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Input-to-State Stability of Homogeneous Infinite Dimensional Systems with Locally Lipschitz Nonlinearities

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Abstract

The Input-to-State Stability (ISS) of homogeneous evolution equations in Banach spaces with unbounded linear operators and locally Lipschitz nonlinearities in the right-hand sides is studied. A new homogeneous converse Lyapunov theorem is presented. Similarly to finite-dimensional models, it is shown that the uniform asymptotic stability of an unperturbed homogeneous evolution equation implies its ISS with respect to homogeneously involved exogenous inputs.

Key words: stability of distributed parameter systems; input-to-state stability; semigroup and operator theory.

1 Introduction

Homogeneity is a dilation symmetry, which can be discovered for functions and vector fields in \mathbb{R}^n (see Zubov (1958), Hermes (1986), Bernuau et al. (2014)) as well as for functionals and operators in Banach and Hilbert spaces (see e.g. Folland (1975), Fischer and Ruzhansky (2016), Polyakov et al. (2016)). All linear and many non-linear models of mathematical physics are homogeneous in a generalized sense or, at least, admit a locally homogeneous approximation. For example, the dilation symmetry can be discovered in heat, wave, Navier–Stokes, Burgers, KdV and Saint-Venant equations (see Polyakov (2020a)).

Homogeneity simplify stability and robustness analysis of control systems (see e.g. Kawski (1991), Ryan (1995), Andrieu et al. (2008)) as well as non-linear controllers/observers design (see Kawski (1990), Coron and Praly (1991), Grune (2000), Andrieu et al. (2008), Polyakov et al. (2018), Lopez-Ramirez et al. (2018)). Homogeneity degree specifies a convergence rate of any asymptotically stable homogeneous system (see e.g. Nakamura et al. (2002)). In particular, the negative homogeneity degree corresponds to finite-time stability.

The method of Lyapunov function is the main tool for stability analysis and stabilization of both finite and infinite dimensional non-linear systems (see Bacciotti and Rosier (2001), Karafyllis and Krstic (2018), Prieur and Mazenc (2012), Deimling (1992) among others). Characterization (necessary and sufficient conditions) of stability in terms of a Lyapunov function is useful for

ISS analysis (see Sontag and Wang (1996), Mironchenko and Wirth (2019), Jacob et al. (2018), Mironchenko and Prieur (2020)). It is known, since Zubov (1958) and Rosier (1992), that any stable homogeneous ODE admits a homogeneous Lyapunov function. The key difference between Zubov’s and Rosier’s theorems is the regularity assumption to the homogeneous Lyapunov function. V. Zubov proved only the existence of a continuous homogeneous Lyapunov function, while L. Rosier constructed a smooth one using the Kurzweil’s converse Lyapunov theorem. Rosier’s theorem allows rather simple ISS analysis of homogeneous finite-dimensional systems to be developed (see e.g. Andrieu et al. (2008), Bernuau et al. (2013)) showing that the asymptotic stability of unperturbed homogeneous ODE implies its ISS with respect to homogeneously involved perturbations. This paper extends the mentioned result to a class of infinite dimensional models.

The theory of strongly continuous semigroups of linear bounded operators in Banach spaces proposes a unified approach to solution existence and uniqueness problem as well as to analysis of distributed parameter dynamical systems such as Partial Differential Equations (PDEs) (see e.g. Pazy (1983); Engel and Nagel (2000), Tucsnak and Weiss (2009)). In this paper we deal with dynamical models described by evolution equations in Banach spaces with unbounded linear closed densely defined operators and locally Lipschitz continuous non-linearities in the right-hand sides. Such models always have the so-called *mild solutions*, which are unique in the regularity domain (see e.g. Pazy (1983), Chapter 6, Theorem 1.4).

The extension of the results known for homogeneous ODEs to evolution equations in Banach spaces is complicated because of the impossibility to use the conventional differential calculus. Solutions of ODEs are, at least, absolutely continuous functions of time, which are differentiable almost everywhere. One-parametric curves in a Banach space \mathbb{B} may be globally Lipschitz but nowhere differentiable (see e.g. Tabor (2006)). The differentiability (in a strong sense) and even local Lipschitz/Hölder continuity of a mild solution $x : \mathbb{R} \rightarrow \mathbb{B}$ cannot be always guaranteed. It is not obvious if the composition $t \rightarrow V(x(t))$ of a smooth Lyapunov function candidate $V : \mathbb{B} \rightarrow \mathbb{R}$ with the mild solution x is differentiable. The smoothness of a Lyapunov function for an infinite-dimensional system seems to be rather restrictive assumption in the general case. However, a successful ISS analysis, even in the finite-dimensional case, Sontag and Wang (1996), requires some regularity of the Lyapunov function. The homogeneous converse Lyapunov theorem for infinite dimensional systems presented in Polyakov (2020b) does not guarantee even continuity of the Lyapunov function. In this paper, we managed to construct a locally Lipschitz continuous of homogeneous Lyapunov function, which allows the ISS results existing for homogeneous ODEs to be successfully extended to a evolution systems in Banach spaces.

Notation. \mathbb{R} is the field of real numbers; $\mathbb{R}_+ = [0, +\infty)$; \mathbb{B} is a real Banach with a norm $\|\cdot\|$; $\mathbf{0}$ is the zero element of a Banach space; I (resp. O) denotes the identity (resp. the zero) operator; $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ denotes the space of linear bounded operators $\mathbb{B}_1 \rightarrow \mathbb{B}_2$ with the norm $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$; S is the unit sphere in \mathbb{B} ; $B(r) \subset \mathbb{B}$ is the ball in \mathbb{B} centered at $\mathbf{0}$ of the radius $r > 0$;

$$K(r) := \{x \in \mathbb{B} : 1/r < \|x\| < r\}, \quad r > 1.$$

$\mathcal{D}(A) \subset \mathbb{B}$ denotes the domain of an operator A ; $C([t_1, t_2], \mathbb{B})$ is the space of continuous functions $x : [t_1, t_2] \rightarrow \mathbb{B}$ with the uniform norm $\|x\|_C = \sup_{t \in [t_1, t_2]} \|x(t)\|$ with $-\infty < t_1 < t_2 < +\infty$; $f_1 \circ f_2$ and $f_1 f_2$ denote a composition of operators f_1 and f_2 ; C_c^∞ is a set of infinitely smooth functions $\mathbb{R} \rightarrow \mathbb{R}$ with compact supports; $L^1((t_1, t_2), \mathbb{B})$ is the space of Bochner integrable functions $(t_1, t_2) \rightarrow \mathbb{B}$, where $-\infty \leq t_1 < t_2 \leq +\infty$; $L^\infty((t_1, t_2), \mathbb{B})$ is a space uniformly essentially bounded locally Bochner integrable functions with the norm $\|q\|_\infty := \text{ess sup}_{s \in (t_1, t_2)} \|q(s)\|$; \mathcal{K} is a set of strictly increasing functions $\sigma : [0, +\infty) \rightarrow [0, +\infty)$ such that $\sigma(0) = 0$; $\mathcal{K}^\infty := \{\sigma \in \mathcal{K} : \sigma(s) \rightarrow +\infty \text{ as } s \rightarrow +\infty\}$; the function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class $\mathcal{K}\mathcal{L}$ if $\beta(\cdot, t) \in \mathcal{K}$ and $t \rightarrow \beta(s, t)$ is strictly decreasing to zero; the symbol $\stackrel{a.e.}{=}$ (resp. $\stackrel{a.e.}{\leq}$ or $\stackrel{a.e.}{\in}$) means that an identity (resp. inequality or inclusion) holds almost everywhere; $\overline{D}^+ V(t) = \limsup_{h \rightarrow 0^+} \frac{v(t+h) - v(t)}{h}$ denotes the right-hand upper Dini derivative of the function $v : \mathbb{R} \rightarrow \mathbb{R}$; $\overline{D}^+ V(x, g) = \limsup_{h \rightarrow 0^+} \frac{V(x+hg) - V(x)}{h}$ de-

notes the right-hand upper directional derivative of the functional $V : \mathbb{B} \rightarrow \mathbb{R}$ in the direction $g \in \mathbb{B}$.

2 Problem Statement

Let us consider the nonlinear system

$$\dot{x} = Ax + f(x, q), \quad t > t_0, \quad x(t_0) = x_0 \quad (1)$$

where $x(t) \in \mathbb{B}$ is a system state, \mathbb{B} is a real Banach space; $t_0 \in \mathbb{R}$ is an initial instant of time; $x_0 \in \mathbb{B}$ is an initial state; $q \in L^\infty(\mathbb{R}, \mathbb{V})$ is an exogenous perturbation; \mathbb{V} is a real Banach space; $A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$ is a linear unbounded closed densely defined operator which generates a strongly continuous semigroup Φ of linear bounded operators on \mathbb{B} ; $f : \mathbb{B} \times \mathbb{V} \rightarrow \mathbb{B}$ is a non-linear mapping such that $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

We assume that, on $\mathbb{B} \setminus \{\mathbf{0}\} \times \mathbb{V}$, the nonlinear mapping f is locally Lipschitz continuous in the first argument and locally γ -Hölder continuous in the second one, i.e. for any $r > 0$ there exists $L_r > 0$ such that

$$\|f(x_1, q_1) - f(x_1, q_2)\| \leq L_r (\|x_1 - x_2\| + \|q_1 - q_2\|^\gamma) \quad (2)$$

for all $x_1, x_2 \in K(r)$ and all $q_1, q_2 \in \mathbb{B}(r)$, where $\gamma \in (0, 1]$.

The non-linear evolution equations are well-studied in the literature (see, for example, Pazy (1983); Engel and Nagel (2000)), where the notion of solution is introduced using the theory of evolution semigroups.

Definition 1 A continuous function $x : [t_0, t_1] \rightarrow \mathbb{B}$ is said to be a mild solution of (1) if $f(x(\cdot), q(\cdot)) \in L^1((t_0, t_1), \mathbb{B})$ and

$$x(t) = \Phi(t-t_0)x_0 + \int_{t_0}^t \Phi(t-s)f(x(s), q(s))ds, \quad \forall t \in [t_0, t_1].$$

If this mild solution satisfies (1) for (almost) all $t \in (t_0, t_1)$ then x is called classical (strong) solution of (1).

The integral in the above definition is understood in the sense of Bochner (see e.g. Driver (2003), page 187). The assumption $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ implies the existence of the zero solution of the non-perturbed system (1). The regularity of f implies the existence and uniqueness of mild solutions on $\mathbb{B} \setminus \{\mathbf{0}\}$ as well as their continuous dependence on initial conditions (see Appendix). Notice that for $\mathbb{B} = \mathbb{R}^n$ the system (1) becomes a nonlinear ODE, $\Phi(t) = e^{tA} \in \mathbb{R}^{n \times n}$, $t \geq 0$ and any mild solution becomes a strong one.

In this paper we deal with operators A and f , which are homogeneous (symmetric in a sense explained below). It is known that an asymptotic stability of the

non-perturbed ($q = 0$) homogeneous ODE ($\mathbb{B} = \mathbb{R}^n$) implies its ISS provided that the perturbation q is homogeneously involved into the system (see e.g. Andrieu et al. (2008), Bernuau et al. (2013) for more details). The first aim of this paper is to extend this result to the infinite dimensional model (1). The corresponding ISS analysis for ODEs was based on the homogeneous Lyapunov function theorem proven in Rosier (1992). Therefore, the second aim of the paper is to extend the converse Lyapunov theorem to homogeneous evolution equations in Banach spaces.

3 Homogeneous systems

3.1 Dilations in Banach Spaces

Dilations in finite-dimensional and infinite-dimensional spaces are studied in the literature (see e.g. Husch (1970), Kawski (1991), Polyakov et al. (2016)). Here we just recall briefly some related definitions and properties.

Definition 2 A mapping $\mathfrak{d} : \mathbb{R} \rightarrow \mathcal{L}(\mathbb{B}, \mathbb{B})$ is said to be a group of linear dilations (or simply dilation) in \mathbb{B} if 1) (Group property) $\mathfrak{d}(0) = I$, $\mathfrak{d}(t+s) = \mathfrak{d}(t)\mathfrak{d}(s)$, $t, s \in \mathbb{R}$; 2) (Limit property) $\lim_{s \rightarrow -\infty} \|\mathfrak{d}(s)z\| = 0$, $\lim_{s \rightarrow +\infty} \|\mathfrak{d}(s)z\| = \infty$ uniformly on the unit sphere $S = \{z \in \mathbb{B} : \|z\| = 1\}$.

Obviously, \mathfrak{d} is the group of linear bounded invertible operators and $\mathfrak{d}(-s) = (\mathfrak{d}(s))^{-1}$, $\forall s \in \mathbb{R}$. The limit property specifies groups being dilations in \mathbb{B} . In particular, it implies $\bigcup_{s \in \mathbb{R}} \mathfrak{d}(s)u = \mathbb{B} \setminus \{0\}$.

Definition 3 A dilation \mathfrak{d} is 1) strongly continuous if $\mathfrak{d}(\cdot)z : \mathbb{R} \rightarrow \mathbb{B}$ is continuous for any $z \in \mathbb{B}$; 2) uniformly continuous if $\mathfrak{d}(\cdot) : \mathbb{R} \rightarrow \mathcal{L}(\mathbb{B}, \mathbb{B})$ is continuous.

Examples of linear dilations in \mathbb{R}^n are

1) Uniform dilation (L. Euler, 18th century): $\mathfrak{d}(s) = e^{sI}$, where I is the identity matrix \mathbb{R}^n (or identity operator in \mathbb{B}).

2) Weighted dilation (Zubov (1958)): $\mathfrak{d}(s) = \begin{pmatrix} e^{r_1 s} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{r_n s} \end{pmatrix} \in \mathbb{R}^{n \times n}$, where $r_i > 0$, $i = 1, 2, \dots, n$.

