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1 **LOW-MACH TYPE APPROXIMATION OF THE NAVIER-STOKES**
2 **SYSTEM WITH TEMPERATURE AND SALINITY FOR FREE**
3 **SURFACE FLOWS***

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6 **Abstract.** We are interested in free surface flows where density variations coming e.g. from
7 temperature or salinity differences play a significant role in the hydrodynamic regime. In water,
8 acoustic waves travel much faster than gravity and internal waves, hence the study of models arising
9 from compressible fluid mechanics often requires a decoupling between these waves. Starting from
10 the compressible Navier-Stokes system, we derive the so-called Navier-Stokes-Fourier system in an
11 "incompressible" regime using the low-Mach scaling, hence filtering the acoustic waves, neglecting
12 the density dependency on the fluid pressure but keeping its variations in terms of temperature and
13 salinity. A slightly modified low-Mach asymptotics is proposed to obtain a model with thermo-
14 mechanical compatibility. The case when the density depends only on the temperature is studied
15 first. Then the variations of the fluid density with respect to temperature and salinity are considered,
16 and it seems to be the first time that salinity dependency is considered in this low Mach limit.

17 **Key words.** Navier-Stokes equations, compressible and incompressible fluids, free surface flows,
18 variable density flows, low-Mach approximation

19 **AMS subject classifications.** 35Q30, 35Q35, 76D05

20 NEW VERSION

21 **1. Introduction.** In oceans and lakes, one of the predominant driving forces is
22 the difference in density, caused by salinity and temperature variations: increasing
23 the salinity and lowering the temperature of a fluid both increase its density [32].
24 Oceans and lakes are stratified: the water density varies along the vertical direction.
25 In the present work, we aim at describing and simulating variable density flows with
26 free surface.

27 The Navier-Stokes equations – or possibly simplified versions of these equations –
28 are the cornerstone of the modeling of hydrodynamics in lakes and oceans. Compared
29 to situations where the density is constant, when considering variable density flows
30 with free surface, two characteristics appear:

- 31 • the variations of the fluid density can lead to acoustic waves,
- 32 • the fluid density adds nonlinearities in the Navier-Stokes equations.

33 These two aspects are detailed hereafter.

34 The density variations are usually small and a common assumption in the case
35 of geophysical flows is the Boussinesq approximation [9]. It is widely used to simplify

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36 the Navier-Stokes equations with variable density and consists in ignoring density
 37 variations in momentum conservation equations except in the buoyancy force term.
 38 The underlying assumptions are 1) the density variations are very small with respect
 39 to the mean density and 2) the vertical scale of the motion is small compared to the
 40 vertical scale of variations in density, and the density and pressure variations due to
 41 motion are not larger than the total static variations of pressure and density [36, 23]
 42 The consequences of the Boussinesq approximation are listed in [4]. Notably, under
 43 the Boussinesq approximation, the volume of the water is conserved, while the mass
 44 is not conserved. With this approximation, the density can be defined as a function
 45 of any given tracer, the temperature or a pollutant for instance. The Boussinesq
 46 approximation is the basis of many ocean models such as ICON [21], NEMO [29] and
 47 POM [31].

48 Several authors have shown the benefits of taking into account non-Boussinesq
 49 effects in lake and ocean models, either for the propagation of internal waves [35] or
 50 for sea level variations induced by expansion/contraction processes [28, 23, 16]. In
 51 the case of ocean water, density is in general a function of pressure, temperature and
 52 salinity. Equations of state for water based on experiments are proposed in [33, 38].

53 Consequently, approaches to take into account the non-Boussinesq effects have
 54 been developed. A possibility is to adopt pressure coordinates [20, 34]. The non-
 55 Boussinesq equations written in pressure coordinates are isomorphic to the Boussi-
 56 nesq equations in z -coordinates, which allows to use the same algorithm for the non-
 57 Boussinesq model as for the Boussinesq model. So far, however, this approach is not
 58 widely used in ocean modeling, though it is available in the code MITgcm ([1]). A
 59 review of the different types of vertical coordinates used for ocean modeling is given in
 60 [18]. In [4], a non-hydrostatic non-Boussinesq model is presented. A non-hydrostatic
 61 pressure anomaly is related to a compressible (non-Boussinesq) density anomaly. Yet
 62 the authors of [4] are primarily concerned with the simulation of acoustic waves and
 63 thermal dilation is not investigated numerically. This model has recently been in-
 64 cluded in CROCO ([11]). In the present work, we focus on the inclusion of dilation
 65 effects in the model.

66 The approach chosen here to propose a non-Boussinesq model is different from the
 67 previous ones. Starting from the compressible Navier-Stokes equations, we propose a
 68 formulation of the Navier-Stokes-Fourier system in the asymptotics of an "incompress-
 69 ible" fluid. The term "Fourier" refers here to Fourier's law of thermal conduction. For
 70 most geophysical flows, the water can be considered as incompressible in the sense
 71 that the variation of its density with respect to the fluid pressure is small [33, 38]
 72 and in this case, the acoustic waves can be filtered out, this is the so called low Mach
 73 number limit. This is advantageous from the computational point of view because a
 74 restrictive condition must be imposed on the time step when the acoustic waves are
 75 included [30]. Even if in some very particular situations [22], a coupling between grav-
 76 ity and acoustic waves can occur in the context of fluid density variations, we consider
 77 here situations (in terms of water depth and wave frequency) when the phenomena
 78 are decoupled.

79 The incompressible limit of the Navier-Stokes-Fourier system has been extensively
 80 studied mathematically, see [2, 13] and the references therein. The incompressible
 81 limit is a low-Mach approximation of the Navier-Stokes-Fourier system. It consists in
 82 neglecting the pressure dependency in the fluid state law (but not the temperature
 83 and salinity dependencies). This limit is singular and requires not necessarily intuitive
 84 scalings, in particular for numerical approximations [8]. Since the studied phenomena
 85 strongly couple mechanics (fluid motion and rheology) with thermodynamics (temper-

86 ature and salinity variations), we pay close attention that the obtained models do not
 87 violate the second principle of thermodynamics. The most simple way of performing
 88 the incompressible limit yields a model for which the energy balance exhibits discrep-
 89 ancies with respect to the energy balance of the original compressible system. To
 90 recover an energy balance that is close to that of the original system, which is impor-
 91 tant for the robustness of numerical methods, some corrections terms are incorporated
 92 in the incompressible model. These do not alter the accuracy of the approximation.
 93 Instead of conserving the volume of fluid and not the mass (which is what the Boussi-
 94 nesq approximation implies), the model derived here strictly conserves the mass and is
 95 enriched by the thermohaline expansion effects i.e. the volume is no longer conserved.
 96 In particular the velocity is not divergence free, but the divergence of the velocity field
 97 is equal to a right-hand side involving temperature and salinity gradients, as well as
 98 viscous dissipation. Our incompressible models include the dependency of density on
 99 salinity additionally to the dependency on temperature, and it seems to be the first
 100 time that this is considered in detail, including Onsager's principles. The proposed
 101 model is close to the one in [6], yet it is rigorously derived here. While solving the
 102 model in [6] required inverting a non-linear system, the numerical resolution of the
 103 present model is more simple, see [7]. Moreover, the model in the present work is 3D
 104 while it was only 2D in [6].

105 A multilayer approximation of the Navier-Stokes-Fourier system with a numerical
 106 scheme – endowed with strong stability properties – and several numerical test cases
 107 will be presented in a forthcoming paper [7].

108 The paper is organized as follows. The incompressible Navier-Stokes-Fourier and
 109 Euler-Fourier models are introduced in Section 2 and their low Mach limits are stud-
 110 ied in Section 3 including the situation where the temperature and the salinity are
 111 considered. Finally in Section 4, the formulation of the considered models with the
 112 hydrostatic assumption is given.

113 2. The 3d Navier-Stokes-Fourier system.

