



HAL
open science

Triangulations in CGAL - To non-Euclidean spaces and beyond!

Monique Teillaud

► **To cite this version:**

Monique Teillaud. Triangulations in CGAL - To non-Euclidean spaces and beyond!. EuroCG 2020 - 36th European Workshop on Computational Geometry, Mar 2020, Würzburg, Germany. hal-02510046

HAL Id: hal-02510046

<https://inria.hal.science/hal-02510046v1>

Submitted on 17 Mar 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Triangulations in



Monique Teillaud

Triangulations in

CGAL

To non-Euclidean spaces and beyond!



By Edo-biscuit - Own work, CC BY-SA 4.0

Monique Teillaud

History of the CGAL project

1995 - Start

Academic project



History of the project

1995 - Start

Academic project

January 2003 - Creation of  Geometry Factory

as an INRIA startup, by Andreas Fabri

sells commercial licenses, support, customized developments

November 2003 - Open Source project

new contributors

November 2019 - CGAL 5.0

C++14

Goals

Gather efforts

PlaGeo, SpaGeo (Utrecht)

XYZ GeoBench (ETH Zürich)

LEDA (MPII Saarbrücken)

C++GAL

(INRIA Sophia Antipolis)

...



Jodelet/Lépinay - CC Attribution-Share Alike 2.0 France

Goals

Gather efforts

Promote research in CG

“ make the large body of geometric algorithms developed in the field of computational geometry available for industrial applications ”

⇒ high quality
review process, . . .

⇒ robust code

Goals

Gather efforts

Promote research in CG

Reward structure for implementations in academia

⇒ high quality
review process, . . .

⇒ robust code

Technical choices

C++

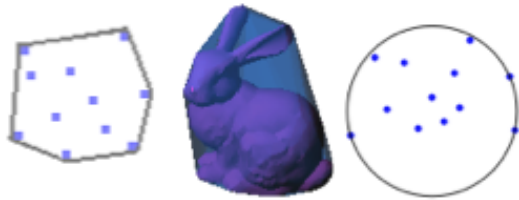
Genericity/flexibility through templates

Exact Geometric Computation [Yap]

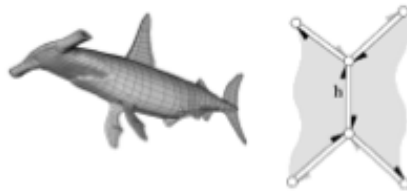
Exact predicates \rightsquigarrow exact decisions

Contents

> 80 chapters in the manual



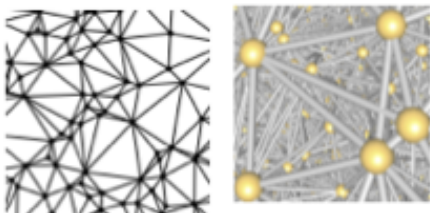
Bounding Volumes



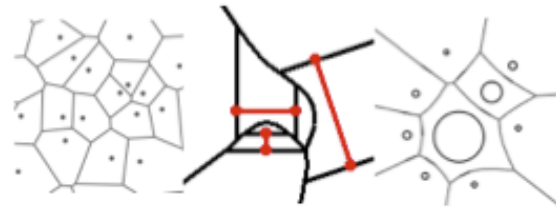
Polyhedral Surface



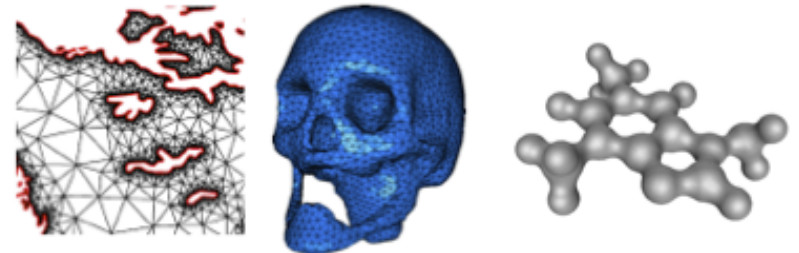
BooleanOperations



Triangulations



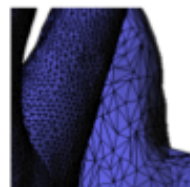
Voronoi Diagrams



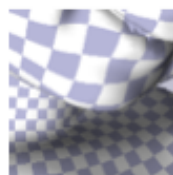
Mesh Generation



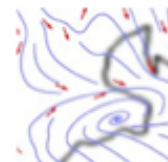
Subdivision



Simplification



Parameterization



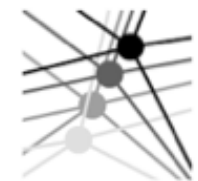
Streamlines



Ridge
Detection



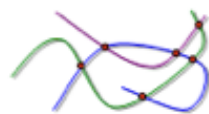
Neighbour
Search



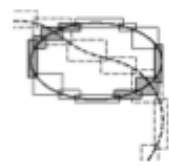
Kinetic
Data structures



Lower Envelope



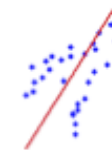
Arrangement



Intersection
Detection



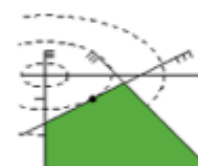
Minkowski
Sum



PCA



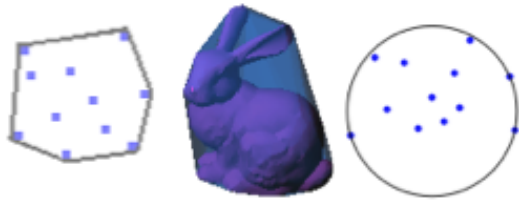
Polytope
distance



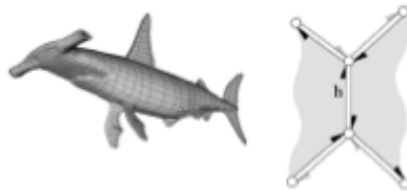
QP Solver

Contents

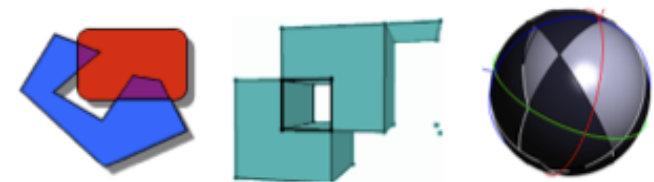
> 80 chapters in the manual



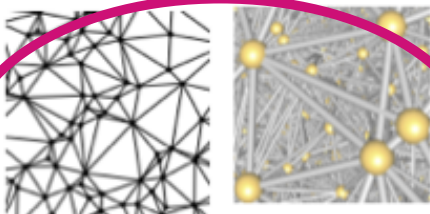
Bounding Volumes



Polyhedral Surface



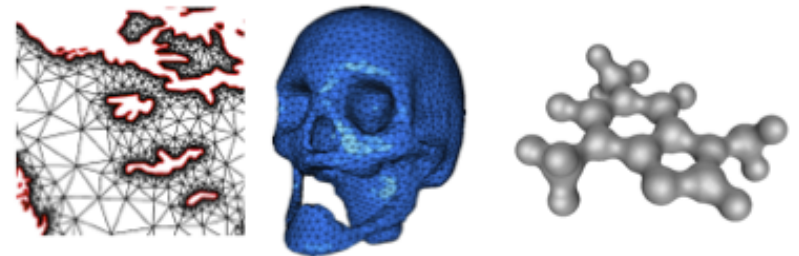
BooleanOperations



Triangulations



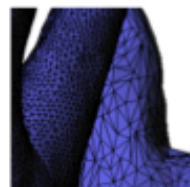
Voronoi Diagrams



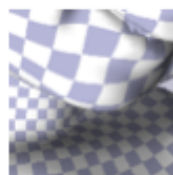
Mesh Generation



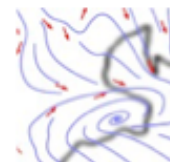
Subdivision



Simplification



Parameterization



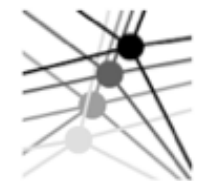
Streamlines



Ridge
Detection



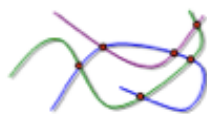
Neighbour
Search



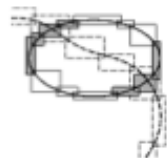
Kinetic
Data structures



Lower Envelope



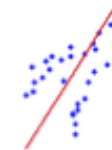
Arrangement



Intersection
Detection



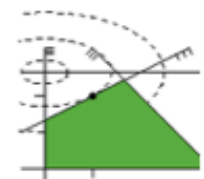
Minkowski
Sum



PCA



Polytope
distance



QP Solver

Triangulations in

2D Triangulations (1997)

3D Triangulations (2000) and meshes (2009)

dD Triangulations (2015)

2D Periodic Triangulations (2013)

3D Periodic Triangulations (2009) and meshes (2018)

2D Hyperbolic Triangulations (2019)

2D Periodic Hyperbolic Triangulations (2019)

General design

Triangulation < Geom_traits, TDS >

General design

Triangulation \langle **Geom_traits** **TDS** \rangle

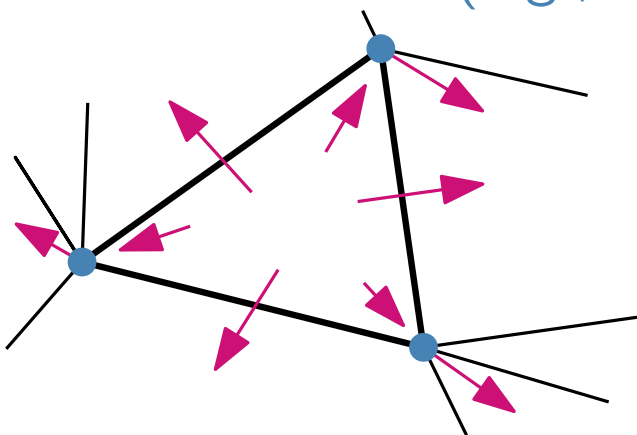
geometry

data structure

objects (points, etc)
predicates (e.g., `in_circle`)
constructions (e.g., `circumcenter`)