3) Linear dilation: $\mathfrak{d}(s) = e^{sG_{\mathfrak{d}}} = \sum_{i=0}^{\infty} \frac{s^i G_{\mathfrak{d}}^i}{i!}$, where $G_{\mathfrak{d}} \in \mathbb{R}^{n \times n}$ is an anti-Hurwitz matrix.

Nonlinear dilations in \mathbb{R}^n are studied in Khomenuk (1961), Kawski (1991), Rosier (1993).

Examples of linear dilations in function spaces are

1) a uniformly continuous dilation $C([0, p], \mathbb{R})$ can be defined as $(\mathfrak{d}(s)z)(y) = e^{s-0.5sy/p}z(y)$, where $s \in \mathbb{R}$, $y \in [0, p]$.

2) a strongly continuous dilation in $L^2(\mathbb{R}^n, \mathbb{R})$ is defined as $(\mathfrak{d}(s)z)(y) = e^{\alpha s}z(e^{\beta s}y)$, where $s \in \mathbb{R}$, $y \in \mathbb{R}^n$, $\alpha > n\beta/2$.

For more details about linear dilations in function spaces we refer the reader to (Polyakov, 2020a, Chapter 6).

Definition 4 A dilation \mathfrak{d} is strictly monotone in \mathbb{B} if $\exists \beta > 0$ such that

$$\|\mathfrak{d}(s)\| \leq e^{\beta s}, \quad \forall s \leq 0. \quad (3)$$

Monotone dilations in \mathbb{R}^n are studied in Polyakov (2018), where, in particular, it is shown that all linear dilations in \mathbb{R}^n are strictly monotone under a proper selection of the weighted Euclidean norm.

Definition 5 A nonempty set $\mathcal{D} \subseteq \mathbb{B}$ is said to be \mathfrak{d} -homogeneous cone in \mathbb{B} if $\mathfrak{d}(s)z \in \mathcal{D}, \forall z \in \mathcal{D}, \forall s \in \mathbb{R}$.

Homogeneous cone is a reasonable notion for a set satisfying the latter definition, since any point $z \in \mathcal{D}$ belongs to \mathcal{D} together with the homogeneous curve $\Gamma_{\mathfrak{d}}(z) = \{y \in \mathbb{B} : y = \mathfrak{d}(s)z, s \in \mathbb{R}\}$. In particular, if \mathfrak{d} is the uniform dilation (i.e. $\mathfrak{d}(s) = e^s I$, $s \in \mathbb{R}$), then the homogeneous curve becomes a ray and \mathcal{D} becomes a positive cone in \mathbb{B} . Homogeneous cones are domains of unbounded homogeneous operators in \mathbb{B} (see below).

3.2 The canonical homogeneous norm

The dilation introduces a new norm topology in \mathbb{B} using the so-called canonical homogeneous norm.

Definition 6 (Polyakov et al. (2018)) The functional $\|\cdot\|_{\mathfrak{d}} : \mathbb{B} \rightarrow \mathbb{R}_+$ given by

$$\|u\|_{\mathfrak{d}} = e^{s_u}, \quad \text{where } s_u \in \mathbb{R} : \|\mathfrak{d}(-s_u)u\| = 1, \quad u \neq 0 \quad (4)$$

is called the canonical homogeneous norm in the Banach space \mathbb{B} , where \mathfrak{d} is a strictly monotone dilation in \mathbb{B} .

Obviously, $\|\mathfrak{d}(s)u\|_{\mathfrak{d}} = e^s \|u\|_{\mathfrak{d}}$ and $\|u\|_{\mathfrak{d}} = \|-u\|_{\mathfrak{d}}$ for $\forall u \in \mathbb{B}$ and $\forall s \in \mathbb{R}$. The homogeneous norm $\|\cdot\|_{\mathfrak{d}}$ is called canonical since it is induced by a canonical norm $\|\cdot\|$ in \mathbb{B} and $\|u\|_{\mathfrak{d}} = 1 \Leftrightarrow \|u\| = 1$. Notice that $\|\cdot\|_{\mathfrak{d}} = \|\cdot\|$ provided that \mathfrak{d} is the standard dilation $\mathfrak{d}(s) = e^s I$, $s \in \mathbb{R}$.

The following theorem refines the some results from Polyakov et al. (2018). In particular, it shows that the functional $\|\cdot\|_{\mathfrak{d}}$ can be prolonged to zero by continuity, i.e. $\|0\|_{\mathfrak{d}} := 0$

Theorem 1 (Polyakov (2020a), Lemmas 7.1, 7.2) If \mathfrak{d} is a strongly continuous strictly monotone dilation then $\|\cdot\|_{\mathfrak{d}}$ is single-valued, positive definite, locally Lipschitz continuous on $\mathbb{B} \setminus \{0\}$ and there exist $\underline{\sigma}, \bar{\sigma} \in \mathcal{K}^{\infty}$ such that

$$\underline{\sigma}(\|u\|) \leq \|u\|_{\mathfrak{d}} \leq \bar{\sigma}(\|u\|), \quad \forall u \in \mathbb{B}.$$

In (Polyakov, 2020a, Theorem 7.1) it is also proven that $\|\cdot\|_{\mathfrak{d}}$ is a norm in a Banach space \mathbb{B} homeomorphic to \mathbb{B} . This justifies the name "norm" for the functional $\|\cdot\|_{\mathfrak{d}}$.

3.3 Homogeneous operators

Homogeneous functionals and operators on \mathbb{B} (see Polyakov et al. (2016)) are defined similarly to homogeneous functions and vector fields (see e.g. Kawski (1991), Grune (2000)) taking into account their possible unboundedness.

Definition 7 An operator $f : \mathcal{D}(f) \subset \mathbb{B} \rightarrow \mathbb{B}$ (a functional $h : \mathcal{D}(h) \subset \mathbb{B} \rightarrow \mathbb{R}$) is said to be \mathfrak{d} -homogeneous of a degree $\nu \in \mathbb{R}$ if the domain $\mathcal{D}(f)$ (resp. $\mathcal{D}(h)$) is a \mathfrak{d} -homogeneous cone and

$$\begin{aligned} e^{\nu s} \mathfrak{d}(s)f(u) &= f(\mathfrak{d}(s)u), \quad \forall s \in \mathbb{R}, \quad \forall u \in \mathcal{D}(f), \\ (\text{resp. } h(\mathfrak{d}(s)u) &= e^{\nu s} h(u), \quad \forall s \in \mathbb{R}, \quad \forall u \in \mathcal{D}(h)) \end{aligned} \quad (5)$$

where \mathfrak{d} is a group of linear invertible operators in \mathbb{B} .

We say that an evolution equation (inclusion) is \mathfrak{d} -homogeneous of a degree $\mu \in \mathbb{R}$ if its right-hand side is a \mathfrak{d} -homogeneous operator of the degree μ . Notice that the identity (5) can be understood in the weak sense as shown below.

Example 1 (Homogeneous of Laplace operator)

Let us consider the Laplace operator $\Delta : \mathcal{D}(\Delta) \subset L^2 \rightarrow L^2$, $\mathcal{D}(\Delta) = \{u \in L^2 : \exists f \in L^2, \int u \cdot \Delta \phi = \int f \cdot \phi, \forall \phi \in C_c^\infty\}$. Let us show that the operator Δ is \mathfrak{d} -homogeneous of degree 2β provided that the dilation \mathfrak{d} is given by $(\mathfrak{d}(s)u)(x) = e^{\alpha s} u(e^{\beta s} x)$, where $s \in \mathbb{R}$, $x \in \mathbb{R}^n$, $\alpha > n\beta/2$.

Since, \mathfrak{d} is a group of linear invertible operators on C_c^∞ and, consequently, $\mathfrak{d}(s)$ maps C_c^∞ onto C_c^∞ . Notice that if $\phi \in C_c^\infty$ then, obviously,

$$(\Delta \circ \mathfrak{d}(s))\phi(x) = e^{(\alpha+2\beta)s} (\Delta \phi)(e^{\beta s} x) = e^{2\beta s} ((\mathfrak{d}(s)\mathfrak{d}\Delta)\phi)(x).$$

In other words, the Laplace operator is \mathfrak{d} -homogeneous as an operator $C_c^\infty \rightarrow C_c^\infty$. Since C_c^∞ is dense in L^2 , H^1 and H^2 then, it is expectable that, Δ is \mathfrak{d} -homogeneous as an operator in $\mathcal{D}(\Delta) \subset L^2 \rightarrow L^2$. Let us prove this rigorously.

Let $u \in \mathcal{D}(\Delta)$ and $\Delta u = f \in L^2$ in the weak sense, i.e.

$$\int u \Delta \phi = \int f \phi, \quad \forall \phi \in C_c^\infty.$$

Since $\mathfrak{d}(s)f \in L^2$ then using the change-of-variable theorem in the Lebesgue integral we derive

$$\begin{aligned} e^{2\beta s} \int (\mathfrak{d}(s)f) \cdot \phi &= e^{(\alpha+2\beta)s} \int f(e^{\beta s} x) \cdot \phi(x) dx = \\ e^{(\alpha+(2-n)\beta)s} \int f(x) \cdot \phi(e^{-\beta s} x) dx &= e^{(2\alpha+(2-n)\beta)s} \int f \cdot \tilde{\phi} = \\ e^{(2\alpha+(2-n)\beta)s} \int u \cdot \Delta \tilde{\phi} &= e^{(2\alpha+(2-n)\beta)s} \int u \cdot (\Delta \circ \mathfrak{d}(-s))\phi = \\ e^{(\alpha-n\beta)s} \int u(x) \cdot \Delta \phi(e^{-\beta s} x) dx &= e^{\alpha s} \int u(e^{\beta s} x) \cdot \Delta \phi(x) dx. \end{aligned}$$

Hence, $\mathfrak{d}(s)u \in \mathcal{D}(\Delta)$ and $(\Delta \circ \mathfrak{d}(s))u = e^{2\beta s} \mathfrak{d}(s)f = e^{2\beta s} (\mathfrak{d}(s) \circ \Delta)u$ in the weak sense for any $s \in \mathbb{R}$, $u \in \mathcal{D}(\Delta)$.

Homogeneity allows local properties of nonlinear operators (such as regularity) to be extended globally (Polyakov, 2020a, Chapter 7).

3.4 Symmetry of solutions of homogeneous systems

A semigroup generated by a closed densely defined linear homogeneous operator in \mathbb{B} is homogeneous as well.

Lemma 1 (Polyakov (2020a), Lemma 8.1) Let a linear closed densely defined operator $A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$ generate a strongly continuous semigroup Φ of linear bounded operators on \mathbb{B} and \mathfrak{d} be a group of linear bounded invertible operators on \mathbb{B} . If the operator A is \mathfrak{d} -homogeneous of a degree $\mu \in \mathbb{R}$ then

$$\Phi(t)\mathfrak{d}(s) = \mathfrak{d}(s)\Phi(e^{\mu s} t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}. \quad (6)$$

Lemma 1 proves the symmetry of solutions of (1) for $f \equiv \mathbf{0}$:

$$x_{\mathfrak{d}(s)x_0}(t) = \mathfrak{d}(s)x_{x_0}(e^{\mu s} t), \quad s \in \mathbb{R}, t \geq 0, \quad (7)$$

where x_z denotes a solution of (1) with the initial data $x(0) = z$. This symmetry of solutions takes a place for a \mathfrak{d} -homogeneous evolution equation (1) even if $f \neq \mathbf{0}$.

Theorem 2 (Polyakov (2020a), Theorem 8.1)

Let \mathfrak{d} be a group of linear bounded invertible operators on \mathbb{B} . Let a linear closed densely defined operator $A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$ generate a strongly continuous semigroup Φ of linear bounded operators on \mathbb{B} . Let A and f be \mathfrak{d} -homogeneous operators of a degree $\mu \in \mathbb{R}$. If $x : [0, T) \rightarrow \mathbb{B}$ is a mild solution of the evolution equation

$$\dot{x} = Ax + f(x), \quad t > 0 \quad (8)$$

then for any $s \in \mathbb{R}$ the function $x^s : [0, e^{-\mu s} T) \rightarrow \mathbb{B}$ given by

$$x^s(t) := \mathfrak{d}(s)x(e^{\mu s} t), \quad t \in [0, e^{-\mu s} T)$$

is also a mild solution of the evolution equation (1).

The latter theorem expands globally any local property of solutions. For example, *if the origin of (1) is locally stable then, from (7) and the limit property of \mathfrak{d} , we immediately derive global stability*. Similarly, the existence of solutions for small initial data implies the existence of solutions for large initial data.

Below we generalize Theorem 2 to homogeneous systems with disturbances (see Theorem 4).

3.5 Homogeneous Lyapunov function theorem

Recall that the origin of the system (8) is said to be

- *globally uniformly Lyapunov stable* if there exists $\varepsilon \in \mathcal{K}^\infty$ such that

$$\|x(t)\| \leq \varepsilon(\|x(0)\|), \quad \forall t \geq 0 \quad (9)$$

for any mild solution x of (8) and any $x_0 \in \mathbb{B}$;

- *globally uniformly asymptotically stable* if it is globally uniformly Lyapunov stable and $\forall \varepsilon > 0, \forall R > \varepsilon$, there exists $\hat{T} = \hat{T}(R, \varepsilon) > 0$ such that the inclusion $x_0 \in B(R)$ implies $x_{x_0}(t) \in B(\varepsilon)$ for $t > \hat{T}$.

Notice that the homogeneity degree specifies the convergence rate of stable homogeneous systems (see e.g. Nakamura et al. (2002), Bhat and Bernstein (2005) for ODE models and (Polyakov, 2020a, Theorem 8.6) for evolution systems in Banach spaces). For instance, any uniformly asymptotically stable \mathfrak{d} -homogeneous system of negative degree is globally uniformly finite-time stable.