114 **2.1. The compressible Navier-Stokes-Fourier system.** We consider the
 115 classical compressible Navier-Stokes system describing a free surface gravitational
 116 flow over a bottom topography $z_b(x, y)$,

$$117 \quad (2.1) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0,$$

$$118 \quad (2.2) \quad \frac{\partial(\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p - \nabla \cdot \sigma = \rho \mathbf{g},$$

$$119 \quad (2.3) \quad \frac{\partial}{\partial t} \left(\rho \frac{|\mathbf{U}|^2}{2} + \rho e \right) + \nabla \cdot \left(\left(\rho \frac{|\mathbf{U}|^2}{2} + \rho e + p - \sigma \right) \mathbf{U} \right) = -\nabla \cdot Q_T + \rho \mathbf{g} \cdot \mathbf{U},$$

120 where $\mathbf{U}(t, x, y, z) = (u, v, w)^T$ is the velocity, ρ is the mass density, p is the fluid
 121 pressure, σ is the viscosity stress and $\mathbf{g} = (0, 0, -g)^T$ represents the gravity forces.
 122 The internal specific (i.e. per mass unit) energy is denoted by e , the temperature by T .
 123 The heat flux Q_T obeys the Fourier law $Q_T = -\lambda \nabla T$, hence the name "Navier-Stokes-
 124 Fourier", λ being the heat conductivity. The quantity ∇ denotes $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)^T$.
 125 In the following, we will also use the notations \mathbf{u} and $\nabla_{x,y}$, $\mathbf{u}(t, x, y, z) = (u, v)^T$ is the
 126 horizontal velocity and $\nabla_{x,y}$ corresponds to the projection of ∇ on the horizontal plane
 127 i.e. $\nabla_{x,y} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)^T$. The square norm of the velocity vector is $|\mathbf{U}|^2 = u^2 + v^2 + w^2$.

We consider a free surface flow (see Fig. 1), therefore we assume

$$z_b(x, y) \leq z \leq \eta(t, x, y) := h(t, x, y) + z_b(x, y)$$

128 with $z_b(x, y)$ the bottom elevation and $h(t, x, y)$ the water depth.

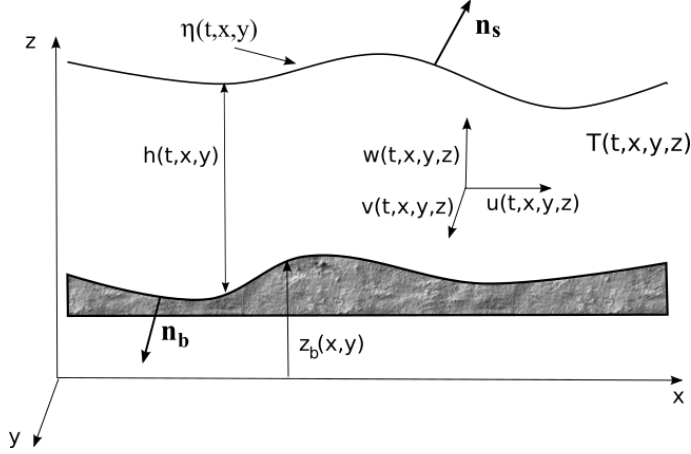


FIG. 1. Flow domain with water height $h(t, x, y)$, free surface $\eta(t, x, y)$ and bottom $z_b(x, y)$.

Notice that, because of the free surface and of the gravitational forces, the term $\rho \mathbf{g} \cdot \mathbf{U} = -\rho g w$ in the right hand side of (2.3) prevents this equation from being directly a local energy conservation law. Nevertheless one can write (see Remark 2.1)

$$\rho g w = \frac{\partial(\rho g z)}{\partial t} + \nabla \cdot (\rho g z \mathbf{U}),$$

129 which leads to a conservative version of (2.3). In the sequel, we will mainly work with
130 the energy balance written under the form (2.3).

131 Regarding constitutive equations, we assume that the fluid is Newtonian i.e. the
132 viscous part of the Cauchy stress depends linearly on the velocity gradient. Hence
133 the stress tensor Σ is given by

$$134 \quad (2.4) \quad \Sigma \equiv -p\mathbf{1} + \sigma = -p\mathbf{1} + \zeta \nabla \cdot \mathbf{U}\mathbf{1} + 2\mu D(\mathbf{U}).$$

135 where μ is the viscosity coefficient, ζ is the second viscosity and $D(\mathbf{U}) = (\nabla \mathbf{U} +$
136 $(\nabla \mathbf{U})^T)/2$. Further on we shall use the notation $A : B$ for the scalar product between
137 two square matrices A, B , and $|A|^2 = A : A$.

138 Among the thermodynamic variables ρ, p, T, e , only two of them are independent.
139 This implies in particular that we have a state law under the form

$$140 \quad (2.5) \quad f(\rho, T, p) = 0.$$

141 The thermodynamic variables are linked by the identity

$$142 \quad (2.6) \quad de = \frac{p}{\rho^2} d\rho + T ds,$$

143 where s is the specific entropy of the fluid. Classically, in order to have a good entropy
144 structure one has to assume that $-s$ is a convex function of $1/\rho, e$. In Section 3.2 the

145 case for which there is an additional thermodynamic variable S , the specific salinity,
146 is described.

147 Energy equations can be deduced from the above equations. Multiplying (2.2) by
148 \mathbf{U} yields the kinetic energy equation

$$149 \quad (2.7) \quad \frac{\partial}{\partial t} \left(\rho \frac{|\mathbf{U}|^2}{2} \right) + \nabla \cdot \left(\left(\rho \frac{|\mathbf{U}|^2}{2} + p - \sigma \right) \mathbf{U} \right) = p \nabla \cdot \mathbf{U} - \sigma : D(\mathbf{U}) + \rho \mathbf{g} \cdot \mathbf{U}.$$

150 Subtracting (2.7) to (2.3) gives the equation for the internal energy

$$151 \quad \frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \mathbf{U}) = -p \nabla \cdot \mathbf{U} + \sigma : D(\mathbf{U}) - \nabla \cdot Q_T,$$

152 or equivalently

$$153 \quad (2.8) \quad \rho \frac{De}{Dt} = -p \nabla \cdot \mathbf{U} + \sigma : D(\mathbf{U}) - \nabla \cdot Q_T,$$

154 with the classical notation $D/Dt \equiv \partial/\partial t + \mathbf{U} \cdot \nabla$. We can write the continuity equation
155 (2.1) as

$$156 \quad (2.9) \quad \rho \frac{D\rho}{Dt} + \rho^2 \nabla \cdot \mathbf{U} = 0.$$

157 With the thermodynamic relation (2.6) one can write $ds = de/T - (p/T\rho^2)d\rho$, thus
158 multiplying (2.8) by $1/T$ and (2.9) by $-p/T\rho^2$ we obtain

$$159 \quad \rho \frac{Ds}{Dt} = \frac{1}{T} \sigma : D(\mathbf{U}) - \frac{1}{T} \nabla \cdot Q_T.$$

160 This can be written also

$$161 \quad (2.10) \quad \frac{\partial \rho s}{\partial t} + \nabla \cdot (\rho s \mathbf{U}) = \frac{1}{T} \sigma : D(\mathbf{U}) - \nabla \cdot \frac{Q_T}{T} - Q_T \cdot \frac{\nabla T}{T^2},$$

162 which gives the increase with time of $\int \rho s$, the second principle of thermodynamics.

163 2.2. Boundary conditions.

