



**HAL**  
open science

# Relating the Entanglement and Optical Nonclassicality of Multimode States of a Bosonic Quantum Field

Anaëlle Hertz, Nicolas Cerf, Stephan De Bièvre

► **To cite this version:**

Anaëlle Hertz, Nicolas Cerf, Stephan De Bièvre. Relating the Entanglement and Optical Nonclassicality of Multimode States of a Bosonic Quantum Field. *Physical Review A*, 2020, 10.1103/PhysRevA.102.032413 . hal-02557297

**HAL Id: hal-02557297**

**<https://inria.hal.science/hal-02557297>**

Submitted on 28 Apr 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Relating the Entanglement and Optical Nonclassicality of Multimode States of a Bosonic Quantum Field

Anaëlle Hertz<sup>1</sup>, Nicolas J. Cerf<sup>2</sup>, Stephan De Bièvre<sup>3</sup>

<sup>1</sup> *Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada*

<sup>2</sup> *Centre for Quantum Information and Communication, École polytechnique de Bruxelles, Université Libre de Bruxelles, CP 165, 1050 Brussels, Belgium*

<sup>3</sup> *Univ. Lille, CNRS, UMR 8524, INRIA - Laboratoire Paul Painlevé, F-59000 Lille, France*

(Dated: April 24, 2020)

The quantum nature of the state of a bosonic quantum field manifests itself in its entanglement, coherence, or optical nonclassicality which are each known to be resources for quantum computing or metrology. We provide quantitative and computable bounds relating entanglement measures with optical nonclassicality measures. These bounds imply that strongly entangled states must necessarily be strongly optically nonclassical. As an application, we infer strong bounds on the entanglement that can be produced with an optically nonclassical state impinging on a beam splitter. For Gaussian states, we analyze the link between the logarithmic negativity and a specific nonclassicality witness called “quadrature coherence scale”.

## I. INTRODUCTION

There are several ways to question the specifically quantum mechanical character of the state of a physical system. First, one may ask how strongly coherent it is. The existence of coherent superpositions of quantum states is at the origin of interference phenomena in matter waves and, as such, is a typically quantum feature for which several measures and witnesses have been proposed (for a recent review, see [1]). Second, when the system under investigation is bi-partite or multi-partite, the entanglement of its components is another intrinsically quantum feature. There exists an extensive literature exploring a wide variety of measures to quantify the amount of entanglement contained in a given state [2–14]. Finally, for modes of a bosonic quantum field, a third notion of nonclassicality arises, which is often referred to as optical nonclassicality. Following Glauber, the coherent states of an optical field (as well as their mixtures) are viewed as “classical” as they admit a positive Glauber-Sudarshan P-function [15]. From there, a variety of measures of optical nonclassicality have been developed over the years, measuring the departure from such optical classical states [15–41].

Each of these three distinct, typically quantum properties of the quantum state of an optical field have been argued to serve as a resource in quantum information or metrology [38, 39, 42–44]. The question then naturally arises what the quantitative relations are between these properties. In [45], for example, bounds are given on how much entanglement can be produced from states with a given amount of coherence using incoherent operations: this links coherence with entanglement. In [46], the coherence and optical nonclassicality of a state are shown to be related to each other: a significant value of far off-diagonal density matrix elements  $\rho(x, x')$  or  $\rho(p, p')$ , called “coherences”, is a witness of the optical nonclassicality of the state. Our purpose here is to establish a relation between optical nonclassicality and bi-partite entanglement for multi-mode bosonic fields.

One expects on intuitive grounds that a strongly entangled state should be strongly optically nonclassical since all optical classical states are separable. Conversely, a state that is only weakly optically nonclassical cannot possibly be highly entangled. To make these statements precise and quantitative, we need both a measure of entanglement and optical nonclassicality. As a natural measure to evaluate bi-partite entanglement, we use the entanglement of formation (EoF) [4]. Regarding optical nonclassicality, we use a recently introduced monotone [38, 39], which we refer to as the monotone of total noise ( $\mathcal{M}_{\text{TN}}$ ). It is obtained by extending to mixed states [through a convex roof construction, see (1)] the so-called total noise  $\Delta x^2 + \Delta p^2$  defined on pure states, for which it is a well established measure of optical nonclassicality [38–41]. Our first main result (Theorems 1 & 1') consists in an upper bound on EoF( $\rho$ ) as a function of  $\mathcal{M}_{\text{TN}}(\rho)$  for an arbitrary state  $\rho$  of a bi-partite system of  $n = n_A + n_B$  modes. In particular, when  $n_A = n_B = n/2$ , this bound implies that states containing  $m$  ebits of entanglement must have an optical nonclassicality – measured via  $\mathcal{M}_{\text{TN}}$  – that grows exponentially with  $m$ . As an application, we show that the maximum entanglement that can be produced when a separable pure state impinges on a balanced beam splitter is bounded by the logarithm of the optical nonclassicality of this in-state, measured by  $\mathcal{M}_{\text{TN}}$ . In other words, while it is well known that beam splitters can produce entanglement [28, 47, 48], the amount of entanglement is shown to be severely constrained by the degree of optical nonclassicality of the in-state.

The bounds in Theorems 1 & 1' can readily be computed for pure states since the EoF then coincides with the von Neumann entropy of the reduced state and  $\mathcal{M}_{\text{TN}}$  coincides with the total noise. For mixed states, however, the bounds relate two quantities that are generally hard to evaluate. Our second main result (Theorem 2) addresses this

issue by considering the special case of (mixed) Gaussian states. It establishes bounds between explicitly computable measures of entanglement (logarithmic negativity – LNeg) and optical nonclassicality (quadrature coherence scale – QCS) for Gaussian states. We also derive an explicit simple formula for the QCS of Gaussian states in terms of their covariance matrix [see Eq. (14)].

## II. BOUNDING EoF BY OPTICAL NONCLASSICALITY.

We consider an  $n$ -mode optical field with annihilation mode operators  $a_i = (X_i + iP_i)/\sqrt{2}$  and corresponding quadratures  $X_i, P_i$ . We set  $\mathbf{R} = (X_1, P_1, \dots, X_n, P_n)$ . The total noise of a pure state  $|\psi\rangle$  is defined as  $\mathcal{N}_{\text{tot}}(\psi) = \sum_j \Delta \mathbf{R}_j^2$  [49]. For a general state  $\rho$ , the convex roof  $\mathcal{M}_{\text{TN}}$  of  $\mathcal{N}_{\text{tot}}$  is

$$\mathcal{M}_{\text{TN}}(\rho) = \frac{1}{n} \inf_{\{p_i, \psi_i\}} \sum_i p_i \mathcal{N}_{\text{tot}}(\psi_i) \geq 1, \quad (1)$$

where the infimum is over all families  $\{p_i, \psi_i\}$  for which  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ ,  $\sum_i p_i = 1$ . It is shown in [38, 39] that  $\mathcal{M}_{\text{TN}}$  belongs to a family of optical nonclassicality monotones and is, as such, a faithful witness of optical nonclassicality:  $\mathcal{M}_{\text{TN}}(\rho) > 1$  iff  $\rho$  is nonclassical. Now consider a bi-partition of the  $n$  modes in two sets of  $n_A$  and  $n_B$  modes, with  $n = n_A + n_B$ . We write  $\rho_A$  (respectively  $\rho_B$ ) for the reduction of the state  $\rho$  to the  $n_A$  ( $n_B$ ) modes. If  $\rho = |\psi\rangle\langle\psi|$ ,  $\text{EoF}(\psi) = -\text{Tr}_A \rho_A \ln \rho_A = -\text{Tr}_B \rho_B \ln \rho_B$ . Then, for a general  $\rho$ , taking the infimum as above [4],

$$\text{EoF}(\rho) = \inf_{\{p_i, \psi_i\}} \sum_i p_i \text{EoF}(\psi_i).$$

We first consider the symmetric case  $n_A = n_B = n/2$ :

### Theorem 1

Let  $\rho$  be a bipartite state with  $n_A = n_B = n/2$ , then

$$\text{EoF}(\rho) \leq \frac{n}{2} g \left( \frac{1}{2} (\mathcal{M}_{\text{TN}}(\rho) - 1) \right), \quad (2)$$

where  $g(x) = (x+1) \ln(x+1) - x \ln x$ .