Theorem 3 *Let \mathfrak{d} be a strictly monotone strongly continuous dilation in \mathbb{B} , the evolution equation (8) be \mathfrak{d} -homogeneous of a degree $\mu \in \mathbb{R}$ and $f : \mathbb{B} \rightarrow \mathbb{B}$ be locally Lipschitz continuous on $\mathbb{B} \setminus \{\mathbf{0}\}$. The origin of (8) is uniformly asymptotically stable if and only if there exists a continuous positive definite functional $V : \mathbb{B} \rightarrow \mathbb{R}$ such that*

- 1) V is \mathfrak{d} -homogeneous of a degree 1 and locally Lipschitz continuous on $\mathbb{B} \setminus \{\mathbf{0}\}$;
- 2) $\exists \underline{k}, \bar{k} > 0$ satisfying

$$\underline{k}\|x\|_{\mathfrak{d}} \leq V(x) \leq \bar{k}\|x\|_{\mathfrak{d}}, \quad \forall x \in \mathbb{B}; \quad (10)$$

- 3) for any mild solution x_{x_0} of (8) with $x(0) = x_0 \in \mathcal{D}(A) \setminus \{\mathbf{0}\}$ the inequality

$$\overline{D}^+ V(x_{x_0}(t)) \stackrel{a.e.}{\leq} -\|x_{x_0}(t)\|_{\mathfrak{d}}^{\mu+1} \quad (11)$$

holds as long as $x_{x_0}(t) \neq \mathbf{0}$.

Proofs of all claims are given in Appendix.

Formally, the latter theorem requires a knowledge of trajectories of the system (8) to check the condition (11). The Lyapunov function method is appreciated in practice if the asymptotic stability can be established using the right-hand side of the evolution system only. This can be done in the case of reflexive Banach spaces.

Corollary 1 *For a reflexive Banach space \mathbb{B} , Theorem 3 remains true even when the condition 3) is replaced with*

$$3') \overline{D}^+ V(x, Ax + f(x)) \leq -\|x\|_{\mathfrak{d}}^{\mu+1}, \quad \forall x \in \mathcal{D}(A) \setminus \{\mathbf{0}\}.$$

This corollary holds, in particular, for Hilbert spaces.

Example 2 *Let us consider the following partial differential equation (PDE)*

$$\frac{\partial x}{\partial t} = \Delta x - \gamma \|x\|^{\mu} x,$$

where $\gamma > 0, \mu \in \mathbb{R}, x : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\Delta : H^2(\mathbb{R}^n, \mathbb{R}) \subset L^2(\mathbb{R}^n, \mathbb{R}) \rightarrow L^2(\mathbb{R}^n, \mathbb{R})$$

is the Laplace operator on $L^2(\mathbb{R}^n, \mathbb{R})$ with the domain $H^2(\mathbb{R}^n, \mathbb{R})$ (Sobolev space); and

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad \langle x, y \rangle = \int_{\mathbb{R}^n} x \cdot y \, dz.$$

The latter PDE admits the representation (1) with $\mathbb{B} = L^2(\mathbb{R}^n, \mathbb{R})$, $A = \Delta$ and $f : \mathbb{B} \rightarrow \mathbb{B}$ given by

$$f(x) = -\gamma \|x\|^{\mu} x.$$

It is well known (see e.g. Pazy (1983), Chapter 7) that A is the generator of a strongly continuous semigroup of linear bounded operators on $L^2(\mathbb{R}^n, \mathbb{R})$.

In Example 1 it is already shown that the Laplace operator Δ is \mathfrak{d} -homogeneous of the degree 2β with the dilation

$$(\mathfrak{d}(s)x)(z) = e^{\alpha s} x(e^{\beta s} z)$$

with $\alpha > n\beta/2$. Simple computations shows

$$\|\mathfrak{d}(s)x\| = \sqrt{\int_{\mathbb{R}^n} \|e^{\alpha s} x(e^{\beta s} z)\|^2 dz} = e^{\alpha - n\beta/2} \|x\| \quad (12)$$

then

$$\|x\|_{\mathfrak{d}} = \|x\|^{\frac{1}{\alpha - n\beta/2}}$$

and f is \mathfrak{d} -homogeneous of the degree $\mu(\alpha - n\beta/2)$. Therefore, for $2\beta = \mu(\alpha - n\beta/2)$ the system is \mathfrak{d} -homogeneous of the degree 2β and the sign of the homogeneity degree is defined by the sign of μ . Simple computations show that a homogeneous Lyapunov functional of the degree 1 for the

considered PDE model is given by $V(x) = \|x\|_{\mathfrak{D}}$. Indeed, for $x \in \mathcal{D}(A) \setminus \{\mathbf{0}\}$ we derive

$$\begin{aligned} \overline{D}^+ V(x, \Delta x + f(x)) &= \frac{1}{\alpha - n\beta/2} \|x\|^{\frac{1}{\alpha - n\beta/2} - 2} \langle x, \Delta x + f(x) \rangle \\ &\leq \frac{1}{\alpha - n\beta/2} \|x\|^{\frac{1}{\alpha - n\beta/2} - 2} \langle x, f(x) \rangle \\ &= -\frac{\gamma}{\alpha - n\beta/2} \|x\|_{\mathfrak{D}}^{1 + \mu(\alpha - n\beta/2)}, \end{aligned}$$

where the dissipative property $\langle x, \Delta x \rangle \leq 0$ of the Laplace operator is utilized on the indeterminate step.

In many other examples, where the canonical homogeneous norm is a homogeneous Lyapunov function of a stable homogeneous system, can be found in Polyakov (2020a).

3.6 Homogeneous systems with perturbations

Certain symmetry of solutions can also be discovered for the perturbed homogeneous system (1).

Theorem 4 Let \mathfrak{d} be a group of linear bounded invertible operators in \mathbb{B} , the operator $A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$ be \mathfrak{d} -homogeneous of a degree μ closed densely defined generator of a strongly continuous semigroup Φ of linear bounded operators on \mathbb{B} , the function $f : \mathbb{B} \times \mathbb{V} \rightarrow \mathbb{B}$ be locally Lipschitz continuous on $\mathbb{B} \setminus \{\mathbf{0}\} \times \mathbb{V}$ and

$$f(\mathfrak{d}(s)x, \tilde{\mathfrak{d}}(s)q) = e^{\mu s} \mathfrak{d}(s) f(x, q), \quad \forall x \in \mathbb{B}, \forall q \in \mathbb{V}, \forall s \in \mathbb{R},$$

where $\tilde{\mathfrak{d}}$ is a group of linear bounded operators in \mathbb{V} .

If $x_{x_0, q} : [t_0, t_0 + T] \rightarrow \mathbb{B}$ is a mild solution of the system (1) then for any $s \in \mathbb{R}$ one has

$$x_{\mathfrak{d}(s)x_0, \tilde{\mathfrak{d}}(s)q}(t) = \mathfrak{d}(s) x_{x_0, q}(t_0 + e^{\mu s}(t - t_0)), \quad \forall t \in [t_0, t_0 + \frac{T}{e^{\mu s}}], \quad (13)$$

where $x_{\mathfrak{d}(s)x_0, \tilde{\mathfrak{d}}(s)q} : [t_0, t_0 + e^{-\mu s}T] \rightarrow \mathbb{B}$ is a solution of the system (1) for $x_0 \in \mathbb{B}$ replaced with $\mathfrak{d}(s)x_0 \in \mathbb{B}$ and $q \in L^\infty((t_0, t_0 + T), \mathbb{V})$ replaced with $\tilde{q} \in L^\infty((t_0, t_0 + e^{-\mu s}T), \mathbb{V})$ given by

$$\tilde{q}(t) = \tilde{\mathfrak{d}}(s)q(t_0 + e^{\mu s}(t - t_0)), \quad \forall t \in [t_0, t_0 + \frac{T}{e^{\mu s}}].$$

Theorem 4 becomes Theorem 2 for $q = \mathbf{0}$. The latter theorem allows rather simple ISS analysis of homogeneous evolution equations to be developed. The definition of the input-to-state stability in this case can be introduced similarly to ODE models (see, examples, Sontag (2007), Karafyllis and Krstic (2018), Mironchenko and Wirth (2019)).

Definition 8 The system (1) is said to be input-to-state stable (ISS) if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\|x_{x_0, q}(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|q\|_{L^\infty((0, t), \mathbb{V})}), \quad \forall t \geq 0,$$

for any mild solution $x_{x_0, q}$ of (1).

Corollary 2 Let all conditions of Theorem 4 are fulfilled for strictly monotone strongly continuous dilations \mathfrak{d} and $\tilde{\mathfrak{d}}$. If the origin of the system (1) with $q = \mathbf{0}$ is uniformly asymptotically stable then this system is ISS.

The obtained corollary introduces the homogeneity based ISS analysis for a class of infinite dimensional systems.

Example 3 (ISS of Reaction Diffusion Equation) In Dashkovskiy and Mironchenko (2011), the local ISS analysis of a nonlinear reaction diffusion equation is presented. The homogeneity allows the global ISS to be established in some cases.

Let us consider the following PDE model

$$\frac{\partial x}{\partial t} = \Delta x + f(x, q), \quad (14)$$

where $x(t) \in L^2(\mathbb{R}^n, \mathbb{R})$ and Δ are defined as in Example 2, $f : L^2(\mathbb{R}^n, \mathbb{R}) \times \mathbb{V} \rightarrow L^2(\mathbb{R}^n, \mathbb{R})$, $q \in L^\infty(\mathbb{R}_+, \mathbb{V})$ can be interpreted as a control input or an exogenous perturbation of the system and \mathbb{V} is a Banach space of any kind. For example, \mathbb{V} could be a function space as L^p , H^p , C^p , a space of sequences l^p , a vector space \mathbb{R}^n or their Cartesian product like $L^p \times \mathbb{R}^m$.

By Corollary 2, if the considered system is homogeneous and uniformly asymptotically stable for $q = 0$, and if f satisfies (2) then it is ISS. For example, let us consider f of the form

$$f(x, q) = -(\|x\| + |q_1|)^\mu x + \sqrt{|xq_2|}$$

where $x \in L^2(\mathbb{R}^n, \mathbb{R})$ and $q = (q_1, q_2) \in \mathbb{V}$,

$$\mathbb{V} := \mathbb{R} \times L^2(\mathbb{R}^n, \mathbb{R}), \quad \|q\| = |q_1| + \|q_2\|.$$

Then $f : L^2(\mathbb{R}^n, \mathbb{R}) \times \mathbb{V} \rightarrow L^2(\mathbb{R}^n, \mathbb{R})$ and for $q = \mathbf{0}$ we derive the system studied in Example 2, that is globally uniformly asymptotically stable and \mathfrak{d} -homogeneous of the degree $\mu(\alpha - n\beta/2)$ provided that the dilation \mathfrak{d} in $L^2(\mathbb{R}^n, \mathbb{R})$ is defined as follows

$$(\mathfrak{d}(s)x)(z) = e^{\alpha s} x(e^{\beta s} z), \quad s \in \mathbb{R}, \quad z \in \mathbb{R}^n$$

with

$$\alpha > n\beta/2, \quad 2\beta = \mu(\alpha - n\beta/2).$$

It is easy to see that f satisfies the regularity assumption (2). Moreover, for

$$(\tilde{\mathfrak{d}}(s)q)(z) = \begin{pmatrix} e^{(\alpha - n\beta/2)s} q_1 \\ e^{(\alpha + 2\mu(\alpha - n\beta/2))s} q_2(e^{\beta s} z) \end{pmatrix}$$

we have

$$f(\mathfrak{d}(s)x, \tilde{\mathfrak{d}}(s)q) = e^{\mu(\alpha - n\beta/2)s} \mathfrak{d}(s)f(x, q).$$

To be a strictly monotone dilation in \mathbb{V} the group $\tilde{\mathfrak{d}}$ must have the parameters (see (12))

$$(\alpha - n\beta/2) > 0 \quad \text{and} \quad 2\mu(\alpha - n\beta/2) + \alpha > n\beta/2.$$

Therefore, the considered system is ISS if $\mu > -1/2$.

4 Conclusions

The main contribution of the paper are Corollaries 1 and 2. The first one provides a characterization (in terms of a locally Lipschitz continuous Lyapunov function) of the uniform asymptotic stability of homogeneous infinite dimensional systems of a certain class, while the second one proposes a simple way for their ISS analysis. As an example, the simplicity of ISS using the obtained results is demonstrated for the nonlinear reaction diffusion equation with a locally Lipschitz continuous nonlinearity.

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5 Appendix

5.1 Auxiliary results

Lemma 2 (Existence of Solutions) *Let $A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$ be closed densely defined linear operator which generates a strongly continuous semigroup Φ of linear bounded operators on \mathbb{B} and $f : \mathbb{B} \times \mathbb{V} \rightarrow \mathbb{B}$ satisfy (2). Let $\delta > 1$ be an arbitrary real number and*

$$f_\delta(x, q) = a_\delta(\|x\|)f(x, q), \quad x \in \mathbb{B}, q \in \mathbb{V}$$

where $a_\delta \in C_c^\infty$ such that $a_\delta(s) = 0$ for $s \notin (1/(2\delta), 2\delta)$ and $a_\delta(s) = 1$ for $s \in [1/\delta, \delta]$. Then for any $x_0 \in \mathbb{B}$ and any $q \in L^\infty([t_0, +\infty), \mathbb{B})$ the initial value problem

$$\dot{x} = Ax + f_\delta(x, q), t \geq t_0 \quad x(t_0) = x_0 \quad (15)$$

has a unique mild solution $x^\delta[0, +\infty) \rightarrow \mathbb{B}$, which, for $x_0 \in K(\delta)$, coincides with the mild solution x of (1) as long as $x(t) \in K(\delta)$. Moreover, for any $x_0 \in \mathbb{B} \setminus \{0\}$ the system (1) has a mild solution x , which is uniquely defined as long as $0 < \|x_{x_0}(t)\| < +\infty$.