Cell = d -simplex
→ d vertices
→ d adjacent cells

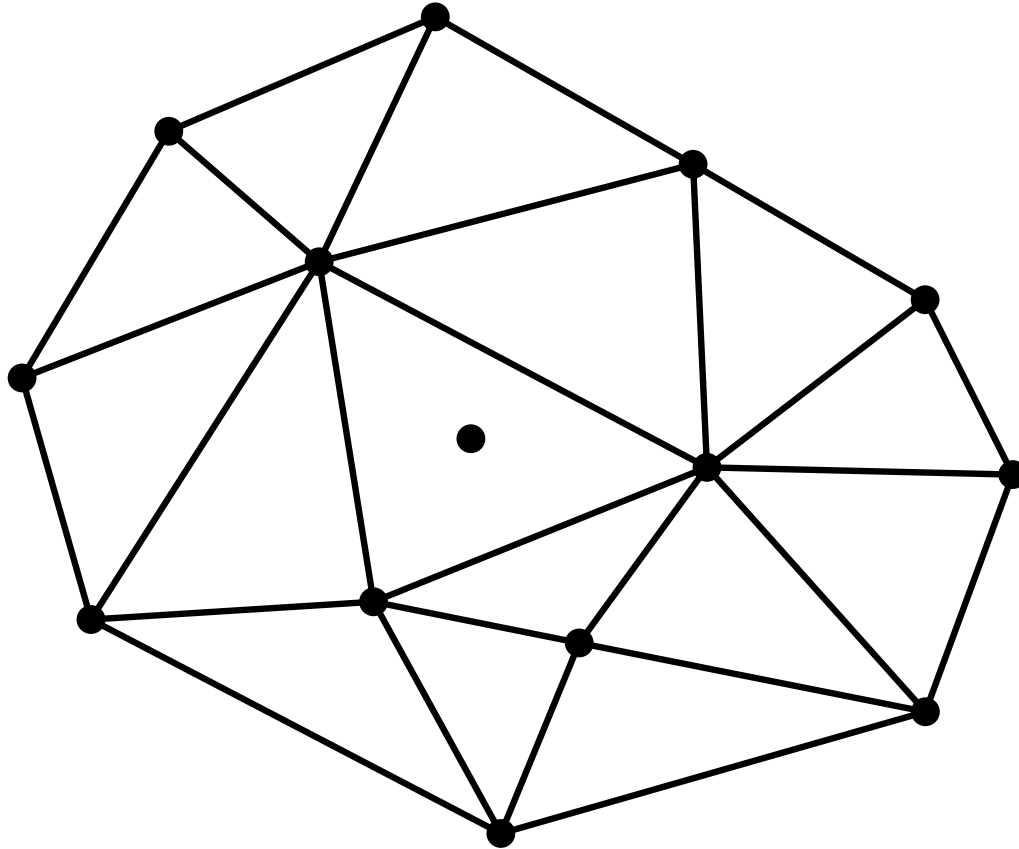
Vertex
point
→ one incident cell



Euclidean Delaunay Triangulations

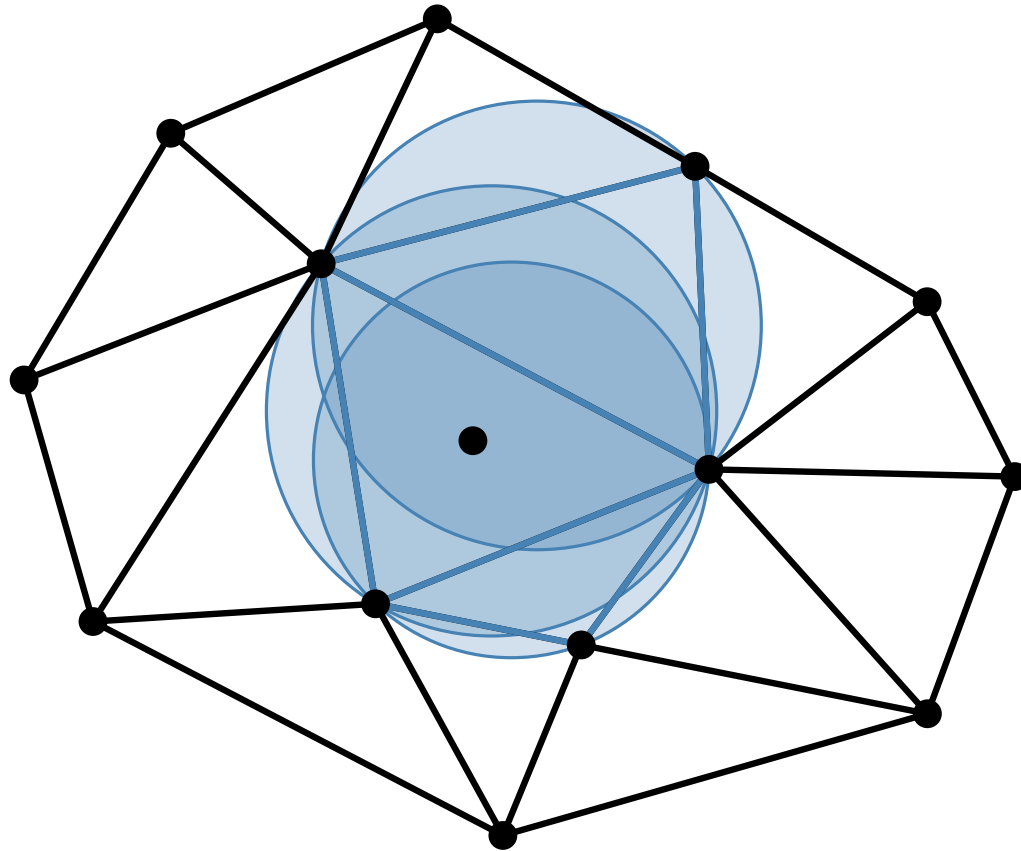
Euclidean Delaunay Triangulations

Bowyer's incremental algorithm



Euclidean Delaunay Triangulations

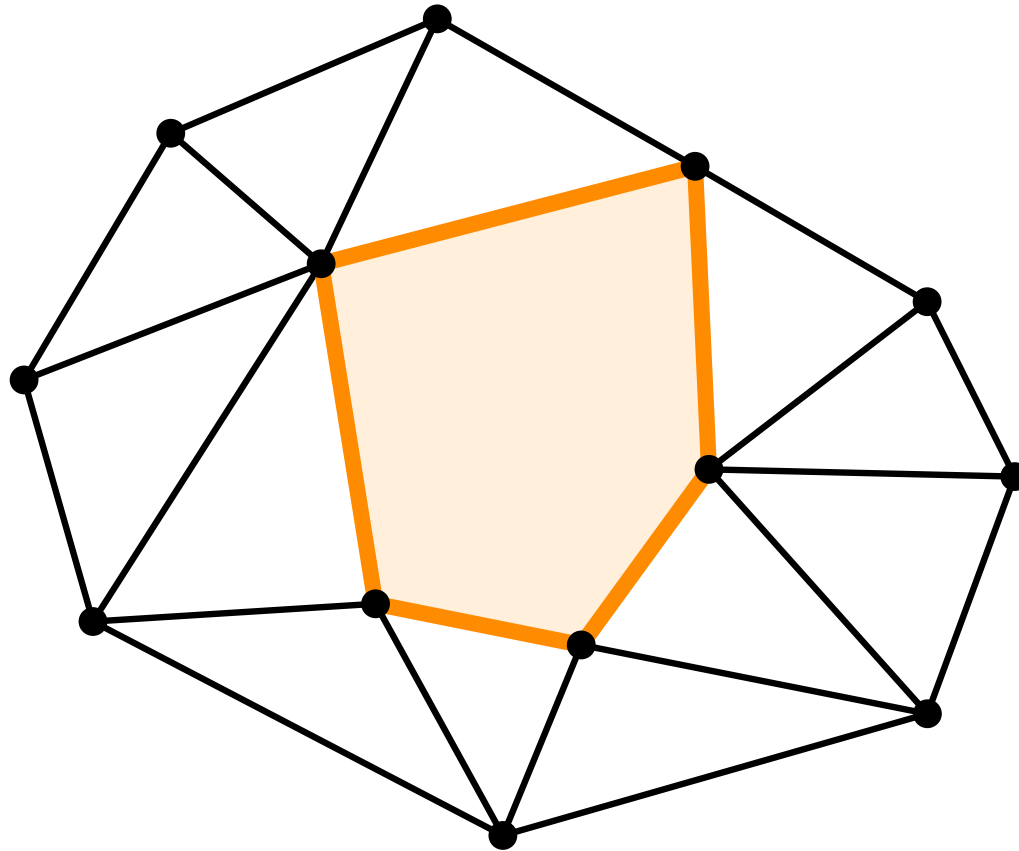
Bowyer's incremental algorithm



find simplices in conflict
Triangulation
using `Geom_traits`

Euclidean Delaunay Triangulations

Bowyer's incremental algorithm

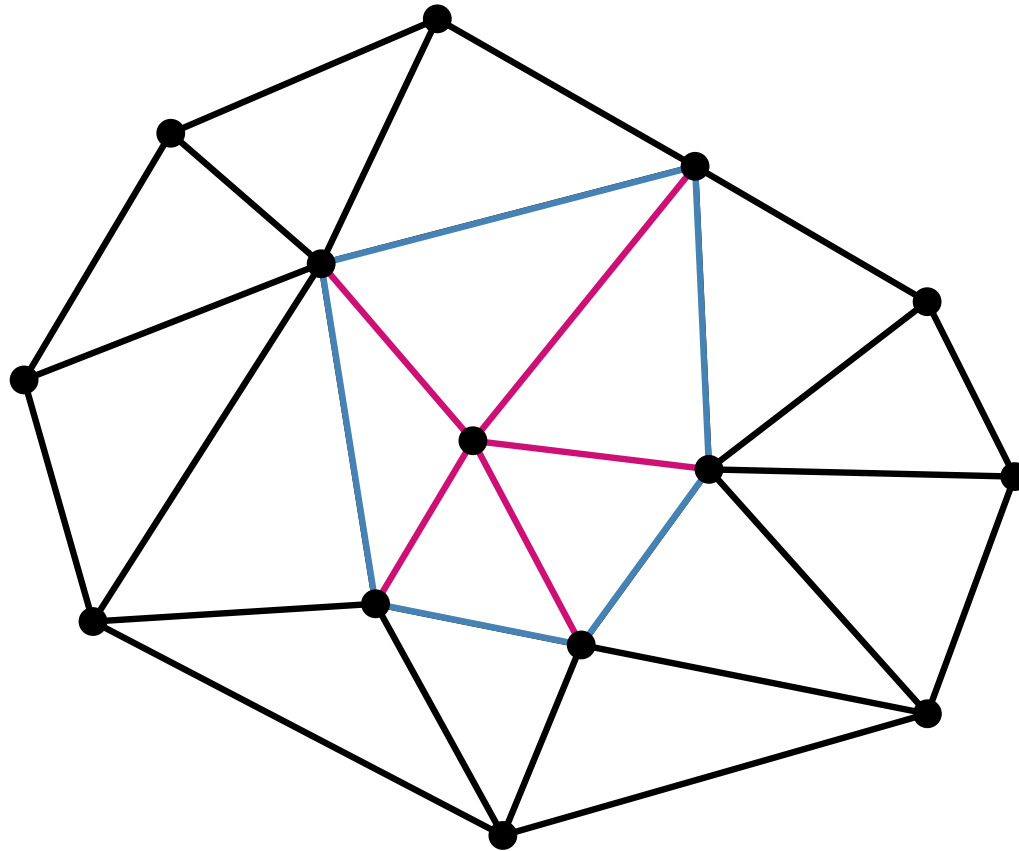


find simplices in conflict
Triangulation
using `Geom_traits`

the conflict region forms a topological ball

Euclidean Delaunay Triangulations

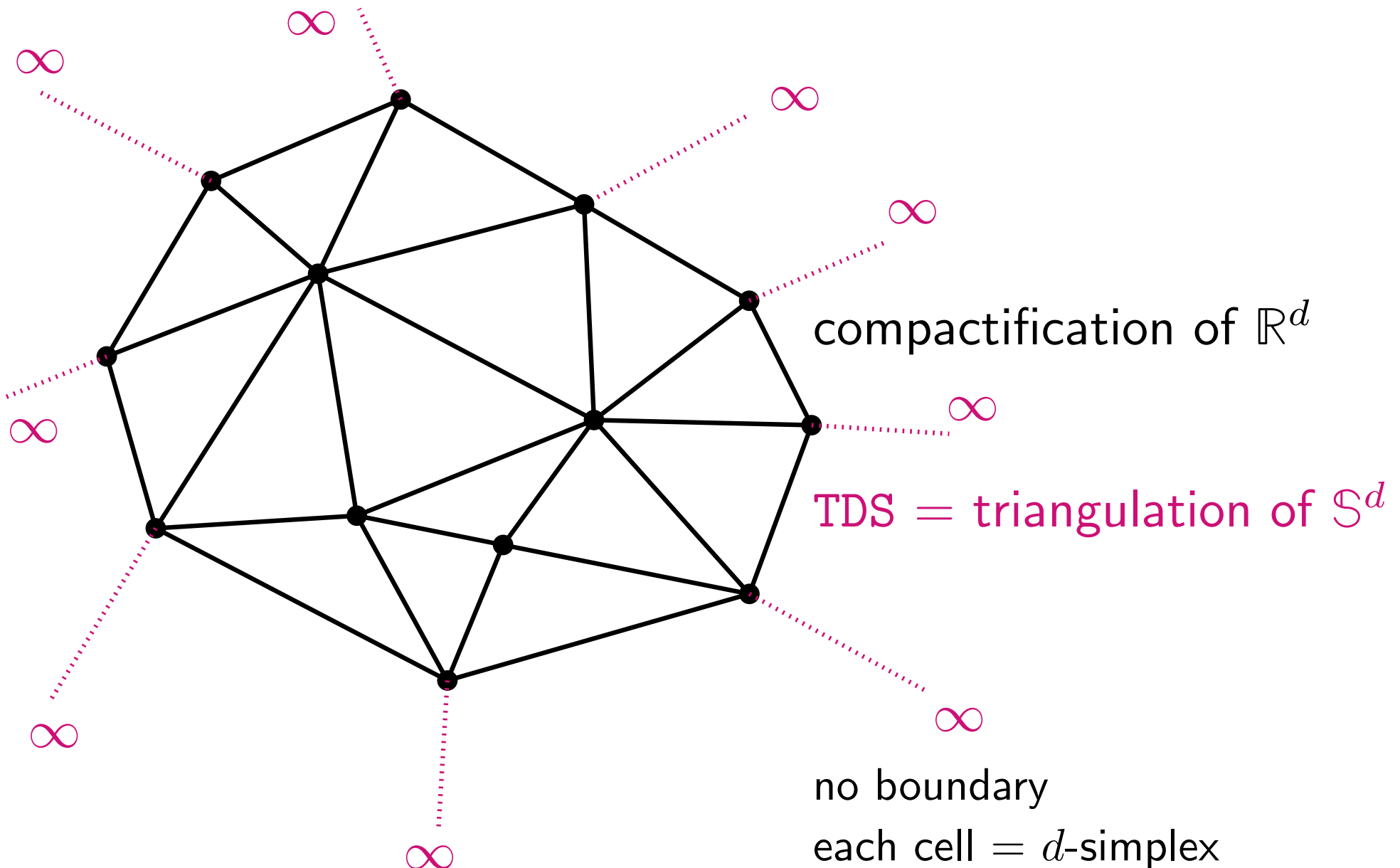
Bowyer's incremental algorithm



find simplices in conflict
Triangulation
using `Geom_traits`

create new simplices
TDS

Euclidean Delaunay Triangulations



Euclidean Delaunay Triangulations

Robustness

Arithmetic issues

Exact Geometric Computation paradigm [Yap]

exact predicates \neq exact arithmetics

Filtering

easy cases are more frequent

\implies cost \simeq cost of approximate (double) computation

Euclidean Delaunay Triangulations

Robustness

Arithmetic issues

Approximate evaluation $P^a(x)$
+ Error ϵ

Filtering

$$P^a(x) > \epsilon$$

?

Yes

No

$$\text{Sign}(P(x)) = \text{Sign}(P^a(x))$$

Exact computation

Euclidean Delaunay Triangulations

Robustness

Degenerate cases

symbolic perturbation

only perturbs the `in_sphere` predicate

[Devillers, T. SODA'03, CGTA'11]

Euclidean Delaunay Triangulations

Tricks to improve efficiency

EuroCG'12 invited talk by Olivier Devillers

<https://hal.inria.fr/hal-00850561>

Euclidean Delaunay Triangulations

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Delaunay_triangulation_3>

typedef CGAL::Exact_predicates_inexact_constructions_kernel K;
typedef CGAL::Delaunay_triangulation_3<K> Delaunay;
typedef Delaunay::Point Point;

int main()
{
    Delaunay T;

    T.insert(Point(0,0,0));
    T.insert(Point(1,0,0));
    T.insert(Point(0,1,0));
    T.insert(Point(0,0,1));
    T.insert(Point(2,2,2));
    T.insert(Point(-1,0,1));

    return 0;
}
```

predefined Geom_traits

default TDS

Euclidean Delaunay Triangulations

Flexibility

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Projection_traits_xy_3.h>
#include <CGAL/Delaunay_triangulation_2.h>
#include <fstream>
```

```
typedef CGAL::Exact_predicates_inexact_constructions_kernel K;
typedef CGAL::Projection_traits_xy_3<K> Gt;
typedef CGAL::Delaunay_triangulation_2<Gt> Terrain;
typedef K::Point_3 Point;
```

```
int main()
{
    Terrain T;

    T.insert(Point(0,0,0));
    T.insert(Point(1,0,0));
    T.insert(Point(0,1,0));
    // etc

    return 0;
}
```

Points : 3D

Predicates :

on their 2D projections

Euclidean Delaunay Triangulations

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Delaunay_triangulation_3.h>
#include <CGAL/Delaunay_triangulation_cell_base_3.h>
#include <CGAL/Triangulation_vertex_base_with_info_3.h>
#include <CGAL/IO/Color.h>

typedef CGAL::Exact_predicates_inexact_constructions_kernel K;
typedef CGAL::Triangulation_vertex_base_with_info_3<CGAL::Color, K> Vb;
typedef CGAL::Delaunay_triangulation_cell_base_3<K> Cb;
typedef CGAL::Triangulation_data_structure_3<Vb, Cb> Tds;
typedef CGAL::Delaunay_triangulation_3<K, Tds> Delaunay;
typedef Delaunay::Point Point;

