On parameter tuning and convergence properties of the DREM procedure

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Abstract— The recently proposed Dynamic Regressor Extension and Mixing (DREM) procedure has been proven to enhance transient performance in online parameter estimation and it has been successfully applied to a variety of adaptive control problems and applications. However, to use this procedure, a linear operator has to be chosen to perform the dynamic extension. A poor choice of the operator can reduce excitation of signals and hence it can compromise convergence properties. This paper presents a systematic selection of operators such that the excitation is always preserved. The paper also studies convergence conditions when the DREM procedure is combined with a least-squares estimator.

I. INTRODUCTION

The linear regression equation (LRE) plays a central role in adaptive parameter estimation and adaptive control. It appears in system identification [1], in model-reference adaptive control [2], [3] and adaptive pole-placement [4], in filtering and prediction [5], in reinforcement learning [6], and in many others areas. The linear regression model is given by

$$y(t) = \phi^\top(t)\theta + w(t), \quad (1)$$

where $y(t) \in \mathbb{R}$ is the output signal, $\phi(t) \in \mathbb{R}^n$ is the regressor, $w(t) \in \mathbb{R}$ is an additive distortion, e.g., a measurement noise, and $\theta \in \mathbb{R}^n$ is the vector of unknown constant parameters. The signals y and ϕ are known, e.g., measured, and the distortion signal w is unknown. The goal is to estimate the vector of parameters θ using the measurements y and ϕ. The model (1) is an LRE with a scalar output, but a more general case of $y(t) \in \mathbb{R}^p$ and $\phi(t) \in \mathbb{R}^{n \times p}$ can be also considered.

Two classical approaches can be deployed to the parameter estimation problem: the gradient and the least-squares estimators. Several modifications of these approaches are known, e.g., the normalized gradient estimator, the least-squares estimator with forgetting factor, and others, see [2], [3]. For both methods, the convergence of the estimates is based on the key assumption of sufficient and uniform (in time) informational richness of the regressor ϕ, which is rigorously defined as the persistence of excitation (PE) condition, see Section II for the definition. The drawback

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of these techniques is that whereas the weak monotonicity of a (weighted) norm of estimation errors is guaranteed, the element-wise estimation transients for each component of the vector θ are hardly predictable and typically exhibit significant oscillations and peaking phenomena. Moreover, tuning procedures for these estimators usually involve the tuning of a gain matrix and they can be somewhat unclear requiring multiple trial-and-error attempts. Notably, it has been shown in [7] that for gradient estimators, the amplification of the gains does not always yield accelerated transients, but it increases the peaking.

The Dynamic Regressor Extension and Mixing (DREM) procedure has been recently proposed in [8]. In [9], the authors propose an interpretation of this procedure as a functional Luenberger observer. The DREM approach has been successfully applied to a variety of adaptive control problems and applications, such as direct model reference control [10], time-varying frequency estimation [11], electrical drives velocity estimation [12], and power system applications [13], [14]. Among many, the DREM procedure has the following advantages:

- it provides a novel convergence condition that differs from the PE requirement, and it is less restrictive in certain scenarios;
- independently of the excitation conditions, it ensures the element-wise transient monotonicity precluding oscillations and peaking;
- each element of the estimate of θ is tuned with a separate scalar gain, which does not affect transients for other elements; this property makes the gain tuning simple and transparent.

Two basic steps are involved in the DREM procedure, namely the *dynamic regressor extension* (DRE) and *mixing* steps. At the first step, a linear dynamic operator is introduced to extend the original linear regression equation (1) and to obtain an extended matrix regressor. At the second step, a nonlinear transformation is applied to the previously generated data to obtain a set of n scalar independent LREs for each component of the vector θ sharing the same new scalar regressor. Finally, the gradient estimator is applied to each of these scalar equations.

The key question of the DREM procedure is the choice of the linear operator for dynamics extension at the first step. As it has been shown in [8], a bad choice of this operator can compromise the convergence even if the original regressor ϕ is PE. In the discrete-time domain, a summation over a fixed window has been proposed in [15]; however, to choose

the window size, a certain *a priori* knowledge about the original regressor ϕ must be available. This problem has also been considered in [16] for a particular class of LRE, where the regressor consists of a finite sum of sinusoidal signals and the upper frequency bound is known. Motivated by the work [17], the authors in [18] and [15] have also proposed to use the Kreisselmeier's regressor extension, where the authors of [18] have used the name *memory regressor extension*. However, these papers do not address properties and applicability conditions of such a choice, and the operator design for the DRE step of the DREM procedure remains an open question.