*Proof.* We first consider pure states  $\rho = |\psi\rangle\langle\psi|$ . Since both sides of (2) are invariant under phase space translations, we may assume that  $\langle\psi|R_j|\psi\rangle = 0, \forall j$ . In that case,  $\mathcal{M}_{\text{TN}}(\psi) = (2\langle\psi|\hat{N}|\psi\rangle + n)/n = 2N/n + 1$ , where  $N = \langle\psi|\hat{N}|\psi\rangle$  is the expectation value of the total photon number operator  $\hat{N} = \sum_j a_j^\dagger a_j$  in the centered state  $|\psi\rangle$ . Similarly, defining  $\hat{N}_A = \sum_{j=1}^{n_A} a_j^\dagger a_j$  and  $\hat{N}_B = \sum_{j=n_A+1}^n a_j^\dagger a_j$ , one has  $N_A = \text{Tr} \hat{N}_A \rho_A$ ,  $N_B = \text{Tr} \hat{N}_B \rho_B$ , and  $N = N_A + N_B$ . Then

$$\text{EoF}(\psi) = -\text{Tr} \rho_A \ln \rho_A = -\text{Tr} \rho_B \ln \rho_B \leq \min \left\{ n_A g \left( \frac{N_A}{n_A} \right), n_B g \left( \frac{N_B}{n_B} \right) \right\},$$

where  $n_A g(N_A/n_A)$  is the von Neumann entropy of the product of  $n_A$  single-mode thermal states with mean photon number  $N_A/n_A$  per mode, which maximizes the von Neumann entropy at fixed mean photon number  $N_A$  [50]. Maximizing over all states  $|\psi\rangle$  with a fixed mean photon number  $N$  then yields

$$\text{EoF}(\psi) \leq \max_{0 \leq N_A \leq N} \min \left\{ n_A g \left( \frac{N_A}{n_A} \right), n_B g \left( \frac{N - N_A}{n_B} \right) \right\}.$$

Since  $g$  is an increasing function, the maximum is, for each  $N$ , reached at a unique value  $N_A^*$  that depends on  $N$  and is the solution of

$$n_A g \left( \frac{N_A^*}{n_A} \right) = n_B g \left( \frac{N - N_A^*}{n_B} \right). \quad (3)$$

Hence

$$\text{EoF}(\psi) \leq n_A g(N_A^*/n_A) := F(N). \quad (4)$$

When  $n_A = n_B$ , then  $N_A^* = N_B^* = N/2$ , so that

$$\text{EoF}(\psi) \leq \frac{n}{2} g\left(\frac{N}{n}\right) = \frac{n}{2} g\left(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\psi) - 1)\right). \quad (5)$$

This implies Eq. (2) for any pure state  $\rho = |\psi\rangle\langle\psi|$ . Now let  $\rho$  be an arbitrary state and consider any set of normalized  $|\psi_i\rangle$  and  $0 \leq p_i \leq 1$  such that  $\sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho$ . Then Eq. (5) and the concavity of  $g$  imply that

$$\sum_i p_i \text{EoF}(\psi_i) \leq \frac{n}{2} \sum_i p_i g\left(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\psi_i) - 1)\right) \leq \frac{n}{2} g\left(\frac{1}{2}\left(\sum_i p_i \mathcal{M}_{\text{TN}}(\psi_i) - 1\right)\right).$$

Since  $g(x)$  is monotonically increasing, taking the infimum over  $\{p_i, \psi_i\}$  on both sides implies Eq. (2).  $\square$

One readily sees that, among all states  $|\psi\rangle$  with a given  $N$ , the upper bound  $\text{EoF}(\psi) = \frac{n}{2} g\left(\frac{N}{n}\right)$  is reached for an  $n/2$ -fold tensor product of two-mode squeezed vacuum states with  $N/n$  photons per mode (note that it is not the unique optimal state, see Appendix A for details).

Since  $g(x)$  is an increasing function, the bound (2) straightforwardly implies that states with a large EoF are necessarily strongly nonclassical (see Appendix B):

### Corollary 1

$$\text{EoF}(\rho) \geq \frac{3}{2} \frac{n}{2} \Rightarrow \mathcal{M}_{\text{TN}}(\rho) \geq 1 + 2e^{\frac{2}{3}\text{EoF}(\rho)-2}. \quad (6)$$

In other words, if we view both entanglement and optical nonclassicality as resources, this inequality shows that the amount of optical nonclassicality of a state  $\rho$ , as measured by  $\mathcal{M}_{\text{TN}}(\rho)$ , grows exponentially fast with its EoF, measured in number of ebits. Conversely, the bound (2) shows that states with a low optical nonclassicality are necessarily weakly entangled.

When  $n_A \leq n_B$ , Theorem 1 can be generalized as follows (the proof is given in Appendix C).

### Theorem 1'

Let  $\rho$  be a bipartite state with  $n_A \leq n_B$ , then

$$\text{EoF}(\rho) \leq n_A g\left(\frac{1}{n_A} N_A^* \left(\frac{n}{2}(\mathcal{M}_{\text{TN}}(\rho) - 1)\right)\right), \quad (7)$$

where  $N_A^*(N)$  is the unique solution of Eq. (3).

An analytic expression for  $N_A^*(N)$  is not available, but for large  $N$ , one finds (see Appendix C)

$$N_A^*(N) = (1 - \delta) N, \quad \text{with } \delta = \frac{(e\nu)^{\mu-1}}{\mu(1 + (e\nu)^{\mu-1})}, \quad (8)$$

where  $\mu = n_A/n_B$  and  $\nu = N/n_A$ . Consequently, using  $g(x) \simeq \ln(x) + 1$  for large  $x$ , one finds approximately that

$$\text{EoF}(\rho) \leq n_A \ln\left(\frac{(1 - \delta)n}{n_A} \frac{n}{2}(\mathcal{M}_{\text{TN}}(\rho) - 1)\right) + n_A, \quad (9)$$

which is valid for large  $\mathcal{M}_{\text{TN}}(\rho)$  and shows a similar logarithmic upper bound on EoF in terms of  $\mathcal{M}_{\text{TN}}$  as above.

For Gaussian pure states, one can show (Appendix D)

$$\text{EoF}(\psi^{\text{G}}) \leq n_A g\left(\frac{n}{4n_A} (\mathcal{M}_{\text{TN}}(\psi^{\text{G}}) - 1)\right). \quad (10)$$

This is the tightest bound for Gaussian pure states only depending on  $\mathcal{M}_{\text{TN}}$ . It is saturated by  $n_A$  two-mode squeezed vacuum states with identical squeezing parameters (involving all  $n_A$  modes of  $A$  and the  $n_A$  first modes of  $B$ ), with the remaining  $n_B - n_A$  modes of  $B$  in the vacuum. When  $n_A = n_B$ , the right-hand sides of (10) and (5) coincide, as expected since the latter inequality is saturated by the above Gaussian state. In contrast, as shown in Fig. 1, when  $n_A < n_B$  (or  $\mu < 1$ ), the right-hand side of (10) is slightly smaller than the one of (4), and there are non-Gaussian pure states inside this gap (see Appendix D). This means that for a fixed total photon number  $N$ , and hence  $\mathcal{M}_{\text{TN}}$ , there exist non-Gaussian pure states with a higher EoF than any Gaussian pure state with the same value of  $\mathcal{M}_{\text{TN}}$ .

Note finally that one cannot expect a lower bound on the EoF in terms of  $\mathcal{M}_{\text{TN}}$  since a product state has vanishing entanglement while it can have an arbitrarily large  $\mathcal{M}_{\text{TN}}$ . The product of a strongly squeezed pure state with the vacuum is an example.