2.2.1. Bottom and free surface. Let \mathbf{n}_b and \mathbf{n}_s be the unit outward normals
at the bottom and at the free surface respectively, defined by (see Fig 1)

$$\mathbf{n}_b = \frac{1}{\sqrt{1 + |\nabla_{x,y} z_b|^2}} \begin{pmatrix} \nabla_{x,y} z_b \\ -1 \end{pmatrix}, \quad \mathbf{n}_s = \frac{1}{\sqrt{1 + |\nabla_{x,y} \eta|^2}} \begin{pmatrix} -\nabla_{x,y} \eta \\ 1 \end{pmatrix}.$$

164 On the bottom we prescribe an impermeability condition

$$165 \quad (2.11) \quad \mathbf{U} \cdot \mathbf{n}_b = 0,$$

166 and a friction condition given e.g. by a Navier law

$$167 \quad (2.12) \quad (\Sigma \cdot \mathbf{n}_b) \cdot \mathbf{t}_i = -\kappa \mathbf{U} \cdot \mathbf{t}_i, \quad i = 1, 2,$$

168 with κ a Navier coefficient and $(\mathbf{t}_i, i = 1, 2)$ two tangential vectors. For some ap-
169 plications, we rather use more specific friction laws and the equation (2.12) is then
170 replaced by

$$171 \quad (\Sigma \cdot \mathbf{n}_b) \cdot \mathbf{t}_i = -\kappa(h, \mathbf{U}) \cdot \mathbf{t}_i, \quad i = 1, 2,$$

172 with $\kappa(h, \mathbf{U}) \cdot \mathbf{U} \geq 0$. On the free surface, we use the kinematic boundary condition

$$173 \quad (2.13) \quad \frac{\partial \eta}{\partial t} + \mathbf{u}(t, x, y, \eta) \cdot \nabla_{x,y} \eta - w(t, x, y, \eta) = 0,$$

174 and the dynamic boundary condition

$$175 \quad (2.14) \quad \boldsymbol{\Sigma} \cdot \mathbf{n}_s = -p^a(t, x, y) \mathbf{n}_s + W(t, x, y) \mathbf{t}_W,$$

176 where $p^a(t, x, y)$, $W(t, x, y)$ are two given quantities, p^a (resp. W) mimics the effects
 177 of the atmospheric pressure (resp. the wind blowing at the free surface) and \mathbf{t}_W is a
 178 unit horizontal vector given by the wind direction. Throughout the paper $p^a = cst$,
 179 $W = 0$. For the temperature, Neumann or Dirichlet boundary conditions can be
 180 taken, see Subsection 2.2.3.

Remark 2.1. Computing the quantity

$$\int_{z_b}^{\eta} z \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) \right) dz,$$

181 and using the boundary conditions (2.13), (2.11) one finds

$$182 \quad \frac{\partial}{\partial t} \int_{z_b}^{\eta} g \rho z dz + \nabla_{x,y} \cdot \int_{z_b}^{\eta} g \rho z \mathbf{u} dz = \int_{z_b}^{\eta} g \rho w dz,$$

183 which is the integrated local conservation of gravitational potential energy.

184 **2.2.2. Fluid boundaries and solid walls.** On solid walls we prescribe a non-
 185 penetration condition (neglecting viscosity)

$$186 \quad \mathbf{U} \cdot \mathbf{n} = 0,$$

187 \mathbf{n} being the outward normal to the considered wall, together with appropriate condi-
 188 tions for the temperature.

189 In this paper we consider fluid boundaries on which we prescribe zero, one or two
 190 of the following conditions depending on the type of the flow (fluvial or torrential):
 191 water level $h + z_b(x, y)$ given, discharge $h \mathbf{U} \cdot \mathbf{n}$ given.

192 **2.2.3. Boundary conditions for the temperature.** .

193 The heat flux in Eq. (2.3) requires to define boundary conditions for the tem-
 194 perature. Moreover when the state law (2.5) will be precised, the definition of the
 195 temperature at each boundary will be mandatory. We can choose either Neumann or
 196 Dirichlet conditions namely at the bottom

$$197 \quad (2.15) \quad \lambda \frac{\partial T}{\partial \mathbf{n}_b} = \phi_T|_{z_b},$$

198 or

$$199 \quad (2.16) \quad T_b = T_b^0,$$

200 and at the free surface

$$201 \quad (2.17) \quad \lambda \frac{\partial T}{\partial \mathbf{n}_s} = \phi_T|_s,$$

202 or

$$203 \quad (2.18) \quad T_s = T_s^0,$$

204 where $\phi_T|_{z_b}$, $\phi_T|_s$ are two given temperature fluxes and T_b^0 , T_s^0 are two given temper-
205 atures.

The system is completed with some initial conditions

$$h(0, x, y) = h^0(x, y), \quad \rho(0, x, y) = \rho^0(x, y), \quad \mathbf{U}(0, x, y, z) = \mathbf{U}^0(x, y, z).$$

206 **3. The low Mach limit.** In this section, the low Mach limit of the compressible
207 Navier-Stokes equations is performed in order to get an incompressible system. As
208 already mentioned in the introduction, one of the motivations for this limit is that the
209 density of the water varies very little with pressure variations, and removing acoustic
210 waves from the model is advantageous from the computational point of view.

211 **3.1. The Navier-Stokes-Fourier with temperature.** Therefore, we now con-
212 sider the state equation of the fluid (2.5) under the form

$$213 \quad (3.1) \quad \tilde{f}(\rho, T, \varepsilon(p - p_{ref})) = 0,$$

214 where $\varepsilon \ll 1$ is a small parameter (related to the sound velocity, see Remark 3.1) and
215 with p_{ref} a reference pressure constant in space and time. In other words, taking into
216 account the implicit function theorem, this means to assume the particular form for
217 the pressure

$$218 \quad (3.2) \quad p = p_{ref} + \frac{p_0}{\varepsilon},$$

219 where the law $p_0(\rho, T)$ has no stiff scale.

220 *Remark 3.1.* When writing equation (3.1), we assume that the density of the
221 water depends very weakly on the pressure, and this is true in practice. Water and
222 seawater compressibility values at various temperatures and pressures are given in
223 [15, 33]. A possible state law for seawater (involving the salinity S , which we will
224 include later on in Section 3.2) is to write (3.1) as

$$225 \quad \rho(S, T, p) = \frac{\rho(S, T, p_{ref})}{1 - \frac{p - p_{ref}}{K(S, T, p - p_{ref})}},$$

226 where in the fraction $(p - p_{ref})/K(S, T, p - p_{ref})$, the denominator is very large with
227 respect to the numerator, so that this law could actually be written

$$228 \quad (3.3) \quad \rho(S, T, p) = \frac{\rho(S, T, p_{ref})}{1 - \varepsilon(p - p_{ref})}.$$

This law was published in [37], where values of the density ρ at different pressures and constant S, T are also given. One can see that the density varies slowly with respect to the pressure. As reported in Appendix A of [26], for $S = 8$ PSU and $T = 10$ °C, the water density for $p = 0$ (atmospheric pressure) is $\rho(8, 10, 0) = 1005.945659$ kg.m⁻³ while for $p = 10$ bar, the density is $\rho(8, 10, 10) = 1006.41797$ kg.m⁻³. **The sound velocity is defined by**

$$c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s,$$

implying after some computations involving relations (2.6) and (3.3) that

$$\varepsilon \approx \frac{1}{\rho(S, T, p_{ref})c^2}.$$

Since in sea water, $c \approx 1500 \text{ m.s}^{-1}$, this justifies the assumption $\varepsilon \ll 1$ done in relation (3.1).

Remark 3.2. One could consider that the reference pressure p_{ref} varies in time, for instance because of changes in the boundary conditions of the system - p_{ref} adapts to temperature fluxes and mass fluxes at the boundaries. Here, for the sake of simplicity, we consider that p_{ref} is constant in space and in time. Yet the model derivation should not be significantly different with $p_{ref} = p_{ref}(t)$.