The use of the gradient estimator in the DREM procedure allows establishing the convergence under the non-square-integrability condition of the new scalar regressor, which differs from the PE condition. An open question is whether the use of the least-squares estimator (LSQ) instead of the gradient estimator will preserve convergence under the same conditions or it will yield different conditions.

Novelty and contribution. The contribution of this paper is twofold. First, we consider the Kreisselmeier's regressor extension in the context of the DREM procedure and we study its properties and applicability conditions. We show that this choice does not require any prior knowledge about the original regressor and that the PE property is *always preserved*. Moreover, we also analyze how the excitation propagates through the proposed dynamics. Second, we consider the DREM procedure when the gradient estimator is replaced with the least-squares estimator, with or without forgetting modification; we compare necessary and sufficient convergence conditions in the context of the DREM procedure.

Results of this paper are presented for continuous time only; however, they can be extended to the discrete time domain in a straightforward way.

The rest of this paper is organized as follows. For the completeness of the presentation, a brief description of the DREM procedure and some background material are given in Section II. The Kreisselmeier's regressor extension is studied in Section III, and the LSQ estimator in the context of the DREM procedure is considered in Section IV. Illustrative simulations are presented in Section V, and the conclusive remarks are in Section VI.

Notation. For a positive integer n, I_n denotes the $n \times n$ identity matrix. For a signal of time $x : \mathbb{R}_+ \to \mathbb{R}$ and a linear operator $\mathcal{H}$, we denote the action of this operator on the signal x as $\mathcal{H}[x]$. For a function $x : \mathbb{R}_+ \to \mathbb{R}$, we say that $x \in \mathcal{L}_2$ if $\int_0^t x^2(s)ds$ converges to a constant as t tends to $+\infty$. If the integral does not converge, we write $x \notin \mathcal{L}_2$.

II. BACKGROUND MATERIAL

The persistence of excitation (PE): First, we present the definition of the (T, μ)-PE property.

Definition 1: A bounded signal $\phi : \mathbb{R}_+ \to \mathbb{R}^n$ is (T, μ)-*persistently exciting* if there exist scalars $T > 0$ and $\mu > 0$

such that for all $t \in \mathbb{R}_+$,

$$\int_t^{t+T} \phi(s)\phi^\top(s)ds \geq \mu I_n.$$

This property is further denoted as $\phi \in \text{PE}$, or ϕ is PE.

The gradient estimators [2].: For all $t \in \mathbb{R}_+$, the gradient estimator for the LRE (1) is given by

$$\dot{\hat{\theta}}(t) = \Gamma\phi(t)\left(y(t) - \phi^\top(t)\hat{\theta}(t)\right), \tag{2}$$

where $\hat{\theta}$ denotes the estimate of θ and $\Gamma > 0$ is the gain matrix. Define the estimation error $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$, $\forall t \in \mathbb{R}_+$. Then the error dynamics is given by

$$\dot{\tilde{\theta}}(t) = -\Gamma\phi(t)\phi^\top(t)\tilde{\theta}(t) + \Gamma\phi(t)w(t), \forall t \in \mathbb{R}_+.$$

In the noise-free scenario, i.e. assuming $w \equiv 0$, the gradient estimator ensures exponential convergence to zero of the error $\tilde{\theta}$ if and only if the regressor ϕ is PE. In this case the gradient estimator is also input-to-state stable with respect to the noise w. Some conditions that are sufficient, and some conditions that are necessary for asymptotic convergence when ϕ is not PE have been discussed in [19]; however, they are very technical and can be hardly applied in practice.

The DREM procedure: To apply the DREM procedure, we start by performing the *dynamic regressor extension* step. For that, we introduce a linear, single-input n-output, bounded-input bounded-output (BIBO)–stable operator $\mathcal{H}$ and define the vector $Y : \mathbb{R}_+ \to \mathbb{R}^n$ and the matrix $\Phi : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ by:

$$Y = \mathcal{H}[y], \quad \Phi = \mathcal{H}[\phi^\top].$$

Due to the linearity of the operator $\mathcal{H}$ and BIBO stability, these signals satisfy

$$Y(t) = \Phi(t)\theta + W(t), \forall t \in \mathbb{R}_+, \tag{3}$$

where $W = \mathcal{H}[w]$. For example, the operator $\mathcal{H}$ can produce an LTI system or can be chosen as a delay operator, as proposed in [8].

