end{array}\ ,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}_E \,\|\,\mathsf{GS} \\ K_E \parr E\end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_D \,\|\,\mathsf{GS} \\ K_D \parr D[A \otimes B]_{S'}\end{array}$$

From $\mathcal{D}_D$, we get $\vdash_{\mathsf{GS}} K_D \parr D \parr (A \otimes B)$ via the $\mathsf{ss}{\downarrow}$-rule. Since $|E| > 0$ we can apply the induction hypothesis again, and get $C''[\cdot]_{R''}$ and K_A and K_B such that

$$\begin{array}{c} C''[K_A \parr K_B]_{R''} \\ \mathcal{D}'_G \,\|\,\mathsf{GS} \\ K_D \parr D \end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}''_C \,\|\,\mathsf{GS} \\ C''\end{array}\ ,\quad \begin{array}{c}\emptyset \\ \mathcal{D}_A\,\|\,\mathsf{GS} \\ K_A \parr A\end{array}\ ,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_B \,\|\,\mathsf{GS} \\ K_B \parr B\end{array}\ .$$

Note that $G = G'' \parr (E \otimes D)$. We therefore can let $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ and obtain $\mathcal{D}_C$ from

Lemma A.2. We obtain $\mathcal{D}_G$ via

$$\mathsf{ss}{\downarrow}\,\dfrac{C'\left[\mathsf{ss}{\downarrow}\,\dfrac{\begin{array}{c} C''[\hat{K}]_{R''}\\ \mathcal{D}'_G\,\|\\ K_D \parr \left(\begin{array}{c}\emptyset\\ \mathcal{D}_E\,\|\\ K_E \parr E\end{array} \otimes D\right)\end{array}}{K_E \parr K_D \parr (E\otimes D)}\right]_{R'}}{\begin{array}{c}C'[K_E \parr K_D]_{R'}\\ \mathcal{D}''_G\,\|\\ G''\end{array} \parr (E\otimes D)}$$

where $\hat{K}$ abbreviates $K_A \parr K_B$.

(iv) $G'[A\otimes B]_S$ is composed via a prime graph Q. Without loss of generality, we can assume that $G'[A\otimes B]_S = Q\llparenthesis N_1[A\otimes B]_{S'}, N_2, \dots, N_m\rrparenthesis$ with $|V_Q| = m \geq 4$. We apply the induction hypothesis to $\mathcal{D}'$ and get one of the following three cases:

(iv.α) We have $C'[\cdot]_{R'}$ and $L_1, \dots, L_m$ such that

$$\begin{array}{c}C'[Q^{\perp}\llparenthesis L_1,\dots,L_m\rrparenthesis]_{R'}\\ \mathcal{D}''_G\,\|\,\mathsf{GS}\\ G''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_C\,\|\,\mathsf{GS}\\ C'\end{array}\;,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}'_1\,\|\,\mathsf{GS}\\ L_1 \parr N_1[A\otimes B]_{S'}\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_i\,\|\,\mathsf{GS}\\ L_i \parr N_i\end{array}$$

for $2\leq i\leq m$. From $\mathcal{D}'_1$, we get (via the $\mathsf{ss}{\downarrow}$-rule) that $\vdash_{\mathsf{GS}} L_1 \parr N_1 \parr (A\otimes B)$, to which we can apply the induction hypothesis again and get $C''[\cdot]_{R''}$ and K_A and K_B such that

$$\begin{array}{c}C''[K_A \parr K_B]_{R''}\\ \mathcal{D}'_G\,\|\,\mathsf{GS}\\ L_1 \parr N_1\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}''_C\,\|\,\mathsf{GS}\\ C''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_A\,\|\,\mathsf{GS}\\ K_A \parr A\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_B\,\|\,\mathsf{GS}\\ K_B \parr B\end{array}\;.$$

As $G = G'' \parr Q\llparenthesis N_1,\dots,N_m\rrparenthesis$, we can (as before) let $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ and obtain $\mathcal{D}_G$ as

$$\mathsf{ss}{\downarrow}\,\dfrac{C'\left[\mathsf{p}{\downarrow}\,\dfrac{\begin{array}{c}C''[\hat{K}]_{R''}\\ \mathcal{D}'_G\,\|\\ L_1 \parr N_1\end{array} \otimes \begin{array}{c}\emptyset\\ \mathcal{D}'_2\,\|\\ L_2 \parr N_2\end{array} \otimes\cdots\otimes \begin{array}{c}\emptyset\\ \mathcal{D}'_m\,\|\\ L_m \parr N_m\end{array}}{Q^{\perp}\llparenthesis L_1,\dots,L_m\rrparenthesis \parr Q\llparenthesis N_1,\dots,N_m\rrparenthesis}\right]_{R'}}{\begin{array}{c}C'[Q^{\perp}\llparenthesis L_1,\dots,L_m\rrparenthesis]_{R'}\\ \mathcal{D}''_G\,\|\\ G''\end{array} \parr Q\llparenthesis N_1,\dots,N_m\rrparenthesis}$$

where $\hat{K}$ abbreviates $K_A \parr K_B$, and $\mathcal{D}_C$ is obtained via Lemma A.2.

(iv.β) We have $C'[\cdot]_{R'}$ and L_X and L_Y such that

$$\begin{array}{c}C'[L_X \parr L_Y]_{R'}\\ \mathcal{D}''_G\,\|\,\mathsf{GS}\\ G''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_C\,\|\,\mathsf{GS}\\ C'\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_X\,\|\,\mathsf{GS}\\ L_X \parr N_1[A\otimes B]_{S'}\end{array}\;,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}'_Y\,\|\,\mathsf{GS}\\ L_Y \parr Q\llparenthesis\emptyset, N_2,\dots,N_m\rrparenthesis\end{array}\;,$$

i.e., we assume $i = 1$ in the second case of (2) in Lemma 6.1. From $\mathcal{D}'_X$ we get (via the $\mathsf{ss}{\downarrow}$) that $\vdash_{\mathsf{GS}} L_X \parr N_1 \parr (A\otimes B)$, to which we can apply the induction hypothesis again to get $C''[\cdot]_{R''}$ and K_A and K_B such that

$$\begin{array}{c}C''[K_A \parr K_B]_{R''}\\ \mathcal{D}'_G\,\|\,\mathsf{GS}\\ L_X \parr N_1\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}''_C\,\|\,\mathsf{GS}\\ C''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_A\,\|\,\mathsf{GS}\\ K_A \parr A\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_B\,\|\,\mathsf{GS}\\ K_B \parr B\end{array}\;.$$

As before, we have $G = G'' \parr Q\llparenthesis N_1,\dots,N_m\rrparenthesis$. We let $C[\cdot]_R = C'[L_Y \parr Q\llparenthesis C''[\cdot]_{R''}, N_2,\dots,N_m\rrparenthesis]_{R'}$, and obtain $\mathcal{D}_G$ as

$$\mathsf{ss}{\downarrow}\,\dfrac{C'\left[\mathsf{ss}{\downarrow}\,\dfrac{L_Y \parr P\llparenthesis \begin{array}{c}C''[\hat{K}]_{R''}\\ \mathcal{D}'_G\,\|\\ L_X \parr N_1\end{array}, N_2,\dots,N_m\rrparenthesis}{L_X \parr L_Y \parr Q\llparenthesis N_1, N_2,\dots,N_n\rrparenthesis}\right]_{R'}}{\begin{array}{c}C'[L_X \parr L_Y]_{R'}\\ \mathcal{D}''_G\,\|\\ G''\end{array} \parr Q\llparenthesis N_1, N_2,\dots,N_m\rrparenthesis}\;,$$

where $\hat{K} = K_A \parr K_B$, and $\mathcal{D}_C$ is obtained via Lemma A.2 from $\mathcal{D}'_C$, $\mathcal{D}'_Y$, and $\mathcal{D}''_C$.

(iv.γ) We have $C'[\cdot]_{R'}$ and L_X and L_Y such that

$$\begin{array}{c}C'[L_X \parr L_Y]_{R'}\\ \mathcal{D}''_G\,\|\,\mathsf{GS}\\ G\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_C\,\|\,\mathsf{GS}\\ C'\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_X\,\|\,\mathsf{GS}\\ L_X \parr N_2\end{array}\;,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}'_Y\,\|\,\mathsf{GS}\\ L_Y \parr Q\llparenthesis N_1[A\otimes B]_{S'}, \emptyset, N_3,\dots,N_m\rrparenthesis\end{array}\;,$$

i.e., we assume $i = 2$ in the second case of (2) in Lemma 6.1 (the cases $i\geq 3$ are similar). From $\mathcal{D}'_Y$ we get $\vdash_{\mathsf{GS}} L_Y \parr Q\llparenthesis N_1, \emptyset, N_3,\dots,N_m\rrparenthesis \parr (A\otimes B)$, to which we can apply the induction hypothesis again, and get $C''[\cdot]_{R''}$ and K_A and K_B such that

$$\begin{array}{c}C''[K_A \parr K_B]_{R''}\\ \mathcal{D}'_G\,\|\,\mathsf{GS}\\ L_Y \parr Q\llparenthesis N_1,\emptyset,N_3,\dots,N_m\rrparenthesis\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}''_C\,\|\,\mathsf{GS}\\ C''\end{array}\;,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}_A \,\|\, \mathsf{GS}\\ K_A \parr A\end{array} \;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_B \,\|\, \mathsf{GS}\\ K_B \parr B\end{array} \;.$$

As before, we have $G = G'' \parr Q(\!|N_1, \ldots, N_m|\!)$ and let $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$. Thus, we can obtain $\mathcal{D}_G$ as follows:

$$\mathsf{ss}{\downarrow}\frac{C'\left[\;\mathsf{ss}{\downarrow}\dfrac{\begin{array}{c}C''[\hat{K}]_{R''}\\ \mathcal{D}'_G \,\|\, \mathsf{GS}\\ L_Y \parr Q(\!|N_1, \begin{array}{c}\emptyset\\ \mathcal{D}_3 \,\|\, \mathsf{GS}\\ L_X \parr N_2\end{array}, N_3, \ldots, N_m|\!)\end{array}}{L_X \parr L_Y \parr Q(\!|N_1, N_2, \ldots, N_m|\!)}\;\right]_{R'}}{\begin{array}{c}C'[L_X \parr L_Y]_{R'}\\ \mathcal{D}''_G \,\|\, \mathsf{GS}\\ G\end{array} \parr Q(\!|N_1, N_2, \ldots, N_m|\!)}$$

where as before $\hat{K} = K_A \parr K_B$ and $\mathcal{D}_C$ is obtained via Lemma A.2.

(d) In the final case to consider we have that $G = G' \parr Q(\!|N_1, \ldots, N_m|\!)$ where N_i non-empty for all $i \in 1, \ldots, m$, and Q is prime with $|V_Q| \geq 4$, and $\mathcal{D}$ is of shape

$$\mathsf{p}{\downarrow}\frac{\begin{array}{c}\emptyset\\ \mathcal{D}' \,\|\\ G' \parr (N_1 \otimes (N_2 \parr (A_2 \otimes B_2)) \otimes \ldots \otimes (N_m \parr (A_m \otimes B_m)))\end{array}}{G' \parr Q(\!|N_1, \ldots, N_m|\!) \parr (A \otimes B)}$$

This is only possible if

$$A \otimes B = Q^{\perp}(\!|\emptyset, A_2 \otimes B_2, \ldots, A_m \otimes B_m|\!) \tag{31}$$