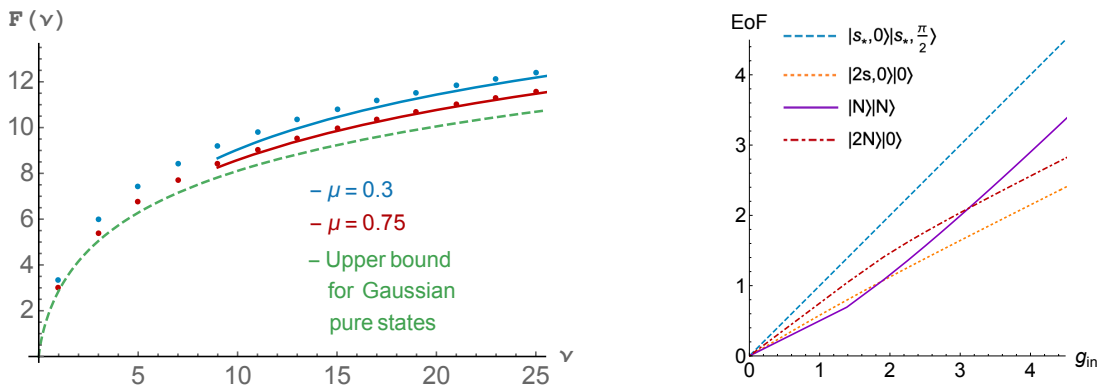


FIG. 1. Left: behaviour of the right hand side of (4) as a function of  $\nu = N/n_A$ , for different values of  $\mu = n_A/n_B$ , as indicated. The dots are obtained from numerical solutions of (3) ( $n_A = 3$ ). Full lines are given by (8) for  $N$  large. The green dashed line represents the Gaussian bound in (10).

Right:  $\text{EoF}(\psi_{\text{out}})$  in function of  $g_{\text{in}} = g(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) - 1))$  for various  $|\psi_{\text{in}}\rangle$ , as indicated.  $|2s, 0\rangle|0\rangle$  and  $|s_*, 0\rangle|s_*, \frac{\pi}{2}\rangle$  are squeezed states with the same  $\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) = \cosh(2s_*)$  ( $s = \frac{1}{4} \cosh^{-1}(2 \cosh(2s_*) - 1)$ ). The Fock states  $|N\rangle|N\rangle$  and  $|2N\rangle|0\rangle$  also have the same  $\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) = 2N + 1$ .

### III. ENTANGLEMENT GENERATION WITH A BEAM SPLITTER.

It is well known that a balanced beam splitter  $\hat{B} = \exp(\frac{\pi}{4}(a_1^\dagger a_2 - a_1 a_2^\dagger))$  applied to a separable in-state  $|\psi_{\text{in}}\rangle$  produces an out-state  $|\psi_{\text{out}}\rangle = \hat{B}|\psi_{\text{in}}\rangle$  that can be entangled provided the in-state is optically nonclassical [28, 34, 47, 48]. By applying Theorem 1, we are able to determine how efficiently a beam splitter can generate entanglement in this manner. Indeed, Eq. (5) implies an upper bound on the EoF of  $|\psi_{\text{out}}\rangle$  given the amount of optical nonclassicality available in  $|\psi_{\text{in}}\rangle$ . Let, for any value of the available nonclassicality  $\mathcal{M}_{\text{TN}*} > 0$ ,

$$S_* = \{|\psi_{\text{in}}\rangle = |\varphi_A, \varphi_B\rangle \mid \mathcal{M}_{\text{TN}}(\psi_{\text{in}}) \leq \mathcal{M}_{\text{TN}*}\}.$$

Since  $\hat{B}$  preserves the total noise ( $\mathcal{N}_{\text{tot}}(\psi_{\text{out}}) = \mathcal{N}_{\text{tot}}(\psi_{\text{in}})$ ), Eq. (5) implies

$$\text{EoF}(\psi_{\text{out}}) \leq g\left(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) - 1)\right) \leq g\left(\frac{1}{2}(\mathcal{M}_{\text{TN}*} - 1)\right),$$

since  $g$  is a monotonically increasing function. To see the bound is reached, let, for  $s \geq 0, \phi \in [0, 2\pi[$ ,  $S(s, \phi) = e^{\frac{s}{2}(e^{-i\phi} a^2 - e^{i\phi} a^{\dagger 2})}$  and define  $|\varphi_A\rangle = |s_*, 0\rangle := S(s_*, 0)|0\rangle$  and  $|\varphi_B\rangle = |s_*, \pi/2\rangle := S(s_*, \frac{\pi}{2})|0\rangle$ , with  $s_*$  chosen so that  $\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) = \cosh(2s_*) = \mathcal{M}_{\text{TN}*}$ . In this case,  $|\psi_{\text{out}}\rangle = \hat{B}|\psi_{\text{in}}\rangle = |\psi_{\text{TMS}}\rangle$ , where  $|\psi_{\text{TMS}}\rangle$  is the two-mode squeezed vacuum state with  $\mathcal{M}_{\text{TN}}(\psi_{\text{TMS}}) = \mathcal{M}_{\text{TN}*}$ , which we saw saturates (5). There is a readily identified family of states that saturate the bound (see Appendix A), but typically states in  $S_*$  do not. Several physically interesting examples are given in Fig. 1; see Appendix E for details on the computations. When  $|\psi_{\text{in}}\rangle = |N, 0\rangle$ ,  $\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) = N + 1$  and the EoF of the out-state satisfies  $\text{EoF}(\psi_{\text{out}})/g(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) - 1)) \simeq \frac{1}{2}$  for large  $N$ . Hence, only one half of the possible maximal amount of entanglement is produced in this manner for a given amount of optical nonclassicality in the in-state. When  $|\psi_{\text{in}}\rangle = |N, N\rangle$ , on the other hand,  $\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) = 2N + 1$ , and, for large  $N$ , the EoF satisfies  $\text{EoF}(\psi_{\text{out}})/g(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) - 1)) \simeq 1$ , hence almost the maximum possible amount of entanglement is produced. It is therefore less efficient to input a  $2N$  photon state on one mode and the vacuum on the other, rather than  $N$  photons on each. A similar phenomenon occurs with squeezed states at the input:  $|\psi_{\text{in}}\rangle = |2s, 0\rangle|0\rangle$  and  $|\psi_{\text{in}}\rangle = |s, 0\rangle|s, \frac{\pi}{2}\rangle$  have similar values of  $\mathcal{M}_{\text{TN}}$ , but the output EoF is, for large  $N$ , twice as large in the second case.

Let us point out that, in terms of resource theory, the beam splitter does not “convert” nonclassicality into entanglement. Indeed, the total noise is conserved and none of the optical nonclassicality resource is lost in the process. Nevertheless, the in-state must have a certain amount of optical nonclassicality for the entanglement production to be possible in this manner.

As mentioned above, Theorems 1 and 1' involve convex roofs, which are hard to exploit for mixed states. This is true even for Gaussian states, for which the EoF remains difficult to evaluate, despite recent progress [51, 52]. This problem can, however, be overcome by using alternative, computable measures of entanglement and optical nonclassicality adapted to Gaussian states.

#### IV. GAUSSIAN STATES: COMPARING LNEG AND QCS.

Consider a Gaussian state  $\rho_G$  with covariance matrix

$$V_{ij} = \langle \{R_i, R_j\} \rangle - 2\langle R_i \rangle \langle R_j \rangle,$$

where  $\langle \cdot \rangle := \text{Tr}(\cdot \rho_G)$ . We will evaluate its nonclassicality using two recently introduced and readily computable quantities: the total quantum Fisher information (TQFI) (see (11)) and the QCS (see (12)). For any state  $\rho$  and observable  $A$ , the quantum Fisher information of  $\rho$  for  $A$  is  $\mathcal{F}(\rho, A) = 4\partial_x^2 D_B^2(\rho, \exp(-ixA)\rho \exp(ixA))|_{x=0}$ , where  $D_B^2(\rho, \sigma) = 2(1 - F(\rho, \sigma))$  is the Bures distance and  $F(\rho, \sigma) = \text{Tr} \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$  the fidelity between  $\rho$  and  $\sigma$ . It is known that  $\mathcal{F}(\rho, A)$  is convex in  $\rho$  and coincides with  $4\Delta A^2$  on pure states [53, 54]. Defining the TQFI of an  $n$ -mode state  $\rho$  as

$$\mathcal{F}_{\text{tot}}(\rho) = \frac{1}{4n} \sum_{j=1}^{2n} \mathcal{F}(\rho, \mathbf{R}_j), \quad (11)$$

we see that it coincides with  $\mathcal{M}_{\text{TN}}(\psi)$  on pure states, and, since it is convex,  $\mathcal{F}_{\text{tot}}(\rho) \leq \mathcal{M}_{\text{TN}}(\rho)$ . In [40, 46] the (squared) QCS is defined as

$$\mathcal{C}^2(\rho) = \frac{1}{2n\mathcal{P}} \left( \sum_{j=1}^{2n} \text{Tr}[\rho, R_j][R_j, \rho] \right), \quad \mathcal{P} = \text{Tr} \rho^2. \quad (12)$$