int main()
{
    Delaunay T;
    T.insert(Point(0,0,0));
    // etc
    Delaunay::Finite_vertices_iterator vit;
    for (Delaunay::Vertex_handle v : T.finite_vertex_handles())
        if (T.degree(v) == 6)
            v->info() = CGAL::red();
    return 0;
}
```

Flexibility

Euclidean Delaunay Triangulations

fully dynamic

fully robust

also weighted

2D (insertion with flips?)

$\simeq 10$ M points / second

[Yvinec CGAL'97]

also constrained

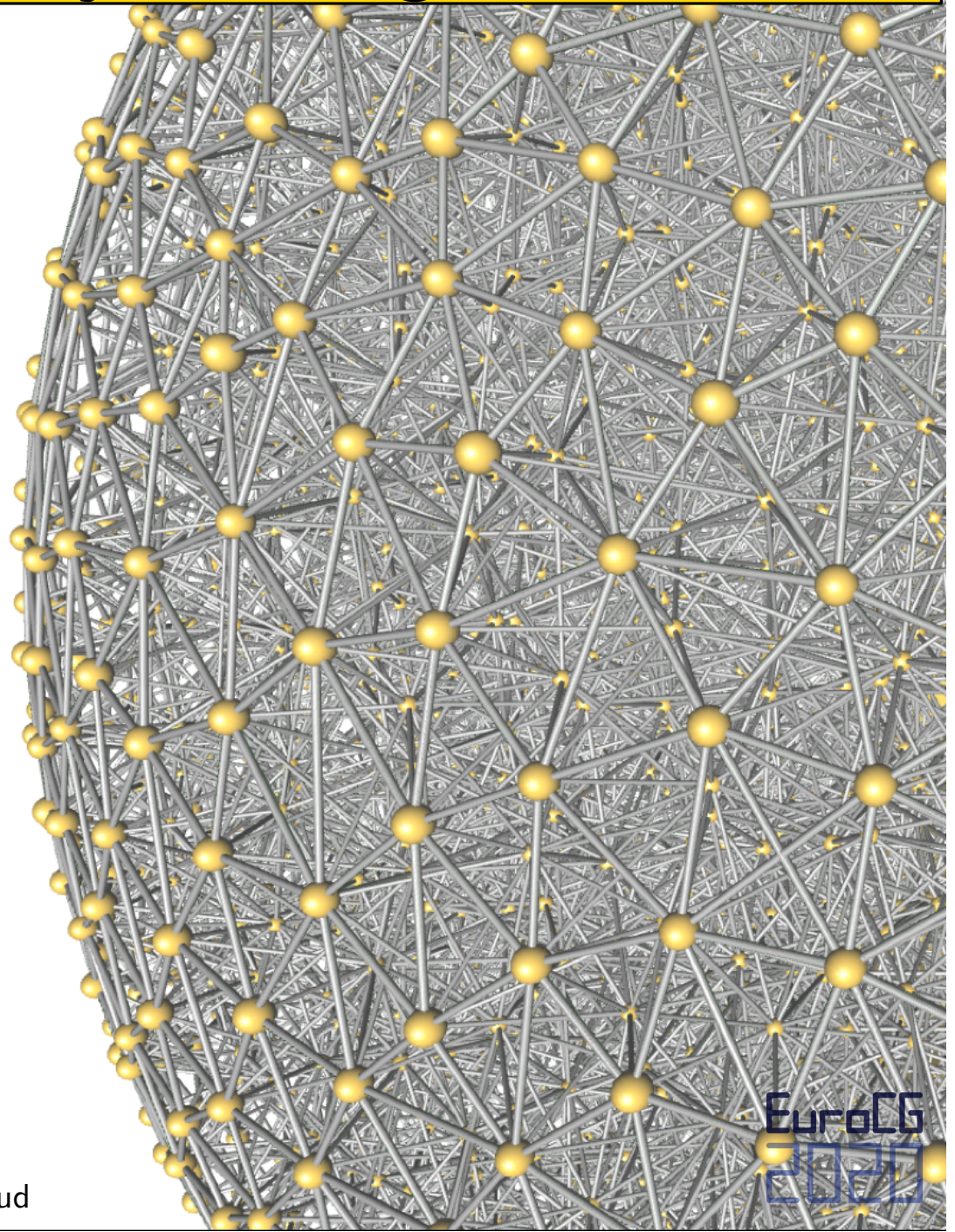
3D

$\simeq 1$ M points / second

[T. CGAL'00] [Pion, T. CGAL'01*]

multicore version

[Jamin CGAL'14]



Hyperbolic triangulations

Hyperbolic triangulations

[Boissonnat 1988]

Once upon a time...

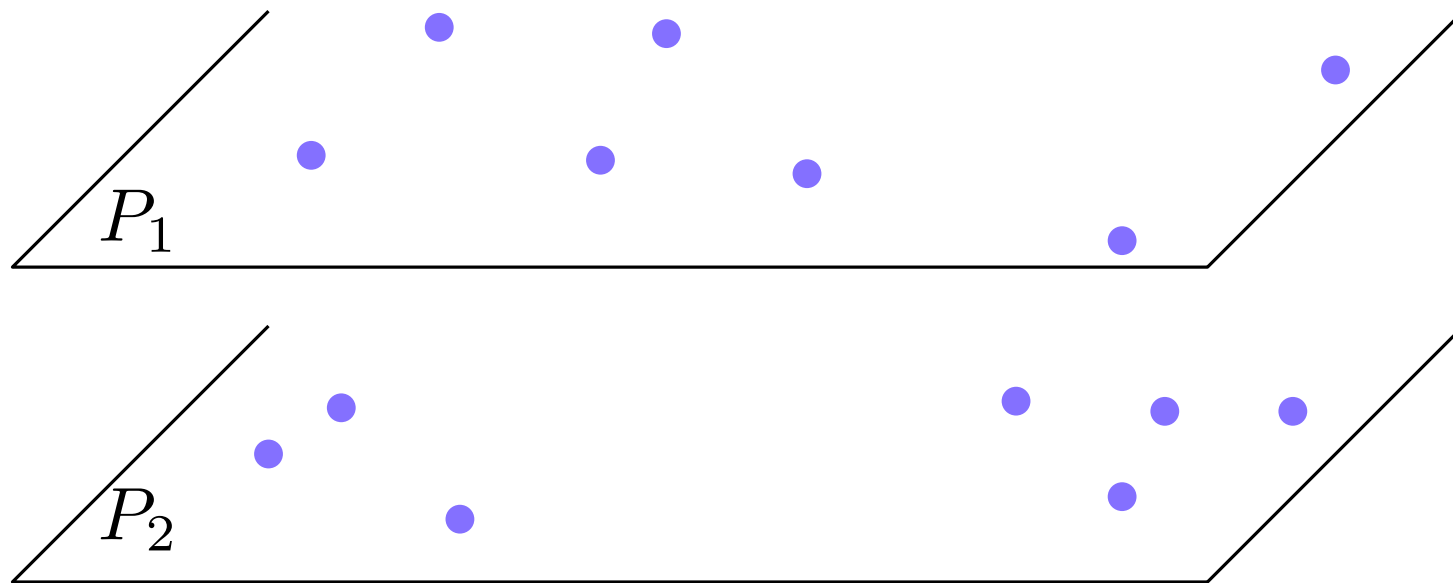
Compute the 3D Delaunay triangulation

of a set of points

lying in two parallel planes

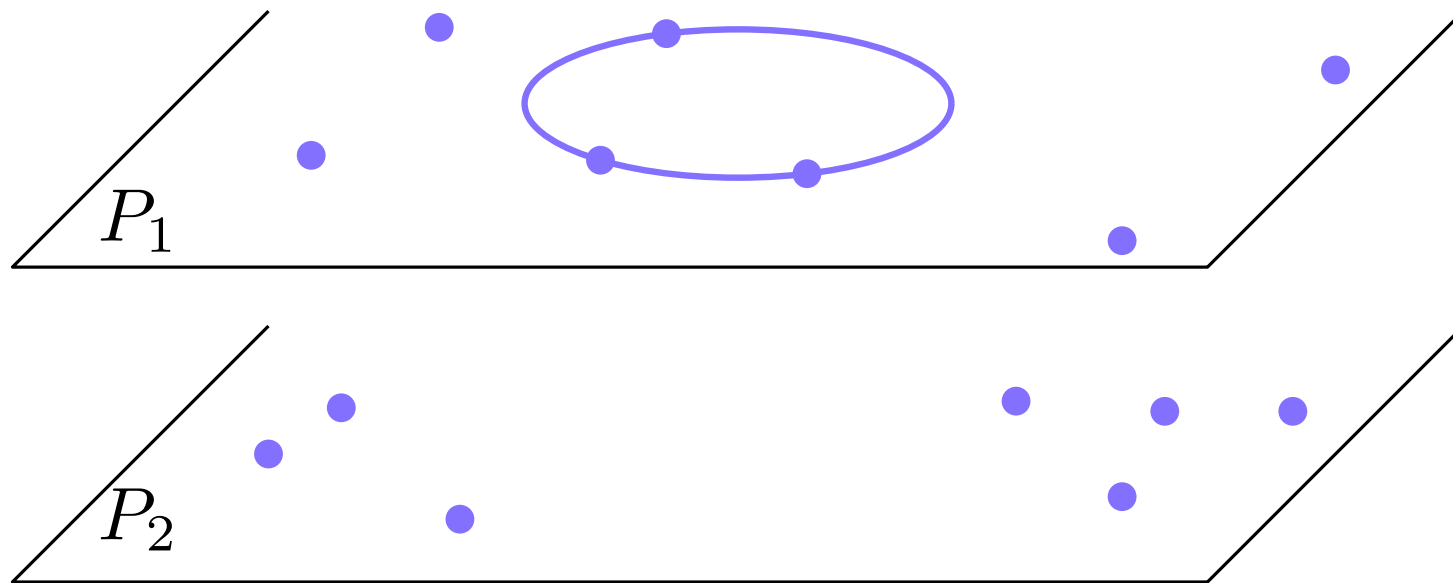
Hyperbolic triangulations

[Boissonnat 1988]



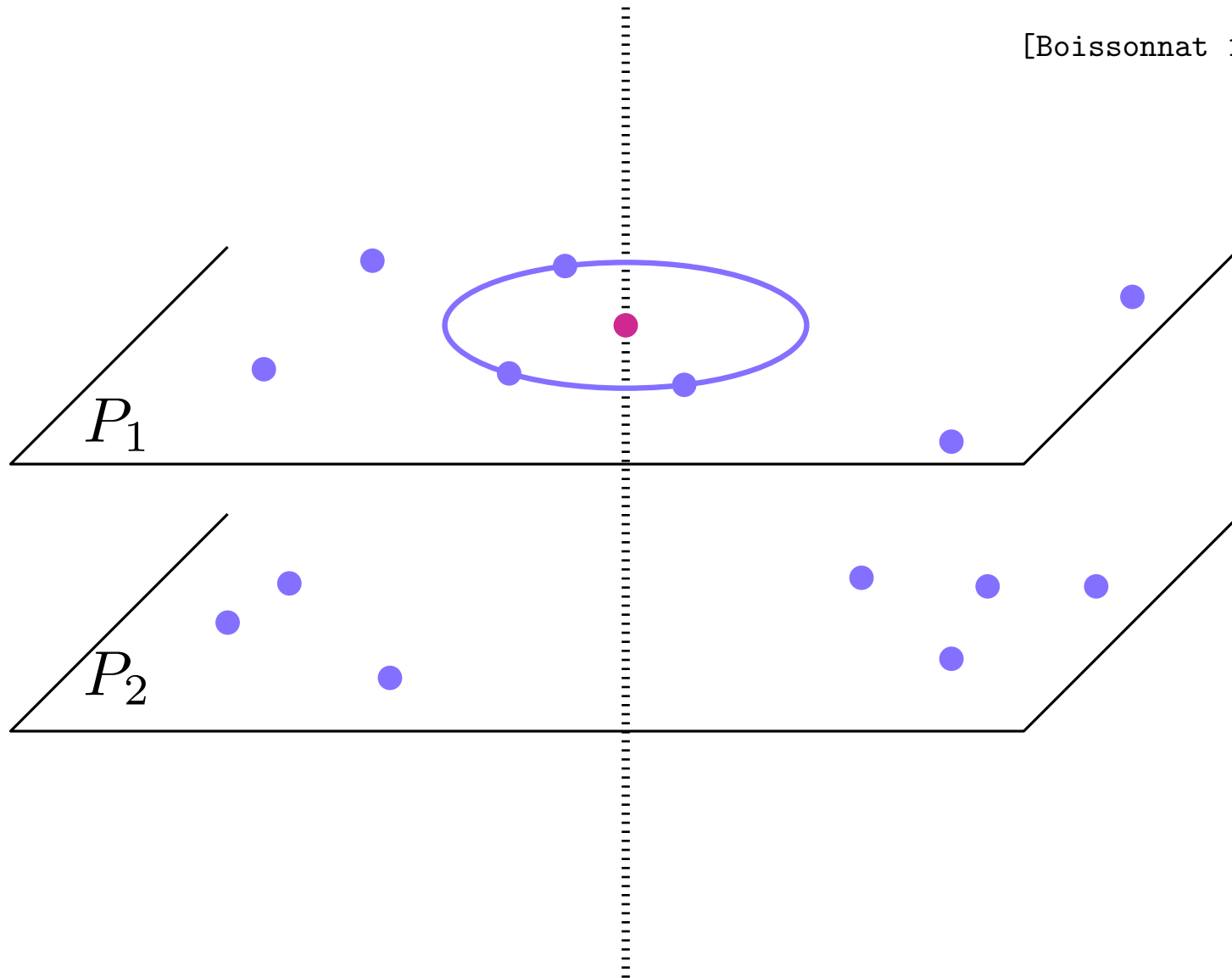
Hyperbolic triangulations