(observe at least one component of the prime connective $Q^{\perp}$ must be empty for this equality to hold and we take without loss of generality the first). We apply the induction hypothesis to $\mathcal{D}'$ and get $C'[\cdot]_{R'}$ and $K_1, \ldots, K_m$, such that

$$\begin{array}{c}C'[K_1 \parr \ldots \parr K_n]_{R'}\\ \mathcal{D}'_G \,\|\\ G'\end{array} \;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_C \,\|\\ C'\end{array} \;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_1 \,\|\\ K_1 \parr N_1\end{array} \;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_i \,\|\\ K_i \parr N_i \parr (A_i \otimes B_i)\end{array}$$

for all $i \in \{2, \ldots, m\}$. Then we can construct the following proof, by using (31):

$$\mathsf{ss}{\downarrow}\frac{\begin{array}{c}\emptyset\\ \mathcal{D}'_C \,\|\\ C'\left[\begin{array}{c}\begin{array}{c}\emptyset\\ \mathcal{D}'_2 \,\|\\ K_2 \parr N_2 \parr (A_2 \otimes B_2)\end{array} \otimes \ldots \otimes \begin{array}{c}\emptyset\\ \mathcal{D}'_m \,\|\\ K_m \parr N_m \parr (A_m \otimes B_m)\end{array}\\ \tilde{\mathcal{D}} \,\|\\ Q(\!|\emptyset, K_2 \parr N_2, \ldots, K_m \parr N_m|\!) \parr (A \otimes B)\end{array}\right]_{R'}\end{array}}{C'[Q(\!|\emptyset, K_2 \parr N_2, \ldots, K_m \parr N_m|\!)]_{R'} \parr (A \otimes B)}$$

where $\tilde{\mathcal{D}}$ is given by Lemma 5.2. To this proof we can apply the induction hypothesis again (since N_1 is non empty), and get $C[\cdot]_R$ and K_A and K_B, such that

$$\begin{array}{c}C[K_A \parr K_B]_R\\ \mathcal{D}''_G \,\|\\ C'[P(\!|\emptyset, K_2 \parr N_2, \ldots, K_m \parr N_m|\!)]_{R'}\end{array} \;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_C \,\|\\ C\end{array} \;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_A \,\|\\ K_A \parr A\end{array} \;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_B \,\|\\ K_B \parr B\end{array}$$

It remains to give $\mathcal{D}_G$ which is as follows:

$$\mathsf{ss}{\downarrow}\frac{\begin{array}{c}C[K_A \parr K_B]_R\\ \mathcal{D}''_G \,\|\\ C'\left[\begin{array}{c}Q(\!|\begin{array}{c}\emptyset\\ \mathcal{D}'_1 \,\|\\ K_1 \parr N_1\end{array} K_2 \parr N_2, \ldots, K_m \parr N_m|\!)\\ \mathcal{D}_s \,\|\\ K_1 \parr \ldots K_m \parr Q(\!|M_1, \ldots, N_m|\!)\end{array}\right]_{R'}\end{array}}{\begin{array}{c}C'[K_1 \parr \ldots \parr K_m]_{R'}\\ \mathcal{D}'_G \,\|\\ G'\end{array} \parr Q(\!|N_1, \ldots, N_m|\!)}$$

where $\mathcal{D}_s$ consist of m applications of the rule $\mathsf{ss}{\downarrow}$.

(2) In this case, we assume $\vdash_{\mathsf{GS}} G \parr P(\!|M_1, \ldots, M_n|\!)$ with P prime and $|V_P| = n \geq 4$ and M_i nonempty for $1 \leq i \leq n$; and aim to construct $C[\cdot]_R$, $\mathcal{D}_G$, $\mathcal{D}_C$, and either K_i, $\mathcal{D}_i$ or K_X, K_Y, $\mathcal{D}_X$, $\mathcal{D}_Y$, as in the statement of the splitting lemma. As before, we make a case analysis on the bottommost rule instance r in $\mathcal{D}$.

(a) If the rule r acts inside G then we can conclude immediately by using the induction hypothesis. Similarly, if the rule r acts inside one of the M_i, i.e., $\mathcal{D}$ is of shape

$$\begin{array}{c}\emptyset\\ \mathcal{D}' \,\|\, \mathsf{GS}\\ G \parr P(\!|M_1, \ldots, M_{i-1}, \mathsf{r}\dfrac{M'_i}{M_i}, M_{i+1}, \ldots, M_n|\!)\end{array}$$

for some $1 \leq i \leq n$, we can apply the induction hypothesis, unless r is ai↓ and $M_i' = \varnothing$. Then we have

$$\begin{array}{c}\varnothing\\ \mathcal{D}' \,\|\, \mathsf{GS}\\ G \parr P\llparenthesis M_1, \dots, M_{i-1}, \ \mathsf{ai}{\downarrow}\dfrac{\varnothing}{a^\perp \parr a}, M_{i+1}, \dots, M_n \rrparenthesis\end{array}$$

and can conclude immediately by letting $C = K_X = \varnothing$, and $K_Y = G$.

(b) $G = G' \parr G''$ and $\mathcal{D}$ is of shape

$$\mathsf{ss}{\downarrow}\dfrac{\begin{array}{c}\varnothing\\ \mathcal{D}'' \,\|\\ G'' \parr P\llparenthesis M_1, \dots, M_n \rrparenthesis[G']_{R_P}\end{array}}{G'' \parr G' \parr P\llparenthesis M_1, \dots, M_n \rrparenthesis}$$

Now consider the possible forms of $P\llparenthesis M_1, \dots, M_n \rrparenthesis[G']_{R_P}$, according to Lemma 3.7:

(i) It cannot be an atom since G' and M_i are non-empty.

(ii) It cannot be a par, due to conditions on ss↓.

(iii) It can only be of the form $P\llparenthesis M_1, \dots, M_n \rrparenthesis \otimes G'$. In this case, we can apply the induction hypothesis to obtain $C'[\cdot]_{R'}$, K_P, K_G'' such that

$$\begin{array}{c}C'[K_P \parr L_G'']_{R'}\\ \mathcal{D}_G' \,\|\\ G'\end{array}, \quad \begin{array}{c}\varnothing\\ \mathcal{D}_C' \,\|\\ C'\end{array}, \quad \begin{array}{c}\varnothing\\ \mathcal{D}_{KG} \,\|\\ K_G'' \parr G''\end{array}, \quad \begin{array}{c}\varnothing\\ \mathcal{D}_P \,\|\\ K_P \parr P\llparenthesis M_1, \dots, M_n \rrparenthesis\end{array},$$

By the induction hypothesis we have context $C''[\cdot]_{R''}$ and graph $\hat{K}$ such that

$$\begin{array}{c}C''[\hat{K}]_{R''}\\ \mathcal{D}'' \,\|\\ K_P\end{array}, \quad \begin{array}{c}\varnothing\\ \mathcal{D}_C'' \,\|\\ C''\end{array}$$

and either $\hat{K} = P^\perp\llparenthesis K_1, \dots, K_n \rrparenthesis$, for some $K_1, \dots, K_n$, where for all $i \in \{1, \dots, n\}$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_i \,\|\\ K_i \parr M_i\end{array}$$

or $\hat{K} = K_X \parr K_Y$ for some K_X and K_Y such that

$$\begin{array}{c}\varnothing\\ \mathcal{D}_X \,\|\\ K_X \parr M_j\end{array}, \quad \begin{array}{c}\varnothing\\ \mathcal{D}_Y \,\|\\ K_Y \parr P\llparenthesis M_1, \dots, M_{j-1}, \varnothing, M_{j+1}, \dots, M_n \rrparenthesis\end{array}$$

for some $j \in \{1, \dots, n\}$.

Using the above, we can construct the following derivation $\mathcal{D}_G$

$$\mathsf{ss}{\downarrow}\dfrac{C'\left[\begin{array}{c}C''[\hat{K}]_{R''}\\ \mathcal{D}'' \,\|\\ K_P\end{array} \parr \begin{array}{c}\varnothing\\ \mathcal{D}_{KG} \,\|\\ K_G'' \parr G''\end{array}\right]_R}{\begin{array}{c}C'[K_P \parr L_G'']_R\\ \mathcal{D}_G' \,\|\\ G'\end{array} \parr G''}$$

Let $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ and obtain $\mathcal{D}_C$ by using Lemma A.2.

(iv) Otherwise, $P\llparenthesis M_1, \dots, M_n \rrparenthesis[G']_{R_P}$ is a prime graph in which case we have the following possibilities.

(I) $\mathcal{D}$ is of the shape

$$\mathsf{ss}{\downarrow}\dfrac{\begin{array}{c}\varnothing\\ \mathcal{D}' \,\|\\ G'' \parr P\llparenthesis M_1, \dots, M_{j-1}, M_j[G']_S, M_{j+1}, \dots, M_n \rrparenthesis\end{array}}{G'' \parr G' \parr P\llparenthesis M_1, \dots, M_n \rrparenthesis}$$

for some $1 \leq j \leq n$. We apply the induction hypothesis to $\mathcal{D}'$ and get one of the following three sub-cases:

(b.(iv).(I).α) We have $C[\cdot]_R$ and L_j and $K_1, \dots, K_{j-1}, K_{j+1}, \dots, K_n$ such that

$$\begin{array}{c}C[P^\perp\llparenthesis K_1, \dots, K_{j-1}, L_j, K_{j+1}, \dots, K_n \rrparenthesis]_R\\ \mathcal{D}_G'' \,\|\\ G''\end{array},$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_C \,\|\\ C\end{array}, \quad \begin{array}{c}\varnothing\\ \mathcal{D}_j' \,\|\\ L_j \parr M_j[G']_S\end{array}, \quad \begin{array}{c}\varnothing\\ \mathcal{D}_i \,\|\\ K_i \parr M_i\end{array}$$

for $i \in \{1, \dots, n\}$ with $i \neq j$. We let $K_j = L_j \parr G'$. Then $\mathcal{D}_G$ is the derivation

$$\mathsf{ss}{\downarrow}\dfrac{C[P^\perp\llparenthesis K_1, \dots, K_{j-1}, L_j \parr G', K_{j+1}, \dots, K_n \rrparenthesis]_R}{\begin{array}{c}C[P^\perp\llparenthesis K_1, \dots, K_{j-1}, L_j, K_{j+1}, \dots, K_n \rrparenthesis]_R\\ \mathcal{D}_G'' \,\|\\ G''\end{array} \parr G'}$$

and $\mathcal{D}_j$ is

$$\mathsf{ss}{\downarrow}\dfrac{\begin{array}{c}\varnothing\\ \mathcal{D}_j' \,\|\\ L_j \parr M_j[G']_S\end{array}}{L_j \parr G' \parr M_j}$$

(b.(iv).(I).β) We have $C[\cdot]_R$ and L_X and K_Y such that

$$\begin{array}{c} C[L_X \parr K_Y]_R \\ \mathcal{D}'_G \Vert \\ G'' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_C \Vert \\ C \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}'_X \Vert \\ L_X \parr M_j[G']_S \end{array} ,$$

$$\begin{array}{c} \varnothing \\ \mathcal{D}_Y \Vert \\ K_Y \parr P(\!|M_1, \dots, M_{j-1}, \varnothing, M_{j+1}, \dots, M_n|\!) \end{array}$$

We let $K_X = L_X \parr G'$ and let $\mathcal{D}_G$ and $\mathcal{D}_X$ be the derivations

$$\mathsf{ss}{\downarrow}\frac{C[L_X \parr G' \parr K_Y]_R}{\begin{array}{c} C[L_X \parr K_Y]_R \\ \mathcal{D}'_G \Vert \\ G'' \end{array} \parr G'} \quad \text{and} \quad \mathsf{ss}{\downarrow}\frac{\begin{array}{c} \varnothing \\ \mathcal{D}'_X \Vert \\ L_X \parr M_j[G']_S \end{array}}{L_X \parr G' \parr M_j}$$

(b.(iv).(I).γ) We have $C[\cdot]_R$ and K_X and L_Y such that

$$\begin{array}{c} C[K_X \parr L_Y]_R \\ \mathcal{D}'_G \Vert \\ G'' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_C \Vert \\ C \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_X \Vert \\ K_X \parr M_i \end{array} ,$$

$$\begin{array}{c} \varnothing \\ \mathcal{D}'_Y \Vert \\ L_Y \parr P(\!|M_1, \dots, M_{i-1}, \varnothing, M_{i+1}, \dots, M_j[G'], \dots, M_n|\!) \end{array}$$

for some $i \neq j$. We let $K_Y = G' \parr L_Y$, we obtain $\mathcal{D}_G$ as in the previous case, and $\mathcal{D}_Y$ is

$$\mathsf{ss}{\downarrow}\frac{\begin{array}{c} \varnothing \\ \mathcal{D}'_Y \Vert \\ L_Y \parr P(\!|M_1, \dots, M_{i-1}, \varnothing, M_{i+1}, \dots, M_j[G'], \dots, M_n|\!) \end{array}}{G' \parr L_Y \parr P(\!|M_1, \dots, M_{i-1}, \varnothing, M_{i+1}, \dots, M_n|\!)} .$$

(II) $\mathcal{D}$ is of the shape

$$\mathsf{ss}{\downarrow}\frac{\begin{array}{c} \varnothing \\ \mathcal{D}' \Vert \\ G'' \parr Q(\!|G', N_2, \dots, N_k|\!) \end{array}}{G'' \parr G' \parr P(\!|M_1, \dots, M_n|\!)}$$

where Q is a prime graph such that $|Q| = k > n$ such that

$$Q(\!|\varnothing, N_2, \dots, N_k|\!) = P(\!|M_1, \dots, M_n|\!) \tag{32}$$

and each N_i for $i \in \{1, \dots k\}$ is a module of some M_j where $j \in \{1, \dots n\}$.