The QCS measures the spread of the coherences of the quadratures of the state [46]. Both the QCS and TQFI are optical nonclassicality witnesses: if  $\mathcal{C}^2(\rho) > 1$  or  $\mathcal{F}_{\text{tot}}(\rho) > 1$ , then  $\rho$  is nonclassical [40]. For a general state  $\rho$ , QCS and TQFI can be very different [40], but they coincide on pure states, namely

$$\mathcal{C}^2(\psi) = \mathcal{F}_{\text{tot}}(\psi) = \mathcal{M}_{\text{TN}}(\psi) = \frac{1}{n} \mathcal{N}_{\text{tot}}(\psi). \quad (13)$$

Moreover, these two optical nonclassicality witnesses  $\mathcal{C}^2$  and  $\mathcal{F}_{\text{tot}}$  coincide and are readily computable on all Gaussian states  $\rho_G$ , including mixed ones (see Appendix F)

$$\mathcal{C}^2(\rho_G) = \mathcal{F}_{\text{tot}}(\rho_G) = \frac{1}{2n} \text{Tr} V^{-1}. \quad (14)$$

We will measure the entanglement of Gaussian states with the logarithmic negativity  $\text{LNeg}(\rho)$  [8]. Let  $\hat{T}_B$  be the partial time-reversal operator applied to the  $n_B$  modes only, and  $\tilde{\rho} = \hat{T}_B \rho \hat{T}_B$  be the partial transpose of an arbitrary state  $\rho$ . It is known that if  $\tilde{\rho}$  is not positive semidefinite, then  $\rho$  is entangled [3]. For a bi-partite system of  $n_A + n_B = n$  modes, the  $\text{LNeg}$  of  $\rho$  is then defined as  $\text{LNeg}(\rho) = \ln(\text{Tr} \sqrt{\tilde{\rho}^2})$ . Note that  $\text{LNeg}(\rho) > 0$  implies that  $\rho$  is entangled. For pure states, but not in general, one has  $\text{EoF}(|\psi\rangle\langle\psi|) \leq \text{LNeg}(|\psi\rangle\langle\psi|)$  [8]. The partial transpose  $\tilde{\rho}_G$  of a Gaussian state is again a Gaussian operator, with covariance matrix  $\tilde{V} = T_B V T_B$ , where  $T_B = \mathbb{I}_{2n_A} \bigoplus_{k=1}^{n_B} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . The  $\text{LNeg}$  of an arbitrary Gaussian state can be expressed in terms of the symplectic spectrum  $\tilde{\nu}_{-,1} \leq \dots \leq \tilde{\nu}_{-,n_-} < 1 \leq \tilde{\nu}_{+,1} \leq \dots \leq \tilde{\nu}_{+,n_+}$  ( $n_- + n_+ = n$ ) of  $\tilde{V}$ , as follows [8]: if  $n_- = 0$ , then  $\text{LNeg}(\rho_G) = 0$ , otherwise if  $n_- \geq 1$ ,

$$\text{LNeg}(\rho_G) = \sum_{i=1}^{n_-} \ln \frac{1}{\tilde{\nu}_{-,i}}.$$

This equation together with Eq. (14) allow us to derive a main bound for arbitrary Gaussian states:

#### Theorem 2

Let  $\rho_G$  be a bipartite Gaussian state, then

$$\text{LNeg}(\rho_G) \leq n_- \left( \ln \mathcal{C}^2(\rho_G) + \ln \frac{n}{n_-} \right). \quad (15)$$

*Proof.* We note that  $\tilde{V}^{-1} = T_B V^{-1} T_B$  so that  $\mathcal{C}^2(\rho_G) = \frac{1}{2n} \text{Tr} \tilde{V}^{-1}$ . We define  $\text{Str} A$  as the symplectic trace of  $A$ , which is twice the sum of its symplectic eigenvalues; we then have  $\text{Tr} A \geq \text{Str} A$  [55]. Therefore,

$$\mathcal{C}^2(\rho_G) \geq \frac{1}{2n} \text{Str} \tilde{V}^{-1} = \frac{1}{n} \left( \sum_{i=1}^{n_-} \frac{1}{\tilde{\nu}_{-,i}} + \sum_{i=1}^{n_+} \frac{1}{\tilde{\nu}_{+,i}} \right) \geq \frac{1}{n} \sum_{i=1}^{n_-} \frac{1}{\tilde{\nu}_{-,i}}.$$

Using the concavity of the logarithm, it implies (15).  $\square$

In the special case  $n_A = n_B = 1$ , a better bound can be obtained when  $\text{LNeg}(\rho_G) > 0$  using the knowledge of  $\det V = \det \tilde{V} = \tilde{\nu}_-^2 \tilde{\nu}_+^2$ :

$$\mathcal{C}^2(\rho_G) \geq \frac{1}{2} \left( \frac{1}{\tilde{\nu}_-} + \frac{1}{\tilde{\nu}_+} \right) = \frac{1}{2} \left( \frac{1}{\tilde{\nu}_-} + \frac{\tilde{\nu}_-}{\sqrt{\det \tilde{V}}} \right)$$

with  $\tilde{\nu}_- = e^{-\text{LNeg}(\rho_G)}$ . This inequality is saturated when the trace and symplectic trace of  $V$  coincide, that is, the Gaussian extractable work from  $\rho_G$  vanishes (see [56]).

Theorem 2 shows a large entanglement implies a large optical nonclassicality, but, in contrast with Theorem 1, both sides of the inequality are readily computable. It is instructive to rework (15) and eliminate  $n_-$  from it (the proof is straightforward and given in Appendix G):

### Corollary 2

Let  $\rho_G$  be a Gaussian state. Then

$$\text{LNeg}(\rho_G) > \frac{n}{e} \Rightarrow \ln \mathcal{C}^2(\rho_G) \geq \frac{1}{n} \text{LNeg}(\rho_G) - \frac{1}{e}, \quad (16)$$

$$\mathcal{C}^2(\rho_G) < e^{-\frac{n}{e}} \Rightarrow \text{LNeg}(\rho_G) = 0. \quad (17)$$

Estimate (16) provides a precise quantitative meaning to the statement that a strongly entangled Gaussian state has a large  $\mathcal{C}^2(\rho_G)$  and is therefore far from optical classicality. One observes here, as in (6), an exponential growth of the optical nonclassicality with the entanglement of  $\rho_G$ . Estimate (17) shows that a Gaussian state with small  $\mathcal{C}^2(\rho_G)$  (far from the nonclassicality threshold 1) cannot be entangled.

## V. CONCLUSIONS.

We have established inequalities relating, for arbitrary states of a multi-mode optical field, several standard measures of entanglement and of optical nonclassicality. In a nutshell, the optical nonclassicality of a strongly entangled state is necessarily large and, in fact, grows exponentially with its entanglement. As an application, we show that entanglement is more efficiently produced in a beam splitter when the nonclassicality is distributed equally among the in-state modes. Furthermore, generating a lot of entanglement requires a large optical nonclassicality, a resource that is hard to generate and preserve due to environmental decoherence [46]. Our results can be interpreted to say that, insofar as the states of a multi-mode bosonic quantum field are concerned, the fragility of their entanglement is a consequence of their large nonclassicality.

## ACKNOWLEDGMENTS

This work was supported in part by the Labex CEMPI (Agence Nationale de Recherche, Grant ANR-11-LABX-0007-01) and by the Nord-Pas de Calais Regional Council and the European Regional Development Fund through the Contrat de Projets État-Région (CPER). The work was also supported by the Fonds de la Recherche Scientifique – FNRS under Projects No. T.0199.13 and No. T.0224.18. A. H. acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC).