For the thermodynamic identity (2.6) to be compatible with (3.2), it is necessary to consider a rescaling for e and s , leading to the following rescaled thermodynamic identities

$$(3.4) \quad e + \frac{p_{ref}}{\rho} = \frac{e_0}{\varepsilon}, \quad s = \frac{s_0}{\varepsilon}, \quad \text{with} \quad de_0 = \frac{p_0}{\rho^2} d\rho + T ds_0,$$

where e_0 and s_0 do not involve stiff scales. The low Mach limit is performed by letting ε go to 0 in the Navier-Stokes-Fourier system with the relations (3.2), (3.4). As p is the physical pressure, it has to remain finite. Therefore according to (3.2) at the limit we get $p_0(\rho, T) = 0$. This can be written

$$T = T^{eq}(\rho),$$

or equivalently $\rho = \rho(T^{eq})$. In all the remainder of the paper, the superscript eq (as in e_0^{eq}) is used for the quantities "at equilibrium", i.e. quantities in which ρ and T are constrained by the relation $p_0(\rho, T) = 0$.

One can define the specific enthalpy $H = e + p/\rho$, and the specific heat capacity at constant pressure $c_p = \left(\frac{\partial H}{\partial T}\right)_p$. Then with (3.2), (3.4) one has $H = H_0/\varepsilon$ with $H_0 = e_0 + p_0/\rho$, and $c_p = c_{p_0}/\varepsilon$ with $c_{p_0} = \left(\frac{\partial H_0}{\partial T}\right)_{p_0}$.

We have the following result.

PROPOSITION 3.3. *The system*

$$(3.5) \quad \nabla \cdot \mathbf{U} = -\frac{\rho'(T^{eq})}{\rho^2 c_p} \nabla \cdot (\lambda \nabla T^{eq}),$$

$$(3.6) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0,$$

$$(3.7) \quad \frac{\partial(\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p - \nabla \cdot \sigma = \rho \mathbf{g},$$

with the relation $T = T^{eq}(\rho)$, $\rho'(T) = \frac{d\rho}{dT}$ and where p is a Lagrange multiplier, is the formal limit of the system (2.1)-(2.3), with (3.2), (3.4) as ε goes to 0. The energy balance verified by (3.5)-(3.7) is

$$(3.8) \quad \begin{aligned} \frac{\partial}{\partial t} \left(\rho \frac{|\mathbf{U}|^2}{2} - p_{ref} + \rho \frac{e_0^{eq}}{\varepsilon} \right) + \nabla \cdot \left(\left(\rho \frac{|\mathbf{U}|^2}{2} - p_{ref} + \rho \frac{e_0^{eq}}{\varepsilon} + p - \sigma \right) \mathbf{U} \right) \\ = \nabla \cdot \left(\frac{\lambda_0}{\varepsilon} \nabla T^{eq} \right) + \rho \mathbf{g} \cdot \mathbf{U} + (p - p_{ref}) \nabla \cdot \mathbf{U} - \sigma : D(\mathbf{U}). \end{aligned}$$

263 The energy balance (3.8) is expressed in the rescaled variables defined by (3.2), (3.4)
 264 in order to show clearly the order of magnitude of each term. The quantity λ_0 is
 265 defined by (3.11).

266 *Proof.* We first rewrite (2.1) under the form

$$267 \quad (3.9) \quad \rho \frac{D\rho}{Dt} = -\rho^2 \nabla \cdot \mathbf{U}.$$

268 When $\varepsilon \rightarrow 0$ we get $\rho = \rho(T^{eq})$, and we can multiply (3.9) by $dT^{eq}/d\rho$ to get an
 269 equation for the temperature

$$270 \quad (3.10) \quad \rho \frac{DT^{eq}}{Dt} = -\rho^2 \frac{dT^{eq}}{d\rho} \nabla \cdot \mathbf{U}.$$

271 In the sequel, we consider that for the heat conduction λ and the fluid viscosity μ , we
 272 are in the following asymptotic regime,

$$273 \quad (3.11) \quad \lambda = \frac{\lambda_0}{\varepsilon}, \quad \text{and} \quad \mu \sim 1.$$

274 Notice that if λ or μ is smaller than these scales, then the corresponding terms in
 275 the incompressible system (3.5)-(3.8) will simply vanish. Hence, the computation of
 276 (2.8) $-p_{ref}/\rho^2 \times$ (2.9), multiplying by ε and taking the limit, we get according to (3.4)

$$277 \quad (3.12) \quad \rho \frac{De_0^{eq}}{Dt} = \nabla \cdot (\lambda_0 \nabla T^{eq}).$$

278 Writing then as above the rescaled enthalpy $H_0 = e_0 + p_0/\rho$, one gets that at equilib-
 279 rium

$$280 \quad H_0^{eq} = e_0^{eq},$$

281 and $H_0^{eq} = H_0^{eq}(T^{eq})$. One has then according to the definition of c_{p_0}

$$282 \quad dH_0^{eq} = c_{p_0} dT^{eq}.$$

283 We thus get a second equation for the temperature

$$284 \quad (3.13) \quad \rho c_{p_0} \frac{DT^{eq}}{Dt} = \nabla \cdot (\lambda_0 \nabla T^{eq}).$$

285 Comparing (3.10) and (3.13) we obtain (3.5), where we have replaced the scaled
 286 quantities c_{p_0} and λ_0 by their physical values εc_p and $\varepsilon \lambda$ respectively. The pressure
 287 p in (3.7) can finally be interpreted as a Lagrange multiplier for the equation (3.5).
 288 The momentum equation (3.7) together with the mass equation (3.9) gives again the
 289 kinetic energy equation (2.7). Adding it to (3.12) divided by ε and to trivial terms in
 290 p_{ref} finally gives the energy balance (3.8). \square

Remark 3.4. At the limit, the thermodynamic identity (3.4) becomes

$$de_0^{eq} = T^{eq} ds_0^{eq}.$$

291 From (3.12) we obtain the equation for the evolution of the entropy

$$292 \quad \frac{\partial}{\partial t} (\rho s_0^{eq}) + \nabla \cdot (\rho s_0^{eq} \mathbf{U}) - \frac{1}{T^{eq}} \nabla \cdot (\lambda_0 \nabla T^{eq}) = 0.$$

293 Written in the conservative/dissipative form, this gives

$$294 \quad \frac{\partial}{\partial t}(\rho s_0^{eq}) + \nabla \cdot (\rho s_0^{eq} \mathbf{U}) - \nabla \cdot \left(\lambda_0 \frac{\nabla T^{eq}}{T^{eq}} \right) = \lambda_0 \frac{|\nabla T^{eq}|^2}{(T^{eq})^2},$$

295 which shows that in accordance with the second law of thermodynamics, the total
296 entropy $\int \rho s_0^{eq}$ can only increase.

297 The energy balance (3.8) exhibits discrepancies of the order ε with respect to the
298 original energy balance (2.3). Namely, the terms $(p - p_{ref})\nabla \cdot \mathbf{U} - \sigma : D(\mathbf{U})$ are of size
299 ε with respect to the leading order terms and they are not present in (2.3). In order
300 to get an energy balance closer to the original compressible equations, we go one step
301 further and some corrections of size ε are incorporated into the system. These do not
302 modify the order of accuracy of the approximation however, since other error terms
303 of order ε are not corrected.