By the induction hypothesis we have $\hat{K}$, $C'[\cdot]_{R'}$ such that

$$\begin{array}{c} C'[\hat{K}]_{R'} \\ \mathcal{D}''_G \Vert \\ G'' \end{array}$$

And one of (α), (β) or (γ) holds as follows.

(α) $\hat{K} = Q^\perp(\!|L, K_2, \dots, K_k|\!)$ and

$$\begin{array}{c} \varnothing \\ \mathcal{D}_C \Vert \\ C \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_L \Vert \\ L \parr G' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_i \Vert \\ K_i \parr N_i \end{array}$$

for $i \in \{1, \dots k\}$.

Hence we have the following derivation, named $\mathcal{D}''$

$$\mathsf{ss}{\downarrow}\frac{C'[Q^\perp(\!| \begin{array}{c} \varnothing \\ \mathcal{D}_L \Vert \\ L \parr G' \end{array}, K_2, \dots, K_k|\!)]_{R'}}{\begin{array}{c} C'[Q^\perp(\!|L, K_2, \dots, K_k|\!)]_{R'} \\ \Vert \\ G'' \end{array} \parr G'}$$

Now, we have the following proof, where $\mathcal{D}_Q$ is given by Lemma 5.2 and by appealing to (32),

$$\mathcal{D}_Q \frac{\begin{array}{c} \varnothing \\ \mathcal{D}_2 \Vert \\ K_2 \parr N_2 \end{array} \otimes \dots \otimes \begin{array}{c} \varnothing \\ \mathcal{D}_k \Vert \\ K_k \parr N_k \end{array}}{Q^\perp(\!|\varnothing, K_2, \dots, K_k|\!) \parr P(\!|M_1, \dots, M_n|\!)}$$

Since N_1 is nonempty, the size of $Q^\perp(\!|\varnothing, K_2, \dots, K_k|\!) \parr Q(\!|G', N_2, \dots, N_k|\!)$ is less than the size of $G \parr P(\!|M_1, \dots, M_n|\!)$. Hence we can apply the induction hypothesis to obtain $\hat{\mathcal{D}}$ such that

$$\begin{array}{c} C''[\hat{K}']_{R''} \\ \hat{\mathcal{D}} \Vert \\ Q^\perp(\!|\varnothing, K_2, \dots, K_k|\!) \end{array}$$

where $\hat{\mathcal{D}}$ satisfies the conditions of the splitting (providing K_i, K_X, K_Y, etc.). $\mathcal{D}_G = C'[\hat{\mathcal{D}}]_{R'} \circ \mathcal{D}''$, $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ and $\mathcal{D}_C$ is given by Lemma A.2.

(β) $\hat{K} = K'_X \parr K'_Y$ such that (appealing to (32))

$$\begin{array}{c} \varnothing \\ \mathcal{D}'_X \Vert \\ K'_X \parr G' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}'_Y \Vert \\ K'_Y \parr P(\!|M_1, \dots, M_n|\!) \end{array}$$

In this case we have derivation $\mathcal{D}''$, defined as follows

$$\mathsf{ss}{\downarrow}\frac{C'[\begin{array}{c} \varnothing \\ \mathcal{D}'_X \Vert \\ gK'_X \parr G' \end{array} \parr K_Y]_{R'}}{\begin{array}{c} C'[K'_X \parr K_Y]_{R'} \\ \Vert \\ G'' \end{array} \parr G'}$$

Now, since G' is nonempty the size of $K'_Y \parr P(\!|M_1, \ldots, M_n|\!)$ is strictly less than the size of $G \parr P(\!|M_1, \ldots, M_n|\!)$. Hence we can apply the induction hypothesis to obtain $\hat{\mathcal{D}}$ such that

$$\begin{array}{c} C''[\hat{K}']_{R''} \\ \hat{\mathcal{D}} \, \Vert \\ Q^\perp(\!|\varnothing, K_2, \ldots, K_k|\!) \end{array}$$

where $\hat{\mathcal{D}}$ satisfies most of the conditions of the splitting (providing K_i, K_X, K_Y, etc.), $\mathcal{D}_G = C'[\hat{\mathcal{D}}]_{R'} \circ \mathcal{D}''$, $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$ and $\mathcal{D}_C$ is given by Lemma A.2.

(γ) $\hat{K} = K'_X \parr K'_Y$ such that for some $\ell \in \{2, \ldots k\}$

$$\begin{array}{c} \varnothing \\ \mathcal{D}'_X \, \Vert \\ K'_X \parr N_\ell \end{array}$$

$$\begin{array}{c} \varnothing \\ \mathcal{D}'_Y \, \Vert \\ K'_Y \parr Q(\!|G', N_2, \ldots, N_{\ell-1}, \varnothing, N_{\ell+1}, \ldots N_k|\!) \end{array}$$

In the case that $N_\ell = M_m$ for some $m \in \{1..k\}$, proof $\mathcal{D}_Y$ is given by (using (32))

$$\mathsf{ss}\!\downarrow \frac{\begin{array}{c} \varnothing \\ \mathcal{D}'_Y \, \Vert \\ K'_Y \parr Q(\!|G', N_2, \ldots, N_{\ell\bar{1}}, \varnothing, N_{\ell+1}, \ldots N_k|\!) \end{array}}{K'_Y \parr G' \parr P(\!|M_1, \ldots, M_{m\bar{1}}, \varnothing, M_{m+1}, \ldots M_n|\!)}$$

Hence we can conclude immediately by taking $K_Y = K'_Y \parr G'$, $K_X = K'_X$, $\mathcal{D}_X = \mathcal{D}'_X$, $\mathcal{D}_G = \mathcal{D}''$ and $C[\cdot]_R = C'[\cdot]_{R'}$.

Otherwise we have to consider the scenario when we have $M_m = M'[N_j]_{R_m}$ for some non-empty M' (recall that N_j must be a module of some such M_m). In this scenario, the following are equivalent (by using (32))

$Q(\!|\varnothing, N_2, \ldots, N_{\ell-1}, \varnothing, N_{\ell+1}, \ldots N_k|\!)$

$P(\!|M_1, \ldots, M_{m-1}, M', M_{m+1}, \ldots M_n|\!)$

$\vdash K'_Y \parr G' \parr P(\!|M_1, \ldots, M_{m-1}, \varnothing, M_{m+1}, \ldots M_n|\!)$ holds, using $\mathcal{D}'_Y$ and $\mathsf{ss}\!\downarrow$, and is strictly smaller than $G \parr P(\!|M_1, \ldots, M_n|\!)$, since N_j is non-empty. Hence we can apply the induction hypothesis to obtain $\hat{K}'$ and $C'[\cdot]_{R'}$ such that

$$\begin{array}{c} C'[\hat{K}']_{R'} \\ \Vert \\ [K'_Y; G'] \end{array}$$

There are then three cases to check (A), (B) and (C) as follows.

(A) Let $\hat{K}' = P^\perp(\!|K_1, \ldots, L, K_{m+1}, \ldots, K_n|\!)$ and

$$\begin{array}{c} \varnothing \\ \mathcal{D}'_m \, \Vert \\ L \parr M' \end{array} \quad , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_i \, \Vert \\ K_i \parr M_i \end{array}$$

for $i \in \{1 \ldots m-1\} \cup \{m+1 \ldots n\}$.

From the above we can construct the following derivation $\mathcal{D}_G$

$$\mathsf{ss}\!\downarrow \frac{C'\left[\begin{array}{c} C'[P^\perp(\!|K_1, \ldots, K'_X \parr L, K_{m+1}, \ldots, K_n|\!)]_{R'} \\ \mathsf{ss}\!\downarrow \Vert \\ K'_X \parr \begin{array}{c} C'[P^\perp(\!|K_1, \ldots, L, K_{m+1}, \ldots, K_n|\!)]_{R'} \\ \Vert \\ K'_Y \parr G' \end{array} \end{array}\right]_{R'}}{\begin{array}{c} C'[K'_X \parr K'_Y]_{R'} \\ \Vert \\ G'' \end{array} \parr G'}$$

Also we can construct proof $\mathcal{D}_m$ as follows

$$\mathsf{ss}\!\downarrow \frac{\begin{array}{c} \varnothing \\ \mathcal{D}'_m \, \Vert \\ L \parr M'\left[\begin{array}{c} \varnothing \\ \mathcal{D}'_X \, \Vert \\ K'_X \parr N_j \end{array}\right]_{R_m} \end{array}}{K'_X \parr L \parr M'[N_j]_{R_m}}$$

We conclude by setting $K_m = K'_X; L$ and $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$, where $\mathcal{D}_C$ is obtained using Lemma A.2.

(B) Let $\hat{K}' = K_X \parr \hat{K}_Y$ where w.l.o.g.

$$\begin{array}{c} \varnothing \\ \mathcal{D}_X \, \Vert \\ K_X \parr M_1 \end{array} \quad , \quad \begin{array}{c} \varnothing \\ \mathcal{D}'_Y \, \Vert \\ K_Y \parr P(\!|\varnothing, \ldots, M_{m-1}, M', M_{m+1}, \ldots M_n|\!) \end{array}$$

In this case we have derivation $\mathcal{D}_G$, defined as follows

$$\mathsf{ss}\!\downarrow \frac{C'\left[\begin{array}{c} K_X \parr K'_X \parr \hat{K}_Y \\ \mathsf{ss}\!\downarrow \Vert \\ K'_X \parr \begin{array}{c} K_X \parr \hat{K}_Y \\ \Vert \\ K'_Y \parr G' \end{array} \end{array}\right]_{R'}}{\begin{array}{c} C'[K'_X \parr K'_Y]_{R'} \\ \Vert \\ G'' \end{array} \parr G'}$$

We also have the following proof, named $\mathcal{D}_Y$

$$\mathsf{ss}{\downarrow}\,\frac{\begin{array}{c}\varnothing\\ \mathcal{D}'_Y \,\|\\ \hat{K}_Y \parr P(\!|\varnothing,\ldots,M_{m\bar{1}}, M'[\begin{array}{c}\varnothing\\ \mathcal{D}'_X\,\|\\ K'_X \parr N_j\end{array}]_{R_m}, M_{m+1},\ldots M_n|\!)\end{array}}{K'_X \parr \hat{K}_Y \parr P(\!|\varnothing,\ldots,M'[N_j]_{R_m}, M_{m+1},\ldots M_n|\!)}$$

We conclude by setting $K_Y = K'_X \parr \hat{K}_Y$ and $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$, where $\mathcal{D}_C$ is obtained using Lemma A.2.

(C) Let $\hat{K}' = \hat{K}_X \parr K_Y$ where w.l.o.g.

$$\begin{array}{c}\varnothing\\ \hat{\mathcal{D}}_X\,\|\\ \hat{K}_X \parr M'\end{array} \quad , \quad \begin{array}{c}\varnothing\\ \mathcal{D}_Y\,\|\\ K_Y \parr P(\!|\varnothing,\ldots,M_{m-1},\varnothing,M_{m+1},\ldots M_n|\!)\end{array}$$

In this case we have derivation $\mathcal{D}_G$, defined as follows

$$\mathsf{ss}{\downarrow}\,\frac{C'[\begin{array}{c}\hat{K}_X \parr K'_X \parr \hat{K}_Y\\ \mathsf{ss}{\downarrow}\,\|\\ K'_X \parr \begin{array}{c}\hat{K}_X \parr K_Y\\ \|\\ K'_Y \parr G'\end{array}\end{array}]_{R'}}{\begin{array}{c}C'[K'_X \parr K'_Y]_{R'}\\ \|\\ G''\end{array} \parr G'}$$

We also have the following proof, named $\mathcal{D}_X$

$$\mathsf{ss}{\downarrow}\,\frac{\begin{array}{c}\varnothing\\ \hat{\mathcal{D}}_X\,\|\\ \hat{K}_X \parr M'[\begin{array}{c}\varnothing\\ \mathcal{D}'_X\,\|\\ K'_X \parr N_j\end{array}]_{R_m}\end{array}}{\hat{K}_X \parr K'_X \parr M'[N_j]_{R_m}}$$

We conclude by setting $K_X = K'_X \parr \hat{K}_X$ and $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$, where $\mathcal{D}_C$ is obtained using Lemma A.2.