[Boissonnat 1988]

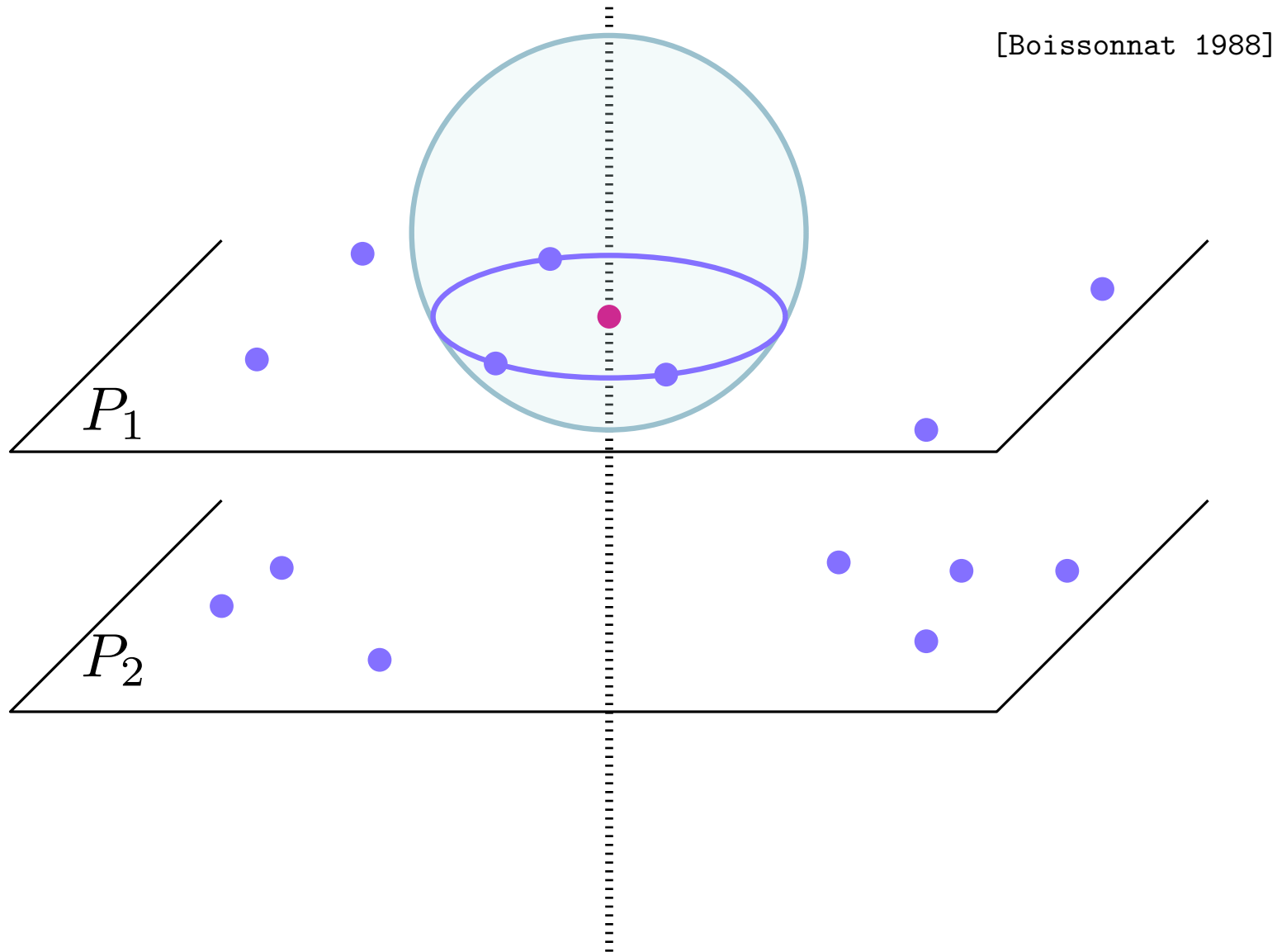


Hyperbolic triangulations

[Boissonnat 1988]

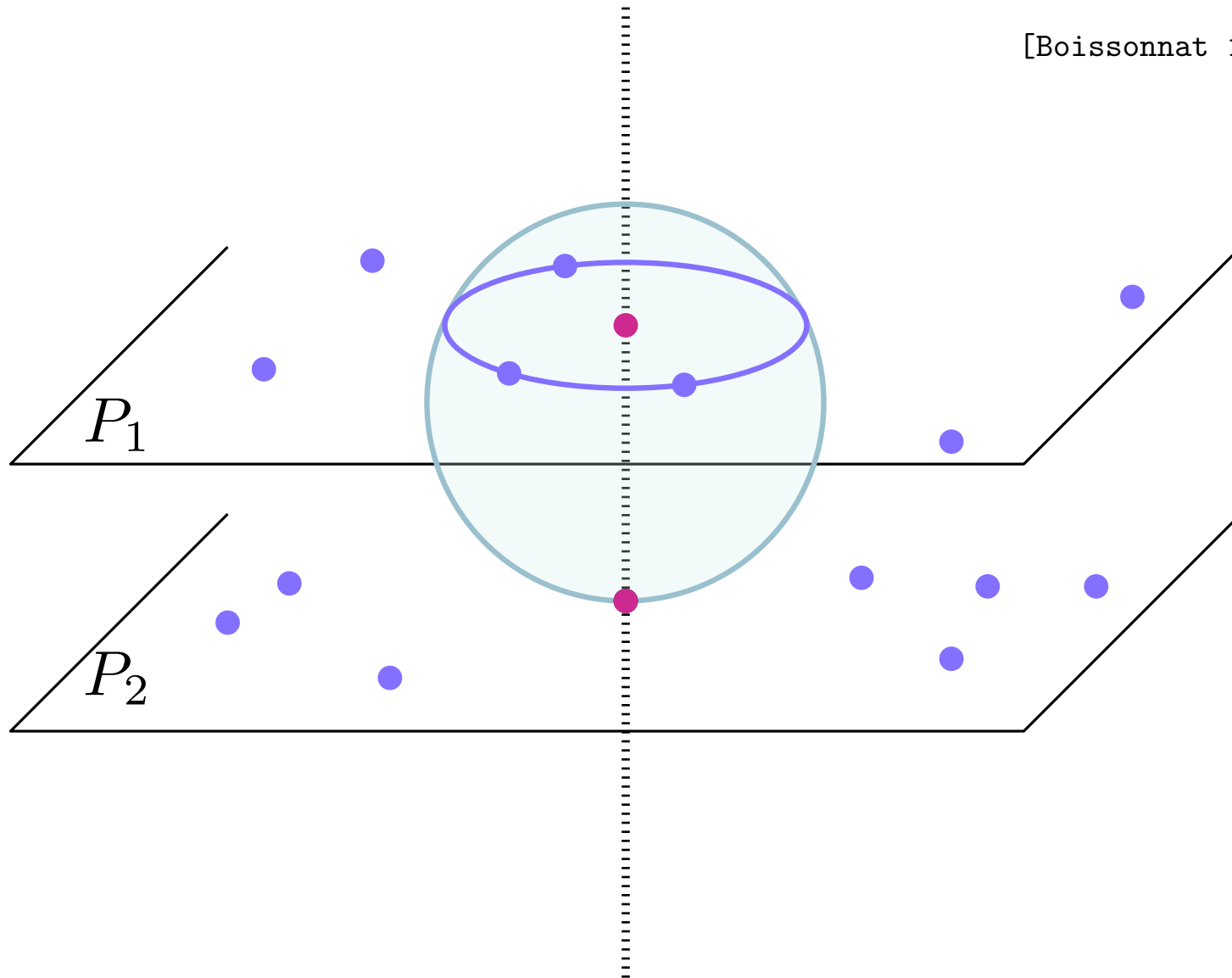


Hyperbolic triangulations



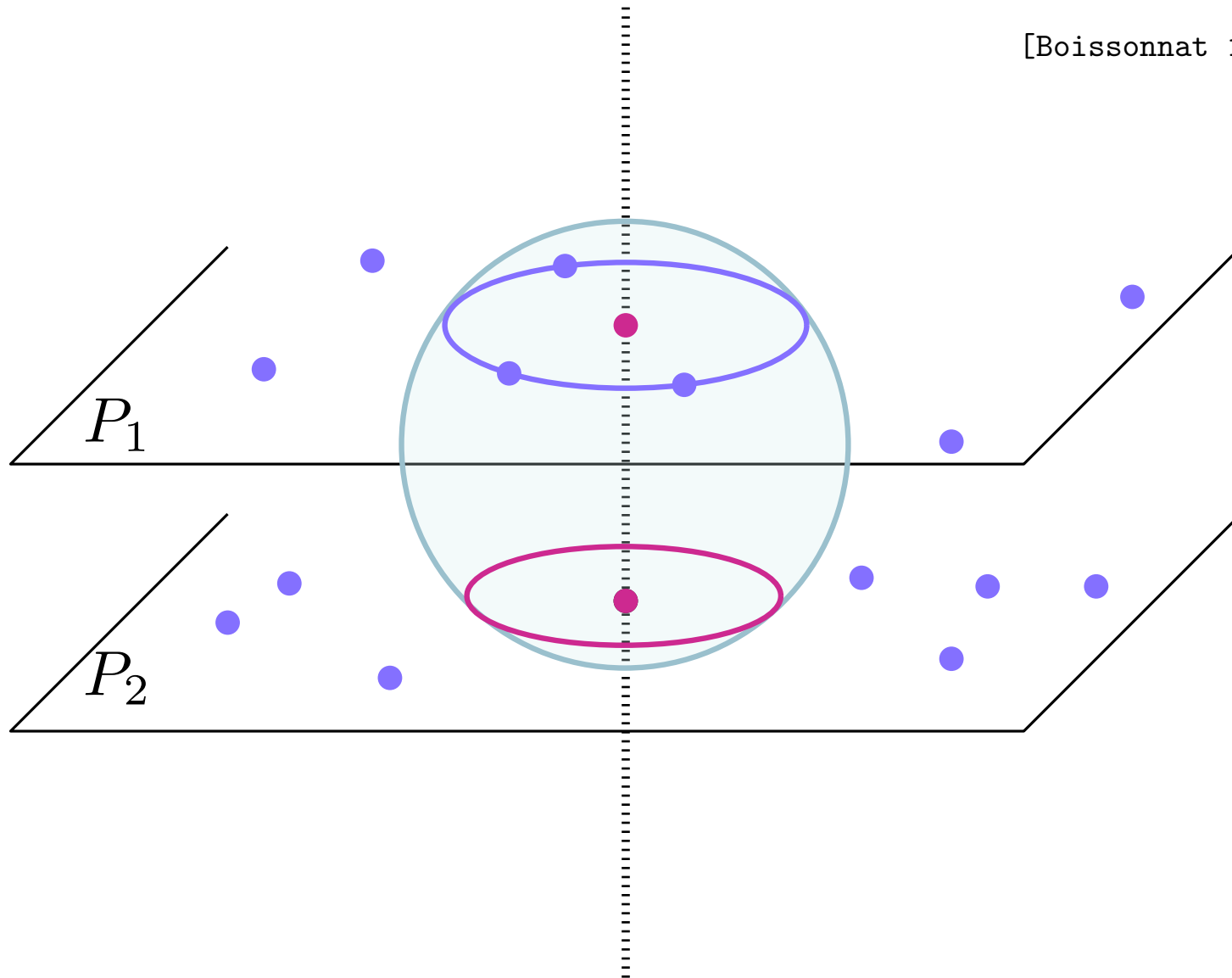
Hyperbolic triangulations

[Boissonnat 1988]



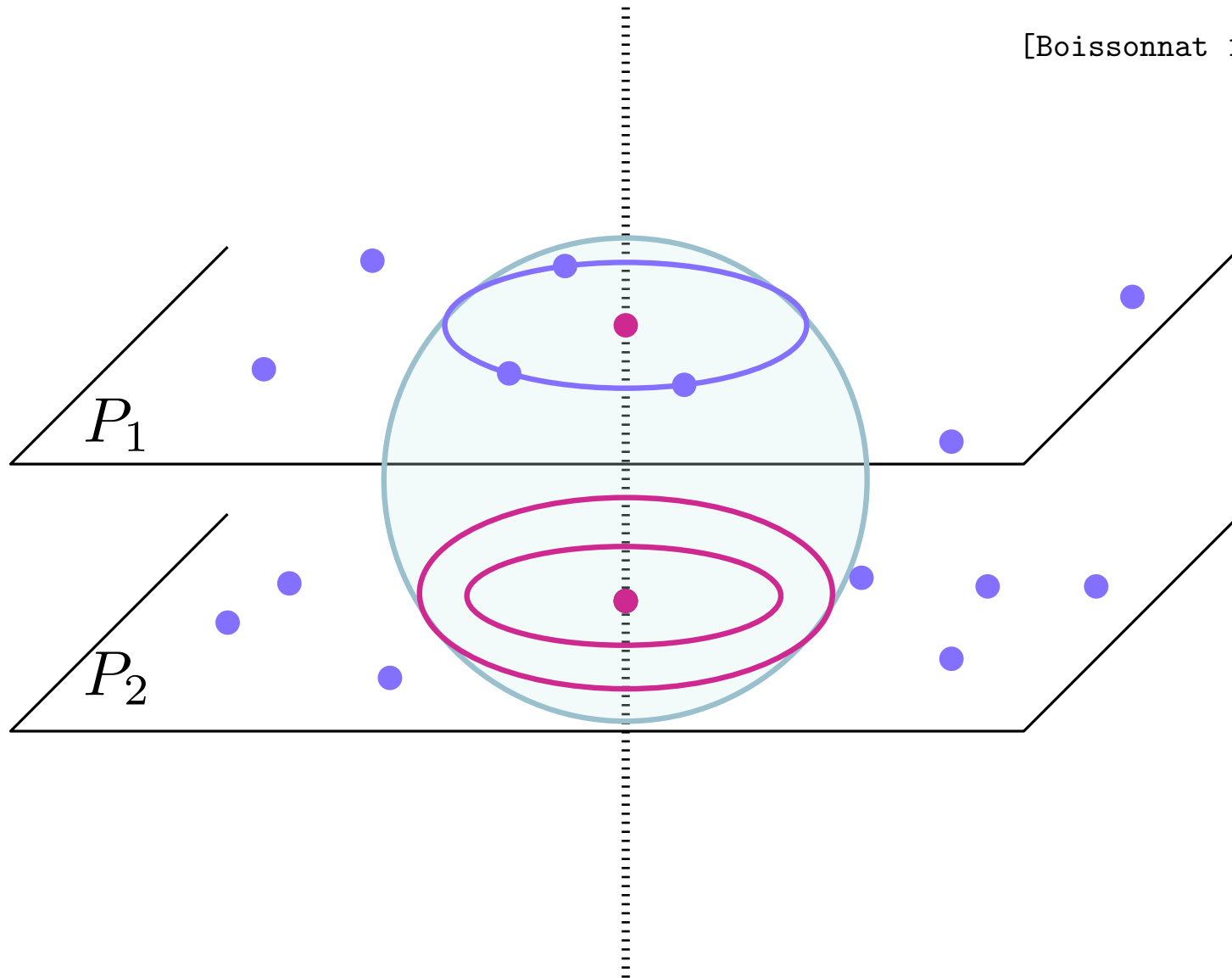
Hyperbolic triangulations

[Boissonnat 1988]



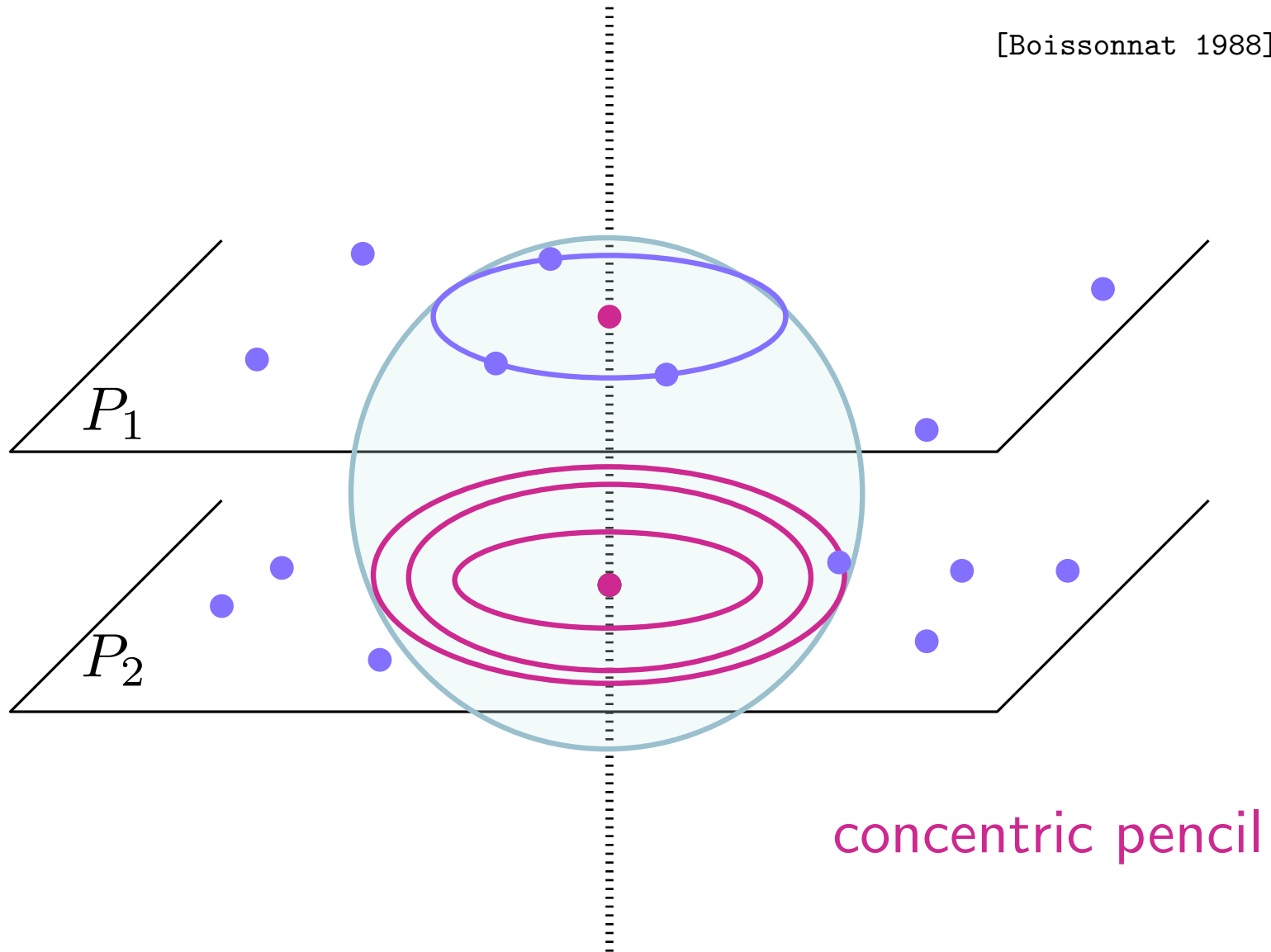
Hyperbolic triangulations

[Boissonnat 1988]



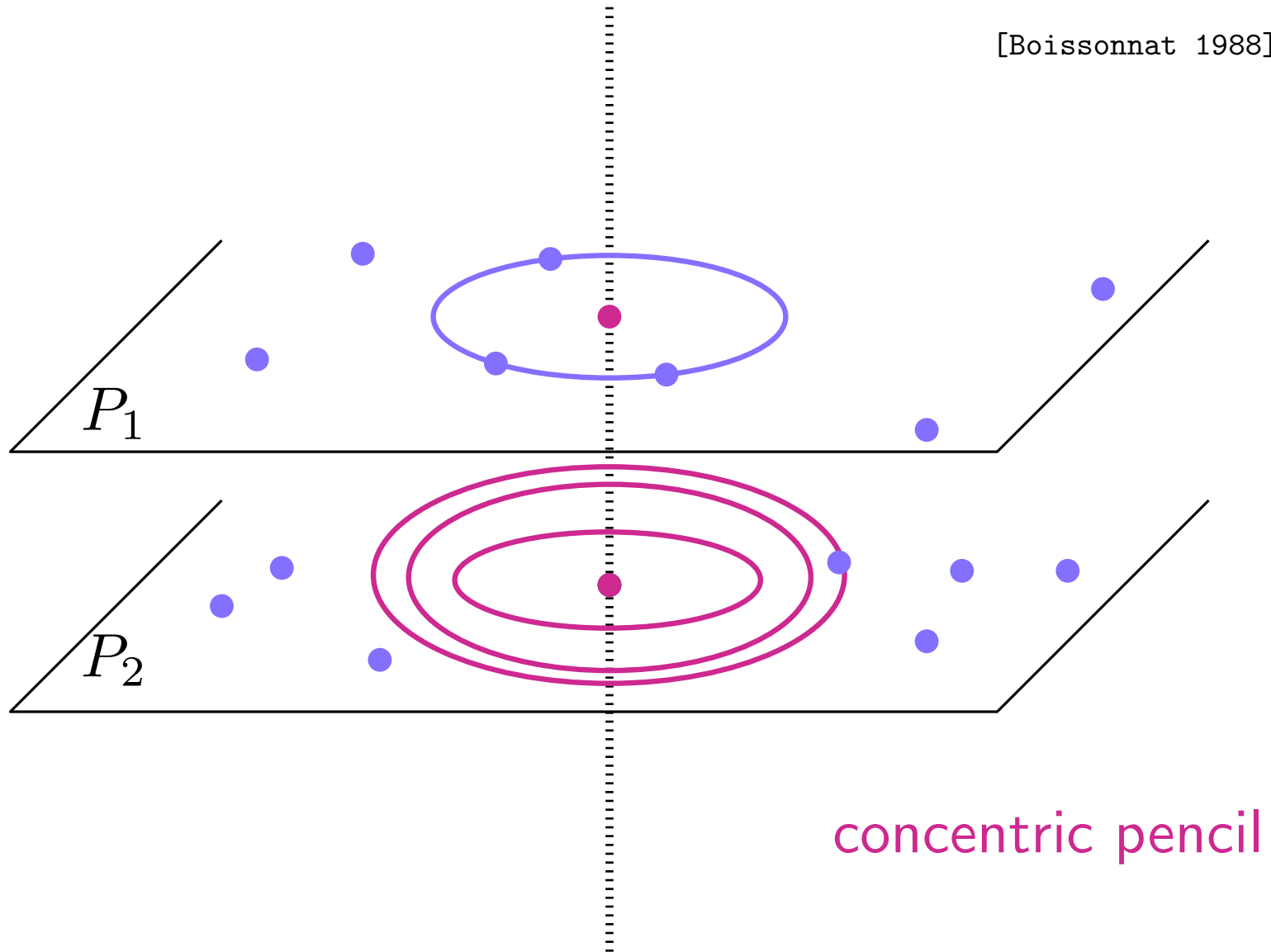
Hyperbolic triangulations

[Boissonnat 1988]



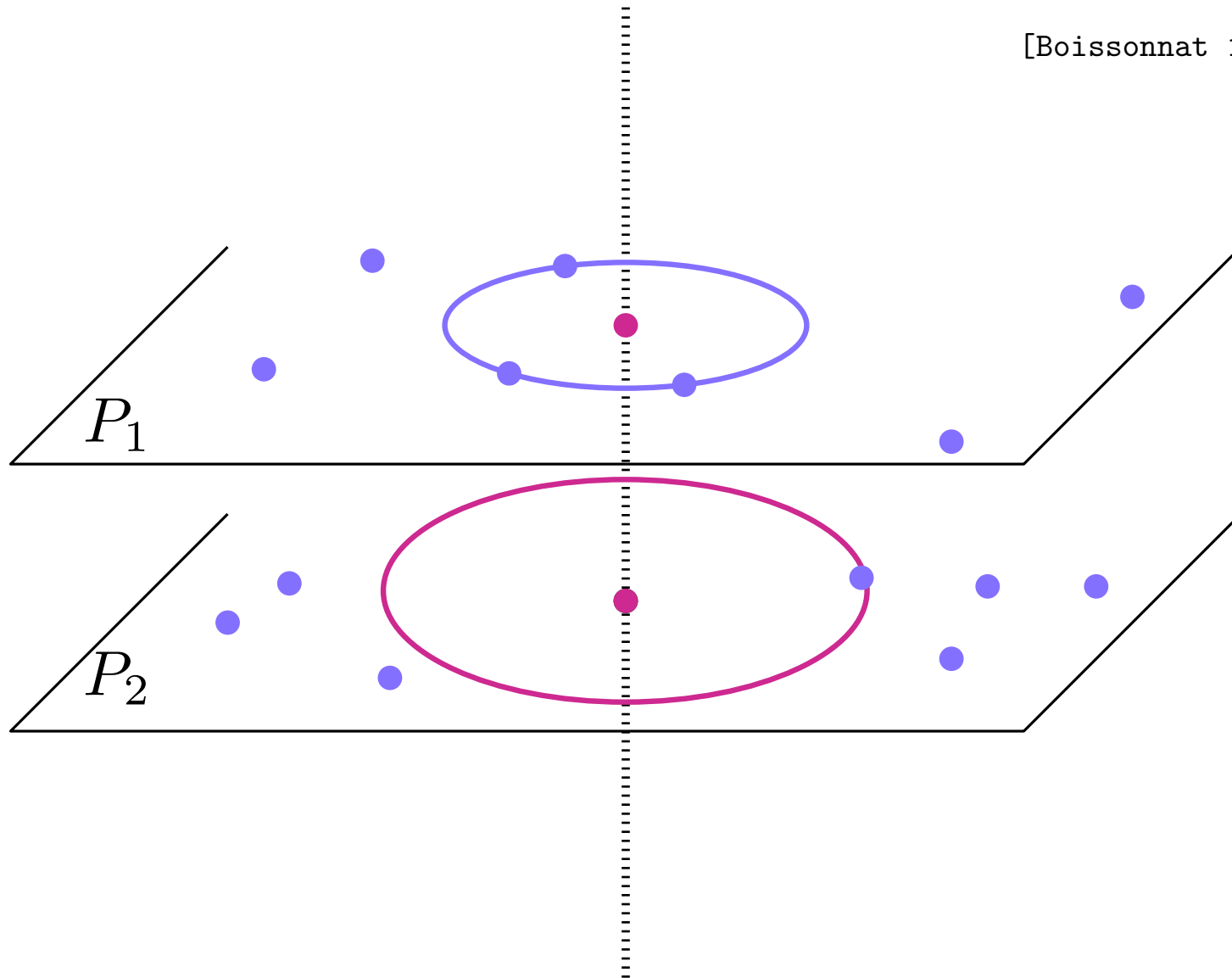
Hyperbolic triangulations

[Boissonnat 1988]



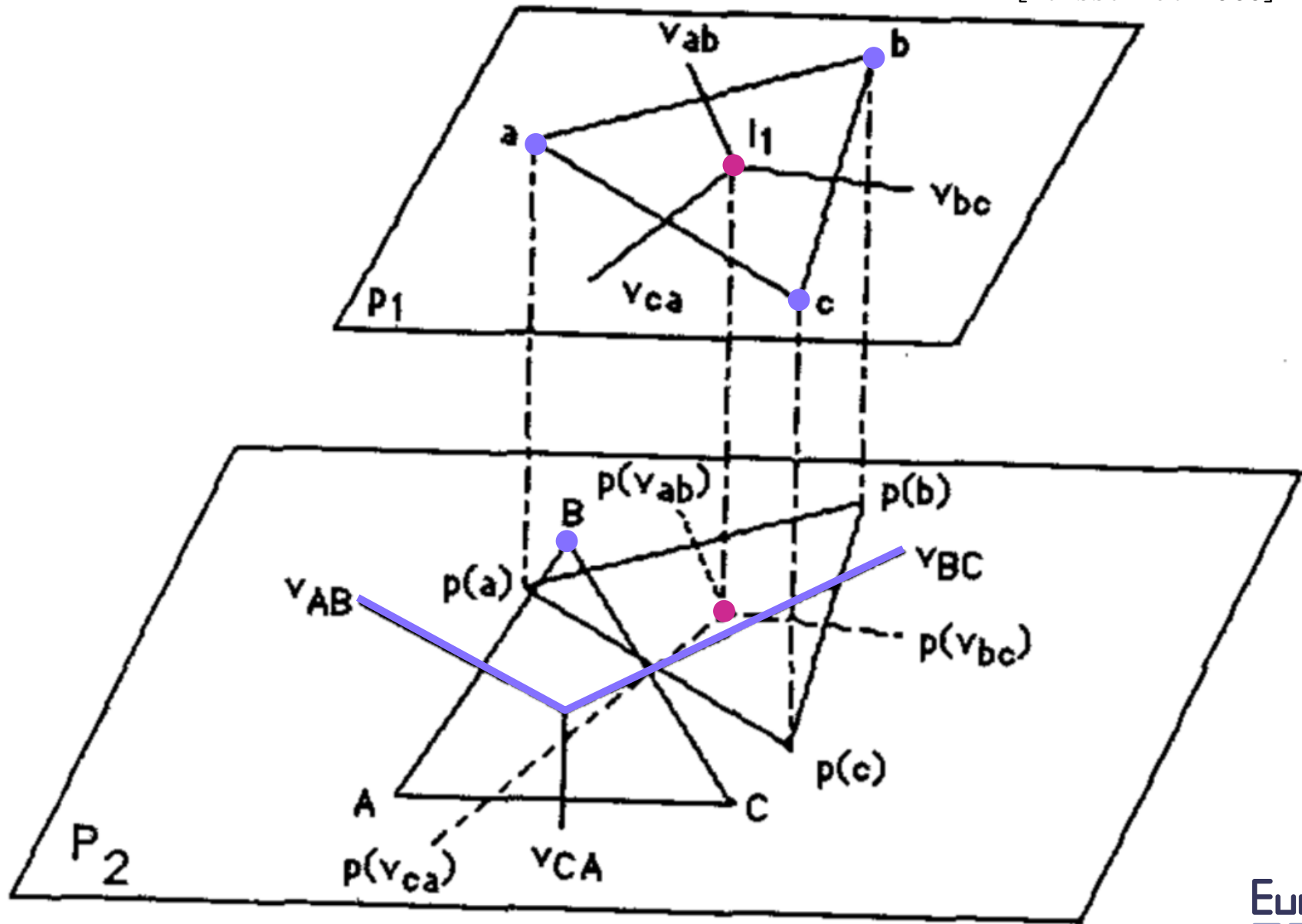
Hyperbolic triangulations

[Boissonnat 1988]



Hyperbolic triangulations

[Boissonnat 1988]



Hyperbolic triangulations

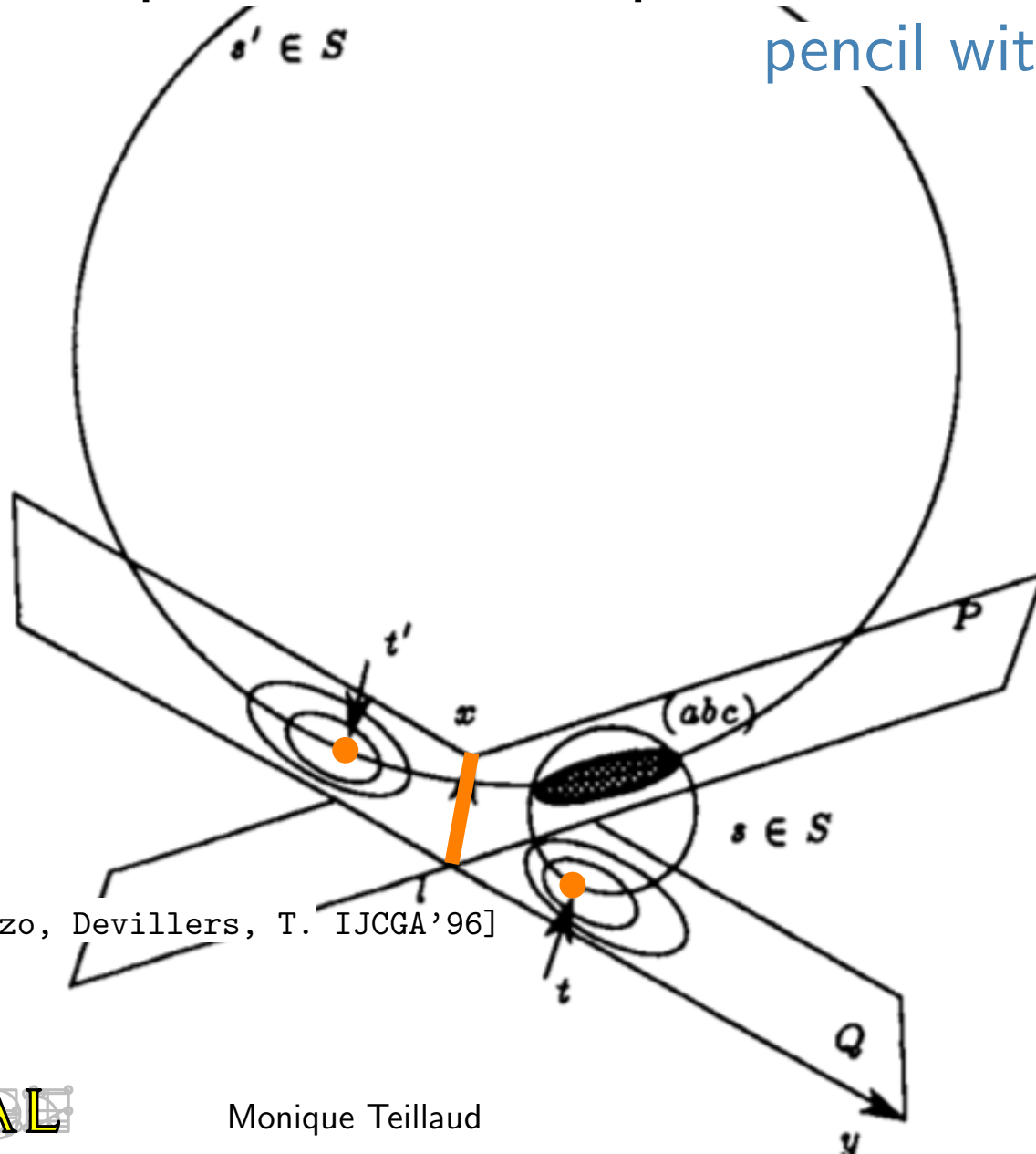
What if the planes are **not** parallel?

[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

Hyperbolic triangulations