### Appendix A: States saturating the bound (5)

We identify here all pure states  $|\psi\rangle$  of  $n$  modes that saturate the bound (5), with  $n_A = n_B = \frac{n}{2}$ ;  $n$  is even. Let us write

$$|\psi\rangle = \sum_{k,l} c_{k,l} |k,l\rangle, \quad \sum_{k,l} |c_{k,l}|^2 = 1, \quad (A1)$$

with  $k = (k_1, \dots, k_{n_A}), l = (l_1, \dots, l_{n_A}) \in \mathbb{N}_*^{n_A}$ . The reduced states on the first (or last)  $n_A$  modes are

$$\rho_A = \sum_{k,k'} (CC^\dagger)_{k,k'} |k\rangle\langle k'|, \quad \rho_B = \sum_{l,l'} (C^\dagger C)_{l,l'} |l\rangle\langle l'|,$$

where  $C$  is the operator defined as  $C = \sum_{k,k'} c_{k,k'} |k\rangle\langle k'|$  and we use the notation  $(\cdot)_{k,k'} = \langle k| \cdot |k'\rangle$ . The right hand side of (5) is the von Neumann entropy of the unique thermal state  $\rho_\beta$  of  $n_A$  modes, determined by

$$\rho_\beta = Z_\beta^{-1} \sum_k e^{-\beta|k|_1} |k\rangle\langle k|,$$

with  $|k|_1 = \sum_{i=1}^{n_A} k_i$ , where  $\beta$  is chosen such that

$$\text{Tr } \hat{N}_A \rho_\beta = \frac{n_A}{e^\beta - 1} = \frac{N}{2}.$$

The bound is therefore saturated iff  $\rho_A = \rho_B = \rho_\beta$ , and hence iff  $C^\dagger C = C C^\dagger = D$ , with  $D$  being a diagonal operator with entries  $d_k = Z_\beta^{-1} e^{-\beta|k|_1}$ . Let  $U = C D^{-1/2}$ , then  $U$  is unitary and, with  $C = U D^{1/2}$ , one finds

$$D = C C^\dagger = U D U^\dagger \Leftrightarrow D U = U D.$$

We conclude that  $|\psi\rangle$  in (A1) saturates the bound iff  $C = U D^{1/2}$ , with  $U$  being a unitary operator commuting with  $D$ . One obvious choice is to take  $U = \mathbb{I}$ , in which case  $|\psi\rangle$  is an  $n/2$ -fold tensor product of two-mode squeezed states ( $\beta = \ln \coth^2(r)$ )

$$|\psi_{\text{TMS}}\rangle = \cosh(r)^{-1} \sum_{k=0}^{+\infty} (\tanh r)^k |k, k\rangle$$

for which  $\mathcal{M}_{\text{TN}}(\psi_{\text{TMS}}) = \cosh(2r)$  and  $\text{EoF}(\psi_{\text{TMS}}) = g(\sinh^2(r))$ . Thus,  $|\psi\rangle$  is a Gaussian state with  $\mathcal{M}_{\text{TN}}(\psi) = \cosh(2r)$  and  $\text{EoF}(\psi) = \frac{n}{2} g(\sinh^2(r))$ , saturating the bound (5).

Note that this is not the unique state saturating the bound since such a state is determined by  $C = U D^{1/2} = D^{1/2} U$ , with  $U$  unitary. Therefore, all saturating states can be obtained from the above choice by applying local unitaries  $U_A$  and  $U_B$  that preserve the photon numbers  $\hat{N}_A$  and  $\hat{N}_B$ , setting  $C = U_A D^{1/2} U_B$ . For example, when  $n = 2$ , they are all states of the form

$$|\psi\rangle = \cosh(r)^{-1} \sum_{k=0}^{+\infty} (\tanh r)^k \exp(i\phi_k) |k, k\rangle$$

with arbitrary phases  $\phi_k$ . If  $\phi_k = k\phi$ , these states are the general two-mode squeezed states obtained when we inject two orthogonal squeezed states in a balanced beam splitter, the angle of the squeezing of the first input state being  $\phi$  (the second input state is squeezed along  $\phi + \pi/2$ ). For general  $n$ , one has

$$|\psi\rangle = Z_\beta^{-1/2} \sum_k \exp(-\beta|k|_1/2) |\varphi_{A,k}, \varphi_{B,k}\rangle$$

where  $|\varphi_{A,k}\rangle = U_A |k\rangle$ ,  $|\varphi_{B,k}\rangle = U_B |k\rangle$ ; note that

$$\hat{N}_A |\varphi_{A,k}\rangle = |k|_1 |\varphi_{A,k}\rangle, \quad \hat{N}_B |\varphi_{B,k}\rangle = |k|_1 |\varphi_{B,k}\rangle.$$

In general, such states are not Gaussian.

## Appendix B: Proof of Corollary 1

Suppose  $\text{EoF}(\rho) \geq \frac{3}{2} \frac{n}{2}$ . Using Eq. (2), it implies that  $g(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\rho) - 1)) \geq \frac{3}{2}$ . Noting that  $g(x) \leq x + \frac{1}{2}$  for all  $x$ , one can conclude that  $\frac{1}{2}(\mathcal{M}_{\text{TN}}(\rho) - 1) \geq 1$ . Now, one also has  $g(x) \leq \ln x + 1 + \frac{1}{x}$  for all  $x > 0$ , and hence  $g(x) \leq \ln x + 2$  for all  $x \geq 1$ . Hence  $g(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\rho) - 1)) \leq \ln(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\rho) - 1)) + 2$ . Using again Eq. (2) implies that  $\text{EoF}(\rho) \leq \frac{n}{2} (\ln(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\rho) - 1)) + 2)$ , from which one concludes  $\mathcal{M}_{\text{TN}}(\rho) \geq 1 + 2e^{\frac{2}{n} \text{EoF}(\rho) - 2}$ , which is Eq. (6).  $\square$



**Appendix C: Proof of Theorem 1' and of (8)**

We may of course consider  $n_A \leq n_B$  with no loss of generality. For the proof of Theorem 1' we first need to show that the function

$$F(N) := n_A g\left(\frac{N_A^*}{n_A}\right)$$

is concave, where  $N_A^*$  is a function of  $N$ , implicitly defined as the solution of (3):

$$n_A g\left(\frac{N_A^*}{n_A}\right) = n_B g\left(\frac{N_B^*}{n_B}\right),$$

where we defined  $N_B^* = N - N_A^*$ . Taking the derivative of both of these equations with respect to  $N$ , one finds

$$g'\left(\frac{N_A^*}{n_A}\right) \frac{dN_A^*}{dN} = g'\left(\frac{N_B^*}{n_B}\right) \frac{dN_B^*}{dN}, \quad \frac{dN_A^*}{dN} + \frac{dN_B^*}{dN} = 1.$$

Since  $g' > 0$ , it follows from these two equations that both  $\frac{dN_A^*}{dN}$  and  $\frac{dN_B^*}{dN}$  are positive, so that both  $N_A^*$  and  $N_B^*$  are increasing functions of  $N$ . One readily finds that

$$\frac{dN_A^*}{dN} = \frac{g'\left(\frac{N_B^*}{n_B}\right)}{g'\left(\frac{N_A^*}{n_A}\right) + g'\left(\frac{N_B^*}{n_B}\right)}, \quad \frac{dN_B^*}{dN} = \frac{g'\left(\frac{N_A^*}{n_A}\right)}{g'\left(\frac{N_A^*}{n_A}\right) + g'\left(\frac{N_B^*}{n_B}\right)}$$

and consequently that

$$F'(N) = \frac{g'\left(\frac{N_A^*}{n_A}\right) g'\left(\frac{N_B^*}{n_B}\right)}{g'\left(\frac{N_A^*}{n_A}\right) + g'\left(\frac{N_B^*}{n_B}\right)} = \frac{1}{g'\left(\frac{N_A^*}{n_A}\right)^{-1} + g'\left(\frac{N_B^*}{n_B}\right)^{-1}}.$$

Now, since  $g$  is concave, it follows that  $g'$  is a decreasing function of its argument. Since  $N_A^*$  is an increasing function of  $N$ , it then follows that  $g'\left(\frac{N_A^*}{n_A}\right)$  is a decreasing function of  $N$ , and similarly for  $g'\left(\frac{N_B^*}{n_B}\right)$ . Hence,  $F'$  is a decreasing function of  $N$ , implying that  $F$  is concave.