(c) If $G = G'' \parr G'$ (with $G' \neq \varnothing$) and $\mathcal{D}$ is of shape

$$\mathsf{ss}{\downarrow}\,\frac{\begin{array}{c}\varnothing\\ \mathcal{D}'\,\|\\ G'' \parr G'[P(\!|M_1,\ldots,M_n|\!)]_S\end{array}}{G'' \parr G' \parr P(\!|M_1,\ldots,M_n|\!)}$$

we proceed as in case (1.c) by a case analysis on the shape of $G'[P(\!|M_1,\ldots,M_n|\!)]_S$ via Lemma 3.7, and for the same reasons as above, there are two cases.

(iii) $G'[P(\!|M_1,\ldots,M_n|\!)]_S$ is a tensor of two graphs. Without loss of generality, we can assume $G'[P(\!|M_1,\ldots,M_n|\!)]_S = E \otimes D[P(\!|M_1,\ldots,M_n|\!)]_{S'}$, with $E \neq \varnothing$. We apply the induction hypothesis to $\mathcal{D}'$ and get $C'[\cdot]_{R'}$ and K_E and K_D such that

$$\begin{array}{c}C'[K_E \parr K_D]_{R'}\\ \mathcal{D}'_G\,\|\,\mathsf{GS}\\ G''\end{array} \quad , \quad \begin{array}{c}\varnothing\\ \mathcal{D}_{C'}\,\|\,\mathsf{GS}\\ C'\end{array} \quad ,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_E\,\|\,\mathsf{GS}\\ K_E \parr E\end{array} \quad , \quad \begin{array}{c}\varnothing\\ \mathcal{D}_D\,\|\,\mathsf{GS}\\ K_D \parr D[P(\!|M_1,\ldots,M_n|\!)]_{S'}\end{array}$$

From $\mathcal{D}_D$, we get $\vdash_{\mathsf{GS}} K_D \parr D \parr P(\!|M_1,\ldots,M_n|\!)$ via the $\mathsf{ss}{\downarrow}$-rule. Since $|E| > 0$ we can apply the induction hypothesis again, giving us one of the following two cases:

- either a context $C''[\cdot]_{R''}$ and graphs $K_1, \ldots, K_n$, such that

$$\begin{array}{c}C''[P^{\perp}(\!|K_1,\ldots,K_n|\!)]_{R''}\\ \mathcal{D}''_G\,\|\,\mathsf{GS}\\ K_D \parr D\end{array} \quad , \quad \begin{array}{c}\varnothing\\ \mathcal{D}'_{C'}\,\|\,\mathsf{GS}\\ C''\end{array} \quad , \quad \begin{array}{c}\varnothing\\ \mathcal{D}_i\,\|\,\mathsf{GS}\\ K_i \parr M_i\end{array}$$

  for all $i \in \{1,\ldots,n\}$,
- or a context $C''[\cdot]_{R''}$ and graphs K_X and K_Y such that

$$\begin{array}{c}C''[K_X \parr K_Y]_{R''}\\ \mathcal{D}''_G\,\|\,\mathsf{GS}\\ K_D \parr D\end{array} \quad , \quad \begin{array}{c}\varnothing\\ \mathcal{D}_{C'}\,\|\,\mathsf{GS}\\ C''\end{array} \quad , \quad \begin{array}{c}\varnothing\\ \mathcal{D}_X\,\|\,\mathsf{GS}\\ K_X \parr M_i\end{array} \quad ,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_Y\,\|\,\mathsf{GS}\\ K_Y \parr P(\!|M_1,\ldots,M_{i-1},\varnothing,M_{i+1},\ldots,M_n|\!)\end{array}$$

  for some $i \in \{1,\ldots,n\}$.

In the first case we let $\hat{K} = P^{\perp}(\!|K_1,\ldots,K_n|\!)$ and in the second $\hat{K} = K_X \parr K_Y$. In both cases we have $G = G'' \parr (E \otimes D)$ and we can let $C[\cdot]_R$ and $\mathcal{D}_G$ and $\mathcal{D}_C$ as in case (1.c.iii) above.[6]

(iv) $G'[P(\!|M_1,\ldots,M_n|\!)]_S$ is composed via a prime graph Q. This case is similar to case (1.c.iv) above. Without loss of generality, we can assume that $G'[P(\!|M_1,\ldots,M_n|\!)]_S = Q(\!|N_1[P(\!|M_1,\ldots,M_n|\!)]_{S'}, N_2,\ldots,N_m|\!)$ for some prime graph Q and $m \geq 4$, and $N_i \neq \varnothing$ for all $1 \leq i \leq m$. Applying the induction hypothesis to $\mathcal{D}'$ gives us one of the following three cases:

(iv.α) We have $C'[\cdot]_{R'}$ and $L_1, \ldots L_m$ such that

$$\begin{array}{c}C'[Q^{\perp}(\!|L_1,\ldots L_m|\!)]_{R'}\\ \mathcal{D}'_G\,\|\\ G''\end{array} \quad , \quad \begin{array}{c}\varnothing\\ \mathcal{D}_{C'}\,\|\\ C'\end{array} \quad ,$$

[6] This is the reason for using the abbreviation $\hat{K}$ in that case.

$$\begin{array}{c}\emptyset\\ \mathcal{D}'_1 \,\Vert\\ L_1 \parr N_1[P\llparenthesis M_1,\dots,M_n\rrparenthesis]_{S'}\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_i \,\Vert\\ L_i \parr N_i\end{array}$$

for $2 \le i \le m$. From $\mathcal{D}'_1$, we get (via the rule ss↓) $\vdash_{\mathsf{GS}} L_1 \parr N_1 \parr P\llparenthesis M_1,\dots,M_n\rrparenthesis$ to which we can apply the induction hypothesis again, to get one of the following two cases:

- either a context $C''[\cdot]_{R''}$ and graphs $K_1, \dots, K_n$, such that

$$\begin{array}{c}C''[P^{\perp}\llparenthesis K_1,\dots,K_n\rrparenthesis]_{R''}\\ \mathcal{D}''_G \,\Vert\, \mathsf{GS}\\ L_1 \parr N_1\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_{C'} \,\Vert\, \mathsf{GS}\\ C''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_i \,\Vert\, \mathsf{GS}\\ K_i \parr M_i\end{array}$$

  for all $i \in \{1,\dots,n\}$,
- or a context $C''[\cdot]_{R''}$ and graphs K_X and K_Y such that

$$\begin{array}{c}C''[K_X \parr K_Y]_{R''}\\ \mathcal{D}''_G \,\Vert\, \mathsf{GS}\\ L_1 \parr N_1\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_{C'} \,\Vert\, \mathsf{GS}\\ C''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_X \,\Vert\, \mathsf{GS}\\ K_X \parr M_i\end{array}\;,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}_Y \,\Vert\, \mathsf{GS}\\ K_Y \parr P\llparenthesis M_1,\dots,M_{i-1},\emptyset,M_{i+1},\dots,M_n\rrparenthesis\end{array}$$

  for some $i \in \{1,\dots,n\}$.

In the first case we let $\hat{K} = P^{\perp}\llparenthesis K_1,\dots,K_n\rrparenthesis$ and in the second $\hat{K} = K_X \parr K_Y$. In both cases we have $G = G'' \parr Q\llparenthesis N_1,\dots,N_m\rrparenthesis$ and can let $C[\cdot]_R$ and $\mathcal{D}_G$ and $\mathcal{D}_C$ as in case (1.c.iv.α) above.[7]

(iv.β) We have $C'[\cdot]_{R'}$ and L_X and L_Y such that

$$\begin{array}{c}C'[L_X \parr L_Y]_{R'}\\ \mathcal{D}'_G \,\Vert\\ G''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_X \,\Vert\\ L_X \parr N_1[P\llparenthesis M_1,\dots,M_n\rrparenthesis]_{S'}\end{array}\;,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}_{C'} \,\Vert\\ C'\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_Y \,\Vert\\ L_Y \parr Q\llparenthesis \emptyset,N_2,\dots,N_m\rrparenthesis\end{array}\;.$$

From $\mathcal{D}_X$, we get $\vdash_{\mathsf{GS}} L_X \parr N_1 \parr P\llparenthesis M_1,\dots,M_n\rrparenthesis$, to which we apply the induction hypothesis again and get one of the following two cases:

- either a context $C''[\cdot]_{R''}$ and graphs $K_1, \dots, K_n$, such that

$$\begin{array}{c}C''[P^{\perp}\llparenthesis K_1,\dots,K_n\rrparenthesis]_{R''}\\ \mathcal{D}''_G \,\Vert\, \mathsf{GS}\\ L_X \parr N_1\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_{C'} \,\Vert\, \mathsf{GS}\\ C''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_i \,\Vert\, \mathsf{GS}\\ K_i \parr M_i\end{array}$$

  for all $i \in \{1,\dots,n\}$,
- or a context $C''[\cdot]_{R''}$ and graphs K_X and K_Y such that

$$\begin{array}{c}C''[K_X \parr K_Y]_{R''}\\ \mathcal{D}''_G \,\Vert\, \mathsf{GS}\\ L_X \parr N_1\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_{C'} \,\Vert\, \mathsf{GS}\\ C''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_X \,\Vert\, \mathsf{GS}\\ K_X \parr M_i\end{array}\;,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}_Y \,\Vert\, \mathsf{GS}\\ K_Y \parr P\llparenthesis M_1,\dots,M_{i-1},\emptyset,M_{i+1},\dots,M_n\rrparenthesis\end{array}$$

  for some $i \in \{1,\dots,n\}$.

As above, we let in the first case $\hat{K} = P^{\perp}\llparenthesis K_1,\dots,K_n\rrparenthesis$ and in the second $\hat{K} = K_X \parr K_Y$. In both cases we have $G = G'' \parr Q\llparenthesis N_1,\dots,N_m\rrparenthesis$ and can let $C[\cdot]_R$ and $\mathcal{D}_G$ and $\mathcal{D}_C$ as in case (1.c.iv.β) above.

(iv.γ) We have $C'[\cdot]_{R'}$ and L_X and L_Y such that

$$\begin{array}{c}C'[L_X \parr L_Y]_{R'}\\ \mathcal{D}'_G \,\Vert\\ G''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_{C'} \,\Vert\\ C'\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_X \,\Vert\\ L_X \parr N_2\end{array}\;,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}_Y \,\Vert\\ L_Y \parr Q\llparenthesis N_1[P\llparenthesis M_1,\dots,M_n\rrparenthesis]_{S'},\emptyset,N_3,\dots,N_m\rrparenthesis\end{array}\;.$$

From $\mathcal{D}_Y$ we get (via ss↓)

$$\vdash_{\mathsf{GS}} L_Y \parr Q\llparenthesis N_1,\emptyset,N_3,\dots,N_m\rrparenthesis \parr P\llparenthesis M_1,\dots,M_n\rrparenthesis$$

to which we apply the induction hypothesis again, and get one of the following two cases:

- either a context $C''[\cdot]_{R''}$ and graphs $K_1, \dots, K_n$, such that

$$\begin{array}{c}C''[P^{\perp}\llparenthesis K_1,\dots,K_n\rrparenthesis]_{R''}\\ \mathcal{D}''_G \,\Vert\, \mathsf{GS}\\ L_Y \parr Q\llparenthesis N_1,\emptyset,N_3,\dots,N_m\rrparenthesis\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_{C'} \,\Vert\, \mathsf{GS}\\ C''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_i \,\Vert\, \mathsf{GS}\\ K_i \parr M_i\end{array}$$

  for all $i \in \{1,\dots,n\}$,
- or a context $C''[\cdot]_{R''}$ and graphs K_X and K_Y such that

$$\begin{array}{c}C''[K_X \parr K_Y]_{R''}\\ \mathcal{D}''_G \,\Vert\, \mathsf{GS}\\ L_Y \parr Q\llparenthesis N_1,\emptyset,N_3,\dots,N_m\rrparenthesis\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_{C'} \,\Vert\, \mathsf{GS}\\ C''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_X \,\Vert\, \mathsf{GS}\\ K_X \parr M_i\end{array}\;,$$

$$\begin{array}{c}\emptyset\\ \mathcal{D}_Y \,\Vert\, \mathsf{GS}\\ K_Y \parr P\llparenthesis M_1,\dots,M_{i-1},\emptyset,M_{i+1},\dots,M_n\rrparenthesis\end{array}$$

  for some $i \in \{1,\dots,n\}$.