Next, a *mixing* step is applied to obtain a set of n scalar equations. Recall that for any square and possibly singular $n \times n$ matrix A, we have $\text{adj}(A)A = \det(A)I_n$, where $\text{adj}(\cdot)$ is the *adjoint* (also called *adjugate*) matrix. Multiplying (3) by $\text{adj}(\Phi(t))$ from the left, we get

$$\mathcal{Y}_i(t) = \Delta(t)\theta_i + \mathcal{W}_i(t), \tag{4}$$

where $i \in \bar{n} = \{1, 2, \dots, n\}$, the scalar function $\Delta : \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$\Delta(t) = \det \Phi(t), \forall t \in \mathbb{R}_+ \tag{5}$$

and $\mathcal{Y}(t) = \text{adj}(\Phi(t))Y(t)$, $\mathcal{W}(t) = \text{adj}(\Phi(t))W(t)$, $\forall t \in \mathbb{R}_+$. It is worth noting that for a bounded regressor ϕ, the vector $\mathcal{W}$ is also bounded, and $w \equiv 0$ implies $\mathcal{W} \equiv 0$.

The set of n scalar LRE (4) sharing the same bounded scalar regressor Δ is the result of the DREM procedure. Applying then the gradient estimator to (4) as

$$\dot{\hat{\theta}}_i(t) = \gamma_i \Delta(t)\left(\mathcal{Y}_i(t) - \Delta(t)\hat{\theta}_i(t)\right), \tag{6}$$

where $\gamma_i > 0$ is a scalar tuning parameter, we obtain

$$\dot{\tilde{\theta}}_i(t) = -\gamma_i \Delta^2(t)\tilde{\theta}_i(t) + \gamma_i \Delta(t)\mathcal{W}_i(t).$$

Assuming the noise-free scenario $w \equiv 0$, the following properties hold:

P1: $\Delta \notin \mathcal{L}_2 \Leftrightarrow |\tilde{\theta}| \to 0$ asymptotically;
P2: Δ is PE $\Leftrightarrow |\tilde{\theta}| \to 0$ exponentially fast;
P3: (element-wise monotonicity) for all $i \in \bar{n}$ for $t_a \leq t_b$ it holds $|\tilde{\theta}_i(t_a)| \leq |\tilde{\theta}_i(t_b)|$;
P4: (element-wise tuning) variations in the gain γ_i affect the transients for $\hat{\theta}_i$ only.

Concerning the case $w \not\equiv 0$, the estimator (6) is input-to-state stable with respect to $\mathcal{W}_i$ if $\Delta \in \text{PE}$, which is a similar result as for the standard gradient estimator discussed above. Moreover, as it has been shown in [20], if $\mathcal{W}_i \in \mathcal{L}_2$ and $\Delta \notin \mathcal{L}_2$, then $\tilde{\theta}_i$ is bounded.

The property P1 illustrates the new convergence condition, namely the non-square-integrability of Δ. As shown in [8], this condition is weaker than PE for ϕ, where the price paid is the asymptotic convergence instead of the exponential one. To get the exponential convergence with the DREM procedure, the PE property of Δ must be satisfied. Therefore, the main design question when applying the DREM procedure is to choose an operator $\mathcal{H}$ such that the PE property of the original regressor ϕ is preserved. Such a choice is discussed in the following section.

Remark 1: Let us remark that the computation of the adjoint matrix $\text{adj}(\Phi)$ can be avoided in numerical implementations of the DREM estimator. So the elements $\mathcal{Y}_i$ in (4) can be computed using the Cramer's rule as

$$\mathcal{Y}_i(t) = \det \Phi_{Y,i}(t), \forall t \in \mathbb{R}_+,$$

where $\Phi_{Y,i}$ is the matrix Φ where the i-th column is replaced with the vector Y, and $i \in \bar{n}$.