Proof. For $x_1 \in \mathbb{B}$, $x_2 \in \mathbb{B}$ and $q \in \mathbb{V}$ we have

$$\begin{aligned} & \|f_\delta(x_1, q) - f_\delta(x_2, q)\| = \\ & \|a_\delta(\|x_1\|)f(x_1, q) - a_\delta(\|x_2\|)f(x_2, q)\| \leq \\ & \|a_\delta(\|x_1\|)f(x_1, q) - a_\delta(\|x_2\|)f(x_1, q)\| + \\ & \|a_\delta(\|x_2\|)f(x_1, q) - a_\delta(\|x_2\|)f(x_2, q)\| \leq \\ & \|f(x_1, q)\| \cdot |a_\delta(\|x_1\|) - a_\delta(\|x_2\|)| + \\ & |a_\delta(\|x_2\|)| \cdot \|f(x_1, q) - f(x_2, q)\| = \\ & \|f(x_1, q)\| \cdot a'_\delta(\theta) \cdot \|\|x_1\| - \|x_2\|\| + \\ & |a_\delta(\|x_2\|)| \cdot \|f(x_1, q) - f(x_2, q)\| \end{aligned}$$

where the mean value theorem is utilized with $\theta \in [\min\|x_i\|, \max\|x_i\|]$, $i = 1, 2$. For any $\bar{q} > 0$, from local Lipschitz continuity of f in the first argument on $\mathbb{B} \setminus \{0\} \times \mathbb{V}$, we conclude that $\exists \tilde{L}_{\delta, \bar{q}} > 0$ such that

$$\|f(x_1, q) - f(x_2, q)\| \leq \tilde{L}_{\delta, \bar{q}} \|x_1 - x_2\|$$

for all $x_1, x_2 \in K(2\delta)$ and all $q \in B(\bar{q})$, and

$$\sup_{(x, q) \in K(\delta) \times B(\bar{q})} \|f(x, q)\| < +\infty.$$

Since $a_\delta \in C_c^\infty$ then $\sup_{s \in \mathbb{R}} a'(s) < +\infty$. Notice also $x_2 \notin K(2\delta) \Rightarrow a_\delta(\|x_2\|) = 0$. Therefore, we conclude $f_\delta(\cdot, q)$ satisfy the Lipschitz condition on \mathbb{B} with the Lipschitz constant

$$L_{\delta, \bar{q}} := \tilde{L}_{\delta, \bar{q}} + \sup_{\theta \in \mathbb{R}} a'_\delta(\theta) \cdot \sup_{(x, q) \in K(\delta) \times B(\bar{q})} \|f(x, q)\| < +\infty$$

for any $q \in B(\bar{q})$. Let us consider the metric space

$$\mathbf{X} = \left\{ x \in C([t_0, t_0 + T], \mathbb{B}) : \begin{array}{l} x(t_0) = x_0, \\ \sup_{t \in [t_0, t_0 + T]} \|x(t)\| < +\infty \end{array} \right\},$$

with $T > 0$ and the metric

$$\rho(x_1, x_2) = \sup_{t \in [t_0, t_0 + T]} \|x_1(t) - x_2(t)\|.$$

Let us consider the operator F defined as follows

$$(Fx)(t) = \Phi(t - t_0)x_0 + \int_{t_0}^t \Phi(t - \tau)f_\delta(x(\tau), q(\tau))d\tau.$$

The function $f_\delta(x(\cdot), q(\cdot))$ is Bochner integrable if and only if $\|f_\delta(x(\cdot), q(\cdot))\|$ is Lebesgue integrable. Since $x \in C([t_0, t_0 + T], \mathbb{B})$, $q \in L^\infty((t_0, t_0 + T), \mathbb{V})$ then $f(x(\cdot), q(\cdot)) \in L^1((t_0, t_0 + T), \mathbb{B})$ and $S : \mathbf{X} \rightarrow \mathbf{X}$. Indeed, for $q \in B(\bar{q})$ we have

$$\|(Fx)(t)\| \leq \|\Phi(t - t_0)\| \cdot \|x_0\| +$$

$$\begin{aligned} & \int_{t_0}^t \|\Phi(t-\tau)\| \cdot \|f_\delta(x(\tau), q(\tau)) - f_\delta(\mathbf{0}, q(\tau))\| + \|f_\delta(\mathbf{0}, q(\tau))\| d\tau \\ & \leq M\|x_0\| + M \int_{t_0}^t L_{\delta, \bar{q}} \|x(\tau)\| d\tau \end{aligned}$$

where $M = \sup_{t \in [0, T-t_0]} \|\Phi(t)\| < +\infty$. For any $q \in L^\infty((t_0, t_0 + T), \mathbb{V}) : \|q\|_\infty \leq \bar{q}$ and any $x \in C([t_0, t_0 + T], \mathbb{B})$ we derive

$$\sup_{t \in [t_0, t_0 + T]} \|(Fx)(t)\| < +\infty.$$

To complete the proof let us show that the operator $F : \mathbf{X} \rightarrow \mathbf{X}$ has the unique fixed point $F(x^*) = x^*$ with $x^* \in \mathbf{X}$. For any $x_1, x_2 \in \mathbf{X}$ we have

$$\begin{aligned} & \|F(x_1)(t) - F(x_2)(t)\| \\ & \leq M \int_{t_0}^t \|f_\delta(x_1(\tau), q(\tau)) - f_\delta(x_2(\tau), q(\tau))\| d\tau \\ & \leq ML_{\delta, \bar{q}}(t - t_0) \sup_{t \in [t_0, t_0 + T]} \|x_1(\tau) - x_2(\tau)\| \\ & = ML_{\delta, \bar{q}}(t - t_0) \rho(x_1, x_2). \end{aligned}$$

Hence, by induction we derive

$$\|F^n(x_1)(t) - F^n(x_2)(t)\| \leq \frac{(ML_\delta(t - t_0))^n}{n!} \rho(x_1, x_2).$$

and $\rho(F^n x_1, F^n x_2) \leq \frac{(ML_\delta T)^n}{n!} \rho(x_1, x_2)$. Hence, for sufficiently large n we have $\rho := \frac{(ML_\delta T)^n}{n!} < 1$ and

$$\|F^{nk}(x_1)(t) - F^{nk}(x_2)(t)\| \leq \rho^k \rho(x_1, x_2), \quad k = 1, 2, \dots$$

Using the contraction principle (Banach Fixed Point Theorem) we deduce the existence of a unique fixed point x^* of the operator F in \mathbf{X} , which, obviously, corresponds to a mild solution x^δ of the nonlinear system (15). This solution can be prolonged for all $t \geq t_0$. Suppose the converse, i.e. exists $t_1 \in \mathbb{R}_+$ such that $\|x^\delta(t)\| \rightarrow +\infty$ as $t \rightarrow t_1$. In this case, due to continuity of the mild solution, there exists $t' \in (t_0, t_1)$ such that $\|x^\delta(t)\| \geq 2\delta$ for all $t \in (t', t_1)$. The latter implies that $f_\delta(x^\delta(t), q(t)) = \mathbf{0}$ and

$$x^\delta(t) = \Phi(t - t')x^\delta(t'), \quad \forall t \geq t'.$$

Since Φ is a strongly continuous semigroup of linear bounded operators on \mathbb{B} then $\sup_{s \in [0, t_1 - t']} \|\Phi(s)\| < +\infty$ provided that $t_1 - t' < +\infty$. Hence, we conclude

$$\limsup_{t \rightarrow t_1} \|x_{x_0}(t)\| < +\infty.$$

Finally, if $x_0 \in K(\delta)$ then, obviously, the mild solution x^δ is the unique mild solution of (1) as long as $x^\delta(t) \in K(\delta)$. Tending $\delta \rightarrow +\infty$ we complete the proof. ■

Corollary 3 (Continuous dependence of initial data)

If all conditions of Lemma 2 are fulfilled then for any $T > 0$ and any $\bar{q} > 0$ there exists $L_{\delta, T, \bar{q}} > 0$ such that for all $x_1, x_2 \in \mathbb{B}$ and all $q \in L^\infty((t_0, +\infty), V) : \|q\|_\infty \leq \bar{q}$ one has

$$\|x_{x_1, q}^\delta(t) - x_{x_2, q}^\delta(t)\| \leq L_{\delta, T, \bar{q}} \|x_1 - x_2\|, \quad \forall t \in [t_0, t_0 + T]$$

where $x_{x_i, q}^\delta(t)$ is a mild solution of (15) with $x(t_0) = x_i$.

Proof. Since we have

$$\|x_{x_1, q}^\delta(t) - x_{x_2, q}^\delta(t)\| \leq \|\Phi(t - t_0)\| \cdot \|x_1 - x_2\| +$$

$$\int_{t_0}^t \|\Phi(t - \tau)\| \cdot \|f_\delta(x_{x_1, q}^\delta(\tau), q(\tau)) - f_\delta(x_{x_2, q}^\delta(\tau), q(\tau))\| d\tau$$

then taking into account Lipschitz continuity of $f_\delta(\cdot, q)$, for any $q \in L^\infty((0, T), \mathbb{V}) : \|q\|_\infty \leq \bar{q}$ we derive

$$\|x_{x_1, q}^\delta(t) - x_{x_2, q}^\delta(t)\| \leq$$

$$M\|x_1 - x_2\| + ML_{\delta, \bar{q}} \int_{t_0}^t \|x_{x_1, q}^\delta(\tau) - x_{x_2, q}^\delta(\tau)\| d\tau,$$

where $M := \sup_{s \in [0, T-t_0]} \|\Phi(s)\| < +\infty$. Hence, using Grönwall-Bellman inequality we derive

$$\|x_{x_1, q}^\delta(t) - x_{x_2, q}^\delta(t)\| \leq M e^{ML_{\delta, \bar{q}}(t-t_0)} \|x_1 - x_2\|$$

for all $t \in [t_0, t_0 + T]$. Taking $L_{\delta, T, \bar{q}} := M e^{ML_{\delta, \bar{q}} T}$ we complete the proof. ■

Corollary 4 Let all conditions of Lemma 2 are fulfilled. If $x_0 \in \mathcal{D}(A)$ then the corresponding mild solution $x_{x_0, q}^\delta$ of (15) is a locally Lipschitz continuous function of time.

Proof. Indeed, if $x_0 \in \mathcal{D}(A)$ then $A\Phi(s)x_0 = \Phi(s)Ax_0$,

$$\Phi(s+h)x_0 - \Phi(s)x_0 = A \int_s^{s+h} \Phi(\tau)x_0 d\tau$$

for all $s \geq 0$ and all $h > 0$ (see Pazy (1983), page 5). Let us denote

$$C(T) := \text{ess sup}_{s \in [0, T]} \|f_\delta(x_{x_0, q}(s), q(s))\|.$$

From Lemma 2 we conclude that $C(T) < +\infty$ for any $x_0 \in \mathbb{B}$. In this case,

$$\|x_{x_0, q}^\delta(t+h) - x_{x_0, q}^\delta(t)\| \leq \|\Phi(t+h-t_0)x_0 - \Phi(t-t_0)x_0\| +$$

$$\int_t^{t+h} \|\Phi(t-\tau) f_\delta(x_{x_0, q}^\delta(\tau), q(\tau))\| d\tau \leq$$

$$\int_{t-t_0}^{t+h-t_0} \|\Phi(\tau)\| \cdot \|Ax_0\| d\tau + \int_t^{t+h} \|f_\delta(x_{x_0, q}^\delta(\tau), q(\tau))\| d\tau \cdot \sup_{s \in [0, h]} \|\Phi(s)\| \leq h(\|Ax_0\| + C(T)) \sup_{s \in [0, h]} \|\Phi(s)\|$$

for all $h \in [0, T)$. The proof is complete. ■

Corollary 5 *Let all conditions of Lemma 2 are fulfilled. Let the origin of (1) with $q = \mathbf{0}$ be globally uniformly Lyapunov stable. Let $r > 1$ be an arbitrary real number and*

$$\delta > \max \left\{ \varepsilon(r), \frac{1}{\varepsilon^{-1}(1/r)} \right\},$$

where $\varepsilon \in \mathcal{K}^\infty$ is given by (9) and ε^{-1} is the inverse function to ε . Then

1) for any $x_0 \in \mathbb{B} : 1/r \leq \|x_0\| \leq r$ we have

$$\|x_{x_0}^\delta(t)\| \leq \varepsilon(\|x_0\|) \leq \delta, \quad \forall t \geq 0$$

2) if $\exists t_1 \geq 0$ such that $\|x_{x_0}^\delta(t_1)\| \leq 1/\delta$ then

$$\|x_{x_0}^\delta(t)\| \leq 1/r, \quad \forall t \geq t_1,$$

where $x_{x_0}^\delta$ is the unique mild solution of (15) with $q = \mathbf{0}$ and $t_0 = 0$.