As above, we let in the first case $\hat{K} = P^{\perp}\llparenthesis K_1,\dots,K_n\rrparenthesis$ and in the second $\hat{K} = K_X \parr K_Y$. In both cases we have $G = G'' \parr Q\llparenthesis N_1,\dots,N_m\rrparenthesis$ and can let $C[\cdot]_R$ and $\mathcal{D}_G$ and $\mathcal{D}_C$ as in case (1.c.iv.γ) above.

[7] This is the reason for using the abbreviation $\hat{K}$ in that case.

(d) Consider when we have $G = G' \parr P^{\perp}\llparenthesis N_1, \ldots, N_n\rrparenthesis$ and $\mathcal{D}$ is of shape

$$\mathsf{p}{\downarrow}\,\frac{\begin{array}{c}\emptyset\\ \mathcal{D}'\,\Vert\\ G' \parr ((N_1 \parr M_1)\otimes\cdots\otimes(N_n\parr M_n))\end{array}}{G'\parr P^{\perp}\llparenthesis N_1,\ldots,N_n\rrparenthesis \parr P\llparenthesis M_1,\ldots,M_n\rrparenthesis}\;.$$

By applying the induction hypothesis $n-1$ times to $\mathcal{D}'$ and some uses of the $\mathsf{ss}{\downarrow}$ rule, we obtain a context $C[\cdot]_R$ and graphs $L_1, \ldots, L_n$ such that

$$\begin{array}{c}C[L_1 \parr \cdots \parr L_n]_R\\ \mathcal{D}'_G\,\Vert\\ G'\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_C\,\Vert\\ C\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}_i\,\Vert\\ L_i \parr N_i \parr M_i\end{array}$$

for $i \in \{1, \ldots, n\}$. We let $K_i = L_i \parr N_i$, and construct $\mathcal{D}_G$ as

$$\mathsf{ss}{\downarrow}\,\frac{C\Big[\begin{array}{c}P^{\perp}\llparenthesis L_1 \parr N_1, \ldots, L_n \parr N_n\rrparenthesis\\ \Vert\,\{\mathsf{ss}{\downarrow}\}\\ L_1 \parr \cdots \parr L_n \parr P^{\perp}\llparenthesis N_1, \ldots, N_n\rrparenthesis\end{array}\Big]_R}{\begin{array}{c}L_1 \parr \cdots \parr L_n\\ \mathcal{D}'_G\,\Vert\\ G'\end{array} \parr P^{\perp}\llparenthesis N_1, \ldots, N_n\rrparenthesis}\;.$$

(e) Finally, consider when $G = G'' \parr Q\llparenthesis N_1, \ldots, N_k\rrparenthesis$ where N_i non-empty for all j, Q is prime, $|V_Q| > |V_P|$, $P\llparenthesis M_1, \ldots, M_n\rrparenthesis = Q^{\perp}\llparenthesis \emptyset, L_2, \ldots L_n\rrparenthesis$ (observe at least one module of the prime connective $Q^{\perp}$ must be empty for this equality to hold and we set that w.l.o.g. to be the first module, otherwise $P^{\perp}$ and Q are isomorphic, contradicting $|V_Q| > |V_P|$), and $\mathcal{D}$ is of shape

$$\mathsf{p}{\downarrow}\,\frac{\begin{array}{c}\emptyset\\ \mathcal{D}'\,\Vert\\ G'' \parr (N_1 \otimes (N_2 \parr L_2) \otimes \ldots \otimes (N_n \parr L_n))\end{array}}{G'' \parr Q\llparenthesis N_1, \ldots, N_k\rrparenthesis \parr P\llparenthesis M_1, \ldots, M_n\rrparenthesis}$$

By applying the induction hypothesis to $\mathcal{D}'$ we can obtain $C'[\cdot]_{R'}$ and K_i such that

$$\begin{array}{c}C'[[K_1; K_2; \ldots K_n]]_{R'}\\ \mathcal{D}''\,\Vert\\ G''\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_1\,\Vert\\ N_1 \parr K_1\end{array}\;,\quad \begin{array}{c}\emptyset\\ \mathcal{D}'_i\,\Vert\\ N_1 \parr K_i \parr M_i\end{array}$$

Now observe we can construct the following proof, where $\mathcal{D}_Q$ is given by Lemma 5.2,

$$\begin{array}{c}\Big(\begin{array}{c}\emptyset\\ \mathcal{D}'_2\,\Vert\\ [K_2; N_2; M_2]\end{array}; \ldots; \begin{array}{c}\emptyset\\ \mathcal{D}'_k\,\Vert\\ [K_k; N_k; M_k]\end{array}\Big)\\ \mathcal{D}_Q\,\Vert\\ [Q\llparenthesis \emptyset, K_2 \parr N_2, \ldots, K_k \parr N_k\rrparenthesis; P\llparenthesis M_1, \ldots, M_n\rrparenthesis]\end{array}$$

Since G'' is nonempty and does not contribute to the conclusion of the above proof the size of the conclusion is strictly less than the size of $G \parr P\llparenthesis M_1, \ldots, M_n\rrparenthesis$. Hence we can apply the induction proof to obtain $C''[\cdot]_{R''}$ and $\hat{K}$ such that

$$\begin{array}{c}C''[\hat{K}]_{R''}\\ \hat{\mathcal{D}}\,\Vert\\ Q\llparenthesis \emptyset, K_2 \parr N_2, \ldots, K_k \parr N_k\rrparenthesis\end{array}$$

where $\hat{K}$ satisfies gives us most conditions of splitting such as K_i or K_X and K_Y. What remains is to set $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$, obtain $\mathcal{D}_C$ using Lemma A.2, and construct derivation $\mathcal{D}_G$ as follows

$$\mathsf{ss}{\downarrow}\,\frac{C'\Bigg[\mathsf{ss}{\downarrow}\,\frac{\begin{array}{c}C''[\hat{K}]_{R''}\\ \hat{\mathcal{D}}\,\Vert\\ Q\Big\llparenthesis \begin{array}{c}\emptyset\\ \mathcal{D}'_1\,\Vert\\ K_1 \parr N_1\end{array}, K_2 \parr N_2, \ldots, K_k \parr N_k\Big\rrparenthesis\end{array}}{K_1 \parr K_2 \parr \ldots K_n \parr Q\llparenthesis N_1, \ldots, N_k\rrparenthesis}\Bigg]_{R'}}{\begin{array}{c}C'[K_1 \parr K_2 \parr \ldots K_n]_{R'}\\ \mathcal{D}''\,\Vert\\ G''\end{array} \parr Q\llparenthesis N_1, \ldots, N_k\rrparenthesis}$$

(3) In this case, we assume $\vdash_{\mathsf{GS}} G \parr a$ and aim to construct $C[\cdot]_R$, $\mathcal{D}_G$, $\mathcal{D}_C$ as in the statement of the splitting lemma. Observe that $G \neq \emptyset$, otherwise $G \parr a$ would not be probable in GS. As in the other two cases, we make a case analysis on the bottommost rule instance r in $\mathcal{D}$.

(a) If r acts completely inside G then we proceed immediately by induction hypothesis, as in case (1.a) above.

(b) As a is an atom, there is no case that corresponds to (1.b) or (2.b), and we leave that out.

(c) In this case we have $G = G'' \parr G'$ for $G' \neq \emptyset$ and $\mathcal{D}$ is of shape

$$\mathsf{ss}{\downarrow}\,\frac{\begin{array}{c}\emptyset\\ \mathcal{D}'\,\Vert\\ G'' \parr G'[a]_S\end{array}}{G'' \parr G' \parr a}$$

We proceed by a case analysis on $G'[a]_S$, using Lemma 3.7. As before, it is not an atom and not a par. Consequently, it is either a tensor or is composed via a prime graph Q with $|V_Q| \geq 4$. The two cases (iii) and (iv) are now almost literally the same as for case (1.c) above, and therefore not repeated here. The only difference is that we replace everywhere $A \otimes B$ by a and $K_A \parr K_B$ by $a^{\perp}$, and omit $\mathcal{D}_A$ and $\mathcal{D}_B$.

(d) Assume P is prime, $|V_P| \geq 4$, and M_i are nonempty in the derivation

$$\mathsf{p}{\downarrow}\,\dfrac{\begin{array}{c}\varnothing\\ \mathcal{D}' \,\| \\ G'' \parr ((a \parr M_1) \otimes M_2 \otimes \ldots \otimes M_n)\end{array}}{G'' \parr P(\!|M_1, \ldots M_n|\!) \parr a}$$

Notice, w.l.o.g., a is ready to interact with the first module of the P connective. By the induction hypothesis, we have $C'[\cdot]_{R'}$, K_i such that

$$\begin{array}{c}C'[K_1 \parr \cdots \parr K_n]_{R'}\\ \mathcal{D}''_G \,\| \\ G''\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_{C1} \,\| \\ C'\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_1 \,\| \\ K_1 \parr M_1 \parr a\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_i \,\| \\ K_i \parr M_i\end{array}$$

for $2 \leq i \leq n$. Since $\vdash K_1 \parr M_1 \parr a$, by the induction hypothesis, we have $C''[\cdot]_{R''}$ such that

$$\begin{array}{c}C''[a^\perp]_{R''}\\ \mathcal{D}_L \,\| \\ K_1 \parr M_1\end{array}$$

From the above, we can construct the following derivation, $\mathcal{D}_G$, as required, where $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R'}$, and $\mathcal{D}_C$ is given by Lemma A.2

$$\mathsf{ss}{\downarrow}\,\dfrac{n{\times}\mathsf{ss}{\downarrow}\,\dfrac{C'[P(\!|\begin{array}{c}C''[a^\perp]_{R''}\\ \mathcal{D}_L \,\| \\ K_1 \parr M_1\end{array}, \begin{array}{c}\varnothing\\ \mathcal{D}_2 \,\| \\ K_2 \parr M_2\end{array}, \ldots \begin{array}{c}\varnothing\\ \mathcal{D}_n \,\| \\ K_n \parr M_n\end{array}|\!)]_{R'}}{C'[K_1 \parr \cdots \parr K_n \parr P(\!|M_1, \ldots M_n|\!)]_{R'}}}{\begin{array}{c}C'[K_1 \parr \cdots \parr K_n]_{R'}\\ \mathcal{D}''_G \,\| \\ G''\end{array} \parr P(\!|M_1, \ldots M_n|\!)}$$

(e) In the final case we have $G = G' \parr a^\perp$ and $\mathcal{D}$ is of shape

$$\begin{array}{c}\varnothing\\ \mathcal{D}'_G \,\| \\ G' \parr \mathsf{ai}{\downarrow}\,\dfrac{\varnothing}{a^\perp \parr a}\end{array}$$

We can immediately conclude by letting $C = \varnothing$ (hence $\mathcal{D}_C$ is trivial) and letting $\mathcal{D}_G$ be

$$\begin{array}{c}\varnothing\\ \mathcal{D}'_G \,\| \\ G'\end{array} \parr a^\perp$$

This completes the proof of the splitting lemma. □

B Proof of Context Reduction (Lemma 7.1)

Proof of Lemma 7.1. We proceed by induction on the size of $G[A]_S$, making a case analysis based on Lemma 3.7.

(i) $G[A]_S$ has only one vertex. This is impossible as it is provable.

(ii) $G[A]_S = G'' \parr G'[A]_S$: In that case we make the same case analysis on $G'[A]_S$. But without loss of generality, we can assume that $G'[A]_S$ is not a par.

(ii.i) $G'[A]_S$ has only one vertex. Then $G'[A]_S = A = a$ for some atom and $G[A]_S = G'' \parr a$. We apply Lemma 6.1.(3) and obtain $C[\cdot]_R$ such that

$$\begin{array}{c}C[a^\perp]_R\\ \mathcal{D}''_G \,\| \\ G''\end{array} \quad\text{and}\quad \begin{array}{c}\varnothing\\ \mathcal{D}_C \,\| \\ C\end{array}$$

We let $K = a^\perp$ and for any X, we can construct

$$\mathsf{ss}{\downarrow}\,\dfrac{C[a^\perp \parr X]_R}{\begin{array}{c}C[a^\perp]_R\\ \mathcal{D}''_G \,\| \\ G''\end{array} \parr X}$$

as $G[X]_S = G'' \parr X$.