III. THE EXCITATION-PRESERVING REGRESSOR EXTENSION

A. Excitation propagation

Let the operator $\mathcal{H}$ be chosen such that the relationship $y_u = \mathcal{H}[u]$ for an input signal $u(t) \in \mathbb{R}$ and an output signal $y_u(t) \in \mathbb{R}^n$ have the following linear time-varying state-space representation:

$$\dot{x}(t) = -ax(t) + \phi(t)u(t),$$
$$y_u(t) = x(t),$$

where $x(t) \in \mathbb{R}^n$ is the internal state vector, and $a > 0$ is the tuning parameter. Then $\Phi = \mathcal{H}[\phi^\top]$ and $Y = \mathcal{H}[y]$ yield the Kreisselmeier's regressor extension (see [15]) and can be found as solutions of

$$\dot{\Phi}(t) = -a\Phi(t) + \phi(t)\phi^\top(t), \qquad (7)$$
$$\dot{Y}(t) = -aY(t) + \phi(t)y(t) \qquad (8)$$

for some initial values $\Phi(0) = \Phi_0 \geq 0$ and $Y(0) = Y_0$.

The regressor extension (7), (8) is widely used in adaptive control. Particularly, it was used in [17] and recently in [21],

[22] to obtain the matrix equation $Y(t) = \Phi(t)\theta, \forall t \in \mathbb{R}_+$. Also in [18] the authors have applied a similar approach under the name *memory regressor extension*. However, to the best of our knowledge, the applicability conditions of (7), (8) for the DREM procedure and the PE property of the determinant of the matrix Φ have not been previously studied. To this end, we present the following theorem showing that (7) preserves the persistence of excitation.

Theorem 1: Let the signal $\phi : \mathbb{R}_+ \to \mathbb{R}^n$ be (T, μ)-PE and let $\Phi : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ be a solution of (7). Then the following implication holds

$$\phi \in \text{PE} \Rightarrow \Delta \in \text{PE},$$

where $\Delta : \mathbb{R}_+ \to \mathbb{R}$ is the determinant of Φ. More precisely, for any positive integer $q \geq 1$ and for all $t \geq qT$, it holds

$$\Delta(t) \geq \mu^n \left(\sum_{k=1}^{q} e^{-akT}\right)^n \qquad (9)$$

and

$$\liminf_{t \to \infty} \Delta(t) \geq \left(\frac{\mu}{e^{aT} - 1}\right)^n. \qquad (10)$$

The proof Theorem is omitted due to the lack of space.

Remark 2: Theorem 1 motivates the use of (7), (8) as a reasonable choice for the dynamic regressor extension step of the DREM procedure. Under this choice, the PE property of the original regressor is always preserved and no prior knowledge about the regressor ϕ (e.g., the excitation period length T) is required. Moreover, after the first excitation interval, the new regressor Δ remains positive for all t, where the lower asymptotic bound on Δ can be computed as (10).

B. Dynamics of the regressor Δ

Theorem 1 provides the lower bounds (9), (10) for the dynamic regressor extension (7). However, it is also possible to derive the dynamics equation of the new regressor signal Δ. Such a result provides bounds on the time derivative of the regressor Δ and allows the estimation of the rate of its variation as it is stated in the following Proposition.

Proposition 1: Let Φ be a solution of (7) and let λ_M denote the maximum eigenvalue of Φ and $\Delta = \det \Phi$. Then:
- if $\lambda_M = 0$, then $\dot{\Delta}(t) = 0$;
- if $\lambda_M > 0$, then

$$\dot{\Delta}(t) \geq \left(-an + \frac{\|\phi(t)\|^2}{\lambda_M(t)}\right)\Delta(t), \ \forall t \in \mathbb{R}_+.$$

The proof of Proposition is omitted due to the lack of space.

It is also worth noting that an upper bound of the maximum eigenvalue λ_M can be estimated given an upper bound of ϕ.

IV. THE LSQ ESTIMATION IN THE DREM PROCEDURE

In the noise-free scenario, the gradient estimator (6) ensures exponential convergence when $\Delta \in \text{PE}$, asymptotic convergence when $\Delta \notin \mathcal{L}_2$, and boundedness of the estimate $\hat{\theta}$ when $\Delta \in \mathcal{L}_2$. It is of interest to study the convergence conditions in the context of the DREM procedure (assuming the noise-free scenario) of the LSQ (Least Squares) estimator.