Proof. 1) Notice that the the inequality (9) and $x_{x_0}(0) = x_0$ yield $\varepsilon(r) \geq r$ and $1/\delta < 1/r < r < \delta$. Since $x_{x_0}^\delta$ coincide with x_{x_0} as long as $\|x_{x_0}^\delta(t)\| \in [1/\delta, \delta]$ then the inequality (9) implies $\|x_{x_0}^\delta(t)\| \leq \varepsilon(\|x_0\|) \leq \delta$ for all $t \geq 0$ and any $x_0 : \|x_0\| \leq [1/r, r]$. 2) Suppose the contrary, i.e. $\exists t_2 > t_1$ such that $\|x_{x_0}^\delta(t_2)\| > 1/r$ and $\|x_{x_0}^\delta(t)\| > 1/\delta$ for $t \in (t_1, t_2]$. In this case, we have $x_{x_0}^\delta(t) = x_{x_0}(t)$ for all $t \in (t_1, t_2]$ and using the inequality (9) we conclude

$$\|x_{x_0}^\delta(t)\| \leq \varepsilon(\|x_{x_0}^\delta(t_1)\|) = \varepsilon(1/\delta) \leq \varepsilon(\varepsilon^{-1}(1/r)) = 1/r$$

for all $t \geq t_1$. ■

5.2 The proof of Theorem 3

Sufficiency. Let $\delta > 1$ be an arbitrary number. Then by Lemma 2 the solution $x_{x_0}^\delta$ of the system

$$\dot{x} = Ax + f_\delta(x), \quad t > 0, \quad x(0) = x_0, \quad (16)$$

with

$$f_\delta(x) = a_\delta(x)f(x).$$

coincides with the solution x_{x_0} of (8) as long as $x_{x_0}^\delta(t) \in K(\delta)$. If $x_0 \in \mathcal{D}(A) \cap K(\delta)$ then by Corollary 4 the

function $t \rightarrow V(x_{x_0}^\delta(t))$ is a locally Lipschitz continuous function of time and differentiable almost everywhere. In this case, using the inequality (11) we conclude

$$\dot{V}(x_{x_0}^\delta(t)) \stackrel{a.e.}{=} \bar{D}^+ V(x_{x_0}^\delta(t)) \stackrel{a.e.}{\leq} -\|x_{x_0}^\delta(t)\|_\delta^{\mu+1},$$

the function $t \rightarrow V(x_{x_0}^\delta(t))$ is strictly decreasing and

$$V(x_{x_0}^\delta(t)) \leq V(x_0) - \int_0^t \|x_{x_0}^\delta(s)\|_\delta^{\mu+1} ds \leq$$

$$V(x_0) - \frac{1}{\bar{k}} \int_0^t V^{\mu+1}(x_{x_0}^\delta(s)) ds$$

as long as $x_{x_0}^\delta(t) \in K(\delta)$

Let $x_0 \in \mathbb{B} \cap K(\delta)$. Since $\mathcal{D}(A)$ dense in \mathbb{B} then there exists $x_i \in \mathcal{D}(A) \cap K(\delta)$ such that $x_i \rightarrow x_0$ as $i \rightarrow +\infty$. Using the continuous dependence of solutions on initial conditions (see Corollary 3), we derive

$$\sup_{t \in [0, T]} \|x_{x_0}^\delta(t) - x_{x_i}^\delta(t)\| \rightarrow 0 \text{ as } i \rightarrow +\infty$$

for any $T > 0$ such that $x_{x_0}^\delta(t) \in K(\delta), \forall t \in [0, T]$.

Tending $\delta \rightarrow +\infty$ we conclude that for any $x_0 \in \mathbb{B}$ the function $t \rightarrow V(x_{x_0}(t))$ is strictly decreasing and

$$V(x_{x_0}(t)) \leq V(x_0) - \frac{1}{\bar{k}} \int_0^t V^{\mu+1}(x_{x_0}(s)) ds$$

as long as $x_{x_0}(t) \neq \mathbf{0}$. Hence, for any $r > 0$ we derive

$$V(x_{x_0}(t)) \leq r, \quad \forall t \geq \max\{0, \bar{k}(V^{-\mu}(x_0) - r^{-\mu})\}.$$

The latter implies that mild solutions of (8) converges to zero uniformly on the initial data, i.e. for any $R > r > 0$ there exists $\hat{T} = \hat{T}(R, r)$ such that $\|x_{x_0}(t)\| \leq r$ for all $x_0 \in B(r)$ and for all $t \geq \hat{T}$. Moreover, for all $x_0 \in \mathbb{B} \setminus \{\mathbf{0}\}$ we have

$$\underline{k}\|x_{x_0}(t)\|_\delta \leq V(x_{x_0}(t)) \leq V(x_0) \leq \bar{k}\|x_0\|_\delta.$$

and

$$\|x_{x_0}(t)\| \leq \varepsilon(\|x_0\|) := \underline{\sigma}^{-1} \left(\frac{\bar{k}}{\underline{k}} \bar{\sigma}(\|x_0\|) \right) \quad (17)$$

as long as $x_{x_0}(t) \neq \mathbf{0}$, where $\underline{\sigma}^{-1}$ is the inverse function to $\underline{\sigma}$ (see Theorem 1). Since $\underline{\sigma}, \bar{\sigma} \in \mathcal{K}^\infty$ then $\varepsilon \in \mathcal{K}^\infty$.

Let us prove that if $x_{x_0}(t_0) = \mathbf{0}$ then $x_{x_0}(t) = \mathbf{0}$ for all $t \geq t_0$. Suppose the converse, i.e. $\exists t_1 > t_0 : \epsilon := \|x(t_1)\| > 0$. Then due to continuity of $t \rightarrow \|x(t)\|$ there exists $t_2 \in (t_0, t_1)$ such that

$\|x(t_2)\| = \delta := \min\{\epsilon/2, \epsilon^{-1}(\epsilon/2)\}$ and $\|x(t)\| > \delta$ for all $t \in (t_2, t_1]$, where $\epsilon^{-1} \in \mathcal{K}^\infty$ is the inverse function to ϵ . In this case, from the inequality (17) we derive

$$\|x_{x_0}(t)\| \leq \epsilon(\|x(t_2)\|) = \epsilon(\delta) \leq \epsilon(\epsilon^{-1}(\epsilon/2)) = \epsilon/2$$

for all $t \in (t_2, t_1]$. This contradicts to the supposition $\|x_{x_0}(t_1)\| = \epsilon$. Therefore, the inequality (17) holds for all $t \geq 0$ and any solution of (8) with $x_0 \in \mathbb{B}$.

Necessity. We design the Lyapunov function in two steps: first, we construct a function $V_r : \mathbb{B} \rightarrow \mathbb{R}_+$ such that $t \rightarrow V_r(x_0(t))$ is strictly decreasing as long as $x_{x_0}(t) \in K(r)$ for some $r > 1$; next, from V_r we derive a \mathfrak{d} -homogeneous Lyapunov function $V : \mathbb{B} \rightarrow \mathbb{R}_+$ by means of a special integral transformation inspired by Rosier (1992).

I. Let $s_0 > 0$,

$$r_0 := \sup_{s \in [-s_0, s_0]} \|\mathfrak{d}(s)\|$$

and $r > r_0^2 > 1$. Let $\delta > r$ and $a_\delta \in C_c^\infty$ be defined as in Corollary 5. Then for any $x_0 \in \mathbb{B}$ the system (16) has the unique mild solution $x_{x_0}^\delta : [0, +\infty) \rightarrow \mathbb{B}$ which, for $x_0 \in K(\delta)$, coincides with the unique solution x_{x_0} of (8) as long as $x_{x_0}^\delta(t) \in K(\delta)$. Moreover, by Corollary 3, for any $T > 0$ there exists $L_{\delta, T} > 0$ such that

$$\|x_{x_1}^\delta(t) - x_{x_2}^\delta(t)\| \leq L_{\delta, T} \|x_1 - x_2\|,$$

for all $t \in [0, T]$.

Since the origin of (8) is globally uniformly asymptotically stable then there exists $T_r > 0$ such that

$$\|x_0\| < r \quad \Rightarrow \quad \|x_{x_0}(t)\| \leq 1/r, \quad \forall t \geq T_r.$$

In the view of Lemma 2 and Corollary 5 we have $x_{x_0}^\delta(t) = x_{x_0}(t)$ for all $t \in [0, T_r]$ provided that $x_0 \in K(r)$. Indeed, the global uniform Lyapunov stability of (8) and the selection of the parameter $\delta > r$ imply that $\|x_{x_0}^\delta(t)\| \leq \delta$ for all $t \geq 0$ and all $x_0 \in K(r)$. Moreover, if there exists $t_1 > 0$ such that $\|x_{x_0}^\delta(t_1)\| \leq 1/\delta$ (resp. $\|x_{x_0}(t_1)\| \leq 1/\delta$) then $\|x_{x_0}^\delta(t)\| \leq 1/r$ (resp. $\|x_{x_0}(t)\| \leq 1/r$) for all $t \geq t_1$. In this case, for the functional $V_0 : \mathbb{B} \rightarrow \mathbb{R}_+$ given by

$$V_0(x_0) := \sup_{t \geq 0} \|x_{x_0}(t)\|$$

we have

$$V_0(x_0) = \sup_{t \in [0, T_r]} \|x_{x_0}^\delta(t)\|$$

for all $x_0 \in K(r)$. Hence, using the triangle inequality we derive

$$|V_0(x_1) - V_0(x_2)| \leq \sup_{t \in [0, T_r]} \|x_{x_1}^\delta(t) - x_{x_2}^\delta(t)\| \leq$$

$$L_{\delta, T_r} (\|x_1 - x_2\|), \quad \forall x_1, x_2 \in K(r).$$

The latter means that the functional V_0 satisfy the Lipschitz condition on $K(r)$.

The global uniform Lyapunov stability of (8) implies that $\|x\| \leq V_0(x) \leq \epsilon(\|x\|)$, where $\epsilon \in \mathcal{K}$ is given by (9). Moreover, the function $t \rightarrow V_0(x_{x_0}(t))$ is decreasing for all $t \geq 0$.

Let us consider the functional

$$V_1(x_0) := \int_0^{+\infty} \tilde{a}(\|x_{x_0}(s)\|) \|x_{x_0}(s)\| ds,$$

where $\tilde{a} \in C_c^\infty$ is such that $\tilde{a}(\rho) = 0$ for $s \notin (1/r, r)$, $0 < \tilde{a}(\rho) \leq 1$ for $\rho \in (1/r, r)$ and $\tilde{a}(\rho) = 1$ for $\rho \in [1/r_0, r_0]$.

The uniform asymptotic stability of the origin implies that $\tilde{a}(\|x_{x_0}(s)\|) \|x_{x_0}(s)\| = 0$ for all $s \geq T_r$ provided that $x_0 \in K(r)$. Repeating the above considerations we conclude that

$$V_1(x_0) = \int_0^{T_r} \tilde{a}(\|x_{x_0}^\delta(s)\|) \|x_{x_0}^\delta(s)\| ds$$

for all $x_0 \in K(r)$. Since $x \rightarrow \tilde{a}(\|x\|) \|x\|$ satisfy the Lipschitz condition on \mathbb{B} with some Lipschitz constant $\tilde{L} > 0$ (see the proof of Lemma 2) then

$$|V_1(x_1) - V_1(x_2)| \leq$$

$$\left| \int_0^{T_r} \tilde{a}(\|x_{x_1}^\delta(s)\|) \|x_{x_1}^\delta(s)\| - \tilde{a}(\|x_{x_2}^\delta(s)\|) \|x_{x_2}^\delta(s)\| ds \right| \leq \int_0^{T_r} |\tilde{a}(\|x_{x_1}^\delta(s)\|) \|x_{x_1}^\delta(s)\| - \tilde{a}(\|x_{x_2}^\delta(s)\|) \|x_{x_2}^\delta(s)\| ds \leq$$

$$\tilde{L} \int_0^{T_r} \|x_{x_1}^\delta(s) - x_{x_2}^\delta(s)\| ds \leq$$

$$T_r L_{\delta, T_r} \tilde{L} \|x_1 - x_2\|, \quad \forall x_1, x_2 \in K(r),$$

i.e. V_1 also satisfies the Lipschitz condition on $K(r)$.

Moreover, if $h > 0$ and $t \geq 0$ are such that $x_{x_0}(t + \theta) \in K(r)$, $\forall \theta \in [0, h]$ then

$$\begin{aligned} V_1(x_{x_0}(t+h)) - V_1(x_{x_0}(t)) &= \\ &= \int_0^{T_r} \tilde{a}(\|x_{x(t+h)}(s)\|) \|x_{x(t+h)}(s)\| ds - \\ &= \int_0^{T_r} \tilde{a}(\|x_{x(t)}(s)\|) \|x_{x(t)}(s)\| ds - \\ &= \int_0^{T_r} \tilde{a}(\|x_{x(t+h)}(s)\|) \|x_{x(t+h)}(s)\| ds - \\ &= \int_0^h \tilde{a}(\|x_{x(t)}(s)\|) \|x_{x(t)}(s)\| ds - \end{aligned}$$

$$\begin{aligned}
& \int_h^{T_r+h} \tilde{a}(\|x_{x(t)}(s)\|) \|x_{x(t)}(s)\| ds = \\
& \int_0^{T_r} \tilde{a}(\|x_{x(t+h)}(s)\|) \|x_{x(t+h)}(s)\| ds - \\
& \int_0^h \tilde{a}(\|x_{x(t)}(s)\|) \|x_{x(t)}(s)\| ds - \\
& \int_0^{T_r} \tilde{a}(\|x_{x(t)}(s+h)\|) \|x_{x(t)}(s+h)\| ds = \\
& - \int_0^h \tilde{a}(\|x_{x(t)}(s)\|) \|x_{x(t)}(s)\| ds = \\
& -\tilde{a}(\|x_{x_0}(t+\theta_h)\|) \|x_{x_0}(t+\theta_h)\| h, \quad \theta_h \in [0, h],
\end{aligned}$$

where the mean value theorem for integrals is utilized on the last step. The latter means that

$$\limsup_{h \rightarrow 0^+} \frac{V_1(x_{x_0}(t+h)) - V_1(x_{x_0}(t))}{h} \leq -\tilde{a}(\|x_{x_0}(t)\|) \|x_{x_0}(t)\|$$

as long as $x(t) \in K(r)$. Notice that $0 \leq V_1(x_0) \leq T_r \varepsilon(\|x_0\|)$.