What if the planes are **not** parallel?

pencil with limit points

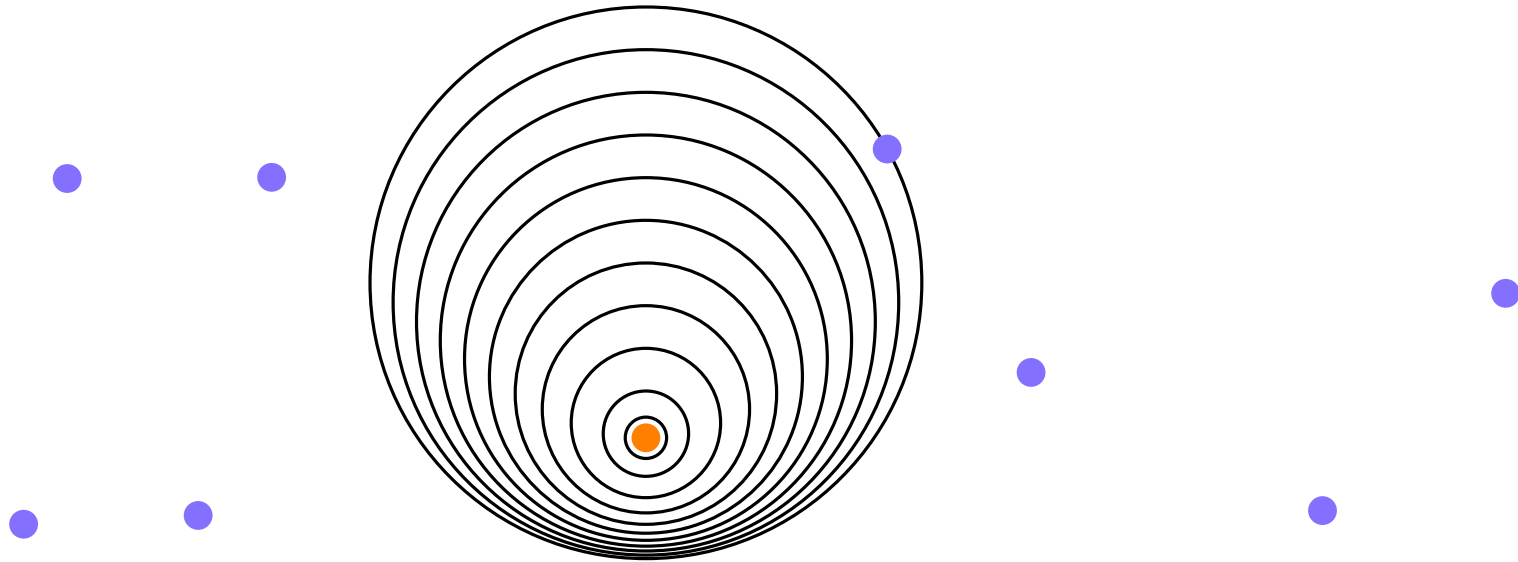


[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

Hyperbolic triangulations

What if the planes are **not** parallel?

- pencil with limit points

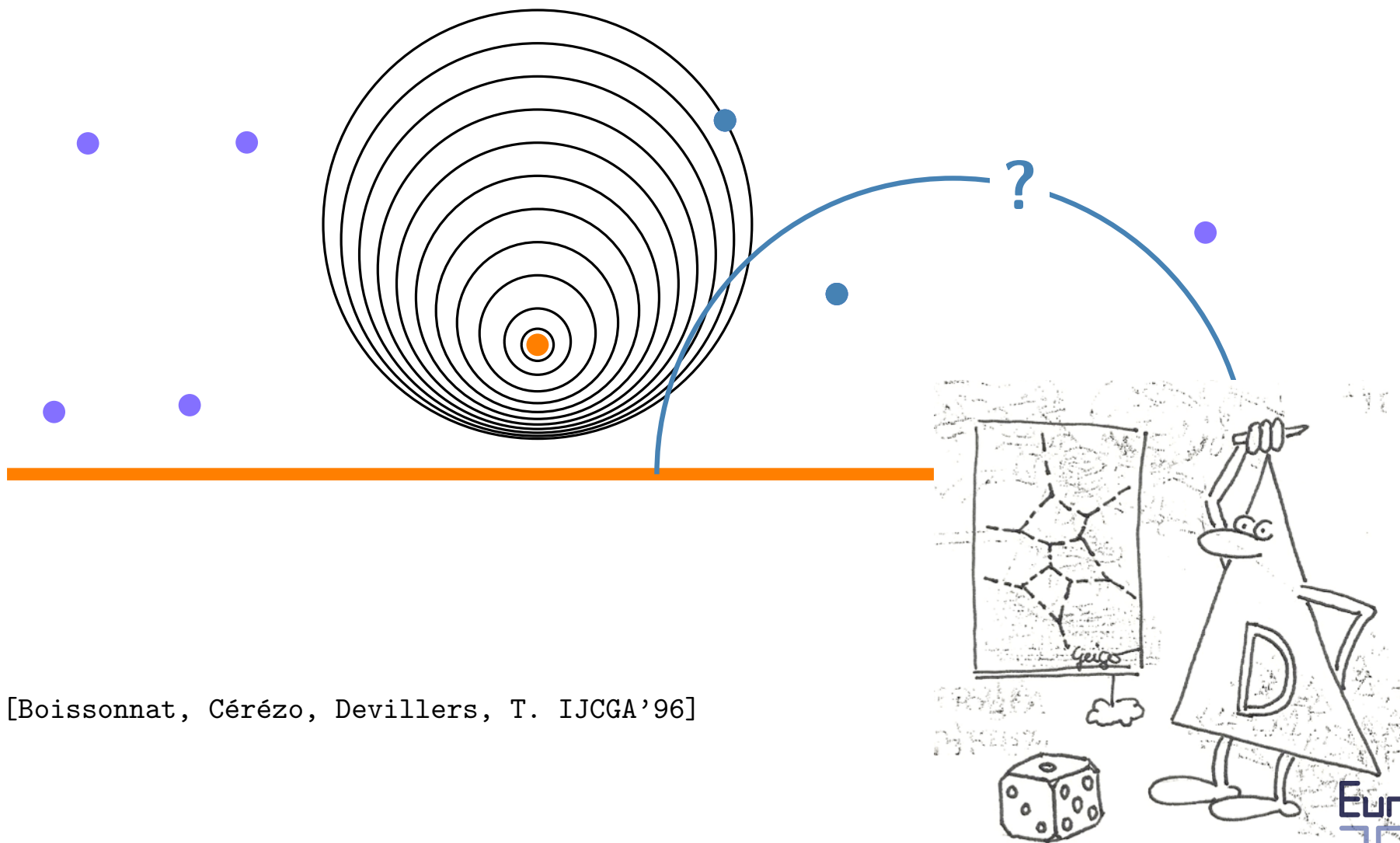


[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

Hyperbolic triangulations

What if the planes are **not** parallel?

- pencil with limit points

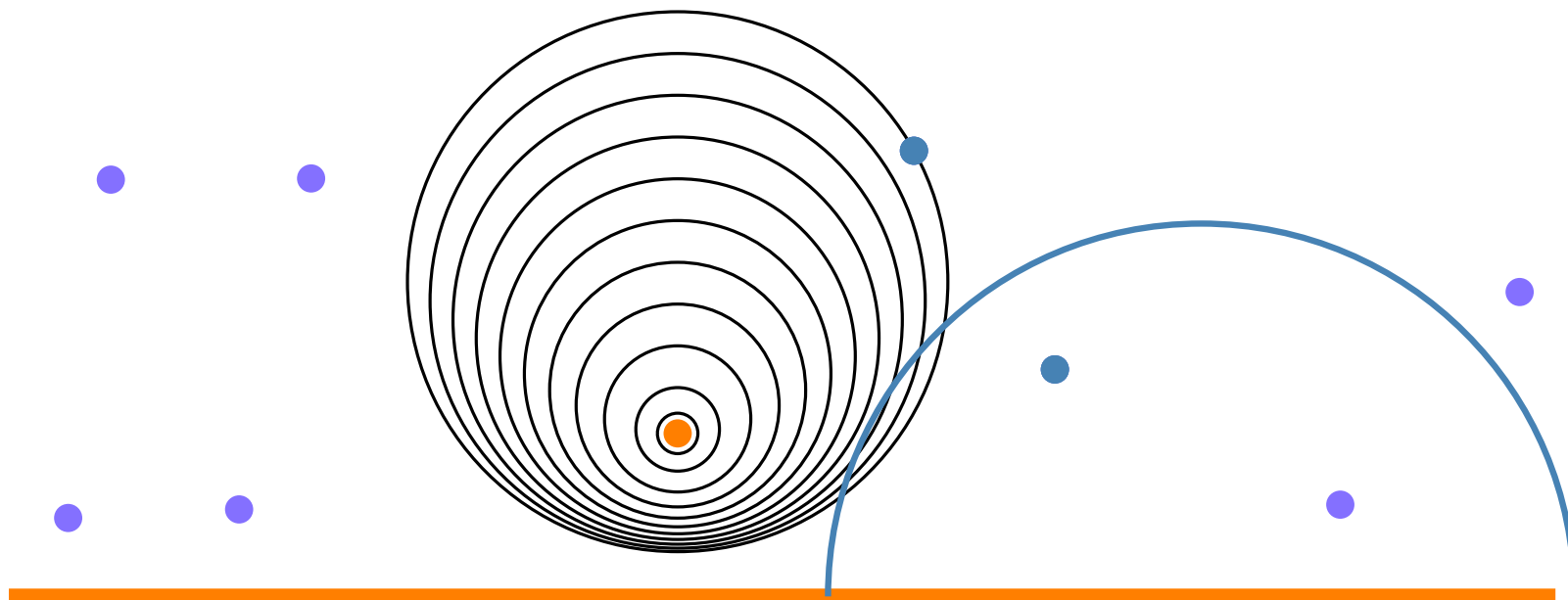


[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

Hyperbolic triangulations

What if the planes are **not** parallel?

- pencil with limit points



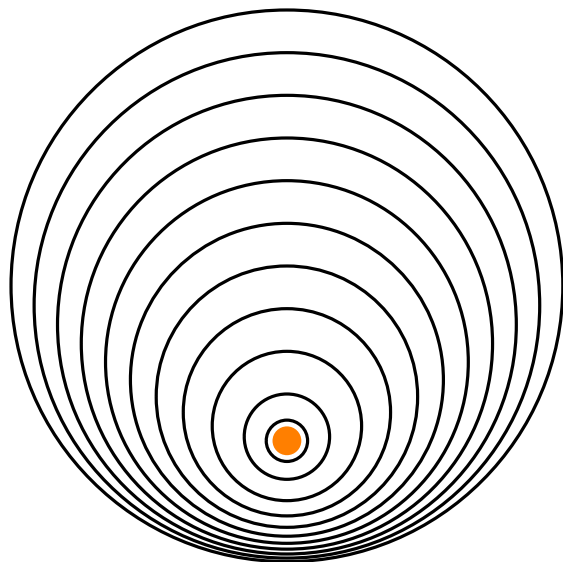
hyperbolic line
in the Poincaré half-plane

[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

Hyperbolic triangulations

What if the planes are **not** parallel?

pencil with limit points

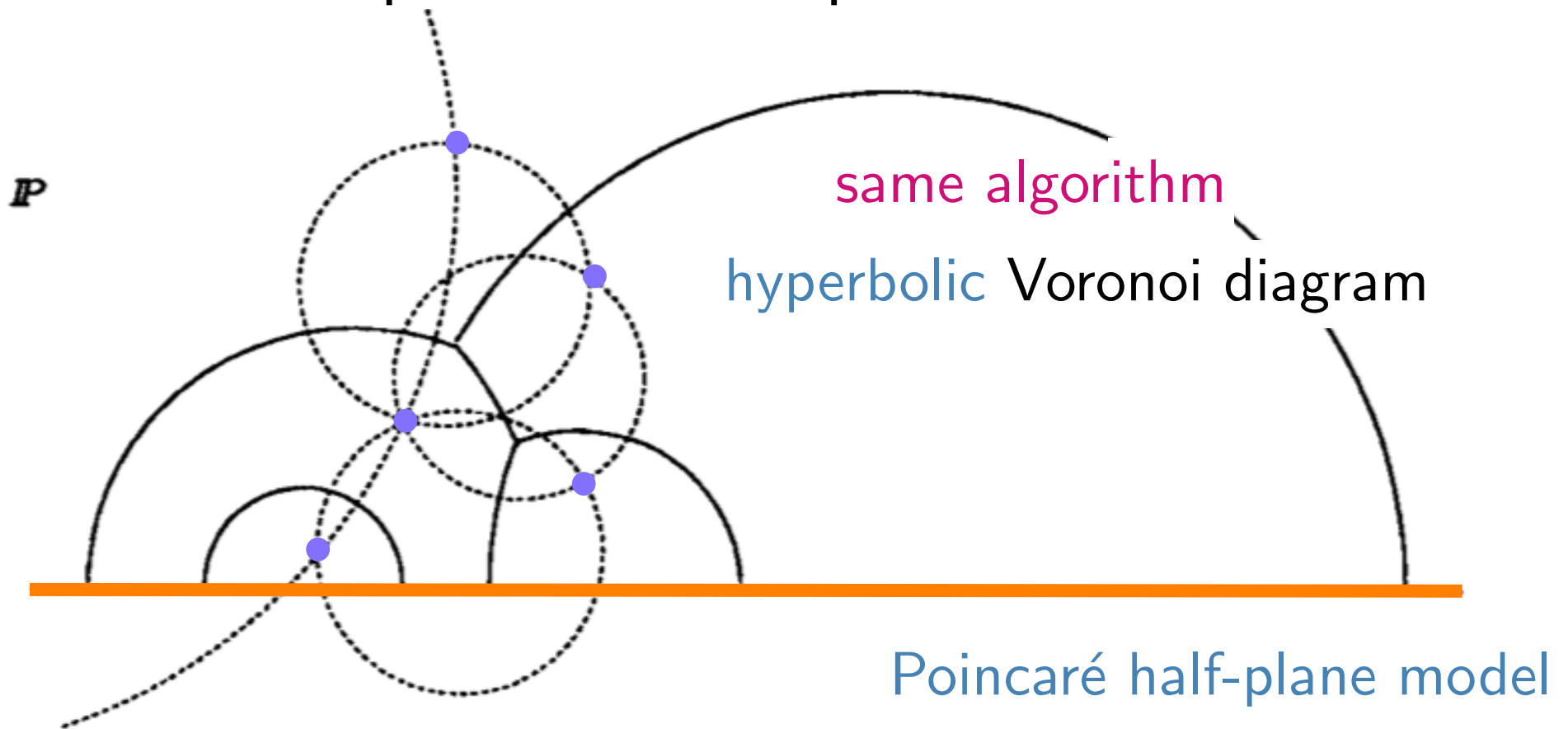


concentric pencil
in the Poincaré half-plane