For $i \in \bar{n}$, the LSQ estimator is given by (see [2])

$$\dot{\hat{\theta}}_i(t) = \gamma_i \Delta(t) p_i(t) \left(\mathcal{Y}_i(t) - \Delta(t)\hat{\theta}_i(t) \right), \quad (11)$$

$$\dot{p}_i(t) = \gamma_i \left(\lambda_i p_i(t) - p_i^2(t)\Delta^2(t) \right), \quad (12)$$

where $p_i(0) > 0$ and $\gamma_i > 0$ are the design parameters, and $\lambda_i \geq 0$ is the forgetting factor. The error dynamics under the LSQ estimator is given by

$$\dot{\tilde{\theta}}_i(t) = -\gamma_i \Delta^2(t) p_i(t)\tilde{\theta}_i(t). \quad (13)$$

The properties of the LSQ estimator are well-studied, and the possible unboundedness of p_i for $\lambda_i > 0$ and Δ not being PE is known, see for example, [3]. However, since the DREM procedure renders the LRE (1) to the set of scalar equations (4), we consider the scalar realization of the LSQ estimator, and thus we can analyze the exact solution of p_i. This provides a conclusion for the case when $\Delta \notin$ PE but $\Delta \notin \mathcal{L}_2$, which is an important situation for the DREM procedure and which is not typically addressed for the LSQ estimators.

Remark 3: It is worth noting that for a bounded scalar signal Δ, if $\Delta \in$ PE then it implies that $\Delta \notin \mathcal{L}_2$. However, $\Delta \notin \mathcal{L}_2$ does not necessarily implies that $\Delta \in$ PE: that can be illustrated by $\Delta(t) = \frac{1}{\sqrt{t+1}}, \forall t$.

Properties of the LSQ estimator in the context of the DREM procedure are given in the following proposition.

Proposition 2: Let $i \in \bar{n}$. Consider the estimation algorithm (11), (12) with $p_i(0) > 0$ and $\gamma_i > 0$.

(i) If $\lambda_i = 0$ (LSQ estimator without forgetting) then
  1) if $\Delta \in \mathcal{L}_2$ then for all nonzero $\tilde{\theta}_i(0)$ the signal $\tilde{\theta}_i$ does not converge to zero;
  2) if $\Delta \notin \mathcal{L}_2$ then $\tilde{\theta}_i$ is monotonic and converges to zero asymptotically;
  3) if Δ is PE, it does not imply exponential convergence.

(ii) If $\lambda_i > 0$ (LSQ estimator with forgetting) then
  1) p_i is bound from below as
  2) if $\Delta \in \mathcal{L}_2$ or

$$\Delta \notin \mathcal{L}_2 \text{ and } \Delta \to 0,$$

  then the estimator is unstable and p_i tends to infinity;
  3) if Δ is PE then p_i is bounded, $\tilde{\theta}_i$ is monotonic and converges to zero exponentially fast.

Proof: Part 1. Consider first the case $\lambda_i = 0$. We obtain

$$\dot{p}_i(t) = -\gamma_i p_i^2(t)\Delta^2(t)$$

giving

$$p_i(t) = \frac{1}{c_1 + \gamma_i \int_0^t \Delta^2(s)ds},$$

where $c_1 = \frac{1}{p_i(0)}$. Obviously, for $\Delta \notin \mathcal{L}_2$ we have $p_i \to 0$, as it is expected for an LSQ estimator without forgetting. The error dynamics can be now written as

$$\dot{\tilde{\theta}}_i(t) = -\beta(t)\tilde{\theta}, \quad (14)$$

where

$$\beta(t) = \frac{\gamma_i \Delta^2(t)}{c_1 + \gamma_i \int_0^t \Delta^2(s)ds} = \frac{\Delta^2(t)}{c_2 + \int_0^t \Delta^2(s)ds}, \quad (15)$$

and $c_2 = \frac{c_1}{\gamma_i}$. For a bounded function Δ, the LTV system (14) has the unique solution

$$\tilde{\theta}_i(t) = \frac{c_3}{c_2 + \int_0^t \Delta^2(s)ds}, \quad (16)$$

where $c_3 = \tilde{\theta}(0)c_2$. Indeed, taking the time derivative of (16) we obtain

$$\dot{\tilde{\theta}}(t) = -\frac{c_3\Delta^2(t)}{(c_2 + \int_0^t \Delta^2(s)ds)^2}$$
$$= -\frac{\Delta^2(t)}{c_2 + \int_0^t \Delta^2(s)ds} \cdot \frac{c_3}{c_2 + \int_0^t \Delta^2(s)ds} = -\beta(t)\tilde{\theta}.$$