In this case, the functional $V_r : \mathbb{B} \rightarrow \mathbb{R}_+$ given by

$$V_r(x) := V_0(x) + V_1(x)$$

satisfies the Lipschitz condition on $K(r)$ and

$$\|x\| \leq V_r(x) \leq (1 + T_r) \varepsilon(\|x\|)$$

for all $x \in \mathbb{B}$. Moreover, for any $x_0 \in K(r)$ the function $t \rightarrow V_r(x_{x_0}(t))$ is strictly decreasing and

$$\limsup_{h \rightarrow 0^+} \frac{V_r(x_{x_0}(t+h)) - V_r(x_{x_0}(t))}{h} \leq -W_r(\|x_{x_0}(t)\|)$$

as long as $x_{x_0}(t) \in K(r)$, where $W_r(\rho) := \tilde{a}(\rho)\rho$ with $\rho \in \mathbb{R}_+$. Moreover, due to homogeneity of the system (8) the solutions x_{x_0} are symmetric (see Theorem 2). Hence, if $\mathfrak{d}(-s)x_{x_0}(t+\theta) \in K(r)$ for all $\theta \in [0, h]$ then using the identity $\mathfrak{d}(-s)x_{x_0}(t) = x_{\mathfrak{d}(-s)x_0}(e^{\mu s}t)$, $\forall s \in \mathbb{R}$ we derive

$$V_r(\mathfrak{d}(-s)x_{x_0}(t+h)) - V_r(\mathfrak{d}(-s)x_{x_0}(t)) =$$

$$\begin{aligned}
& V_r(x_{\mathfrak{d}(-s)x_0}(e^{\mu s}(t+h))) - V_r(x_{\mathfrak{d}(-s)x_0}(e^{\mu s}t)) \leq \\
& -\tilde{a}(\|x_{\mathfrak{d}(-s)x_0}(e^{\mu s}t + \theta_h)\|) \|x_{\mathfrak{d}(-s)x_0}(e^{\mu s}t + \theta_h)\| e^{\mu s} h,
\end{aligned}$$

where $\theta_h \in [0, e^{\mu s}h]$, and $\mu \in \mathbb{R}$ is the homogeneity degree of the system (8), i.e. the function $t \rightarrow V_r(\mathfrak{d}(-s)x_{x_0}(t))$ is strictly decreasing and

$$\limsup_{h \rightarrow 0^+} \frac{V_r(\mathfrak{d}(-s)x_{x_0}(t+h)) - V_r(\mathfrak{d}(-s)x_{x_0}(t))}{h} \leq$$

$$-\tilde{a}(\|\mathfrak{d}(-s)x_{x_0}(t)\|) \|\mathfrak{d}(-s)x_{x_0}(t)\| e^{\mu s}$$

as long as $\mathfrak{d}(-s)x_{x_0}(t) \in K(r)$.

II. Inspired by Rosier (1992) let us consider the functional $V : \mathbb{B} \rightarrow \mathbb{R}_+$ given by

$$V(x) := \int_{-\infty}^{+\infty} e^{s-q} \hat{a}(V_r(\mathfrak{d}(-s)x)) ds$$

where $\hat{a} \in C^\infty$ is an increasing function such that $\hat{a}(\rho) = 1$ for $\rho \geq (1+T_r)\varepsilon(r)$, $\hat{a}(\rho) = 0$ for $\rho \leq 1/r$ and $\hat{a}'(\rho) > 0$ for $1/r < \rho < (1+T_r)\varepsilon(r)$,

$$q := \ln \frac{\min\{e^{(1+\mu)s_0}, e^{-(1+\mu)s_0}\}}{r_0} \frac{\min_{1/r_0 \leq \rho \leq (1+T_r)\varepsilon(r_0)} \hat{a}'(\rho)}{r_0}.$$

By construction, the function $V : \mathbb{B} \rightarrow \mathbb{R}_+$ is \mathfrak{d} -homogeneous of the degree 1. Moreover,

$$V(x) \leq \int_{-\infty}^{+\infty} e^{s-q} \hat{a}((1+T_r)\varepsilon(\|\mathfrak{d}(-s)x\|)) ds$$

$$\leq \int_{-\infty}^{\bar{s}(x)} e^{s-q} \hat{a}((1+T_r)\varepsilon(\|\mathfrak{d}(-s)x\|)) ds \leq e^{-q} e^{\bar{s}(x)}$$

where $\bar{s}(x) := \ln \left(\|x\|_{\mathfrak{d}} / \underline{\sigma} \left(\varepsilon^{-1} \left(\frac{1}{r(1+T_r)} \right) \right) \right)$ and $\underline{\sigma} \in \mathcal{K}^\infty$ is defined in Theorem 1. Indeed, if $s \geq \bar{s}(x)$ then from Theorem 1 we derive

$$\underline{\sigma}(\|\mathfrak{d}(-s)x\|) \leq \|\mathfrak{d}(-s)x\|_{\mathfrak{d}} = e^{-s} \|x\|_{\mathfrak{d}} \leq$$

$$e^{-\bar{s}(x)} \|x\|_{\mathfrak{d}} = \underline{\sigma} \left(\varepsilon^{-1} \left(\frac{1}{r(1+T_r)} \right) \right)$$

and taking into account $\underline{\sigma}, \varepsilon \in \mathcal{K}^\infty$ we conclude $\|\mathfrak{d}(-s)x\| \leq \varepsilon^{-1} \left(\frac{1}{r(1+T_r)} \right)$ or, equivalently, $(1+T_r)\varepsilon(\|\mathfrak{d}(-s)x\|) \leq 1/r$ for all $s \geq \bar{s}(x)$. The latter means $V_r(\mathfrak{d}(-s)x) \leq 1/r$ and $\hat{a}(V_r(\mathfrak{d}(-s)x)) = 0$ for all $s \geq \bar{s}(x)$.

Similarly, we derive

$$V(x) \geq \int_{-\infty}^{+\infty} e^{s-q} \hat{a}(\|\mathfrak{d}(-s)x\|) ds \geq$$

$$\int_{-\infty}^{\underline{s}(x)} e^{s-q} ds = e^{-q} e^{\underline{s}(x)},$$

where $\underline{s}(x) := \ln(\|x\|_{\mathfrak{d}} / \bar{\sigma}((1+T_r)\varepsilon(r)))$. Indeed, if $s \leq \underline{s}(x)$ then from Theorem 1 we have

$$\bar{\sigma}(\|\mathfrak{d}(-s)x\|) \geq \|\mathfrak{d}(-s)x\|_{\mathfrak{d}} = e^{-s} \|x\|_{\mathfrak{d}} \geq$$

$$e^{-\underline{s}(x)} \|x\|_{\mathfrak{d}} = \bar{\sigma}((1+T_r)\varepsilon(r)).$$

Since $\bar{\sigma} \in \mathcal{K}^\infty$ then $\|\mathfrak{d}(-s)x\| \geq (1+T_r)\varepsilon(r)$ and $\hat{a}(\|\mathfrak{d}(-s)x\|) = 1$ for all $s \leq \underline{s}(x)$. Therefore, we have proven that the inequality (10) holds for

$$\underline{k} := \frac{e^{-q}}{\bar{\sigma}((1+T_r)\varepsilon(r))}, \quad \bar{k} := \frac{e^{-q}}{\underline{\sigma} \left(\varepsilon^{-1} \left(\frac{1}{r(1+T_r)} \right) \right)}.$$

If $x(t+\theta) \in K(r_0)$ for all $\theta \in [0, h]$ then $\mathfrak{d}(-s)x(t+\theta) \in K(r)$ for all $\theta \in [0, h]$ and for all $s \in [-s_0, s_0]$. Indeed,

$$\|\mathfrak{d}(-s)x(t+\theta)\| \leq \|\mathfrak{d}(-s)\| \cdot \|x(t+\theta)\| \leq r_0^2 = r$$

and

$$1/r_0 \leq \|x(t+\theta)\| = \|\mathfrak{d}(s)\| \cdot \|\mathfrak{d}(-s)x(t+\theta)\|.$$

The latter means that

$$1/r \leq V_r(\mathfrak{d}(-s)x_{x_0}(t+\theta)) \leq (1+T_r)\varepsilon(r)$$

for all $s \in [-s_0, s_0]$ and all $\theta \in [0, h]$. Using

$$\begin{aligned} V_r(\mathfrak{d}(-s)x_{x_0}(t+h)) - V_r(\mathfrak{d}(-s)x_{x_0}(t)) &\leq \\ -W_r(x_{\mathfrak{d}(-s)x_0}(e^{\mu s}t + \theta_h))e^{\mu s}h &= \\ -W_r(\mathfrak{d}(-s)x_{x_0}(t + e^{-\mu s}\theta_h))e^{\mu s}h & \end{aligned}$$

we derive

$$V(x_{x_0}(t+h)) - V(x_{x_0}(t)) =$$

$$\begin{aligned} &\int_{-\infty}^{+\infty} e^{s-q} [\hat{a}(V_r(\mathfrak{d}(-s)x_{x_0}(t+h))) - \hat{a}(V_r(\mathfrak{d}(-s)x_{x_0}(t)))] ds = \\ &\int_{-s_0}^{s_0} e^{s-q} [\hat{a}(V_r(\mathfrak{d}(-s)x_{x_0}(t+h))) - \hat{a}(V_r(\mathfrak{d}(-s)x_{x_0}(t)))] ds = \\ &\int_{-s_0}^{s_0} e^{s-q} \hat{a}'(\theta) [V_r(\mathfrak{d}(-s)x_{x_0}(t+h)) - V_r(\mathfrak{d}(-s)x_{x_0}(t))] ds \leq \\ &-h \int_{-s_0}^{s_0} e^{s-q} \hat{a}'(\theta) W_r(\mathfrak{d}(-s)x_{x_0}(t + e^{-\mu s}\theta_h)) e^{\mu s} ds, \end{aligned}$$

where the mean value theorem is utilized with $\theta \in [V_r(\mathfrak{d}(-s)x_{x_0}(t+h)), V_r(\mathfrak{d}(-s)x_{x_0}(t))]$. Therefore, the function $t \rightarrow V(x_{x_0}(t))$ is strictly decreasing and

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{V(x_{x_0}(t+h)) - V(x_{x_0}(t))}{h} &\leq \\ - \int_{-\infty}^{+\infty} e^{(1+\mu)s-q} \hat{a}'(V_r(\mathfrak{d}(-s)x_{x_0}(t))) W_r(\|\mathfrak{d}(-s)x_{x_0}(t)\|) ds &= \\ -W(x_{x_0}(t)), & \end{aligned}$$

as long as $x_{x_0}(t) \in K(r_0)$, where

$$W(x) := \int_{-\infty}^{+\infty} e^{(1+\mu)s-q} \hat{a}'(V_r(\mathfrak{d}(-s)x)) W_r(\|\mathfrak{d}(-s)x\|) ds.$$

The functional W is, obviously, nonnegative and \mathfrak{d} -homogeneous of the degree $\mu + 1$. If $\|x\| = 1$ then $\mathfrak{d}(-s)x \in K(r_0)$ and $W_r(\|\mathfrak{d}(-s)x\|) = \|\mathfrak{d}(-s)x\|$ for all $s \in [-s_0, s_0]$. Using the mean value theorem for integrals we derive

$$W(x) \geq \int_{-s_0}^{s_0} e^{(1+\mu)s-q} \hat{a}'(V_r(\mathfrak{d}(-s)x)) \|\mathfrak{d}(-s)x\| ds \geq$$

$$\frac{1}{r_0} e^{(1+\mu)s^*} \hat{a}'(V_r(\mathfrak{d}(-s^*)x)),$$

where $s^* \in [-s_0, s_0]$. Since $\mathfrak{d}(-s^*)x \in K(r_0)$ then

$$\frac{1}{r} < \frac{1}{r_0} \leq V_r(\mathfrak{d}(-s^*)x) \leq (1+T_r)\varepsilon(r_0) < (1+T_r)\varepsilon(r).$$

Hence, we derive

$$\inf_{\|x\|=1} W(x) \geq \frac{\min\{e^{(1+\mu)s_0}, e^{-(1+\mu)s_0}\} \min_{1/r_0 \leq \rho \leq (1+T_r)\varepsilon(r_0)} \hat{a}'(\rho)}{e^q r_0} = 1$$

and using \mathfrak{d} -homogeneity of W we conclude

$$W(x) = W(\mathfrak{d}(\ln \|x\|_{\mathfrak{d}}) \mathfrak{d}(-\ln \|x\|_{\mathfrak{d}}) x) =$$

$$\|x\|_{\mathfrak{d}}^{1+\mu} W(\mathfrak{d}(\ln \|x\|_{\mathfrak{d}}) \mathfrak{d}(-\ln \|x\|_{\mathfrak{d}}) x) \leq \|x\|_{\mathfrak{d}}^{1+\mu}.$$

We have proven that

$$\limsup_{h \rightarrow 0^+} \frac{V(x_{x_0}(t+h)) - V(x_{x_0}(t))}{h} \leq -\|x_{x_0}(t)\|_{\mathfrak{d}}^{1+\mu}$$

as long as $x_{x_0}(t) \in K(r_0)$. If $x_{x_0}(t) \in \mathfrak{d}(\tau)K(r_0)$, where $\tau \in \mathbb{R}$, then $\mathfrak{d}(-\tau)x_{x_0}(t) \in K(r_0)$. Using homogeneity of V we derive

$$\begin{aligned} V(x_{x_0}(t)) &= V(\mathfrak{d}(-\tau) \mathfrak{d}(\tau)x_{x_0}(t)) = \\ &e^{-\tau} V(\mathfrak{d}(\tau)x_{x_0}(t)). \end{aligned}$$

Since by Theorem 2 we have $x_{\mathfrak{d}(\tau)x_0}(e^{-\mu\tau}t) = \mathfrak{d}(\tau)x_{x_0}(t)$ then the function $t \rightarrow V(x_{x_0}(t))$ is strictly decreasing as well and

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{V(x_{x_0}(t+h)) - V(x_{x_0}(t))}{h} &= \\ e^{-\tau s} \limsup_{h \rightarrow 0^+} \frac{V(x_{\mathfrak{d}(\tau)x_0}(e^{-\mu\tau}(t+h))) - V(x_{\mathfrak{d}(\tau)x_0}(e^{-\mu\tau}t))}{h} &\leq \\ -e^{-\tau(1+\mu)} \|x_{\mathfrak{d}(\tau)x_0}(e^{-\mu\tau}t)\|_{\mathfrak{d}}^{1+\mu} &= \\ -\|\mathfrak{d}(-\tau)x_{\mathfrak{d}(\tau)x_0}(e^{-\mu\tau}t)\|_{\mathfrak{d}}^{1+\mu} &= -\|x_{x_0}(t)\|_{\mathfrak{d}}^{1+\mu} \end{aligned}$$

as long as $x_{x_0}(t) \in \mathfrak{d}(\tau)K(r_0)$, where $\tau \in \mathbb{R}$ is any real number. Taking into account $\bigcup_{\tau \in \mathbb{R}} \mathfrak{d}(\tau)K(r_0) = \mathbb{B} \setminus \{\mathbf{0}\}$ we complete the proof.