[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

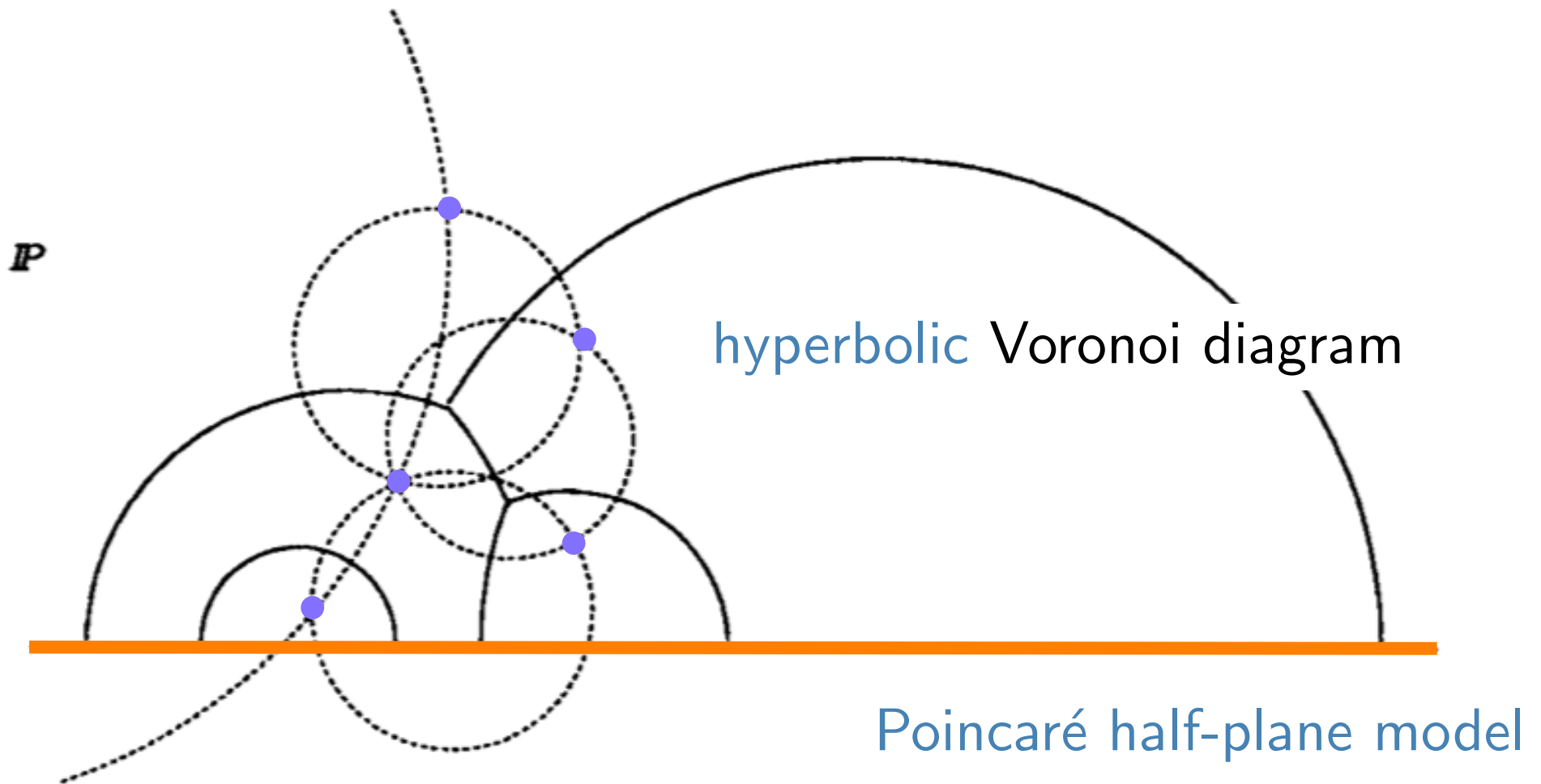
Hyperbolic triangulations

What if the planes are **not** parallel?



[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

Hyperbolic triangulations

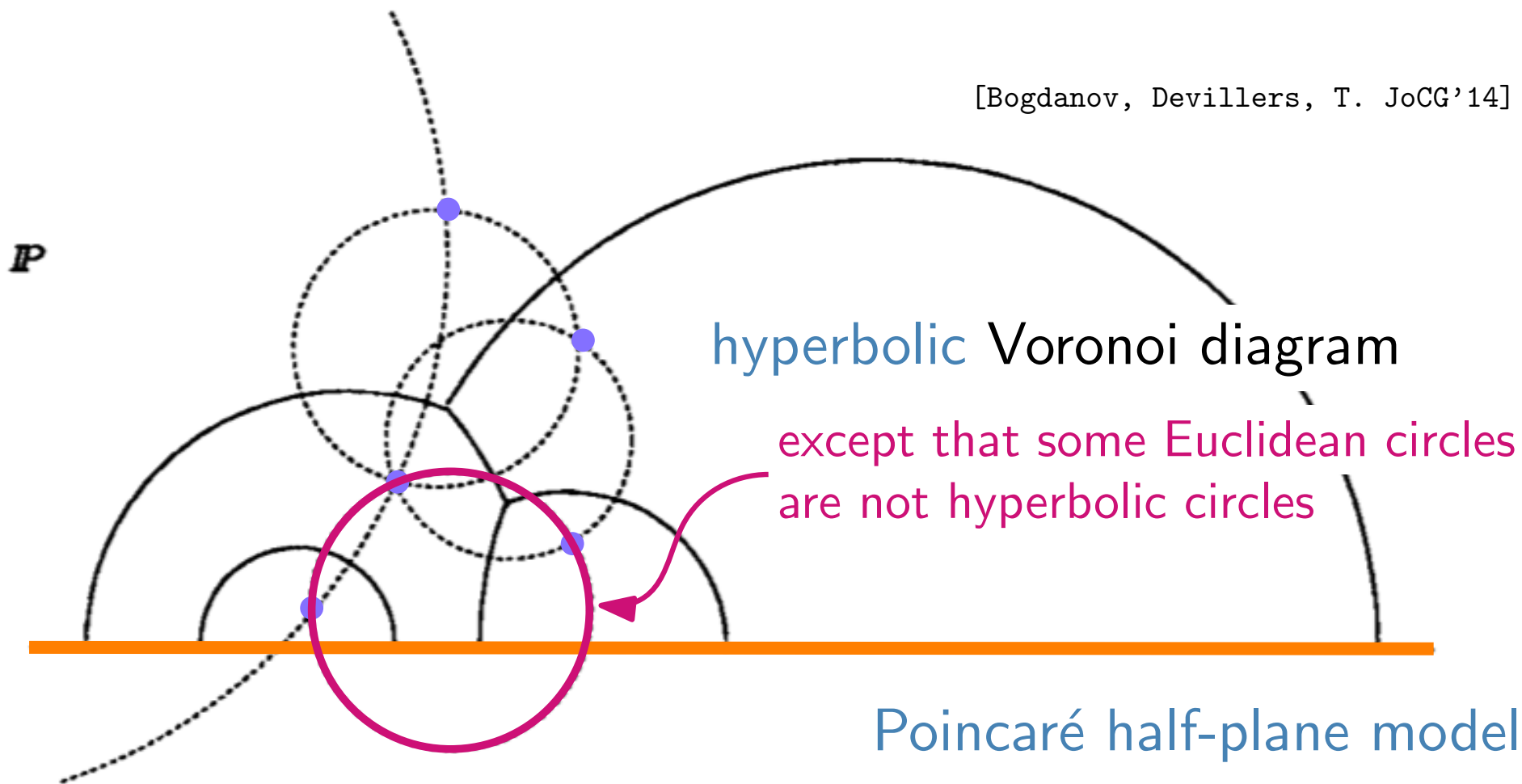


hyperbolic circles are Euclidean circles

↳ same combinatorics as Euclidean Voronoi diagram

Hyperbolic triangulations

[Bogdanov, Devillers, T. JoCG'14]



hyperbolic Voronoi diagram

except that some Euclidean circles
are not hyperbolic circles

Poincaré half-plane model

hyperbolic circles are Euclidean circles

↳ same combinatorics as Euclidean Voronoi diagram

Hyperbolic triangulations

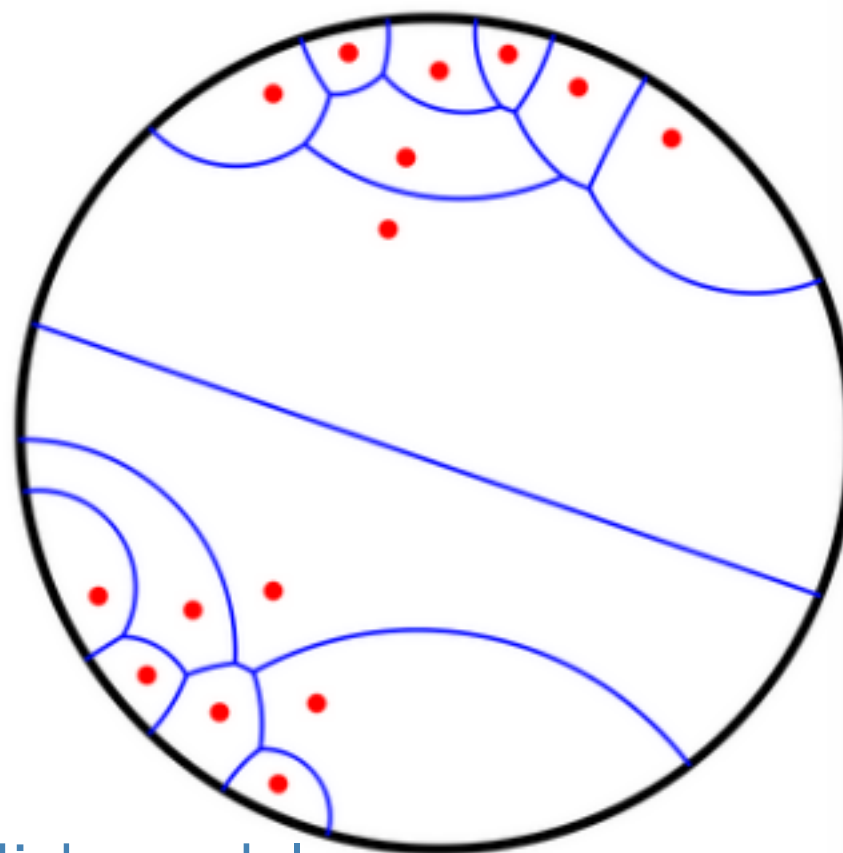
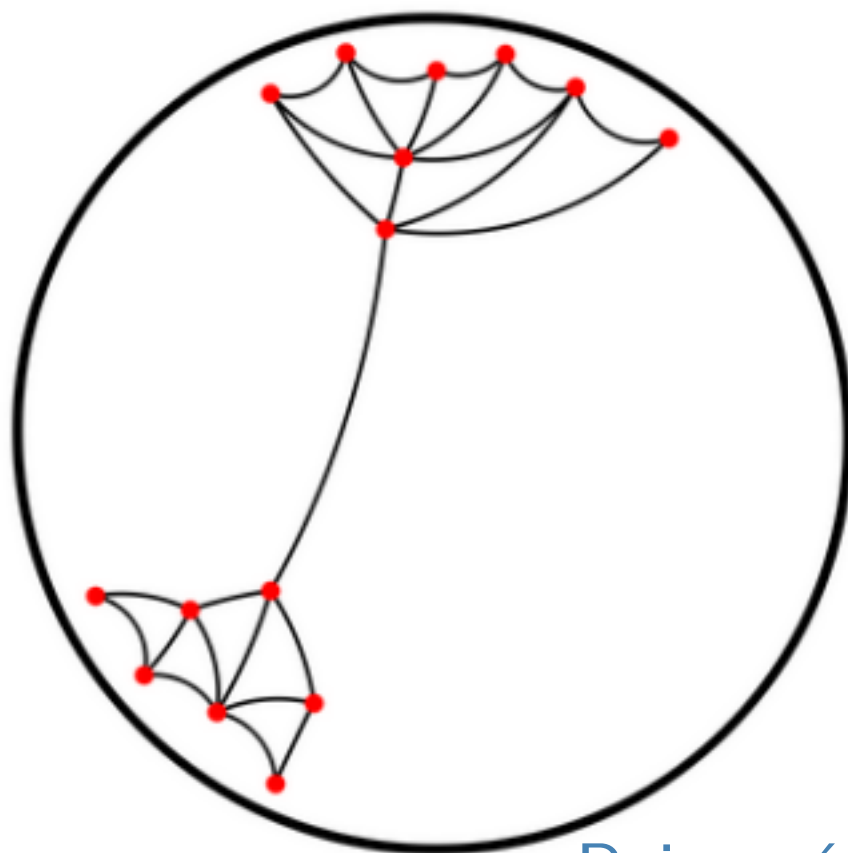
Hyperbolic_Delaunay_triangulation_2

inherits from

Delaunay_triangulation_2

[Bogdanov, Devillers, T. JoCG'14]

[Bogdanov, Iordanov, T. CGAL'19]



Poincaré disk model

Hyperbolic triangulations