We now use this fact to conclude the proof of Theorem 1'. We initially follow the same lines as in the proof of Theorem 1. For a centered pure state  $\psi$ , Eq. (4) reads

$$\text{EoF}(\psi) \leq F(N) = F\left(\frac{n}{2}(\mathcal{M}_{\text{TN}}(\psi) - 1)\right).$$

Since both EoF and  $\mathcal{M}_{\text{TN}}$  are invariant under phase space translations, one then has, for all  $\psi$ ,

$$\text{EoF}(\psi) \leq F\left(\frac{n}{2}(\mathcal{M}_{\text{TN}}(\psi) - 1)\right).$$

The concavity of the function  $F$  further implies that

$$\sum_i p_i \text{EoF}(\psi_i) \leq \sum_i p_i F\left(\frac{n}{2}(\mathcal{M}_{\text{TN}}(\psi_i) - 1)\right) \leq F\left(\sum_i p_i \frac{n}{2}(\mathcal{M}_{\text{TN}}(\psi_i) - 1)\right).$$

Taking the infimum on both sides and using the fact that  $F$  is a monotonically increasing function of its argument, one finds

$$\text{EoF}(\rho) \leq F\left(\frac{n}{2}(\mathcal{M}_{\text{TN}}(\rho) - 1)\right).$$

Recalling the definition of  $F$  in (4), one sees this is Eq. (7).

Finally, it remains to prove the asymptotic expression for  $N_A^*(N)$ , namely Eq. (8). We rewrite Eq. (3) as

$$\mu g(\nu_*) = g(\mu(\nu - \nu_*)), \tag{C1}$$

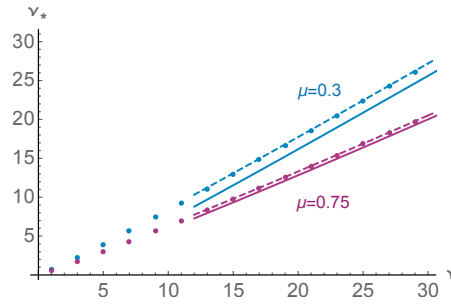


FIG. 2. Numerically computed solution  $\nu_*$  of (C1) (dots) and asymptotic expression (8) (plain line) and (C2) (dashed line) of  $\nu_* = N_A^*/n_A$ , for different values of  $\mu$ , as indicated.

where  $\nu_* = N_A^*/n_A$ ,  $\mu = n_A/n_B$ , and  $\nu = N/n_A$ . Since we are mostly interested in states with a large optical nonclassicality, we consider the case where  $\nu \gg 1$ . Writing  $\nu_*/\nu = 1 - \delta$  and using that for large  $x$ ,  $g(x) \simeq \ln(ex) + \frac{1}{2x}$ , Eq. (C1) becomes

$$\mu \ln(e\nu(1 - \delta)) + \frac{\mu}{2\nu(1 - \delta)} = \ln(e\mu\nu\delta) + \frac{1}{2\mu\nu\delta}.$$

Suppose now that  $\delta \ll 1$ ,  $\mu\nu\delta \gg 1$ , and  $\nu(1 - \delta) \gg 1$ . Then, we find

$$N_A^* = (1 - \delta)N, \quad \text{with } \delta = \frac{1}{\mu} \left[ 1 - \frac{e^{1-\mu}}{2\nu\mu} \right] \frac{1}{(e\nu)^{1-\mu} + 1}. \quad (\text{C2})$$

Keeping only the dominant term, one finds (8). In Fig. 2 the numerically computed solution to (C1) is compared to the asymptotic expressions (C2). The agreement is seen to be excellent, even for relatively small values of  $\nu$ . The asymptotic expression (8) is also shown for comparison.

#### Appendix D: An improvement on Theorem 1' for Gaussian pure states

Let us consider a pure Gaussian state  $|\psi^G\rangle$  of an  $n = n_A + n_B$  mode system. We will assume without loss of generality that  $n_A \leq n_B$  and that the state is centered. Applying local Gaussian unitaries  $U_A^G, U_B^G$ , such a state can always be transformed into a state  $|\psi_\nu\rangle$ , in which Alice and Bob share  $n_A$  two-mode squeezed vacuum states, while Bob's remaining  $n_B - n_A$  modes are in the vacuum state [57]. Here  $\nu = (\nu_1, \nu_2, \dots, \nu_{n_A})$  and the state  $|\psi_\nu\rangle$  is characterized by its covariance matrix

$$V_\nu = \begin{pmatrix} V_{A,\nu} & C \\ C^T & V_{B,\nu} \end{pmatrix}, \quad V_{A,\nu} = \begin{pmatrix} \nu_1 \mathbb{1}_2 & & 0 \\ & \ddots & \\ 0 & & \nu_{n_A} \mathbb{1}_2 \end{pmatrix}_{2n_A \times 2n_A},$$

$$V_{B,\nu} = \begin{pmatrix} \nu_1 \mathbb{1}_2 & & 0 & & \\ & \ddots & & & \\ 0 & & \nu_{n_A} \mathbb{1}_2 & & \\ & & & \mathbb{1}_2 & 0 \\ 0 & & & & \ddots & \\ & & & & & \mathbb{1}_2 \end{pmatrix}_{2n_B \times 2n_B}, \quad C = \begin{pmatrix} \mu_1 \sigma_z & & 0 & \\ & \ddots & & \\ 0 & & \mu_{n_A} \sigma_z & \\ & & & 0 \end{pmatrix}_{2n_B \times 2n_A}$$

where  $\nu_i = \cosh 2r_i$ ,  $\mu_i = \sinh 2r_i$  and  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Since the local unitaries do not change the EoF, we find

$$\text{EoF}(\psi^G) = \text{EoF}(\psi_\nu) = \sum_{i=1}^{n_A} g\left(\frac{1}{2}(\nu_i - 1)\right).$$

Since  $g$  is concave, one has

$$\text{EoF}(\psi^{\text{G}}) \leq n_A g \left( \frac{1}{2n_A} \sum_{i=1}^{n_A} (\nu_i - 1) \right).$$

On the other hand,

$$\mathcal{N}_{\text{tot}}(\psi^{\text{G}}) = \frac{1}{2} \text{Tr} V_\psi = \frac{1}{2} (\text{Tr} V_{A,\psi} + \text{Tr} V_{B,\psi}) \geq \frac{1}{2} (\text{Str} V_{A,\psi} + \text{Str} V_{B,\psi}).$$

Since the local symplectic transformations do not change the symplectic spectrum of the reduced states  $\rho_A, \rho_B$ , we also have

$$\text{Str} V_{A,\psi} + \text{Str} V_{B,\psi} = \text{Str} V_{A,\nu} + \text{Str} V_{B,\nu} = 4 \sum_{i=1}^{n_A} \nu_i + 2(n_B - n_A).$$

Hence

$$\mathcal{N}_{\text{tot}}(\psi^{\text{G}}) \geq 2 \sum_{i=1}^{n_A} \nu_i + (n_B - n_A) = 2 \sum_{i=1}^{n_A} (\nu_i - 1) + n_B + n_A.$$

It follows that

$$\text{EoF}(\psi^{\text{G}}) \leq n_A g \left( \frac{1}{4n_A} (\mathcal{N}_{\text{tot}}(\psi^{\text{G}}) - n_B - n_A) \right) = n_A g \left( \frac{n}{4n_A} (\mathcal{M}_{\text{TN}}(\psi^{\text{G}}) - 1) \right).$$

Note that equality is obtained if  $|\psi^{\text{G}}\rangle = |\psi^\nu\rangle$ , with  $\nu_1 = \nu_2 = \dots = \nu_{n_A}$ .