5.3 The proof of Corollary 1

Proof. Since \mathbb{B} is a reflexive Banach space then by Theorem 1.6, Chapter 6, Pazy (1983) any mild solution x of (8) with $x_0 \in \mathcal{D}(A)$ is a strong solution and by Theorem 2.9, Chapter 4, Pazy (1983) we have $x(t) \stackrel{a.e.}{\in} \mathcal{D}(A)$. In this case, using the representation

$$x(t+h) = \Phi(t+h)x_0 + \int_0^t \Phi(t-s)f(x(s))ds =$$

$$\begin{aligned}
& x(t) + (\Phi(h) - I)\Phi(t)x_0 + \\
& \int_0^{t+h} \Phi(t+h-s)f(x(s))ds - \int_0^t \Phi(t-s)f(x(s))ds = \\
& (\Phi(h) - I) \left(\Phi(t)x_0 + \int_0^t \Phi(t-s)f(x(s))ds \right) + \\
& \int_t^{t+h} \Phi(t+h-s)f(x(s))ds = \\
& x(t) + (\Phi(h) - I)x(t) + \int_0^h \Phi(s)f(x(t+h-s))ds
\end{aligned}$$

we derive for $x(t) \in \mathcal{D}(A)$

$$\begin{aligned}
\frac{\|x(t+h)-x(t)-h(Ax(t)+f(x(t)))\|}{h} & \leq \left\| Ax(t) - \frac{(\Phi(h)-I)x(t)}{h} \right\| + \\
& \left\| f(x(t)) - \frac{1}{h} \int_0^h \Phi(s)f(x(t+h-s))ds \right\| \leq \\
\left\| Ax(t) - \frac{(\Phi(h)-I)x(t)}{h} \right\| + \left\| f(x(t)) - \frac{1}{h} \int_0^h \Phi(s)f(x(t))ds \right\| & + \\
\frac{1}{h} \int_0^h \|\Phi(s)\| \|f(x(t)) - f(x(t+h-s))\| ds & \rightarrow 0 \text{ as } h \rightarrow 0,
\end{aligned}$$

where the properties of the semigroup Φ (see Pazy (1983), page 5), the local Lipschitz continuity of f and continuity of x are utilized on the last step. Since

$$\begin{aligned}
\frac{V(x(t+h))-V(x(t))}{h} & = \frac{V(x(t+h))-V(x(t)+h(Ax(t)+f(x(t))))}{h} + \\
& \frac{V(x(t)+h(Ax(t)+f(x(t))))-V(x(t))}{h}
\end{aligned}$$

and V is locally Lipschitz continuous then, taking into account

$$\frac{|V(x(t+h))-V(x(t)+h(Ax(t)+f(x(t))))|}{h} \leq$$

$$\frac{L\|x(t+h)-x(t)-h(Ax(t)+f(x(t)))\|}{h} \rightarrow 0 \text{ as } h \rightarrow 0, \quad x(t) \in \mathcal{D}(A),$$

we derive

$$\overline{D}^+ V(x(t)) \stackrel{a.e.}{=} \overline{D}^+ V(x(t), Ax(t)+f(x(t))), \quad \forall x_0 \in \mathcal{D}(A) \setminus \{\mathbf{0}\}$$

if \mathbb{B} is a reflexive Banach space.

Sufficiency. The latter identity and the condition 3') imply the fulfillment of the condition 3) of Theorem 3.

Necessity. In the proof of Theorem 3 we design a Lyapunov function satisfying 1), 2) and

$$\overline{D}^+ V(x(t)) \leq -\|x(t)\|_{\mathfrak{D}}^{1+\mu},$$

as long as $x(t) \in \mathbb{B} \setminus \{\mathbf{0}\}$.

Since \mathbb{B} is a reflexive Banach space then for any $x_0 \in \mathcal{D}(A)$ the mild solution x is the strong one and $x(t) \stackrel{a.e.}{\in}$

$\mathcal{D}(A)$. Let $t_j \rightarrow 0, t_j > 0$ as $j \rightarrow +\infty$ be an arbitrary be a sequence of time instances such that the latter inclusion holds. Since any mild solution is continuous function of time then for any $h > 0$ there exists $i : \|x(t_i) - x_0\| < h^2$ and $\|x(t_i + h) - x(t_i)\| < h^2$. Obviously, $i \rightarrow +\infty$ as $h \rightarrow 0$.

Let us consider

$$\begin{aligned}
& \frac{V(x_0+h(Ax_0+f(x_0)))-V(x_0)}{h} = \\
& \frac{V(x_0+h(Ax_0+f(x_0)))-V(x_0+h(Ax(t_i)+f(x(t_i))))}{h} + \\
& \frac{V(x_0+h(Ax(t_i)+f(x(t_i))))-V(x(t_i)+h(Ax(t_i)+f(x(t_i))))}{h} + \\
& \frac{V(x(t_i)+h(Ax(t_i)+f(x(t_i))))-V(x(t_i+h))}{h} + \\
& \frac{V(x(t_i+h))-V(x(t_i))}{h} + \frac{V(x(t_i))-V(x_0)}{h}.
\end{aligned}$$

The third term in the above sum tends to zero as $h \rightarrow 0^+$ (see above). Since the operator A is closed then $x(t_i) \rightarrow x_0, x(t_i) \in \mathcal{D}(A)$ implies $Ax(t_i) \rightarrow Ax_0$ and

$$\frac{|V(x_0+h(Ax_0+f(x_0)))-V(x_0+h(Ax(t_i)+f(x(t_i))))|}{h} \leq$$

$$\|Ax_0 - Ax(t_i)\| + \|f(x_0) - f(x(t_i))\| \rightarrow 0 \text{ as } h \rightarrow 0.$$

Using the local Lipschitz continuity of V we derive

$$\frac{|V(x_0+h(Ax(t_i)+f(x(t_i))))-V(x(t_i)+h(Ax(t_i)+f(x(t_i))))|}{h} \leq$$

$$\frac{L\|x_0-x(t_i)\|}{h} \leq Lh \rightarrow 0 \text{ as } h \rightarrow 0.$$

Similarly, we obtain

$$\frac{|V(x(t_i+h))-V(x(t_i))|}{h} \leq \frac{Lx_0\|x_0-x(t_i)\|}{h} \leq Lh \rightarrow 0 \text{ as } h \rightarrow 0.$$

Therefore,

$$\limsup_{h \rightarrow 0^+} \frac{V(x_0+h(Ax_0+f(x_0)))-V(x_0)}{h} =$$

$$\limsup_{h \rightarrow 0^+} \frac{V(x(t_i))-V(x_0)}{h} \leq -\|x_0\|_{\mathfrak{D}}^{\mu+1}, \quad \forall x_0 \in \mathcal{D}(A).$$

The proof is complete. ■

5.4 The proof of Theorem 4

Since the \mathfrak{D} -homogeneous operator A generates a strongly continuous semigroup Φ , then according Lemma 1 we have

$$\Phi(\tau)\mathfrak{D}(s) = \mathfrak{D}(s)\Phi(e^{\mu s}\tau), \quad \forall \tau \geq 0, \quad \forall s \in \mathbb{R}.$$

Recall (see e.g. Driver (2003), page 187) that $K \int_a^b \xi(s)ds = \int_a^b K\xi(s)ds$ for any bounded linear operator $K : \mathbb{B} \rightarrow \mathbb{B}$ and any Bochner integrable function $\xi \in L^1((a, b), \mathbb{B})$.

Let $x_{x_0,q}$ be a solution of (1). Then using $f(x_{x_0,q}(\cdot), q(\cdot)) \in L^1((t_0, t_0 + T), \mathbb{B})$ we derive

$$\begin{aligned} \mathfrak{d}(s)x_{x_0,q}(t_0 + e^{\mu s}(t - t_0)) &= (\mathfrak{d}(s)\Phi(e^{\mu s}(t - t_0)))x_0 + \\ &\int_{t_0}^{t_0 + e^{\mu s}(t - t_0)} \mathfrak{d}(s)\Phi(t_0 + e^{\mu s}(t - t_0) - \tau)f(x_{x_0,q}(\tau), q(\tau)) d\tau = \\ &\Phi(t - t_0)\mathfrak{d}(s)x_0 + e^{\mu s} \int_{t_0}^t \mathfrak{d}(s)\Phi(e^{\mu s}(t - \sigma)) \circ \\ &f(x_{x_0,q}(t_0 + e^{\mu s}(\sigma - t_0)), q(t_0 + e^{\mu s}(\sigma - t_0))) d\sigma = \\ &\Phi(t - t_0)\mathfrak{d}(s)x_0 + e^{\mu s} \int_{t_0}^t \Phi(t - \sigma) \circ \\ \mathfrak{d}(s)f(x_{x_0,q}(t_0 + e^{\mu s}(\sigma - t_0)), q(t_0 + e^{\mu s}(\sigma - t_0))) d\sigma = \\ &\Phi(t)\mathfrak{d}(s)x_0 + \int_{t_0}^t \Phi(t - \sigma) \circ \end{aligned}$$

$f(\mathfrak{d}(s)x_{x_0,q}(t_0 + e^{\mu s}(\sigma - t_0)), \tilde{\mathfrak{d}}(s)q(t_0 + e^{\mu s}(\sigma - t_0)))d\sigma$, where the linearity of the operator $\Phi(t - \sigma)$ and the \mathfrak{d} -homogeneity of Φ , f are utilized on the last two steps. The proof is complete.

5.5 The proof of Corollary 2

I. Let $\bar{q} > 0$ be an arbitrary positive number. Let $r > 1$ and $\delta > r$ be defined for $f(\cdot, \mathbf{0})$ as in Corollary 5. Since for $q = \mathbf{0}$ the origin of (1) is uniformly asymptotically stable then exists $T_r > 0$ such that

$$x(t_0) \in K(r) \Rightarrow \|x_{x_0,\mathbf{0}}(t)\| \leq 1/r, \quad \forall t > t_0 + T_r.$$

and

$$x(t_0) \in K(r) \Rightarrow \|x_{x_0,\mathbf{0}}(t)\| \leq \delta, \quad \forall t \geq t_0,$$

Moreover, Corollary 5 yields

$$x_{x_0,\mathbf{0}}(t) = x_{x_0,\mathbf{0}}^\delta(t),$$

as long as $x_{x_0,\mathbf{0}}(t) \in K(\delta)$.

For any $x_0 \in \mathbb{B}$ we have

$$\begin{aligned} x_{x_0,q}^\delta(t) &= \Phi(t)x_0 + \int_{t_0}^t \Phi(t - s)f_\delta(x_{x_0,q}^\delta(\tau), q(\tau))d\tau = \\ x_{x_0,\mathbf{0}}^\delta(t) &+ \int_{t_0}^t \Phi(t - s)(f_\delta(x_{x_0,q}^\delta(\tau), q(\tau)) - f_\delta(x_{x_0,\mathbf{0}}^\delta(\tau), \mathbf{0}))d\tau, \end{aligned}$$

Let $M = \sup_{s \in [0, T_r]} \|\Phi(s)\|$ and $L_{\delta, T_r, \bar{q}} > 0$ be defined as in Corollary 3. Then for any $q \in L^\infty((0, T_r), \mathbb{V}) : \|q\|_\infty \leq \bar{q}$ we have

$$\|x_{x_0,q}^\delta(t) - x_{x_0,\mathbf{0}}^\delta(t)\| \leq ML_{\delta, T_r, \bar{q}} \int_0^t \|x_{x_0,q}^\delta(\tau) - x_{x_0,\mathbf{0}}^\delta(\tau)\| +$$

$$+ M \int_{t_0}^t a_\delta(\|x_{x_0,\mathbf{0}}^\delta(\tau)\|) \|f(x_{x_0,\mathbf{0}}^\delta(\tau), q(\tau)) - f(x_{x_0,\mathbf{0}}^\delta(\tau), \mathbf{0})\| d\tau$$

for all $t \in [t_0, t_0 + T_r]$ such that $\|x_{x_0,q}(t)\| < +\infty$. By assumption on f there exist $\hat{L}_{\delta, \bar{q}} > 0$ such that

$$\|f(x_1, q_1) - f(x_1, q_2)\| \leq \hat{L}_{\delta, \bar{q}} (\|x_1 - x_2\| + \|q_1 - q_2\|^\gamma)$$

for all $x_1, x_2 \in K(2\delta)$ and for all $q_1, q_2 \in \mathbb{B}(\bar{q})$. Since $a_\delta(\|x\|) = 0$ for $x \notin K(2\delta)$ then

$$\begin{aligned} \|x_{x_0,q}^\delta(t) - x_{x_0,\mathbf{0}}^\delta(t)\| &\leq ML_{\delta, T_r, \bar{q}} \int_0^t \|x_{x_0,q}^\delta(\tau) - x_{x_0,\mathbf{0}}^\delta(\tau)\| + \\ &+ M \hat{L}_{\delta, \bar{q}} \int_{t_0}^t \|q(\tau)\|^\gamma d\tau \end{aligned}$$

and using Grönwall-Bellman inequality we derive

$$\|x_{x_0,q}^\delta(t) - x_{x_0,\mathbf{0}}^\delta(t)\| \leq \hat{L}_{\delta, \bar{q}} M e^{L_{\delta, T_r, \bar{q}} M t} \int_0^t \|q(\tau)\|^\gamma d\tau,$$

for all $t \in [0, T_r]$ and any $q \in L^\infty((0, T_r), \mathbb{V}) : \|q\|_\infty \leq \bar{q}$.