304 PROPOSITION 3.5. *The system*

$$305 \quad (3.14) \quad \nabla \cdot \mathbf{U} = -\frac{\rho'(T^{eq})}{\rho^2 c_p} (\nabla \cdot (\lambda \nabla T^{eq}) + \sigma : D(\mathbf{U})),$$

$$306 \quad (3.15) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0,$$

$$307 \quad (3.16) \quad \frac{\partial(\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p - \nabla \cdot \sigma = \rho \mathbf{g},$$

308 with T a given function of ρ

$$309 \quad (3.17) \quad T = T^{eq}(\rho),$$

310 and where p is a Lagrange multiplier, is an approximation of order ε of the formal
311 limit (3.5)-(3.7) of the system (2.1)-(2.3) with (3.2), (3.4). The system (3.14)-(3.16)
312 satisfies the energy balance equation

$$313 \quad (3.18) \quad \frac{\partial}{\partial t} \left(\rho \frac{|\mathbf{U}|^2}{2} - p_{ref} + \rho \frac{e_0^c}{\varepsilon} \right) + \nabla \cdot \left(\left(\rho \frac{|\mathbf{U}|^2}{2} - p_{ref} + \rho \frac{e_0^c}{\varepsilon} + p - \sigma \right) \mathbf{U} \right) \\ = \nabla \cdot \left(\frac{\lambda_0}{\varepsilon} \nabla T^{eq} \right) + \rho \mathbf{g} \cdot \mathbf{U},$$

314 where e_0^c is an independent (rescaled) corrected energy variable such that $e_0^c - e_0^{eq} =$
315 $O(\varepsilon)$.

316 *Proof.* The proof is similar to that of Proposition 3.3. At the limit, the variables
317 are constrained by the relation (3.17). To obtain an energy balance close to (2.3), we
318 define an independent corrected energy variable e_0^c ($e_0^c \neq e_0^{eq}$) satisfying

$$319 \quad (3.19) \quad \rho \frac{De_0^c}{Dt} = \nabla \cdot (\lambda_0 \nabla T^{eq}) - \varepsilon(p - p_{ref})\nabla \cdot \mathbf{U} + \varepsilon \sigma : D(\mathbf{U}).$$

320 With this correction, the energy balance is now (3.18). The internal energy at equi-
321 librium e_0^{eq} satisfies (3.19) but without the term $\varepsilon(p - p_{ref})\nabla \cdot \mathbf{U}$, which implies that
322 $e_0^c - e_0^{eq}$ is of order ε . Recalling that $H_0^{eq} = e_0^{eq}$, we get the equation for the enthalpy
323 at equilibrium H_0^{eq}

$$324 \quad (3.20) \quad \rho \frac{DH_0^{eq}}{Dt} = \nabla \cdot (\lambda_0 \nabla T^{eq}) + \varepsilon \sigma : D(\mathbf{U}).$$

Eq. (3.20) appears as a correction of (3.12). Notice that if the correction $-\varepsilon(p - p_{ref})\nabla \cdot \mathbf{U}$ were incorporated, the model obtained would be even more accurate, but it would contain derivatives of p , which makes the equations much more difficult to handle since p is a Lagrange multiplier. Here we restrict ourselves to the correction $\varepsilon\sigma : D(\mathbf{U})$, which is viscous dissipation, to keep the model simple.

From equation (3.20), we get a corrected equation on the temperature

$$\rho c_{p0} \frac{DT^{eq}}{Dt} = \nabla \cdot (\lambda_0 \nabla T^{eq}) + \varepsilon\sigma : D(\mathbf{U}).$$

Consequently, comparing with (3.10), we get the equation on $\nabla \cdot \mathbf{U}$ (in the rescaled variables)

$$\nabla \cdot \mathbf{U} = -\frac{\rho'(T^{eq})}{\rho^2 c_{p0}} (\nabla \cdot (\lambda_0 \nabla T^{eq}) + \varepsilon\sigma : D(\mathbf{U})),$$

which is (3.14). In physical variables, the equation for the temperature can also be written

$$(3.21) \quad \rho c_p \frac{DT^{eq}}{Dt} = -\nabla \cdot Q_T + \sigma : D(\mathbf{U}). \quad \square$$

Remark 3.6. The entropy equation for the model (3.14)-(3.16) is

$$\frac{\partial}{\partial t} (\rho s_0^{eq}) + \nabla \cdot (\rho s_0^{eq} \mathbf{U}) - \nabla \cdot \left(\lambda_0 \frac{\nabla T^{eq}}{T^{eq}} \right) = \lambda_0 \frac{|\nabla T^{eq}|^2}{(T^{eq})^2} + \frac{\varepsilon}{T^{eq}} \sigma : D(\mathbf{U}).$$

As $\sigma : D(\mathbf{U}) \geq 0$, we obtain again that the total entropy $\int \rho s_0^{eq}$ can only increase. Here there is no discrepancy with the original entropy equation (2.10).

Remark 3.7. In models (3.5)-(3.7) and (3.14)-(3.16), the temperature T^{eq} is no longer an independent variable of the system. It is recovered by inverting the state law $\rho = \rho(T^{eq})$.

Remark 3.8. The model (3.14)-(3.16) is very similar to what is classically obtained in the literature when the low-Mach limit of the Navier-Stokes equations with thermal conduction is taken, see for instance [30]. The limit is usually performed by expanding the variables of the system in power series of the Mach number. The method we have used here is different. As we have given ourselves only a generic state law, we cannot express the Mach number, let alone make it appear in the equations. Instead, the result is obtained via a rescaling of the variable part of the pressure, and this enables to consider general state laws.

Remark 3.9. Even if a gas and water behave very differently, an example of equation of state to which the previous asymptotics can be applied is the stiffened gas law [19]

$$p = (\gamma - 1)\rho e - \gamma p_\infty, \quad \frac{c_p}{\gamma} T = e - \frac{p_\infty}{\rho},$$

with the constraint $e - p_\infty/\rho > 0$, where $\gamma > 1$, $p_\infty > 0$, $c_p > 0$ are constants. Here the entropy is given by $s = \frac{c_p}{\gamma} \log(T/\rho^{\gamma-1})$. Then the scaling assumptions (3.2), (3.4) are satisfied when

$$p_\infty = -p_{ref} + \frac{p_{\infty 0}}{\varepsilon}, \quad c_p = \frac{c_{p0}}{\varepsilon},$$

361 with $p_{\infty 0}$ and c_{p0} constants independent of ε . At equilibrium we get the relation
 362 $\rho T^{eq} = \frac{\gamma}{\gamma-1} \frac{p_{\infty 0}}{c_{p0}}$, thus T^{eq} is inversely proportional to ρ .

363 *Remark 3.10.* In [6] the authors focused on the following 2D (x, z) model

$$364 \quad (3.22) \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0,$$

$$365 \quad (3.23) \quad \frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u w}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z},$$

$$366 \quad (3.24) \quad \frac{\partial p}{\partial z} = -\rho g + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z},$$

$$367 \quad (3.25) \quad \rho = \rho(T),$$

$$368 \quad (3.26) \quad \frac{\partial \rho T}{\partial t} + \frac{\partial \rho u T}{\partial x} + \frac{\partial \rho w T}{\partial z} = \frac{\lambda}{c_p} \frac{\partial^2 T}{\partial x^2} + \frac{\lambda}{c_p} \frac{\partial^2 T}{\partial z^2}.$$

369 Rewriting equation (3.26) in the non-conservative form, we get

$$370 \quad \rho \frac{\partial T}{\partial t} + \rho u \frac{\partial T}{\partial x} + \rho w \frac{\partial T}{\partial z} = \frac{\lambda}{c_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right).$$

371 Multiplying this equation by $\rho'(T)/\rho$ gives an equation for ρ

$$372 \quad (3.27) \quad \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} = \frac{\rho'(T)}{\rho} \frac{\lambda}{c_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right).$$

373 Finally, subtracting (3.22) to (3.27) and rearranging the terms gives a compatibility
 374 condition similar to (3.5) for a constant λ

$$375 \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{\rho'(T)}{\rho^2 c_p} \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right).$$

376 Indeed (3.26) is similar to (3.13). This shows that the model (3.22)-(3.26) corresponds
 377 to the model (3.5)-(3.7) with a hydrostatic assumption, see paragraph 4.

378 Because of the stability inherited from the energy balance (3.18), that is consistent
 379 with (2.3), in the sequel we consider the system (3.14)-(3.16) instead of (3.5)-(3.7).