We now show that there exist non Gaussian states that violate the Gaussian bound (10). For that purpose, we consider the case  $n_A = 1, n_B = 2$ . The Gaussian states that saturate the bound are then explicitly given by ( $0 < q < 1$ )

$$|\psi_q\rangle = (1 - q)^{1/2} \sum_n q^{n/2} |n; n, 0\rangle.$$

Here we wrote  $|n; m_1, m_2\rangle$  for the Fock state with  $n$  photons in the single  $A$  mode and  $m_1$ , respectively  $m_2$  photons in the two modes of  $B$ . For this state, we have explicitly

$$\langle \psi_q | \hat{N} | \psi_q \rangle = \frac{2q}{1 - q}, \quad \mathcal{M}_{\text{TN}}(\psi_q) = \frac{2}{3} \langle \psi_q | \hat{N} | \psi_q \rangle + 1.$$

We will now exhibit a non Gaussian local transformation  $U_B$  that, when applied to  $|\psi_q\rangle$ , yields a state  $|\psi'\rangle = U_B |\psi_q\rangle$  that has the same EoF as  $|\psi_q\rangle$  (since  $U_B$  is local) but that lowers its  $\mathcal{M}_{\text{TN}}$ . In other words, we will show that

$$\mathcal{M}_{\text{TN}}(\psi') < \mathcal{M}_{\text{TN}}(\psi_q). \tag{D1}$$

This implies that

$$\text{EoF}(\psi') = \text{EoF}(\psi_q) = g(\langle \psi_q | \hat{N} | \psi_q \rangle / 2) > g(\langle \psi' | \hat{N} | \psi' \rangle / 2),$$

and therefore shows  $|\psi'\rangle$  does not satisfy the Gaussian bound (10). Of course, it does satisfy the bound (2). The local transformation  $U_B$  is constructed as follows. Let  $k > 1$  be fixed. Then

$$\begin{aligned} U_B |m_1, m_2\rangle &= |m_1, m_2\rangle, \text{ if } m_1 m_2 \neq 0, \\ U_B |0, 0\rangle &= |0, 0\rangle, \\ U_B |m_1, 0\rangle &= |m_1, 0\rangle, \text{ if } m_1 \neq k, \\ U_B |k, 0\rangle &= |0, 1\rangle, \\ U_B |0, 1\rangle &= |k, 0\rangle \\ U_B |0, m_2\rangle &= |0, m_2\rangle, \text{ if } m_2 > 1. \end{aligned}$$

With  $|\psi'\rangle = U_B |\psi_q\rangle$  one then easily checks that, for  $i = 1, 2, 3$ ,

$$\langle \psi' | a_i | \psi' \rangle = 0 = \langle \psi' | a_i^\dagger | \psi' \rangle,$$

so that  $\langle \psi' | X_i | \psi' \rangle = 0 = \langle \psi' | P_i | \psi' \rangle$ . It follows that

$$\mathcal{M}_{\text{TN}}(\psi') = \frac{2}{3} \langle \psi' | \hat{N} | \psi' \rangle + 1.$$

It will therefore suffice to prove  $\langle \psi' | \hat{N} | \psi' \rangle < \langle \psi_q | \hat{N} | \psi_q \rangle$ . One readily finds

$$\langle \psi' | \hat{N} | \psi' \rangle = \langle \psi | \hat{N} | \psi \rangle + (1 - q)q^k(1 - k).$$

so that (D1) follows since  $k > 1$ .

### Appendix E: Beam splitters

For many states commonly considered the entanglement produced by the beam splitter is considerably lower than the maximal value possible. For example, when  $|\psi_{\text{in}}\rangle = |N, 0\rangle$ , one obtains

$$|\psi_{\text{out}}\rangle = \hat{B}|N, 0\rangle = \sum_{m=0}^N \sqrt{\frac{n! 2^{-N}}{m!(N-m)!}} |m, N-m\rangle.$$

Then  $\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) = N + 1$  and  $\text{EoF}(\psi_{\text{out}})$  is given by the entropy of the binomial distribution  $P(k) = \frac{N!}{k!(N-k)!} 2^{-N}$ , which for large  $N$  is approximately given by  $\text{EoF}(\psi_{\text{out}}) \simeq \frac{1}{2} \ln(2\pi eN)$ . Hence in this case  $\text{EoF}(\psi_{\text{out}})/g(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) - 1)) \simeq \frac{1}{2}$ , as can be observed in Fig. 1. When  $|\psi_{\text{in}}\rangle = |N, N\rangle$ , one has  $\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) = 2N + 1$  and  $|\psi_{\text{out}}\rangle$  is [47]

$$|\psi_{\text{out}}\rangle = \hat{B}|N, N\rangle = \sum_{m=0}^N c_m |2m, 2N - 2m\rangle,$$

with

$$c_m = \frac{1}{2^N} \frac{\sqrt{(2N - 2m)!(2m)!}}{m!(N - m)!}.$$

For large  $N$ , choosing  $m = N/2 + \delta$ , one can apply the Stirling approximation  $N! \rightarrow \sqrt{2\pi N}(N/e)^N$  to find that the coefficients  $|c_m|^2$  converge to  $\frac{f(m/N)}{N}$  with  $f(x) = \frac{1}{\pi} \frac{1}{\sqrt{\frac{m}{N}(1 - \frac{m}{N})}}$ . This result coincides with the one obtained in [58].

The Von Neumann entropy is thus given by

$$\begin{aligned} -\text{Tr} \rho_1 \log \rho_1 &= - \sum_{m=0}^N |c_m|^2 \log |c_m|^2 \\ &= - \frac{1}{N} \sum_{m=0}^N N |c_m|^2 \log |c_m|^2 \\ &= - \frac{1}{N} \sum_{m=0}^N f(m/N) \log f(m/N) + \log N \\ &\approx - \int_0^1 f(x) \log f(x) dx + \log N \\ &= \log \frac{\pi}{4} + \log N. \end{aligned}$$

Hence, in this case  $\text{EoF}(\psi_{\text{out}})/g(\frac{1}{2}(\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) - 1)) \simeq 1$  which means that, asymptotically, the maximal possible amount of entanglement can be produced in this manner. It is therefore more efficient to input a state with  $N$  photon on each mode than  $2N$  photons on one mode and the vacuum on the other, as in both cases  $\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) = 2N + 1$ , but the output EoF is, for large  $N$ , twice as large in the first case.

If  $|\psi_{\text{in}}\rangle = |2s, 0\rangle|0\rangle$ , one finds the out-state is a TMS of parameter  $s$  on which we add some squeezing  $s'$  on the first mode and  $-s'$  on the second. Since those squeezing are local, they do not modify the value of the EoF, which is thus the one of the TMS. Hence,  $\text{EoF}(\psi_{\text{out}}) = g(\sinh^2(s))$ . Note, nevertheless, that while a TMS of parameter  $s$  has a

total noise of  $\cosh(2s)$ , the total noise of the in-state is given by  $\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) = \frac{\cosh(4s)+1}{2}$ . Only about one half of the possible maximal amount of entanglement is produced in this manner. On the other hand, if  $|\psi_{\text{in}}\rangle = |s_*, 0\rangle_{|s_*, \frac{\pi}{2}}\rangle$ , with  $s = \frac{1}{4} \cosh^{-1}(2 \cosh(2s_*) - 1) \simeq \frac{s_*}{2}$  the total noise of the in-state is also given by  $\mathcal{M}_{\text{TN}}(\psi_{\text{in}}) = \frac{\cosh(4s)+1}{2}$  but yields, after the beam splitter, the maximum entanglement possible, namely  $g(\sinh^2(s_*)) \geq g(\sinh^2(s))$ . So in this instance too it is more efficient, in terms of entanglement creation, to insert a symmetric input in the beam splitter .

### Appendix F: Proof of (14)

To prove (14), we first note that, since both  $\mathcal{C}(\rho_G)$  and  $\mathcal{F}_{\text{tot}}(\rho_G)$  are invariant under phase space translations, we will assume that  $\langle R_i \rangle = 0$  for all  $i$ . The characteristic function of  $\rho_G$  is

$$\chi_G(\xi) = \text{Tr} \rho_G D(\xi) = \exp\left\{-\frac{1}{2} \xi^T \Omega V \Omega^T \xi\right\}, \quad \text{where} \quad \Omega = \bigoplus_{k=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and  $\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2})$ . It was shown in [40, 59] that the right hand side of (12) can be written in terms of the characteristic function  $\chi(\xi)$  of the state as follows:

$$\mathcal{C}^2(\rho) = \frac{\|\xi\| \|\chi\|_2^2}{n \|\chi\|_2^2}. \quad (\text{F1})$$

Here  $\|\cdot\|_2$  designates the  $L^2$ -norm, meaning for example  $\|\chi\|_2^2 := \int |\chi|^2(\xi) d^2\xi$ . From (F1), one finds with a direct computation

$$\mathcal{C}^2(\rho_G) = \frac{1}{n} \int (|\xi_1|^2 + \dots + |\xi_n|^2) f(\xi) d^2\xi = \frac{1}{n} \text{Tr} \Sigma = \frac{1}{2n} \text{Tr} V^{-1}$$

where  $f(\xi)$  is a Gaussian function with 0 mean value and covariance matrix  $\Sigma = \frac{1}{2} \Omega V^{-1} \Omega^T$ . It was on the other hand proven in [38] that for Gaussian states  $\mathcal{F}_{\text{tot}}(\rho_G) = \frac{1}{2n} \text{Tr} V^{-1}$ , so that (14) follows.