From (16), we observe that $\tilde{\theta}_i$ does not converge to zero if $\Delta \in \mathcal{L}_2$ and $c_3 \neq 0$. On the other hand, for $\Delta \notin \mathcal{L}_2$ it follows from (16) that $\tilde{\theta}$ converges to zero asymptotically. Moreover, since the function β defined in (15) is non-negative, the convergence is monotonic. Finally, notice that a PE Δ implies that β in (14) converges to zero that prevents $\tilde{\theta}$ to converge exponentially; for the exponential convergence the PE property of β is required.

Part 2. Consider now the case $\lambda_i > 0$. The nonlinear ODE (12) has the solution

$$p_i(t) = \frac{e^{\lambda_i \gamma_i t} p_i(0)}{1 + p_i(0)\gamma_i \int_0^t e^{\lambda_i \gamma_i s}\Delta^2(s)ds}.$$

The rest of proof is performed in three steps. First, we show that for all bounded Δ, p_i is bounded from below by a positive constant. Second, we show that p_i is bounded from above since Δ being PE implies exponential convergence of $\tilde{\theta}$. Finally, we show that p_i tends to infinity if $\Delta \in \mathcal{L}_2$ or if Δ tends to zero.

Step 1. Consider the inverse function

$$\frac{1}{p_i(t)} = \frac{1}{p_i(0)}e^{-\lambda_i \gamma_i t} + \gamma_i z(t), \quad (17)$$

where

$$z(t) = e^{-\lambda_i \gamma_i t} \int_0^t e^{\lambda_i \gamma_i s}\Delta^2(s)ds.$$

Recalling that Δ is bounded, say $\Delta^2(t) \leq \bar{\Delta}$, then the function z is bounded as

$$z(t) \leq \bar{\Delta}e^{-\lambda_i \gamma_i t} \int_0^t e^{\lambda_i \gamma_i s}ds = \frac{\bar{\Delta}}{\lambda_i \gamma_i}\left(1 - e^{-\lambda_i \gamma_i t}\right) \leq \frac{\bar{\Delta}}{\lambda_i \gamma_i}.$$

It follows that $\frac{1}{p_i(t)} \leq \frac{1}{p_i(0)} + \frac{\bar{\Delta}}{\lambda_i}$, and p_i is bounded from below as

$$p_i(t) \geq \frac{p_i(0)\lambda_i}{\lambda_i + \bar{\Delta}p_i(0)} = p_m. \quad (18)$$

Step 2. Assume that Δ is (T, μ)-PE. Then for $t \geq T$ the function z is bounded from below as

$$z(t) \geq e^{-\lambda_i \gamma_i t} \int_{t-T}^t e^{\lambda_i \gamma_i s}\Delta^2(s)ds$$
$$\geq e^{-\lambda_i \gamma_i t}e^{\lambda_i \gamma_i(t-T)} \int_{t-T}^t \Delta^2(s)ds \geq \mu e^{-\lambda_i \gamma_i T}.$$

Then we have the following bounds:

$$\frac{1}{p_i(t)} \geq \frac{1}{p_i(0)} e^{-\lambda_i \gamma_i T} \qquad \text{for } 0 \leq t \leq T,$$

$$\frac{1}{p_i(t)} \geq \gamma_i z(t) \geq \gamma_i \mu e^{-\lambda_i \gamma_i T} \qquad \text{for } t > T.$$

Hence the function p_i is bounded by

$$p_i(t) \leq e^{\lambda_i \gamma_i T} \max \left(p_i(0), \frac{1}{\mu \gamma_i} \right) = p_M. \qquad (19)$$

From (18) and (19) it follows that if Δ is PE, then the signal $t \mapsto \sqrt{p_i(t)} \Delta(t)$ is bounded and PE as well. Therefore the exponential convergence of $\tilde{\theta}$ follows from (13). Moreover, the convergence is monotonic since $p_i(t) \Delta^2(t) \geq 0$, $\forall t \geq 0$.