By Theorem 3 (see also its proof), for any $x_0 \in \mathbb{B} \setminus \{\mathbf{0}\}$ we have

$$V(x_{x_0,\mathbf{0}}(h)) - V(x_{x_0,\mathbf{0}}(0)) \leq - \int_0^h \|x_{x_0,\mathbf{0}}(\tau)\|_\delta^{\mu+1} d\tau$$

as long as $x_{x_0,\mathbf{0}}(h) \neq \mathbf{0}$.

For any $h > 0 : x_{x_0,\mathbf{0}}(\theta), x_{x_0,q}(\theta) \in K(\delta), \forall \theta \in [0, h]$ we have

$$x_{x_0,\mathbf{0}}(\theta) = x_{x_0,\mathbf{0}}^\delta(\theta), \quad x_{x_0,q}(\theta) = x_{x_0,q}^\delta(\theta), \quad \theta \in [0, h].$$

Then taking into account the local Lipschitz continuity of V and $x_{x_0,q}(0) = x_{x_0,\mathbf{0}}(0) = x_0$ we derive

$$\begin{aligned} V(x_{x_0,q}(h)) - V(x_{x_0,q}(0)) &= \\ V(x_{x_0,\mathbf{0}}(h)) - V(x_{x_0,\mathbf{0}}(0)) &+ V(x_{x_0,q}(h)) - V(x_{x_0,\mathbf{0}}(h)) \leq \\ &- \int_0^h \|x_{x_0,\mathbf{0}}(\tau)\|_\delta^{\mu+1} d\tau + L_{V,\delta} \|x_{x_0,q}(h) - x_{x_0,\mathbf{0}}(h)\| \leq \\ &- \int_0^h \|x_{x_0,\mathbf{0}}(\tau)\|_\delta d\tau + L_{V,\delta} \hat{L}_{\delta, \bar{q}} M e^{L_{\delta, T_r, \bar{q}} M h} \int_0^h \|q(\tau)\|^\gamma d\tau, \end{aligned}$$

for any $q \in L^\infty((0, h), \mathbb{V}) : \|q\|_\infty \leq \bar{q}$. Therefore, one has

$$\limsup_{h \rightarrow 0^+} \frac{V(x_{x_0,q}(h)) - V(x_{x_0,q}(0))}{h} \leq$$

$$-\|x_0\|_\delta^{1+\mu} + C(\bar{q}) \limsup_{h \rightarrow 0^+} \frac{1}{h} \int_0^h \|q(\tau)\|^\gamma d\tau,$$

where $C(\bar{q}) := L_{V,\delta} \hat{L}_{\delta,\bar{q}} M e^{L_{\delta,T_r,\bar{q}} M T_r}$. The function $\bar{q} \rightarrow C(\bar{q})$ is uniformly bounded on any compact from \mathbb{R}_+ .

Since $x_{x_0,q}(t+h) = x_{x^*,q^*}(h)$ with $x^* = x_{x_0,q}(t)$ and $q^*(h) := q(t+h)$ then the function $t \rightarrow V(x_{x_0,q}(t))$ is strictly decreasing and

$$\limsup_{h \rightarrow 0^+} \frac{V(x_{x_0,q}(h)) - V(x_{x_0,q}(0))}{h} \leq$$

$$- \|x_{x_0,q}(t)\|_{\mathfrak{d}}^{\mu+1} + C(\bar{q}) \limsup_{h \rightarrow 0^+} \frac{\int_0^h \|q(t+\tau)\|^\gamma d\tau}{h}$$

as long as $x_{x_0,q}(t) \in K(\delta)$ and

$$\|x_{x_0,q}(t)\|_{\mathfrak{d}}^{\mu+1} > C(\bar{q}) \limsup_{h \rightarrow 0^+} \frac{1}{h} \int_0^h \|q(t+\tau)\|^\gamma d\tau.$$

The latter conclusion holds for any $q \in L^\infty(\mathbb{R}, \mathbb{V})$ such that $\|q\|_{L^\infty((t_0,t), \mathbb{V})} \leq \bar{q}$.

II. Let $\mathfrak{d}(s)x_{x_0,q}(t) \in K(r)$ or, equivalently, $x_{x_0,q}(t) \in \mathfrak{d}(-s)K(r)$ for all $t \in [t_0, t_0 + T]$ and some $T > 0$. From Theorem 4 we derive

$$x_{\mathfrak{d}(s)x_0,\bar{q}}(\tilde{t}) = \mathfrak{d}(s)x_{x_0,q}(t_0 + e^{\mu s}(\tilde{t} - t_0)), \quad \forall \tilde{t} \in [t_0, t_0 + \frac{T}{e^{\mu s}}],$$

where $\tilde{t} := t_0 + e^{-\mu s}(t - t_0)$, $x_{\mathfrak{d}(s)x_0,\bar{q}} : [t_0, t_0 + e^{-\mu s}T] \rightarrow \mathbb{B}$ is a solution of the system (1) for x_0 replaced with $\mathfrak{d}(s)x_0$ and $q \in L^\infty((t_0, t_0 + T), \mathbb{V})$ replaced with $\tilde{q} \in L^\infty((t_0, t_0 + e^{-\mu s}T), \mathbb{V})$ given by

$$\tilde{q}(\tilde{t}) = \tilde{\mathfrak{d}}(s)q(t_0 + e^{\mu s}(\tilde{t} - t_0)), \quad \forall \tilde{t} \in [t_0, t_0 + \frac{T}{e^{\mu s}}].$$

Since $x_{\mathfrak{d}(s)x_0,\bar{q}}(\tilde{t}) = \mathfrak{d}(s)x_{x_0,q}(t_0 + e^{\mu s}(\tilde{t} - t_0)) \in K(r)$ then

$$\limsup_{h \rightarrow 0^+} \frac{V(x_{\mathfrak{d}(s)x_0,\bar{q}}(\tilde{t}+h)) - V(x_{\mathfrak{d}(s)x_0,\bar{q}}(\tilde{t}))}{h} \leq$$

$$- \|x_{\mathfrak{d}(s)x_0,\bar{q}}(\tilde{t})\|_{\mathfrak{d}}^{1+\mu} + C(\bar{q}) \limsup_{h \rightarrow 0^+} \frac{1}{h} \int_0^h \|\tilde{q}(\tilde{t} + \tau)\|^\gamma d\tau,$$

for all $\tilde{t} \in [t_0, t_0 + e^{-\mu s}T]$.

Taking into account homogeneity of V and $\|\cdot\|_{\mathfrak{d}}$ we derive

$$V(x_{\mathfrak{d}(s)x_0,\bar{q}}(\tilde{t})) = V(\mathfrak{d}(s)x_{x_0,q}(t_0 + e^{\mu s}(\tilde{t} - t_0))) =$$

$$e^s V(x_{x_0,q}(t_0 + e^{\mu s}(\tilde{t} - t_0))).$$

and

$$\|x_{\mathfrak{d}(s)x_0,\bar{q}}(\tilde{t})\|_{\mathfrak{d}}^{\mu+1} = e^{(\mu+1)s} \|x_{x_0,q}(t_0 + e^{\mu s}(\tilde{t} - t_0))\|_{\mathfrak{d}}.$$

The latter means that for any $q \in L^\infty(\mathbb{R}, \mathbb{V})$: $\|q\|_{L^\infty((t_0,t), \mathbb{V})} \leq \bar{q}$ and any $x_0 \in \mathbb{B} \setminus \{\mathbf{0}\}$ we have

$$\limsup_{h \rightarrow 0^+} \frac{V(x_{x_0,q}(t+h)) - V(x_{x_0,q}(t))}{h} \leq$$

$$- \|x_{x_0,q}(t)\|_{\mathfrak{d}}^{1+\mu} + e^{-(\mu+1)s} C(\bar{q}) \limsup_{h \rightarrow 0^+} \frac{\int_0^h \|\tilde{\mathfrak{d}}(s)q(t+e^{\mu s}\tau)\|^\gamma d\tau}{h},$$

as long as $x_{x_0,q}(t) \in \mathfrak{d}(-s)K(r)$.

Since the dilation $\tilde{\mathfrak{d}}$ is strictly monotone then there exists $\tilde{\beta} > 0$ such that

$$\|\tilde{\mathfrak{d}}(s)\| = e^{\tilde{\beta}s}, \quad \forall s \leq 0.$$

By Theorem 1 there exists $\sigma \in \mathcal{K}^\infty$ such that

$$\|q\|_{\tilde{\mathfrak{d}}} \leq \sigma(\|q\|) \leq \sigma(\bar{q})$$

for all $q \in \mathbb{V}$: $\|q\| \leq \bar{q}$. Hence, using the semigroup property pf $\tilde{\mathfrak{d}}$, we derive

$$\|\tilde{\mathfrak{d}}(s)q\| = \|\tilde{\mathfrak{d}}(s + \ln \sigma(\bar{q}))\tilde{\mathfrak{d}}(\ln \|q\|_{\tilde{\mathfrak{d}}} - \ln \sigma(\bar{q}))\tilde{\mathfrak{d}}(-\ln \|q\|_{\tilde{\mathfrak{d}}})q\| \leq$$

$$\|\tilde{\mathfrak{d}}(s + \ln \sigma(\bar{q}))\| \cdot \|\tilde{\mathfrak{d}}(\ln \|q\|_{\tilde{\mathfrak{d}}} - \ln \sigma(\bar{q}))\| \cdot \|\tilde{\mathfrak{d}}(-\ln \|q\|_{\tilde{\mathfrak{d}}})q\| \leq$$

$$\|\tilde{\mathfrak{d}}(s + \ln \sigma(\bar{q}))\| \leq e^{\tilde{\beta}(s + \ln \sigma(\bar{q}))}, \quad \forall s \leq -\ln \sigma(\bar{q}).$$

The latter means that

$$\limsup_{h \rightarrow 0^+} \frac{V(x_{x_0,q}(t+h)) - V(x_{x_0,q}(t))}{h} \leq$$

$$- \|x_{x_0,q}(t)\|_{\mathfrak{d}}^{1+\mu} + e^{(\tilde{\beta}\gamma - (\mu+1))s} C(\bar{q})\sigma(\bar{q}),$$

as long as $x_{x_0,q}(t) \in \mathfrak{d}(-s)K(r)$ and $s \leq -\ln \sigma(\bar{q})$.

Selecting $s = -\ln \|x_{x_0,q}(t)\|_{\mathfrak{d}}$ we derive

$$\|\mathfrak{d}(-\ln \|x_{x_0,q}(t)\|_{\mathfrak{d}})x_{x_0,q}(t)\| = 1$$

and $x_{x_0,q}(t) \in \mathfrak{d}(-s)K(r)$. For any $q \in L^\infty(\mathbb{R}, \mathbb{V})$: $\|q\|_{L^\infty((t_0,t), \mathbb{V})} \leq \bar{q}$ and any $x_0 \in \mathbb{B} \setminus \{\mathbf{0}\}$ we have

$$\limsup_{h \rightarrow 0^+} \frac{V(x_{x_0,q}(t+h)) - V(x_{x_0,q}(t))}{h} \leq$$

$$- \|x_{x_0,q}(t)\|_{\mathfrak{d}}^{1+\mu} \left(1 - \|x_{x_0,q}(t)\|_{\mathfrak{d}}^{-\tilde{\beta}\gamma} C(\bar{q})\sigma(\bar{q})\right),$$

as long as $\|x_{x_0,q}(t)\|_{\mathfrak{d}} \geq \sigma(\bar{q})$. Since $C : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is locally uniformly bounded on compacts from \mathbb{R}_+ and $\sigma \in \mathcal{K}^\infty$ then then there exists $\xi \in \mathcal{K}^\infty$ such that

$$\xi(\bar{q}) \geq \max\{\sigma(\bar{q}), (2C(\bar{q})\sigma(\bar{q}))^{1/(\tilde{\beta}\gamma)}\}.$$

Therefore, we have

$$\limsup_{h \rightarrow 0^+} \frac{V(x_{x_0,q}(t+h)) - V(x_{x_0,q}(t))}{h} \leq -0.5 \|x_{x_0,q}(t)\|_{\mathfrak{d}}^{1+\mu}$$

as long as

$$\|x_{x_0,q}(t)\|_{\mathfrak{d}} \geq \xi(\bar{q}),$$

where $x_{x_0,q}$ is any solution of (1) with an arbitrary $q \in L^\infty(\mathbb{R}, \mathbb{V})$: $\|q\|_{L^\infty((t_0,t), \mathbb{V})} \leq \bar{q}$. The latter means that V is the ISS Lyapunov function of the system. Repeating the conventional arguments (see e.g. Sontag and Wang (1996) for more details) we complete the proof.