380 **3.2. The Navier-Stokes-Fourier system with salinity.** We now consider the
 381 situation where the fluid density depends on the temperature T and on another inter-
 382 nal variable, the specific salinity S . This is the case of sea water. The compressible
 383 Navier-Stokes-Fourier system with temperature and salinity can be written

$$384 \quad (3.28) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0,$$

$$385 \quad (3.29) \quad \frac{\partial(\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p - \nabla \cdot \sigma = \rho \mathbf{g},$$

$$386 \quad (3.30) \quad \frac{\partial}{\partial t} \left(\rho \frac{|\mathbf{U}|^2}{2} + \rho e \right) + \nabla \cdot \left(\left(\rho \frac{|\mathbf{U}|^2}{2} + \rho e + p - \sigma \right) \mathbf{U} \right) = -\nabla \cdot \mathbf{F}^T + \rho \mathbf{g} \cdot \mathbf{U},$$

$$387 \quad (3.31) \quad \frac{\partial(\rho S)}{\partial t} + \nabla \cdot (\rho S \mathbf{U}) = -\nabla \cdot \mathbf{F}^S.$$

388 The local mass and momentum conservation equations are identical to (2.1) and
 389 (2.2), whereas the energy equation is slightly modified: the heat flux is now \mathbf{F}^T . The

457 PROPOSITION 3.11. *The system*

$$458 \quad \nabla \cdot \mathbf{U} = \frac{1}{\rho^2 c_p} \left(\frac{\partial \rho}{\partial T^{eq}} \right)_S \left((T^{eq})^2 \left(\frac{\partial (\mu_S / T^{eq})}{\partial T^{eq}} \right)_S \nabla \cdot \mathbf{F}^S + \nabla \cdot \mathbf{F}^T \right. \\ 459 \quad (3.46) \quad \left. - \sigma : D(\mathbf{U}) + \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial S} \right)_{T^{eq}} \nabla \cdot \mathbf{F}^S, \right.$$

$$460 \quad (3.47) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0,$$

$$461 \quad (3.48) \quad \frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p - \nabla \cdot \sigma = \rho \mathbf{g},$$

$$462 \quad (3.49) \quad \frac{\partial (\rho S)}{\partial t} + \nabla \cdot (\rho S \mathbf{U}) = -\nabla \cdot \mathbf{F}^S,$$

$$463 \quad (3.50) \quad \mathbf{F}^S = -\rho k^S \nabla S - \left(\frac{\rho k^S T}{\partial_S \mu_S} \partial_T \left(\frac{\mu_S}{T} \right) + \frac{B}{T^2} \right) \nabla T,$$

$$464 \quad (3.51) \quad \mathbf{F}^T = -\rho c_p k^T \nabla T + \frac{B \partial_S \mu_S}{\rho k^S T} \mathbf{F}^S,$$

465 *with* $T = T^{eq}(\rho, S)$ *and where* p *is a Lagrange multiplier, is an approximation of order*
466 ε *of the formal limit of the system* (3.28)-(3.31), (3.40), (3.41), (3.45).

467 *Proof.* At the limit there remain only two independent thermodynamic variables,
468 and one can take T and S . We consider that \mathbf{F}^S is bounded but $\mathbf{F}^T \sim 1/\varepsilon$. With
469 (3.33), (3.34) this means that $A \sim \varepsilon$, $B \sim 1$, $C \sim 1/\varepsilon$, and using (3.42) that

$$470 \quad (3.52) \quad k^S \sim 1, \quad c_p k^T \sim 1/\varepsilon.$$

471 At equilibrium the term in ∇p in (3.40) disappears in the expression of \mathbf{F}^S since μ_S
472 depends only weakly on p (this is a consequence of the scaling assumption (3.44)) and
473 ∇p remains bounded, giving (3.50), (3.51). The mass conservation equation (3.28), the
474 momentum equation (3.29) and the salinity equation (3.31) are unchanged. Consider
475 then the enthalpy $H = e + p/\rho = H_0/\varepsilon$. Taking (3.36)– $p_{ref}/\rho^2 \times (2.9)$, multiplying
476 by ε and taking the limit, we get according to (3.45) and using the correction as in
477 Section 3 the equation for the rescaled enthalpy at equilibrium $H_0^{eq} = e_0^{eq}$,

$$478 \quad (3.53) \quad \rho \frac{DH_0^{eq}}{Dt} = -\varepsilon \nabla \cdot \mathbf{F}^T + \varepsilon \sigma : D(\mathbf{U}).$$

479 One can write at equilibrium

$$480 \quad (3.54) \quad dH^{eq} = \left(\frac{\partial H^{eq}}{\partial T^{eq}} \right)_S dT^{eq} + \left(\frac{\partial H^{eq}}{\partial S} \right)_{T^{eq}} dS.$$

481 Combining (3.53) (written in the physical variables) with (3.54) gives

$$482 \quad (3.55) \quad \rho \left(\frac{\partial H^{eq}}{\partial T^{eq}} \right)_S \frac{DT^{eq}}{Dt} - \left(\frac{\partial H^{eq}}{\partial S} \right)_{T^{eq}} \nabla \cdot \mathbf{F}^S + \nabla \cdot \mathbf{F}^T - \sigma : D(\mathbf{U}) = 0.$$

483 We have similarly for the density

$$484 \quad (3.56) \quad d\rho = \left(\frac{\partial \rho}{\partial T^{eq}} \right)_S dT^{eq} + \left(\frac{\partial \rho}{\partial S} \right)_{T^{eq}} dS.$$

514 *Remark 3.13.* A criterion of well-posedness of our incompressible system (3.46)-
 515 (3.51) can be derived as follows. We write that the second-order terms in the coupled
 516 S and T^{eq} equations (3.49), (3.55) give a diffusion matrix with positive eigenvalues.
 517 With (3.50), (3.51) we have at equilibrium (using (3.60))

$$518 \quad (3.62) \quad \mathbf{F}^S = -\rho k^S \nabla S - E \nabla T, \quad \text{with } E = \frac{\rho k^S T}{\partial_S \mu_S} \partial_T \left(\frac{\mu_S}{T} \right) + \frac{B}{T^2},$$

519

$$520 \quad (3.63) \quad \mathbf{F}^T - \left(\frac{\partial H^{eq}}{\partial S} \right)_T \mathbf{F}^S = -\rho c_p k^T \nabla T + \left(\frac{B \partial_S \mu_S}{\rho k^S T} + T^2 \partial_T (\mu_S / T) \right) \mathbf{F}^S \\ = -\rho c_p k^T \nabla T + E \frac{T \partial_S \mu_S}{\rho k^S} \mathbf{F}^S.$$

521 The diffusion matrix of the system is thus, taking into account (3.59),

$$522 \quad \begin{pmatrix} \rho k^S & E \\ E \frac{T \partial_S \mu_S}{c_p} & E^2 \frac{T \partial_S \mu_S}{\rho k^S c_p} + \rho k^T \end{pmatrix}.$$

523 We obtain positive eigenvalues under the natural conditions also mentioned in [38]

$$524 \quad (3.64) \quad k^S > 0, \quad k^T > 0, \quad \partial_S \mu_S > 0, \quad c_p > 0.$$

525 *Remark 3.14.* For the particular choice of B such that $E = 0$ (the relation between
 526 E and B is (3.62)) we have a diagonal diffusion matrix in S, T . In this case we have
 527 particular formulas

$$528 \quad (3.65) \quad \mathbf{F}^S = -\rho k^S \nabla S,$$

529 and from (3.63), (3.60)

$$530 \quad (3.66) \quad \mathbf{F}^T + T^2 \partial_T \left(\frac{\mu_S}{T} \right) \mathbf{F}^S = -\rho c_p k^T \nabla T.$$

531 We deduce that

$$532 \quad (3.67) \quad \nabla \cdot \mathbf{F}^T + T^2 \partial_T \left(\frac{\mu_S}{T} \right) \nabla \cdot \mathbf{F}^S = -\nabla \cdot (\rho c_p k^T \nabla T) - \mathbf{F}^S \cdot \nabla \left(T^2 \partial_T \left(\frac{\mu_S}{T} \right) \right),$$