```
#include <CGAL/Hyperbolic_Delaunay_triangulation_2.h>
#include <CGAL/Hyperbolic_Delaunay_triangulation_traits_2.h>
#include <vector>

typedef CGAL::Hyperbolic_Delaunay_triangulation_traits_2<> Gt;
typedef Gt::Point_2 Point_2;
typedef CGAL::Hyperbolic_Delaunay_triangulation_2<Gt> Dt;

int main(int argc, char** argv)
{
    std::vector<Point_2> pts;    \\ pts filled in some way

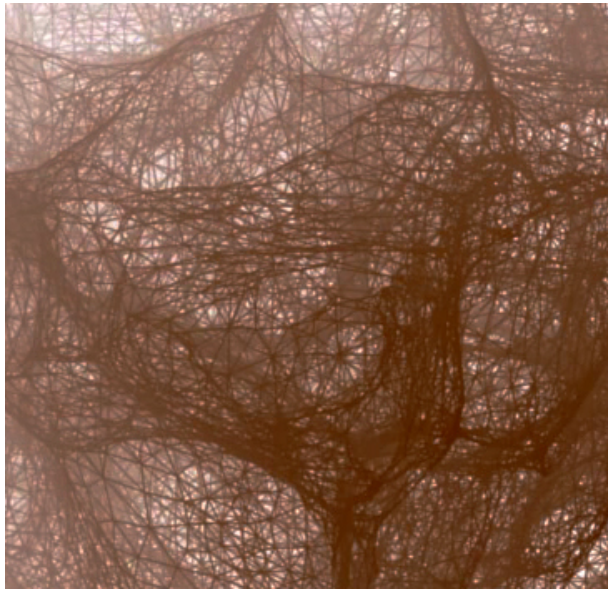
    Dt dt_during;    // hyperbolic filtering at each step:
    std::vector<Point_2>::iterator ip;
    for(ip = pts.begin(); ip != pts.end(); ++ip) dt_during.insert(*ip);

    Dt dt_end;    // hyperbolic filtering only once at the end:
    dt_end.insert(pts.begin(),pts.end());