Step 3. Assume now that $\Delta \in \mathcal{L}_2$. Rewriting z as

$$z(t) = \int_0^t e^{-\lambda_i \gamma_i (t-s)} \Delta^2(s) ds,$$

it can be noted that z is the solution of the differential equation

$$\dot{z}(t) + \lambda_i \gamma_i z(t) = \Delta^2(t), \ z(0) = 0. \qquad (20)$$

Note that $\Delta \in \mathcal{L}_2$ implies $\Delta^2 \in \mathcal{L}_1$. It is known that the considered stable first order linear time-invariant (LTI) system (20) has a finite $\mathcal{L}_1$ gain, thus $z \in \mathcal{L}_1$. Noting also that $\dot{z}$ is bounded and applying Barbalat's lemma we conclude $z \to 0$. Then from (17) it follows that p_i tends to infinity and the estimator (11), (12) is unstable.

For the case when $\Delta \notin \mathcal{L}_2$ but $\Delta \to 0$, we note that the LTI system (20) is exponentially stable, therefore for $\Delta^2 \to 0$ we have $z \to 0$ and p_i tends to infinity. $\blacksquare$

Remark 4: Proposition 2 shows that (within the context of the DREM procedure) the use of the LSQ estimator (11), (12) does not provide any improvement in the convergence conditions with respect to the gradient estimator (6) and can yield unboundedness of the trajectories when $\Delta \to 0$.

V. SIMULATIONS

A. Preservation of the excitation

To illustrate the results obtained in Section III we consider the problem of parameter estimation for a sinusoidal signal with a known frequency. Consider $y(t) = B + A \sin(t + \psi)$, where $B, A > 0$, and $\psi \in [-\pi, \pi)$ are the unknown parameters. This signal can be rewritten as the LRE (1) with
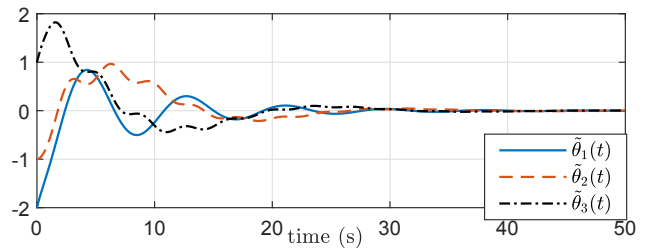
$$\phi(t) = \begin{bmatrix} 1 & \sin(t) & \cos(t) \end{bmatrix}^\top,$$

$$\theta = \begin{bmatrix} B & A \cos(\psi) & A \sin(\psi) \end{bmatrix}^\top,$$

and $w \equiv 0$. For simulations, we set $B = 2$, $A = \sqrt{2}$, and $\psi = -\frac{1}{4}\pi$ providing $\theta = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}^\top$.
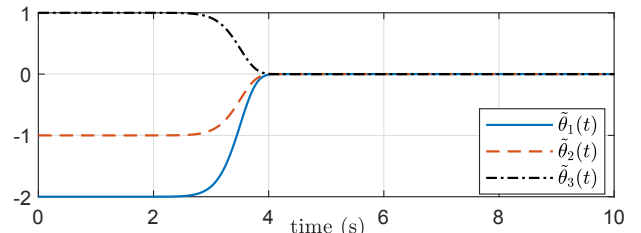
It is straightforward to verify that the regressor ϕ is $(2\pi, \pi)$-PE, i.e., for all $t \geq 0$

$$\int_t^{t+2\pi} \phi(s) \phi^\top(s) ds \geq \pi I_3.$$

First we apply the standard gradient algorithm (2) with $\Gamma = I_3$ and $\hat{\theta}(0) = 0$; the simulation results are depicted

(a) The standard gradient algorithm (2).

(b) The DREM procedure (7), (8) and the gradient estimator (6).

Fig. 1: The estimation error $\tilde{\theta}$.

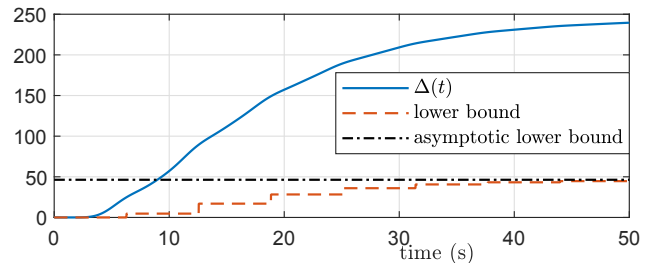
Fig. 2: The new regressor Δ, the lower bound (9), and the asymptotic lower bound (10).

in Fig. 1a. Since ϕ is PE, the estimation error $\tilde{\theta}$ converges to zero; however, the transients exhibit oscillations. Then we apply the dynamic regressor extension (7), (8), where the only tuning parameter is chosen as $a = 0.1$. The estimation error transient $\tilde{\theta}$ for the DREM procedure with the gradient estimator (6) with $\gamma_i = 1$, $i = 1, 2, 3$, are depicted in Fig. 1b and illustrate performance improvement with respect to the standard gradient estimator; note the difference in the time scale and the monotonicity of the transients.