(ii.iii) $G'[A]_S$ is a tensor. Then we can assume without loss of generality that $G[A]_S = G'' \parr (G'_1[A]_{S'} \otimes G_2)$. By Lemma 6.1.(1), we get $C'[\cdot]_{R'}$ and K_1 and K_2, such that

$$\begin{array}{c}C'[K_1 \parr K_2]_{R'}\\ \mathcal{D}''_G \,\| \\ G''\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}'_C \,\| \\ C'\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_1 \,\| \\ K_1 \parr G'_1[A]_{S'}\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_2 \,\| \\ K_2 \parr G'_2\end{array}\ .$$

We apply the induction hypothesis to $\mathcal{D}_1$ and get K and $C''[\cdot]_{R''}$, such that

$$\begin{array}{c}\varnothing\\ \mathcal{D}''_C \,\| \\ C''\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_A \,\| \\ K \parr A\end{array}\ ,\quad\text{and}\quad \begin{array}{c}C''[K \parr X]_{R''}\\ \mathcal{D}'_G \,\|\,\mathsf{GS} \\ K_1 \parr G'_1[X]_{S'}\end{array}$$

for any X. We now let $C[\cdot]_R = C'[C''[\cdot]_{R''}]R'$. We obtain $\mathcal{D}_C$ from Lemma A.2 and construct $\mathcal{D}_G$ as

$$\mathsf{ss}{\downarrow}\,\dfrac{C'\Big[\mathsf{ss}{\downarrow}\,\dfrac{\begin{array}{c}C''[K \parr X]_{R''}\\ \mathcal{D}'_G \,\| \\ K_1 \parr (G'_1[X]_{S'} \otimes \begin{array}{c}\varnothing\\ \mathcal{D}_2 \,\| \\ K_2 \parr G'_2\end{array})\end{array}}{K_1 \parr K_2 \parr (G'_1[X]_{S'} \otimes G'_2)}\Big]_{R'}}{\begin{array}{c}C'[K_1 \parr K_2]_{R'}\\ \mathcal{D}''_G \,\| \\ G''\end{array} \parr (G'_1[X]_{S'} \otimes G_2)}\ ,$$

since $G[X]_S = G'' \parr (G'_1[X]_{S'} \otimes G_2)$.

(ii.iv) $G'[A]_S$ is composed via a prime graph P with $|V_P| \geq 4$. Then, without loss of generality we can assume that $G[A]_S = G'' \parr P(\!|M_1[A]_{S'}, M_2, \ldots, M_n|\!)$. Applying Lemma 6.1.(2) gives us one of the following three cases:

(α) We have $C'[\cdot]_{R'}$ and $K_1, \ldots, K_n$, such that

$$\begin{array}{c} C'[P^\perp(\!|K_1,\ldots,K_n|\!)]_{R'} \\ \mathcal{D}''_G \Vert \\ G'' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}'_C \Vert \\ C' \end{array} ,$$

$$\begin{array}{c} \varnothing \\ \mathcal{D}_1 \Vert \\ K_1 \parr M_1[A]_{S'} \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_i \Vert \\ K_i \parr M_i \end{array}$$

for $2 \leq i \leq n$. We can apply the induction hypothesis to $\mathcal{D}_1$ and obtain K and $C''[\cdot]_{R''}$, such that

$$\begin{array}{c} \varnothing \\ \mathcal{D}''_C \Vert \\ C'' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_A \Vert \\ K \parr A \end{array} , \quad \begin{array}{c} C''[K \parr X]_{R''} \\ \mathcal{D}'_G \Vert \mathsf{GS} \\ K_1 \parr M_1[X]_{S'} \end{array}$$

for any X. We let $C[\cdot]_R$ and $\mathcal{D}_C$ as in case (ii.iii) above and obtain $\mathcal{D}_G$ as

$$\mathsf{ss}{\downarrow}\frac{C'\left[\mathsf{p}{\downarrow}\dfrac{\begin{array}{c} C''[K \parr X]_{R''} \\ \mathcal{D}'_G \Vert \\ K_1 \parr M_1[X]_{S'} \end{array} \otimes \begin{array}{c} \varnothing \\ \mathcal{D}_2 \Vert \\ K_2 \parr M_2 \end{array} \otimes \cdots \otimes \begin{array}{c} \varnothing \\ \mathcal{D}_n \Vert \\ K_n \parr M_m \end{array}}{P^\perp(\!|K_1,\ldots,K_n|\!) \parr K_1 \parr P(\!|M_1[X]_{S'}, M_2, \ldots, M_n|\!)}\right]_{R'}}{\begin{array}{c} C'[P^\perp(\!|K_1,\ldots,K_n|\!)]_{R'} \\ \mathcal{D}''_G \Vert \\ G'' \end{array} \parr P(\!|M_1[X]_{S'}, M_2, \ldots, M_n|\!)} ,$$

where $G[X]_S = G'' \parr P(\!|M_1[X]_{S'}, M_2, \ldots, M_n|\!)$.

(β) We have $C'[\cdot]_{R'}$ and K_X and K_Y, such that

$$\begin{array}{c} C'[K_X \parr K_Y]_{R'} \\ \mathcal{D}''_G \Vert \\ G'' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}'_C \Vert \\ C' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_X \Vert \\ K_X \parr M_1[A]_{S'} \end{array} ,$$

$$\begin{array}{c} \varnothing \\ \mathcal{D}_Y \Vert \\ K_Y \parr P(\!|\varnothing, M_2, \ldots, M_n|\!) \end{array} .$$

We apply the induction hypothesis to $\mathcal{D}_X$ and get K and $C''[\cdot]_{R''}$, such that

$$\begin{array}{c} \varnothing \\ \mathcal{D}''_C \Vert \\ C'' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_A \Vert \\ K \parr A \end{array} , \quad \begin{array}{c} C''[K \parr X]_{R''} \\ \mathcal{D}'_G \Vert \mathsf{GS} \\ K_X \parr M_1[X]_{S'} \end{array}$$

for any X. We let $C[\cdot]_R = C'[K_Y \parr P(\!|C''[\cdot]_{R''}, M_2, \ldots, M_n|\!)]_{R'}$ and obtain $\mathcal{D}_C$ from Lemma A.2, and $\mathcal{D}_G$ is as follows:

$$\mathsf{ss}{\downarrow}\frac{C'\left[\mathsf{ss}{\downarrow}\dfrac{K_Y \parr P(\!|\begin{array}{c} C''[K \parr X]_{R''} \\ \mathcal{D}'_G \Vert \\ K_X \parr M_1[X]_{S'} \end{array}, M_2, \ldots, M_n|\!)}{K_X \parr K_Y \parr P(\!|M_1[X]_{S'}, M_2, \ldots, M_n|\!)}\right]_{R'}}{\begin{array}{c} C'[K_X \parr K_Y]_{R'} \\ \mathcal{D}''_G \Vert \\ G'' \end{array} \parr P(\!|M_1[X]_{S'}, M_2, \ldots, M_n|\!)} .$$

(γ) We have $C'[\cdot]_{R'}$ and K_X and K_Y, such that

$$\begin{array}{c} C'[K_X \parr K_Y]_{R'} \\ \mathcal{D}''_G \Vert \\ G'' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}'_C \Vert \\ C' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_X \Vert \\ K_X \parr M_2 \end{array} ,$$

$$\begin{array}{c} \varnothing \\ \mathcal{D}_Y \Vert \\ K_Y \parr P(\!|M_1[X]_{S'}, \varnothing, M_3, \ldots, M_n|\!) \end{array} .$$

We apply the induction hypothesis to $\mathcal{D}_Y$ and get K and $C''[\cdot]_{R''}$, such that

$$\begin{array}{c} \varnothing \\ \mathcal{D}''_C \Vert \\ C'' \end{array} , \quad \begin{array}{c} \varnothing \\ \mathcal{D}_A \Vert \\ K \parr A \end{array} , \quad \begin{array}{c} C''[K \parr X]_{R''} \\ \mathcal{D}'_G \Vert \\ K_Y \parr P(\!|M_1[X]_{S'}, \varnothing, M_3, \ldots, M_n|\!) \end{array}$$

for any X. We let $C[\cdot]_R$ and $\mathcal{D}_C$ as in case (ii.iii) above and obtain $\mathcal{D}_G$ as

$$\mathsf{ss}{\downarrow}\frac{C'\left[\mathsf{ss}{\downarrow}\dfrac{\begin{array}{c} C''[K \parr X]_{R''} \\ \mathcal{D}'_G \Vert \\ K_Y \parr P(\!|M_1[X]_{S'}, \begin{array}{c} \varnothing \\ \mathcal{D}_X \Vert \\ K_X \parr M_2 \end{array}, M_3, \ldots, M_n|\!) \end{array}}{K_X \parr K_Y \parr P(\!|M_1[X]_{S'}, M_2, M_3, \ldots, M_n|\!)}\right]_{R'}}{\begin{array}{c} C'[K_X \parr K_Y]_{R'} \\ \mathcal{D}''_G \Vert \\ G'' \end{array} \parr P(\!|M_1[X]_{S'}, M_2, M_3, \ldots, M_n|\!)} .$$

(iii) $G[A]_S$ is a tensor. This case is as case (ii.iii) above, with $G'' = \varnothing$, and consequently $C' = K_1 = K_2 = \varnothing$.

(iv) $G[A]_S$ is composed via a prime graph P with $|V_P| \geq 4$. This case is as case (ii.iv) above, with $G'' = \varnothing$. Consequently, C' and $K_1, \ldots, K_n$ (resp. K_X and K_Y) are empty as well. □

C Proof of admissibility of the up-rules (Section 8)

In this section we show how splitting and context reduction are used to show the admissibility of all rules in the up-fragment of SGS.

Theorem C.1. *The rule* ai↑ *is admissible for* GS.

Proof. Assume we have a proof of $G[a \otimes a^\perp]_S$. By Lemma 7.1 we have a graph L and a context $C_1[\cdot]_{R_1}$, such that there are derivations

$$\begin{array}{c}\varnothing\\ \mathcal{D}_1 \,\|\, \mathsf{GS}\\ C_1\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_2 \,\|\, \mathsf{GS}\\ L \parr (a \otimes a^\perp)\end{array}\ ,\quad \begin{array}{c}C_1[L \parr X]_{R_1}\\ \mathcal{D}_3 \,\|\, \mathsf{GS}\\ G[X]_S\end{array}$$

for any graph X. We apply Lemma 6.1.(1) to $\mathcal{D}_2$ and get K_a and $K_{a^\perp}$ and a context $C_2[\cdot]_{R_2}$ such that

$$\begin{array}{c}C_2[K_a \parr K_{a^\perp}]_{R_2}\\ \mathcal{D}_4 \,\|\, \mathsf{GS}\\ L\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_5 \,\|\, \mathsf{GS}\\ C_2\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_6 \,\|\, \mathsf{GS}\\ K_a \parr a\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_7 \,\|\, \mathsf{GS}\\ K_{a^\perp} \parr a^\perp\end{array}$$

Applying Lemma 6.1.(3) to $\mathcal{D}_6$ and $\mathcal{D}_7$ gives us $C_3[\cdot]_{R_3}$ and $C_4[\cdot]_{R_4}$ such that

$$\begin{array}{c}C_3[a^\perp]_{R_3}\\ \mathcal{D}_8 \,\|\, \mathsf{GS}\\ K_a\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_9 \,\|\, \mathsf{GS}\\ C_3\end{array}\quad \text{and}\quad \begin{array}{c}C_4[a]_{R_4}\\ \mathcal{D}_{10} \,\|\, \mathsf{GS}\\ K_{a^\perp}\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_{11} \,\|\, \mathsf{GS}\\ C_4\end{array}$$

We can now give the following derivation

$$\begin{array}{c}\varnothing\\ \mathcal{D}_1 \,\|\\ C_1\left[\begin{array}{c}\varnothing\\ \mathcal{D}_5 \,\|\\ C_2\left[\begin{array}{c}\varnothing\\ \mathcal{D}_8 \,\|\\ \mathsf{ss}{\downarrow}\,\dfrac{C_3\left[\begin{array}{c}\varnothing\\ \mathcal{D}_{11} \,\|\\ \mathsf{ss}{\downarrow}\,\dfrac{C_4\left[\mathsf{ai}{\downarrow}\,\dfrac{\varnothing}{a^\perp \parr a}\right]_{R_4}}{a^\perp \parr C_4[a]_{R_4}}\end{array}\right]_{R_3}}{\begin{array}{c}C_3[a^\perp]_{R_3}\\ \mathcal{D}_8 \,\|\\ K_a\end{array} \parr \begin{array}{c}C_4[a]_{R_4}\\ \mathcal{D}_{10} \,\|\\ K_{a^\perp}\end{array}}\end{array}\right]_{R_2}\\ \mathcal{D}_4 \,\|\\ L\end{array}\right]_{R_1}\\ \mathcal{D}_3 \,\|\\ G[\varnothing]_S\end{array}$$

that proves $G = G[\varnothing]_S$ in GS. □

Theorem C.2. *The rule* ss↑ *is admissible for* GS.