533 that can be used in (3.46). Next we write (3.55) using (3.59), (3.60) and (3.67) to get

$$534 \quad (3.68) \quad \rho c_p \frac{DT}{Dt} = \nabla \cdot (\rho c_p k^T \nabla T) + \mathbf{F}^S \cdot \nabla \left(T^2 \partial_T \left(\frac{\mu_S}{T} \right) \right) + \sigma : D(\mathbf{U}).$$

535 We can write

$$536 \quad \begin{aligned} \nabla \left(T^2 \partial_T \left(\frac{\mu_S}{T} \right) \right) &= \nabla \left(-\mu_S + T \partial_T \mu_S \right) \\ &= -\partial_T \mu_S \nabla T - \partial_S \mu_S \nabla S + \nabla \left(T \partial_T \mu_S \right) \\ &= -\partial_S \mu_S \nabla S + T \nabla \left(\partial_T \mu_S \right), \end{aligned}$$

537 thus we obtain

$$538 \quad (3.69) \quad \rho c_p \frac{DT}{Dt} = \nabla \cdot (\rho c_p k^T \nabla T) + T \mathbf{F}^S \cdot \nabla \left(\partial_T \mu_S \right) - \partial_S \mu_S \mathbf{F}^S \cdot \nabla S + \sigma : D(\mathbf{U}),$$

539 that generalizes (3.21). We therefore obtain the equations (3.49) and (3.69) on S and
 540 T , with the value of \mathbf{F}^S given by (3.65). With the conditions (3.64) we can see that
 541 T remains nonnegative and S verifies the maximum principle (i.e. it remains within
 542 its initial lower and upper bounds). Notice that with (3.59), (3.60), the model finally
 543 relies on the knowledge of $\rho^{eq}(T, S)$, $H^{eq}(T, S)$, $k^T(T, S)$, $k^S(T, S)$.

544 **4. Hydrostatic approximation of the Navier-Stokes-Fourier system.** We
 545 consider the incompressible Navier-Stokes-Fourier system (3.14)-(3.16) derived in para-
 546 graph 3, for the sake of simplicity we consider that the density only depends on the
 547 temperature $\rho = \rho(T)$, and we neglect the salinity S . For the sake of lightness, the
 548 exponent eq is dropped in this part and in the rest of the present document. The
 549 system is completed with the boundary conditions (2.11)-(2.14).

550 The hydrostatic assumption consists in neglecting the vertical acceleration of the
 551 fluid

$$552 \quad (4.1) \quad \rho \left(\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} \right) \approx 0,$$

553 where we recall that $\mathbf{U} = (\mathbf{u}, w)$ with $\mathbf{u} = (u, v)$. We refer to [10, 17, 25] for the
 554 analysis of hydrostatic models and for their asymptotic derivation [5, 14, 24].

555 **4.1. The incompressible and hydrostatic Navier-Stokes-Fourier system.**
 556 Using the assumption (4.1) and the definition (2.4), the hydrostatic approximation of
 557 the incompressible Navier-Stokes-Fourier system (3.14)-(3.16) therefore reads

$$558 \quad \nabla \cdot \mathbf{U} = -\frac{\rho'(T)}{\rho^2 c_p} \left(\nabla \cdot (\lambda \nabla T) + \zeta (\nabla_{x,y} \cdot \mathbf{u} + \partial_z w)^2 \right. \\
 559 \quad \left. + 2\mu \left(|D_{x,y} \mathbf{u}|^2 + \frac{1}{2} |\nabla_{x,y} w + \partial_z \mathbf{u}|^2 + (\partial_z w)^2 \right) \right),$$

$$560 \quad (4.3) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0,$$

$$561 \quad \frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla_{x,y} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial(\rho \mathbf{u} w)}{\partial z} + \nabla_{x,y} p = \nabla_{x,y} \left(\zeta (\nabla_{x,y} \cdot \mathbf{u} + \partial_z w) \right) \\
 562 \quad + \nabla_{x,y} \cdot \left(2\mu D_{x,y} \mathbf{u} \right) + \partial_z \left(\mu (\nabla_{x,y} w + \partial_z \mathbf{u}) \right),$$

$$563 \quad \partial_z p = -\rho g + \partial_z \left(\zeta (\nabla_{x,y} \cdot \mathbf{u} + \partial_z w) \right) + \nabla_{x,y} \cdot \left(\mu (\nabla_{x,y} w + \partial_z \mathbf{u}) \right) \\
 564 \quad + \partial_z (2\mu \partial_z w).$$

565 with $T = T(\rho)$ a given function. According to Remark 3.2, p_{ref} is taken to be the
 566 atmospheric pressure p^a that is supposed constant.

567 The energy balance for the hydrostatic system (4.2)-(4.5) is

$$\frac{\partial}{\partial t} \left(\rho \frac{|\mathbf{u}|^2}{2} + \rho e^c \right) + \nabla \cdot \left(\left(\rho \frac{|\mathbf{u}|^2}{2} + \rho e^c + p - p^a \right) \mathbf{U} \right) \\
 568 \quad + \nabla_{x,y} \cdot \left(-\zeta (\nabla_{x,y} \cdot \mathbf{u} + \partial_z w) \mathbf{u} - 2\mu (D_{x,y} \mathbf{u}) \mathbf{u} - \mu (\nabla_{x,y} w + \partial_z \mathbf{u}) w \right) \\
 + \partial_z \left(-\zeta (\nabla_{x,y} \cdot \mathbf{u} + \partial_z w) w - \mu (\nabla_{x,y} w + \partial_z \mathbf{u}) \cdot \mathbf{u} - 2\mu w \partial_z w \right) = \nabla \cdot (\lambda \nabla T) - \rho g w,$$

569 with the corrected internal energy e^c governed by

$$570 \quad (4.6) \quad \frac{\partial}{\partial t} (\rho e^c) + \nabla \cdot (\rho e^c \mathbf{U}) = -(p - p^a) (\nabla_{x,y} \cdot \mathbf{u} + \partial_z w) + \nabla \cdot (\lambda \nabla T) \\
 + \zeta (\nabla_{x,y} \cdot \mathbf{u} + \partial_z w)^2 + 2\mu \left(|D_{x,y} \mathbf{u}|^2 + \frac{1}{2} |\nabla_{x,y} w + \partial_z \mathbf{u}|^2 + (\partial_z w)^2 \right).$$

571 Notice that (4.6) is similar to (3.19) divided by ε . The energy $e(T)$ defined by $de/dT =$
 572 c_p (e corresponds to e^{eq} in paragraph 3) satisfies the same equation (4.6) except the
 573 term $-(p - p^a)\nabla \cdot \mathbf{U}$ in the right-hand side. It is obtained by multiplying (4.3) by
 574 $c_p dT/d\rho$ and by using (4.2).

575 In order to make the numerical approximation easier and without loss of applica-
 576 bility we propose a few usual simplifications for the viscous terms. First, let us notice
 577 that for water the second viscosity cannot be neglected compared to the dynamic
 578 viscosity, see [12]. But in practice, the chosen value for the coefficient μ (typically
 579 $\mu \approx 0.1 \text{ kg.m}^{-1}.\text{s}^{-1}$) is much bigger than the value of the dynamic viscosity ($\approx 10^{-3}$
 580 $\text{kg.m}^{-1}.\text{s}^{-1}$) in order to model the turbulence effects. Hence hereafter, we can neglect
 581 the second viscosity i.e. we consider $\zeta = 0$.