The new regressor Δ computed as (5) is depicted in Fig. 2 with the lower bound (9) and with the asymptotic lower bound (10).

B. DREM with the LSQ estimator

To illustrate Proposition 2 and possible unboundedness of the LSQ estimator, we consider two regression models of the form (1) with the regressors $\phi_1 \in$ PE and $\phi_2 \notin$ PE where

$$\phi_1(t) = \begin{bmatrix} 1 & \sin(t) \end{bmatrix}^\top, \ \phi_2(t) = \begin{bmatrix} 1 & \frac{\sin(t)}{\sqrt{t+2\pi}} \end{bmatrix}^\top, \ \forall t \geq 0$$

and the vector of unknown parameters $\theta = \begin{bmatrix} 2 & 1 \end{bmatrix}^\top$.

To apply the DREM procedure, we consider time-delay

(a) $\Delta_1 \in$ PE.

(b) $\Delta_2 \notin \mathcal{L}_2$.
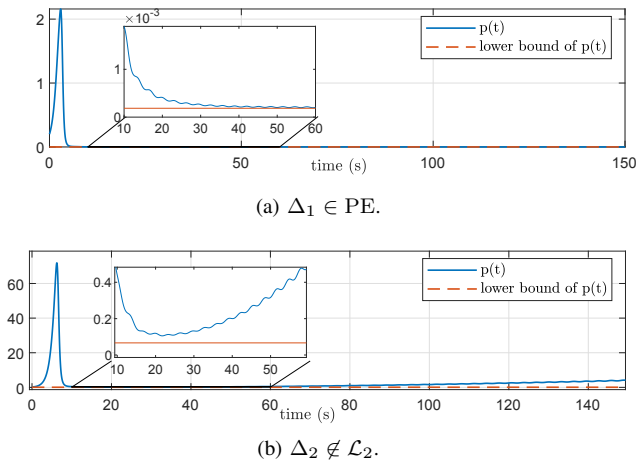
Fig. 3: The signal p and the lower bound (18).

operators and define

$$\Phi_i(t) = \begin{bmatrix} \phi_i(t) & \phi_i(t - \pi) \end{bmatrix}^\top, \quad \Delta_i(t) = \det \Phi_i(t),$$

where $i = 1, 2$. It is straightforward to show that $\Delta_1 \in$ PE and $\Delta_2(t) \to 0$ as $t \to \infty$.

The LSQ estimator (11)–(12) with $\lambda_1 = 0.5$ and $\gamma_1 = 0.5$ is applied for both models. For the persistently excited regressor Δ_1 the corresponding signal p is bounded and tends to the lower bound (18) as shown in Fig. 3a. The behavior of the LSQ estimator for the decaying regressor Δ_2 is depicted in Fig. 3b and illustrates that the signal p is not bounded and tends to infinity.

VI. CONCLUSION

The recently proposed DREM procedure provides significant performance improvement in linear regression parameter estimation, where the main degree of freedom is the choice of the dynamic extension operator. In this paper, we have studied a special choice of this operator as the Kreisselmeier's regressor extension (7), (8) that ensures the preservation of the persistence of excitation property. In particular, we proved that the determinant of the extended matrix is persistently exciting if the original regressor is. This result alleviates the main design question of the DREM procedure.

We have also considered the use of the LSQ estimator within the DREM procedure. It is shown that for the resulting scalar linear regression equations, the LSQ estimator does not provide any advantage in the convergence properties and it can be unbounded if the regressor Δ decays.

Our further researches are oriented towards the extension of Theorem 1 for the case when $\phi \notin$ PE. We expect that eigenvalue conditions similar to the ones reported in [19] can be formulated. We also intend to study the propagation of (the non-uniform in time) interval excitation.

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