### Appendix G: Proof of Corollary 2.

First note that  $n_- \ln(n/n_-) \leq n/e$ , so that (15) implies

$$n_- \ln \mathcal{C}^2(\rho_G) \geq \text{LNeg}(\rho_G) - \frac{n}{e}.$$

Hence, if  $\text{LNeg}(\rho_G) - \frac{n}{e} > 0$ , then  $n_- \geq 1$  and  $\mathcal{C}^2(\rho_G) > 1$ . Equation (16) then follows.

To prove (17), note that (15) implies that

$$n_- \ln \frac{1}{\mathcal{C}^2(\rho_G)} \leq -\text{LNeg}(\rho_G) - n \frac{n_-}{n} \ln \frac{n_-}{n} \leq \frac{n}{e}.$$

Hence  $\mathcal{C}^2(\rho_G) \geq \exp(-\frac{n}{n_-e})$ . Therefore, if  $\mathcal{C}^2(\rho_G) \leq e^{-\frac{n}{e}}$  then  $n_- < 1$  which implies (17).

- 
- [1] A. Streltsov, G. Adesso, and M. B. Plenio, *Rev. Mod. Phys.* **89**, 041003 (2017).
  - [2] M. Horodecki, P. Horodecki, and R. Horodecki, *Physics Letters A* **223**, 1 (1996).
  - [3] A. Peres, *Phys. Rev. Lett.* **77**, 1413 (1996).
  - [4] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).
  - [5] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **84**, 2722 (2000).
  - [6] R. Simon, *Phys. Rev. Lett.* **84**, 2726 (2000).
  - [7] R. F. Werner and M. M. Wolf, *Phys. Rev. Lett.* **86**, 3658 (2001).
  - [8] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002).
  - [9] A. Serafini, G. Adesso, and F. Illuminati, *Phys. Rev. A* **71**, 032349 (2005).
  - [10] E. Shchukin and W. Vogel, *Phys. Rev. Lett.* **95**, 230502 (2005).
  - [11] O. Rudolph, *Quant. Inf. Process.* **4**, 219 (2005).

- [12] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
- [13] S. P. Walborn, B. G. Taketani, A. Salles, F. Toscano, and R. L. de Matos Filho, *Phys. Rev. Lett.* **103**, 160505 (2009).
- [14] C. Zhang, S. Yu, Q. Chen, and C. H. Oh, *Phys. Rev. Lett.* **111**, 190501 (2013).
- [15] U. M. Titulaer and R. J. Glauber, *Phys. Rev.* **140**, B676 (1965).
- [16] M. Hillery, *Phys. Lett.* **111 A**, 409 (1985).
- [17] A. Bach and U. Lüxmann-Ellinghaus, *Commun.Math. Phys.* **107**, 553 (1986).
- [18] M. Hillery, *Phys. Rev. A* **35**, 725 (1987).
- [19] M. Hillery, *Phys. Rev. A* **39**, 2994 (1989).
- [20] C. T. Lee, *Phys. Rev. A* **44**, R2775 (1991).
- [21] G. S. Agarwal and K. Tara, *Phys. Rev. A* **46**, 485 (1992).
- [22] C. T. Lee, *Phys. Rev. A* **52**, 3374 (1995).
- [23] N. Lütkenhaus and S. M. Barnett, *Phys. Rev. A* **51**, 3340 (1995).
- [24] V. Dodonov, O. Man'ko, A. O. Man'ko, and A. Wünsche, *J. Mod. Opt.* **47**, 633 (2000).
- [25] P. Marian, T. A. Marian, and H. Scutaru, *Phys. Rev. Lett.* **88**, 153601 (2002).
- [26] T. Richter and W. Vogel, *Phys. Rev. Lett.* **89**, 283601 (2002).
- [27] A. Kenfack and K. Zyczkowski, *J. Opt. B: Quantum Semiclass. Opt.* **6**, 396 (2004).
- [28] J. K. Asbóth, J. Calsamiglia, and H. Ritsch, *Phys. Rev. Lett.* **94**, 173602 (2005).
- [29] A. A. Semenov, D. Vasylyev, and B. I. Lev, *J. Phys. B: At. Mol. Opt. Phys.* **39**, 905 (2006).
- [30] A. Zavatta, V. Parigi, and M. Bellini, *Phys. Rev. A* **75**, 052106 (2007).
- [31] W. Vogel and J. Sperling, *Phys. Rev. A* **89**, 052302 (2014).
- [32] S. Ryl, J. Sperling, E. Agudelo, M. Mraz, S. Köhnke, B. Hage, and W. Vogel, *Phys. Rev. A* **92**, 011801 (2015).
- [33] J. Sperling and W. Vogel, *Phys. Scr.* **90**, 074024 (2015).
- [34] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio, *Phys. Rev. Lett.* **116**, 080402 (2016).
- [35] M. Alexanian, *Journal of Modern Optics* (2017), 10.1080/09500340.2017.1374481.
- [36] R. Nair, *Phys. Rev. A* **95**, 063835 (2017).
- [37] S. Ryl, J. Sperling, and W. Vogel, *Phys. Rev. A* **95**, 053825 (2017).
- [38] B. Yadin, F. C. Binder, J. Thompson, V. Narasimhachar, M. Gu, and M. S. Kim, *Phys. Rev. X* **8**, 041038 (2018).
- [39] H. Kwon, K. C. Tan, T. Volkoff, and H. Jeong, *Phys. Rev. Lett.* **122**, 040503 (2019).
- [40] S. De Bièvre, D. B. Horoshko, G. Patera, and M. I. Kolobov, *Phys. Rev. Lett.* **122**, 080402 (2019).
- [41] S. Luo and Y. Zhang, *Phys. Rev. A* **100**, 032116 (2019).
- [42] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, *Phys. Rev. A* **92**, 022106 (2015).
- [43] J. Sahota and N. Quesada, *Phys. Rev. A* **91**, 013808 (2015).
- [44] W. Ge, K. Jacobs, Z. Eldredge, A. V. Gorshkov, and M. Foss-Feig, *Phys. Rev. Lett.* **121**, 043604 (2018).
- [45] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, *Phys. Rev. Lett.* **115**, 020403 (2015).
- [46] A. Hertz and S. De Bièvre, *Phys. Rev. Lett.* **124**, 090402 (2020).
- [47] M. S. Kim, W. Son, V. Bužek, and P. L. Knight, *Phys. Rev. A* **65**, 032323 (2002).
- [48] W. Xiang-bin, *Phys. Rev. A* **66**, 024303 (2002).
- [49] B. L. Schumaker, *Physics Reports* **135**, 317 (1986).
- [50] A. Wehrl, *Rev. Mod. Phys.* **50**, 221 (1978).
- [51] S. Tserkis and T. C. Ralph, *Phys. Rev. A* **96**, 062338 (2017).
- [52] S. Tserkis, S. Onoe, and T. C. Ralph, *Phys. Rev. A* **99**, 052337 (2019).
- [53] G. Tóth and D. Petz, *Phys. Rev. A* **87**, 032324 (2013).
- [54] S. Yu, arXiv:1302.5311 (2013).
- [55] R. Bhatia and T. Jain, *Journal of Mathematical Physics* **56**, 112201 (2015), <https://doi.org/10.1063/1.4935852>.
- [56] U. Singh, M. G. Jabbour, Z. Van Herstraeten, and N. J. Cerf, *Phys. Rev. A* **100**, 042104 (2019).
- [57] A. Serafini, *Quantum Continuous Variables* (Boca Raton: CRC Press, 2017).
- [58] H. Nakazato, S. Pascazio, M. Stobińska, and K. Yuasa, *Phys. Rev. A* **93**, 023845 (2016).
- [59] Y. Gu, *Phys. Lett.* **149**, 95 (1990).