582 Notice that we have from Eq. (4.2) $\nabla \cdot \mathbf{U} = \mathcal{O}((\lambda + \mu)\rho'/(\rho^2 c_p))$ and there-
 583 fore the terms in $\mathcal{O}(\mu(\lambda + \mu)\rho'/(\rho^2 c_p)) \ll 1$ appearing in Eqs. (4.2),(4.4) can be
 584 neglected. Likewise, neglecting the viscous terms in Eq. (4.5) and in the third com-
 585 ponent of (2.14), an integration between z and η of (4.5) gives the relation

$$586 \quad (4.7) \quad p = p^a + \int_z^\eta \rho g dz_1.$$

587 Hence we propose to replace the system (4.2)-(4.5) by the following simplified system
 588 with constant μ

$$589 \quad (4.8) \quad \nabla \cdot \mathbf{U} = -\frac{\rho'(T)}{\rho^2 c_p} \left(\nabla \cdot (\lambda \nabla T) + \mu |\nabla_{x,y} \mathbf{u}|^2 + \mu \left| \frac{\partial \mathbf{u}}{\partial z} \right|^2 \right),$$

$$590 \quad (4.9) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0,$$

$$591 \quad \frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla_{x,y} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial(\rho \mathbf{u} w)}{\partial z} + \nabla_{x,y} \int_z^\eta \rho g dz_1 = \mu \nabla_{x,y} \cdot \nabla_{x,y} \mathbf{u}$$

$$592 \quad (4.10) \quad + \mu \frac{\partial^2 \mathbf{u}}{\partial z^2}.$$

593 The system (4.8)-(4.10) is completed with the energy balance

$$594 \quad \frac{\partial}{\partial t} \left(\rho \frac{|\mathbf{u}|^2}{2} + \rho e \right) + \nabla_{x,y} \cdot \left(\mathbf{u} \left(\rho \frac{|\mathbf{u}|^2}{2} + \int_z^\eta \rho g dz_1 + \rho e \right) - \mu \nabla \frac{|\mathbf{u}|^2}{2} \right) = \nabla \cdot (\lambda \nabla T),$$

595 with e is the internal energy of the fluid governed by

$$596 \quad \frac{\partial(\rho e)}{\partial t} + \nabla_{x,y} \cdot (\rho e \mathbf{u}) = \left(\int_z^\eta \rho g dz_1 \right) \nabla_{x,y} \cdot \mathbf{u} + \mu |\nabla_{x,y} \mathbf{u}|^2 + \mu \left| \frac{\partial \mathbf{u}}{\partial z} \right|^2 + \nabla \cdot (\lambda \nabla T).$$

597 Therefore the formulation of the rheology terms appearing in (4.10) is similar to what
 598 has been studied by some of the authors in [3].

599 For the system (4.8)-(4.10), the boundary conditions (2.11),(2.13) and (2.15)-
 600 (2.18) remain valid. Concerning the dynamical boundary conditions for Eq. (4.10),
 601 the assumptions used above for the viscous terms imply that the boundary condi-
 602 tion (2.12) becomes

$$603 \quad (4.11) \quad \mu \sqrt{1 + |\nabla_{x,y} z_b|^2} \frac{\partial \mathbf{u}}{\partial \mathbf{n}_b} = -\kappa \mathbf{u},$$

604 with κ a Navier coefficient. Notice that up to terms in $\mathcal{O}(\mu|\nabla_{x,y}z_b|)$, relations (2.12)
 605 and (4.11) are similar. For some applications, one can choose $\kappa = \kappa(h, \mathbf{u}|_b)$.

606 Likewise, the modified expression of the viscous terms leads to a simplified version
 607 of the dynamic boundary condition at the free surface and Eq. (2.14) becomes

$$608 \quad (4.12) \quad \mu \frac{\partial \tilde{\mathbf{u}}}{\partial \mathbf{n}_s} = W(t, x, y) \mathbf{t}_W.$$

609 where $\tilde{\mathbf{u}} = (\mathbf{u}, 0)^T$. The third component of (2.14) corresponds to $p_s = p^a$ that has
 610 been used to obtain (4.7)

611 **4.2. The hydrostatic Euler-Fourier system.** Assuming that the viscous terms
 612 are small, the incompressible Navier-Stokes-Fourier system (3.14)-(3.16) becomes the
 613 incompressible Euler-Fourier system

$$614 \quad (4.13) \quad \nabla \cdot \mathbf{U} = -\frac{\rho'(T)}{\rho^2 c_p} \nabla \cdot (\lambda \nabla T),$$

$$615 \quad (4.14) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0,$$

$$616 \quad (4.15) \quad \frac{\partial(\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{g}.$$

617 Therefore, the hydrostatic approximation of the system (4.13)-(4.15) consists in the
 618 hydrostatic Euler-Fourier model

$$619 \quad (4.16) \quad \nabla \cdot \mathbf{U} = -\frac{\rho'(T)}{\rho^2 c_p} \nabla \cdot (\lambda \nabla T),$$

$$620 \quad (4.17) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0,$$

$$621 \quad (4.18) \quad \frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla_{x,y} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial(\rho \mathbf{u} w)}{\partial z} + \nabla_{x,y} \int_z^\eta \rho g dz = 0,$$

622 completed with the relation $\rho = \rho(T)$ and the boundary conditions (2.11), (2.13).

4.3. Remark on the Boussinesq assumption. In geophysical water flows,
 density variations are often considered as small and this allows justifying the Boussi-
 nesq assumption, which consists in considering the density variations only in the
 gravitational forces. More precisely, assuming

$$\rho = \rho(T) = \rho_0 + f(T),$$

623 with $f(T) \ll \rho_0$, this leads to writing the incompressible hydrostatic Euler sys-
 624 tem (4.16)-(4.18) under the form

$$625 \quad (4.19) \quad \nabla \cdot \mathbf{U} = 0,$$

$$626 \quad (4.20) \quad \rho_0 c_p \left(\frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T \right) = \nabla \cdot (\lambda \nabla T),$$

$$627 \quad (4.21) \quad \rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla_{x,y} \cdot (\mathbf{u} \otimes \mathbf{u}) + \frac{\partial(\mathbf{u} w)}{\partial z} \right) + \nabla_{x,y} \int_z^\eta \rho g dz = 0.$$

628 Notice that whereas in (4.16), the divergence of the velocity field equals the expansion
 629 due to the temperature effects, the Boussinesq assumption implies the divergence free
 630 condition (4.19).

631 The Boussinesq assumption is valid in various regimes [27, 16] but

- 632 • it does not ensure a conservation of the kinetic energy since $\rho_0 \frac{|\mathbf{u}|^2}{2}$ is conserved
- 633 instead of $\rho \frac{|\mathbf{u}|^2}{2}$,
- 634 • for long time phenomena (sloshing, wave propagation,...) significant differ-
- 635 ences appear when the Boussinesq assumption is made, see [6, paragraph 6.2].
- 636 In this work, the Boussinesq assumption is not done and some remarks about its
- 637 validity are given in the forthcoming paper [7], see also [27, 16].

638 **5. Conclusion.** In this paper we have derived, using a low-Mach approxima-
 639 tion, an incompressible model for variable density flows. This model is obtained from
 640 compressible Navier-Stokes equations by assuming that the density depends weakly
 641 on the pressure, thus involving a small parameter ε , that can eventually be identified
 642 as the Mach number. As $\varepsilon \rightarrow 0$, the pressure dependency is lost, while the depen-
 643 dency of the density on temperature and eventually salinity remains. It leads to an
 644 "incompressible" model where the divergence of the velocity is equal to source terms
 645 involving temperature and salinity fluxes. In order to obtain a more accurate model
 646 in terms of energy balance, a correction of order ε is incorporated, which is indeed the
 647 viscous dissipation. The resulting model does not rely on the Boussinesq approxima-
 648 tion. It is mass-conservative; expansion or contraction can be observed as a result of
 649 the variation of a tracer concentration. Our models generalize classical incompressible
 650 models to quite general state laws that can depend on salinity. They include entropy
 651 consistency via Onsager's principles.

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