Proof. Assume we have a proof of $G[B \otimes A]_S$ in GS. By Lemma 7.1 we have a graph L and a context $C_1[\cdot]_{R_1}$, such that there are derivations

$$\begin{array}{c}\varnothing\\ \mathcal{D}_1 \,\|\\ C_1\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_2 \,\|\\ L \parr (B \otimes A)\end{array}\ ,\quad \begin{array}{c}C_1[L \parr X]_{R_1}\\ \mathcal{D}_3 \,\|\\ G[X]_S\end{array}$$

for any graph X. We apply Lemma 6.1.(1) to $\mathcal{D}_2$ and get K_B and K_A and a context $C_2[\cdot]_{R_2}$ such that

$$\begin{array}{c}C_2[K_B \parr K_A]_{R_2}\\ \mathcal{D}_4 \,\|\\ L\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_5 \,\|\\ C_2\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_6 \,\|\\ K_B \parr B\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_7 \,\|\\ K_A \parr A\end{array}\ .$$

We can now give a proof of $G[B[A]_T]_S$ as follows:

$$\begin{array}{c}\varnothing\\ \mathcal{D}_1 \,\|\\ C_1\left[\mathsf{ss}{\downarrow}\,\dfrac{C_2\left[\begin{array}{c}\varnothing\\ \mathcal{D}_5 \,\|\\ \begin{array}{c}\varnothing\\ \mathcal{D}_6 \,\|\\ K_B \parr \mathsf{ss}{\downarrow}\,\dfrac{B\left[\begin{array}{c}\varnothing\\ \mathcal{D}_7 \,\|\\ K_A \parr A\end{array}\right]_T}{K_A \parr B[A]_T}\end{array}\end{array}\right]_{R_2}}{\begin{array}{c}C_2[K_B \parr K_A]_{R_2}\\ \mathcal{D}_4 \,\|\\ L\end{array} \parr B[A]_T}\right]_{R_1}\\ \mathcal{D}_3 \,\|\\ G[B[A]_T]_S\end{array}$$

for any $\varnothing \subseteq T \subset |V_B|$. □

The proof of the next theorem is different from the others, and also different from what usually happens in a deep inference cut elimination proof. Even though we still use splitting and context reduction, we additionally need an induction on the size of the cut.

Theorem C.3. *The rule* p↑ *is admissible for* GS.

Proof. We define the size of an instance of p↑ (see Figure 1) as

$$\sum_{1 \le i \le n} (|M_i| + |N_i|)$$

i.e., the number of vertices in the subgraph that are modified by the rule. We now proceed as in the previous two proofs. Assume we have a proof of $G[P(\!|M_1, \ldots, M_n|\!) \otimes P^\perp(\!|N_1, \ldots, N_n|\!)]_S$. We apply Lemma 7.1 and get a graph L and a context $C_1[\cdot]_{R_1}$, such that there are derivations

$$\begin{array}{c}\varnothing\\ \mathcal{D}_1 \,\|\\ C_1\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_2 \,\|\\ L \parr (P(\!|M_1, \ldots, M_n|\!) \otimes P^\perp(\!|N_1, \ldots, N_n|\!))\end{array}\ ,\quad \begin{array}{c}C_1[L \parr X]_{R_1}\\ \mathcal{D}_3 \,\|\\ G[X]_S\end{array}$$

for any graph X. We apply Lemma 6.1.(1) to $\mathcal{D}_2$ and get graphs L_P and $L_{P^\perp}$ and a context $C_2[\cdot]_{R_2}$ such that

$$\begin{array}{c}C_2[L_P \parr L_{P^\perp}]_{R_2}\\ \mathcal{D}_4 \,\|\\ L\end{array}\ ,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_5 \,\|\\ C_2\end{array}\ ,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_6\,\Vert\\ L_P \parr P(\!|M_1,\dots,M_n|\!)\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_7\,\Vert\\ L_{P^\perp} \parr P^\perp(\!|N_1,\dots,N_n|\!)\end{array}\;.$$

Applying Lemma 6.1.(2) to $\mathcal{D}_6$ and $\mathcal{D}_7$ gives us four different cases.

(a) We get $K_1,\dots,K_n$ and $H_1,\dots,H_n$ and contexts $C_3[\cdot]_{R_3}$ and $C_4[\cdot]_{R_4}$, such that

$$\begin{array}{c}C_3[P^\perp(\!|K_1,\dots,K_n|\!)]_{R_3}\\ \mathcal{D}_8\,\Vert\\ L_P\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_9\,\Vert\\ C_3\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}'_i\,\Vert\\ K_i \parr M_i\end{array}\;,$$

$$\begin{array}{c}C_4[P(\!|H_1,\dots,H_n|\!)]_{R_4}\\ \mathcal{D}_{10}\,\Vert\\ L_{P^\perp}\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_{11}\,\Vert\\ C_4\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}''_j\,\Vert\\ H_j \parr N_j\end{array}\;,$$

for all $i,j \in \{1,\dots,n\}$. Then, our proof of $G[(M_1 \otimes N_1) \parr \cdots \parr (M_n \otimes N_n)]_S$ is shown in Figure 4, where $\mathcal{D}^*$ exist by Lemma 5.2.

(b) We get $K_1,\dots,K_n$ and H_X and H_Y and contexts $C_3[\cdot]_{R_3}$ and $C_4[\cdot]_{R_4}$, such that

$$\begin{array}{c}C_3[P^\perp(\!|K_1,\dots,K_n|\!)]_{R_3}\\ \mathcal{D}_8\,\Vert\\ L_P\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_9\,\Vert\\ C_3\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}'_i\,\Vert\\ K_i \parr M_i\end{array}\;,$$

$$\begin{array}{c}C_4[H_X \parr H_Y]_{R_4}\\ \mathcal{D}_{10}\,\Vert\\ L_{P^\perp}\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_{11}\,\Vert\\ C_4\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_X\,\Vert\\ H_X \parr N_j\end{array}\;,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_Y\,\Vert\\ H_Y \parr P^\perp(\!|N_1,\dots,N_{j-1},\varnothing,N_{j+1},\dots N_n|\!)\end{array}$$

for all $i \in \{1,\dots,n\}$ and some $j \in \{1,\dots,n\}$. The derivation for $G[(M_1 \otimes N_1) \parr \cdots \parr (M_n \otimes N_n)]_S$ is shown in Figure 5, where $\mathcal{D}^*$ consists of $n+1$ instances of $\mathsf{ss}{\downarrow}$ and the double-line in the center indicates two instances of $\mathsf{ss}{\downarrow}$.

(c) We get K_Z and K_W and $H_1,\dots,H_n$ and contexts $C_3[\cdot]_{R_3}$ and $C_4[\cdot]_{R_4}$, such that

$$\begin{array}{c}C_3[K_Z \parr K_W]_{R_3}\\ \mathcal{D}_8\,\Vert\\ L_P\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_9\,\Vert\\ C_3\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_Z\,\Vert\\ K_Z \parr M_i\end{array}\;,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_W\,\Vert\\ K_W \parr P(\!|M_1,\dots,M_{i-1},\varnothing,M_{i+1},\dots M_n|\!)\end{array}\;,$$

$$\begin{array}{c}C_4[P(\!|H_1,\dots,H_n|\!)]_{R_4}\\ \mathcal{D}_{10}\,\Vert\\ L_{P^\perp}\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_{11}\,\Vert\\ C_4\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}''_i\,\Vert\\ H_j \parr N_j\end{array}\;,$$

for some $i \in \{1,\dots,n\}$ and all $j \in \{1,\dots,n\}$. This case is similar to the previous one.

(d) We get K_Z and K_W and H_X and H_Y and contexts $C_3[\cdot]_{R_3}$ and $C_4[\cdot]_{R_4}$, such that

$$\begin{array}{c}C_3[K_Z \parr K_W]_{R_3}\\ \mathcal{D}_8\,\Vert\\ L_P\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_9\,\Vert\\ C_3\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_Z\,\Vert\\ K_Z \parr M_i\end{array}\;,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_W\,\Vert\\ K_W \parr P(\!|M_1,\dots,M_{i-1},\varnothing,M_{i+1},\dots M_n|\!)\end{array}\;,$$

$$\begin{array}{c}C_4[H_X \parr H_Y]_{R_4}\\ \mathcal{D}_{10}\,\Vert\\ L_{P^\perp}\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_{11}\,\Vert\\ C_4\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_X\,\Vert\\ H_X \parr N_j\end{array}\;,$$

$$\begin{array}{c}\varnothing\\ \mathcal{D}_Y\,\Vert\\ H_Y \parr P^\perp(\!|N_1,\dots,N_{j-1},\varnothing,N_{j+1},\dots N_n|\!)\end{array}$$

for some $i,j \in \{1,\dots,n\}$. In this case we use the derivation in Figure 6 to prove $G[(M_1 \otimes N_1) \parr \cdots \parr (M_n \otimes N_n)]_S$. More precisely, Figure 6 shows the case $i < j$, the cases $i = j$ and $i > j$ are similar. The derivation $\mathcal{D}^*$ exist by the second statement in Lemma 5.2. If $i \neq j$, this derivation consists of a single $\mathsf{p}{\uparrow}$ instance. If $i = j$, it can be a longer derivation containing all rules of $\mathsf{SGS}{\uparrow}$ (where the instances of $\mathsf{ai}{\uparrow}$ and $\mathsf{ss}{\uparrow}$ can be eliminated by the previous two theorems). The important observation to make is that all instances of $\mathsf{p}{\uparrow}$ occurring in $\mathcal{D}^*$ have smaller size than the one we started with. Therefore we can invoke the induction hypothesis. □

D Proof of Conservativity (Lemma 9.1)

Proof of Lemma 9.1. By way of contradiction, assume there is a cograph that is not provable with passing through a non-cograph. Let A be a minimal such graph, where we define the size of A as the lexicographic pair $\langle |V_A|, |E_{A^\perp}| \rangle$. The only way to create a non-cograph from a cograph while going up in a derivation is via the $\mathsf{ss}{\downarrow}$ as in

$$\mathsf{ss}{\downarrow}\,\frac{P(\!|M_1,\dots,M_{i_1},M_i,M_{i+1},\dots,M_n|\!)}{M_i \parr P(\!|M_1,\dots,M_{i_1},\varnothing,M_{i+1},\dots,M_n|\!)}$$

where M_i and $P(\!|M_1,\dots,M_{i_1},\varnothing,M_{i+1},\dots,M_n|\!)$ are cographs and $P(\!|M_1,\dots,M_n|\!)$ is not. Without loss of generality, we assume $i = 1$. By minimality of A, we can assume that this $\mathsf{ss}{\downarrow}$ occurs as bottommost rule instance in $\mathcal{D}$, and we can also assume that it occurs in a shallow context, i.e., we have

$$A = G \parr M_1 \parr P(\!|\varnothing,M_2,\dots,M_n|\!)$$

for some G. Otherwise $A = G \parr C[M_1 \parr P(\!|\varnothing,M_2,\dots,M_n|\!)]_R$ for some nontrivial context $C[\cdot]_R$, and we could apply context reduction to get a K with $\vdash K \parr M_1 \parr P(\!|\varnothing,M_2,\dots,M_n|\!)$

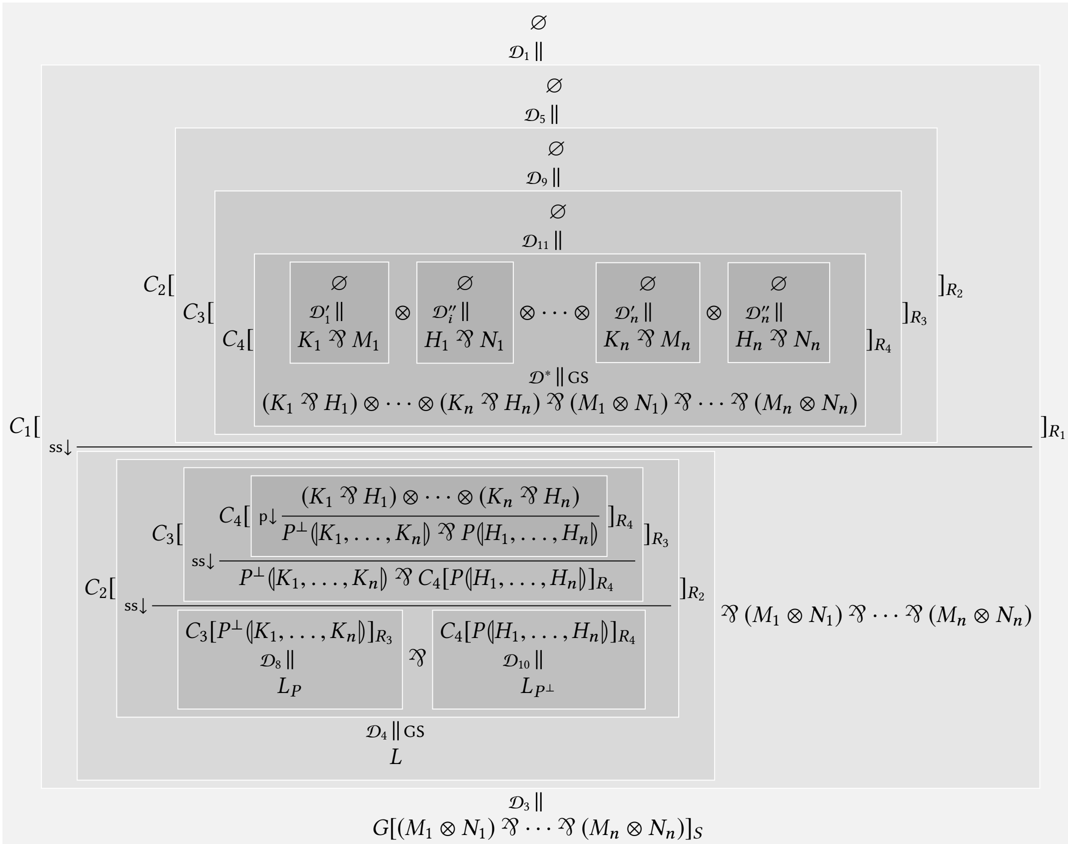

Figure 4. Derivation for case (a) in the proof of Theorem C.3

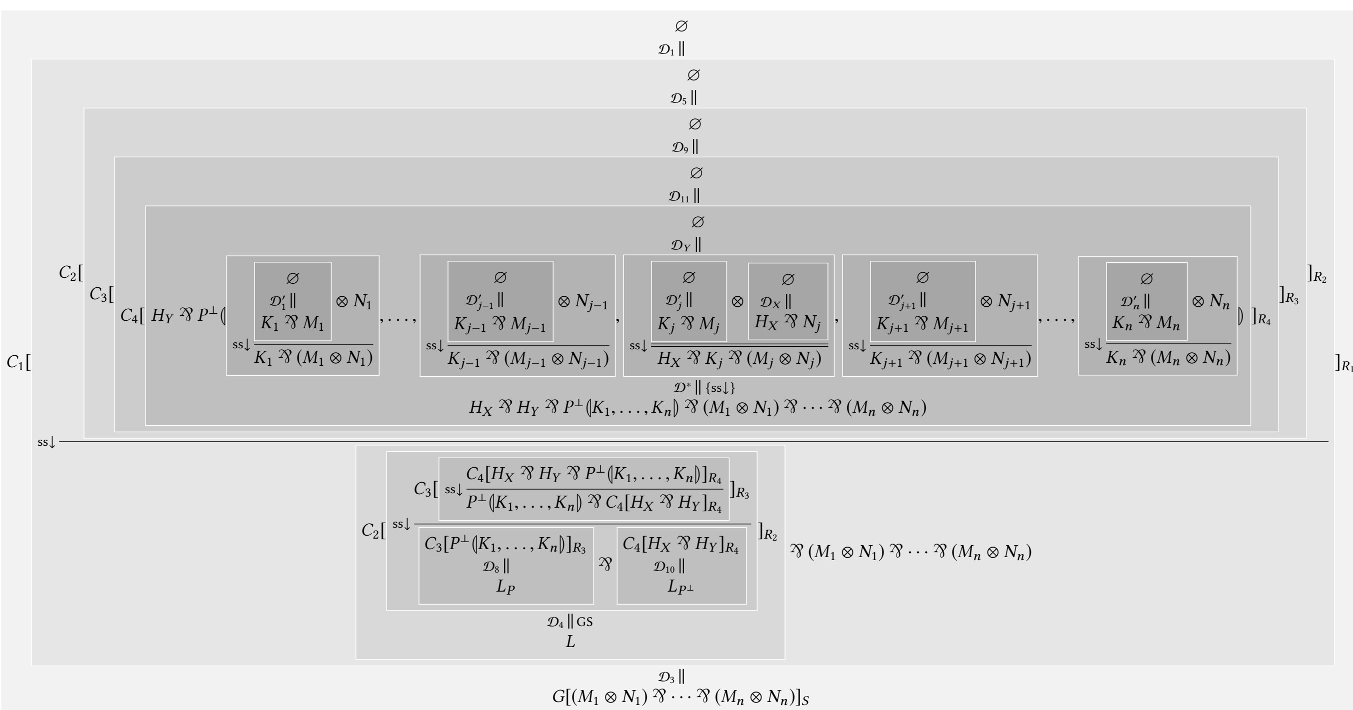

Figure 5. Derivation for case (b) in the proof of Theorem C.3

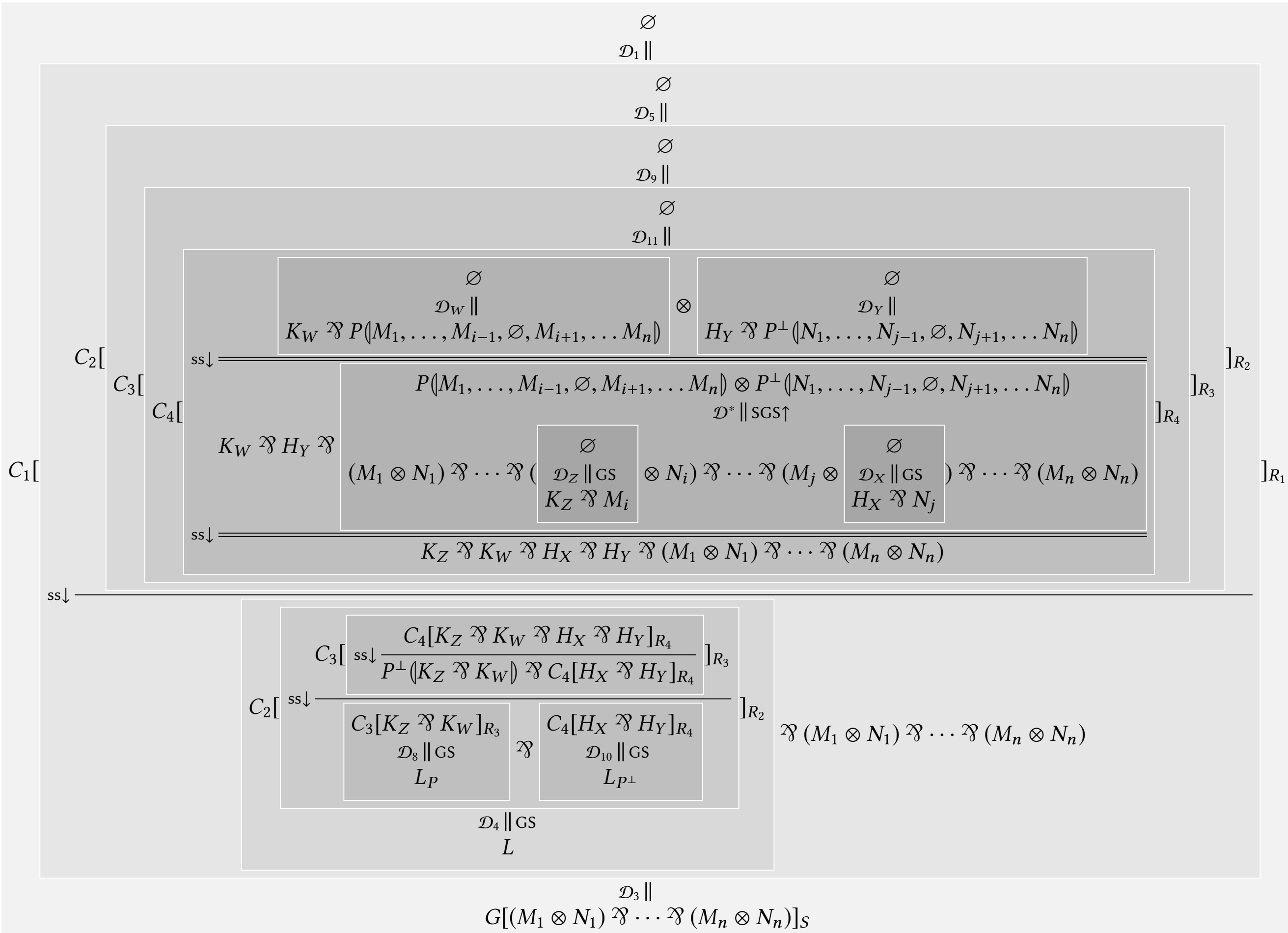

Figure 6. Derivation for case (d) in the proof of Theorem C.3.

contradicting the minimality of A. Hence, $\mathcal{D}$ is of shape

$$\begin{array}{c}\varnothing\\ \mathcal{D}'\,\|\,\mathsf{GS}\\ \mathsf{ss}{\downarrow}\,\dfrac{G \parr P(\!|M_1, M_2, \ldots, M_n|\!)}{G \parr M_1 \parr P(\!|\varnothing, M_2, \ldots, M_n|\!)}\end{array}$$

and we apply splitting (Lemma 6.1.(2)) to $\mathcal{D}'$, yielding 2 possibilities, of which we show here only the first, the second one being simpler.

- there is a context $C[\cdot]_R$ and graphs $K_1, \ldots, K_n$, such that

$$\begin{array}{c}C[P^\perp(\!|K_1, \ldots, K_n|\!)]_R\\ \mathcal{D}_G\,\|\,\mathsf{GS}\\ G\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_C\,\|\,\mathsf{GS}\\ C\end{array}\;,\quad \begin{array}{c}\varnothing\\ \mathcal{D}_i\,\|\,\mathsf{GS}\\ K_i \parr M_i\end{array}$$

for all $i \in \{1, \ldots, n\}$. If $|\mathcal{D}_G| = 0$ then $G = C[P^\perp(\!|K_1, \ldots, K_n|\!)]_R$ and we can have a derivation

$$\mathsf{ss}{\downarrow}\,\dfrac{C[P^\perp(\!|\begin{array}{c}\varnothing\\ \mathcal{D}_1\,\|\,\mathsf{GS}\\ K_1 \parr M_1\end{array} K_2, \ldots, K_n|\!)]_R}{C[P^\perp(\!|K_1, \ldots, K_n|\!)]_R \parr M_1} \parr P(\!|\varnothing, M_2, \ldots, M_n|\!) \tag{33}$$

contradicting the minimality of A. If $|\mathcal{D}_G| \neq 0$ and there is a cograph G' occurring in $\mathcal{D}_G$, then G' has smaller size than G, contradicting the minimality of $G \parr M_1 \parr P(\!|\varnothing, M_2, \ldots, M_n|\!)$. If there is no smaller cograph G' occurring in $\mathcal{D}_G$ in, then the bottommost rule instance in $\mathcal{D}_G$ is a $\mathsf{ss}{\downarrow}$ creating a non-cograph. By the same argument as above, we can conclude that it must be in a shallow context. Then $G = K_i \parr C[P^\perp(\!|K_1, \ldots, K_{i-1}, \varnothing, K_{i+1}, \ldots, K_n|\!)]_R$. If $i = 1$ we apply $\mathsf{ss}{\downarrow}$ to move M_1 and K_1 inside $C[\cdot]_R$ and conclude by a similar reasoning as with (33). If $i \neq 1$ we have $P^\perp(\!|K_1, \ldots, K_{i-1}, \varnothing, K_{i+1}, \ldots, K_n|\!)$ and

$P(\!|\varnothing, M_2, \ldots, M_n|\!)$ are cographs and again we get a contradiction to the minimality of A. □