    return 0;
}
```

Periodic Delaunay triangulations

Periodic Delaunay triangulations



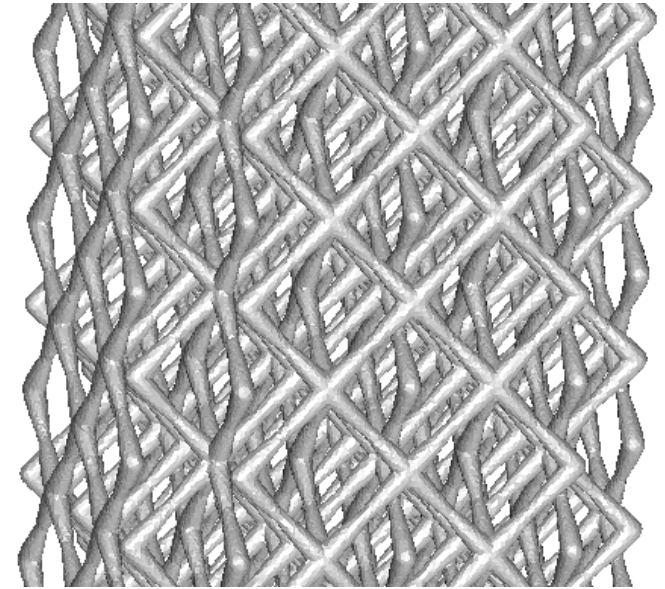
cosmic web

[v.d. Weijgaert, Groningen]

motivation:

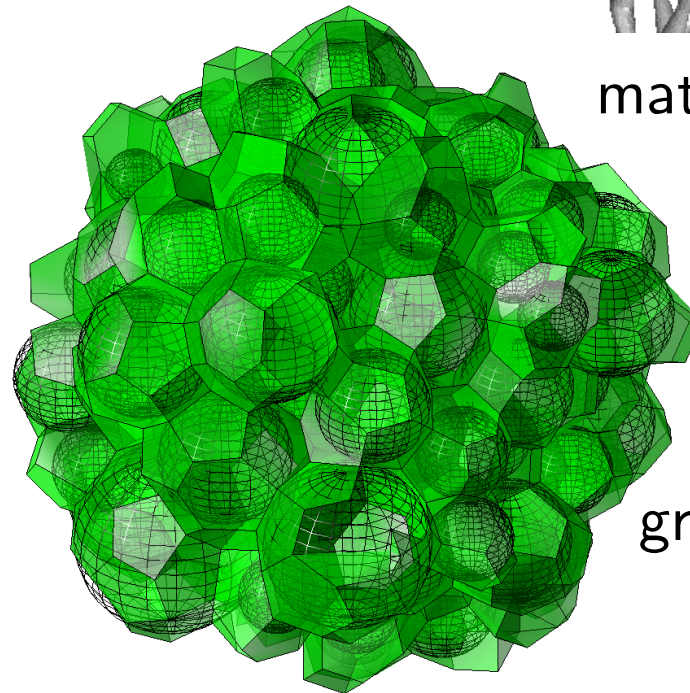


users



material for bone scaffold

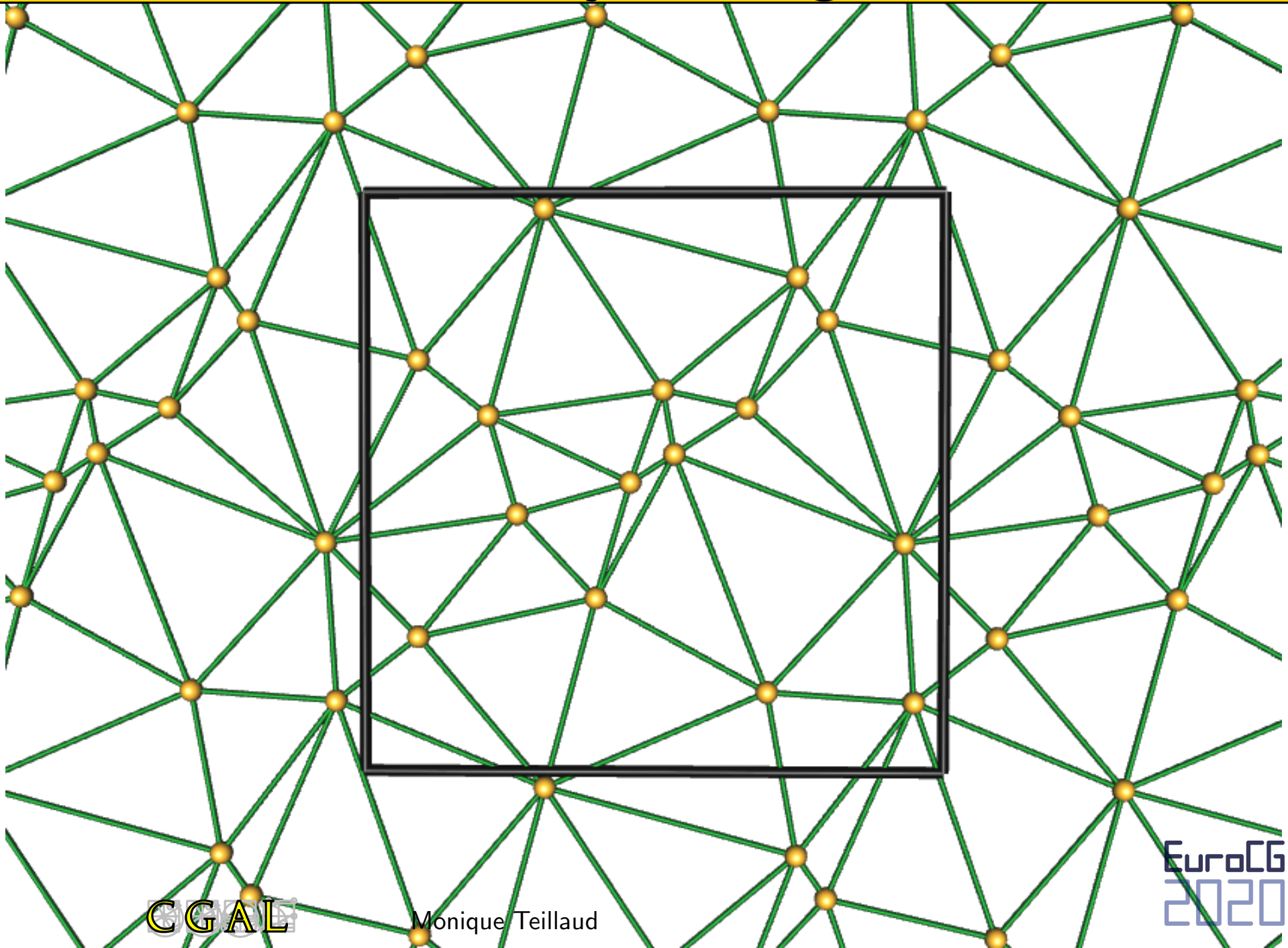
[Moesen⁺, Leuven]



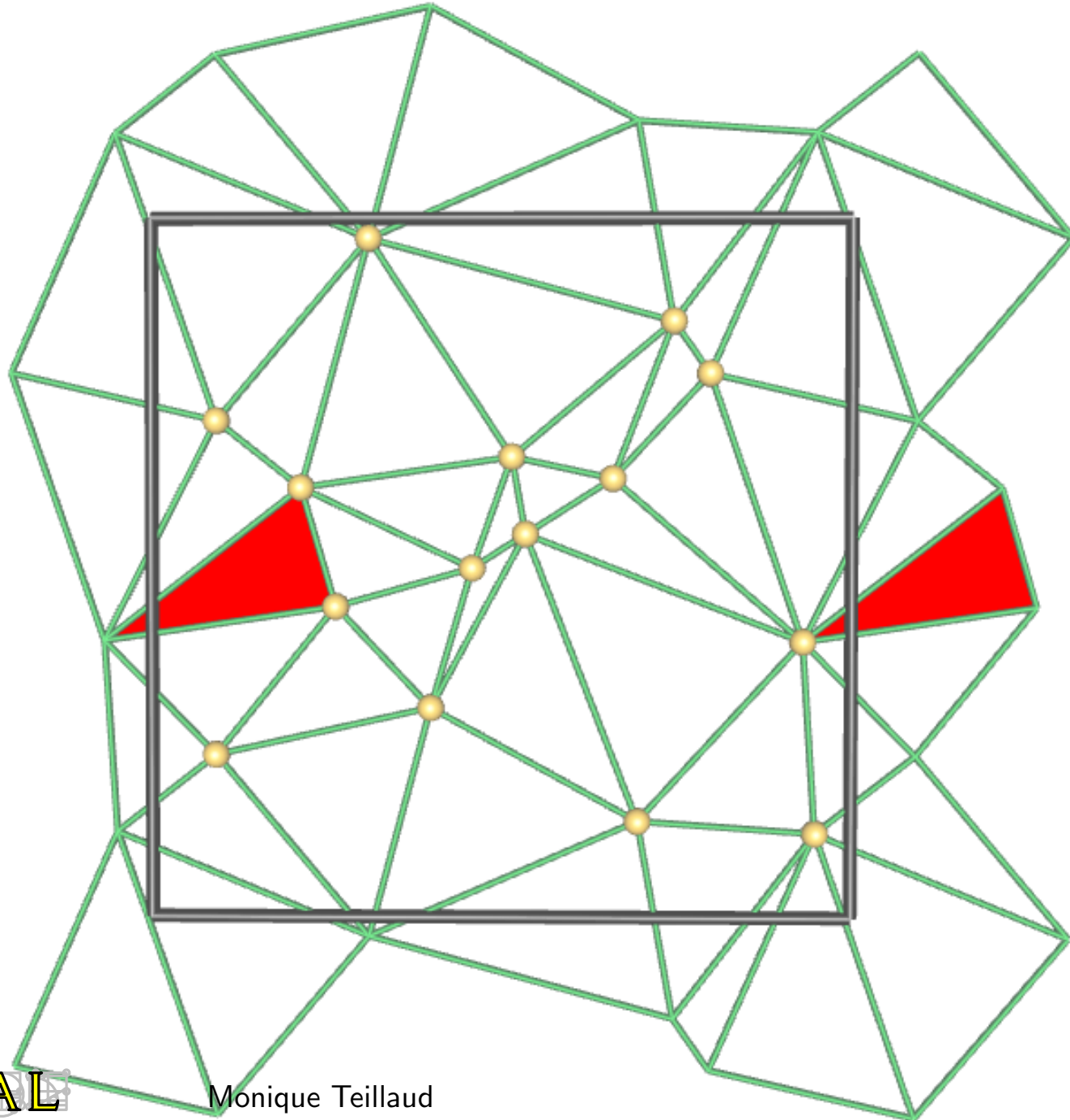
granular material

[Ludig⁺, Twente]

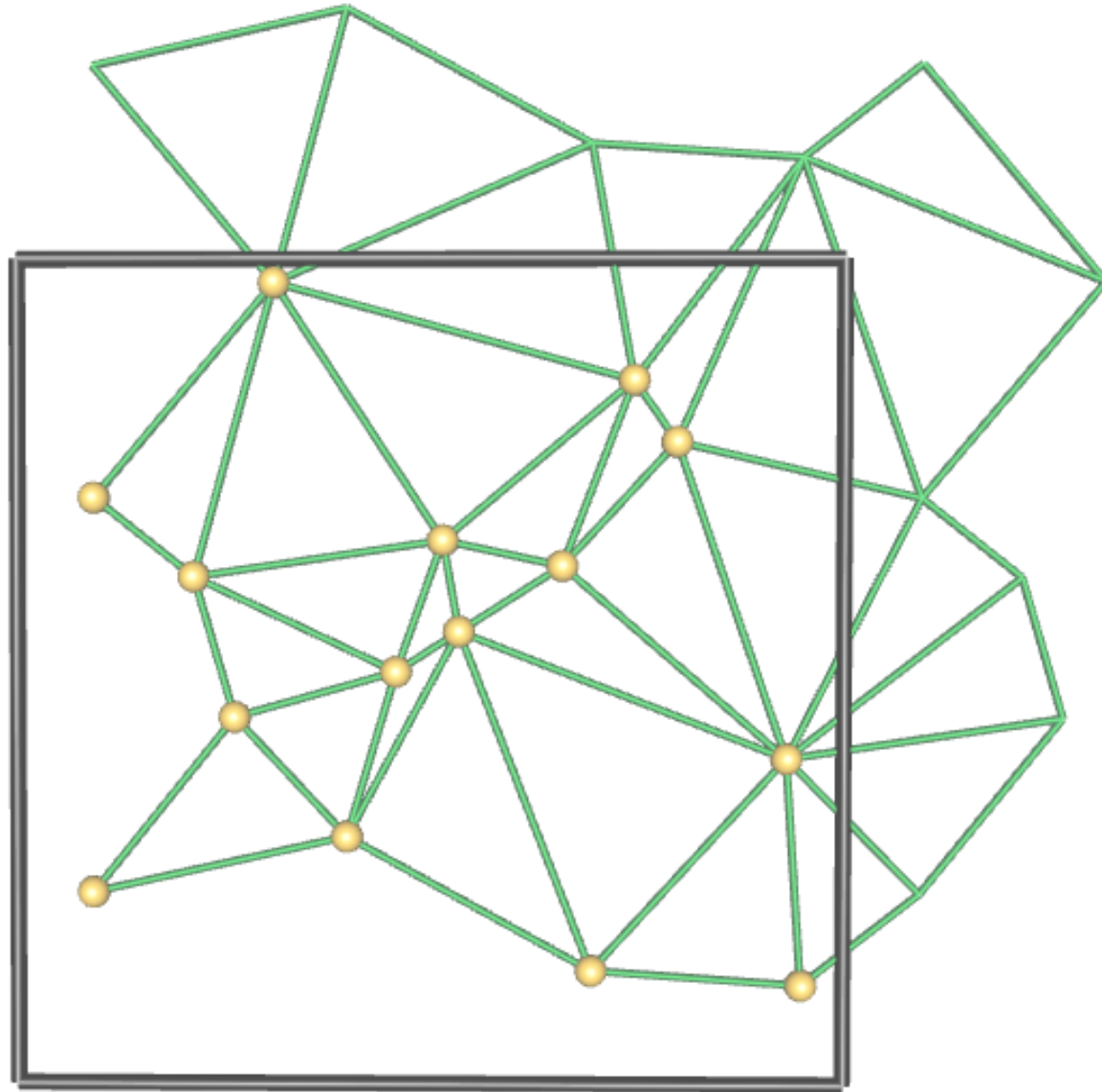
Periodic Delaunay triangulations



Periodic Delaunay triangulations



Periodic Delaunay triangulations



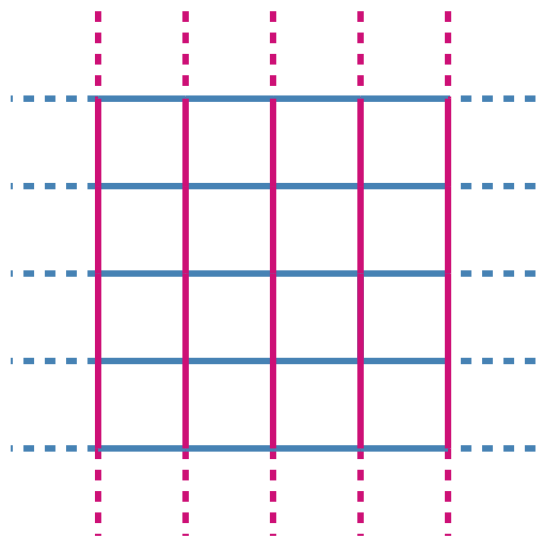
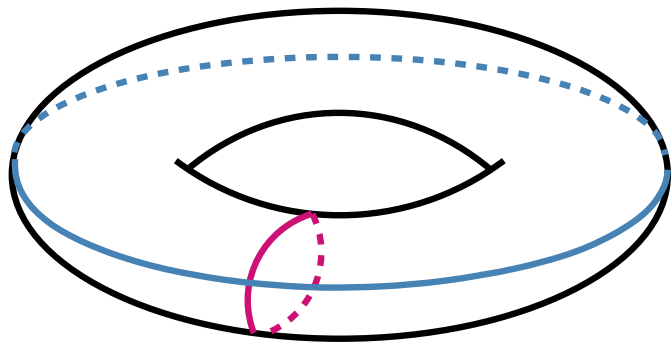
[Caroli, T. SoCG'08 video]

Periodic Delaunay triangulations

flat torus

$$\mathbb{T}^2 \sim \mathbb{R}^2 / G$$

$$G = \langle t_x, t_y \rangle$$



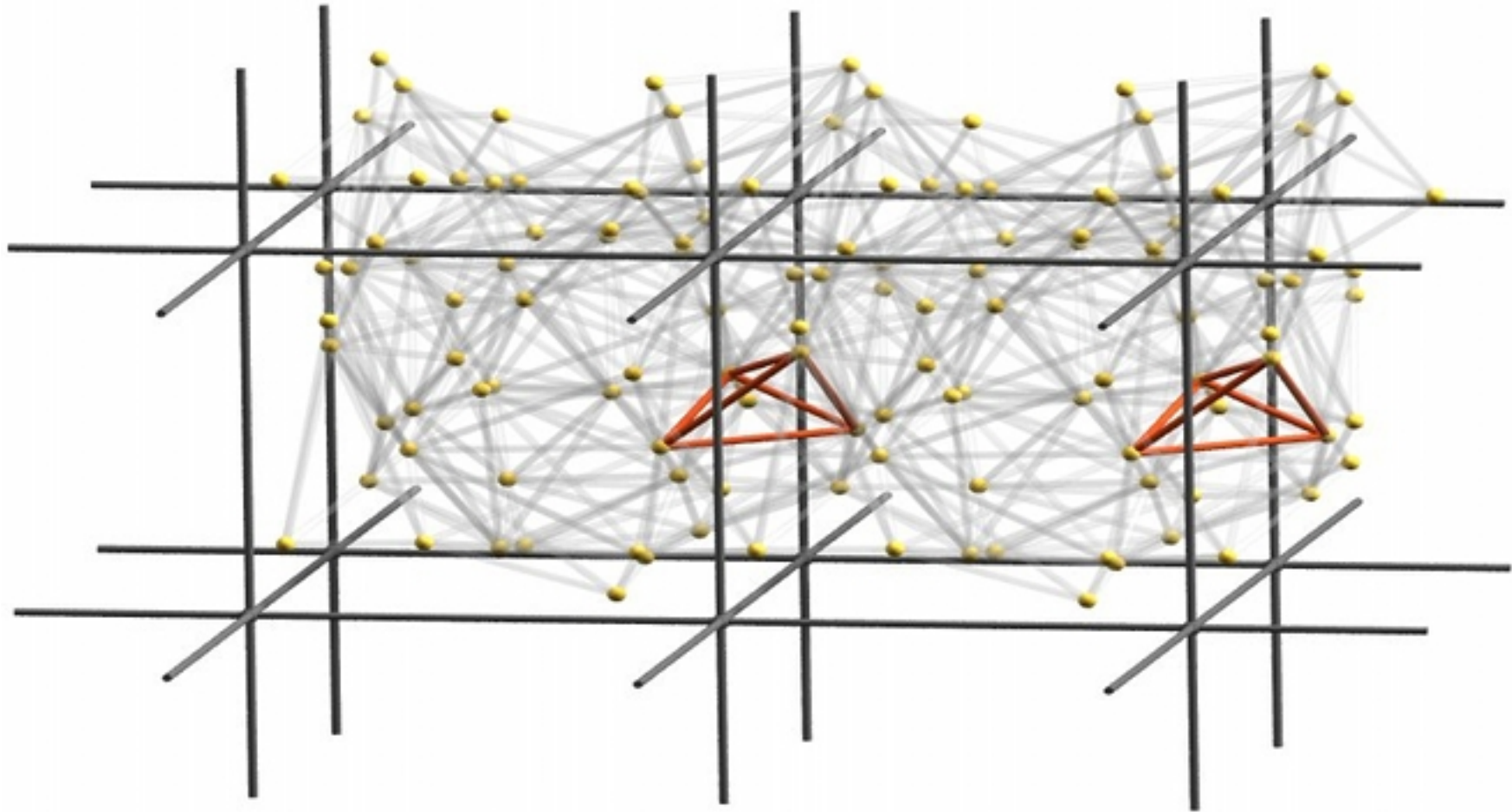
universal covering space

Periodic Delaunay triangulations

flat torus

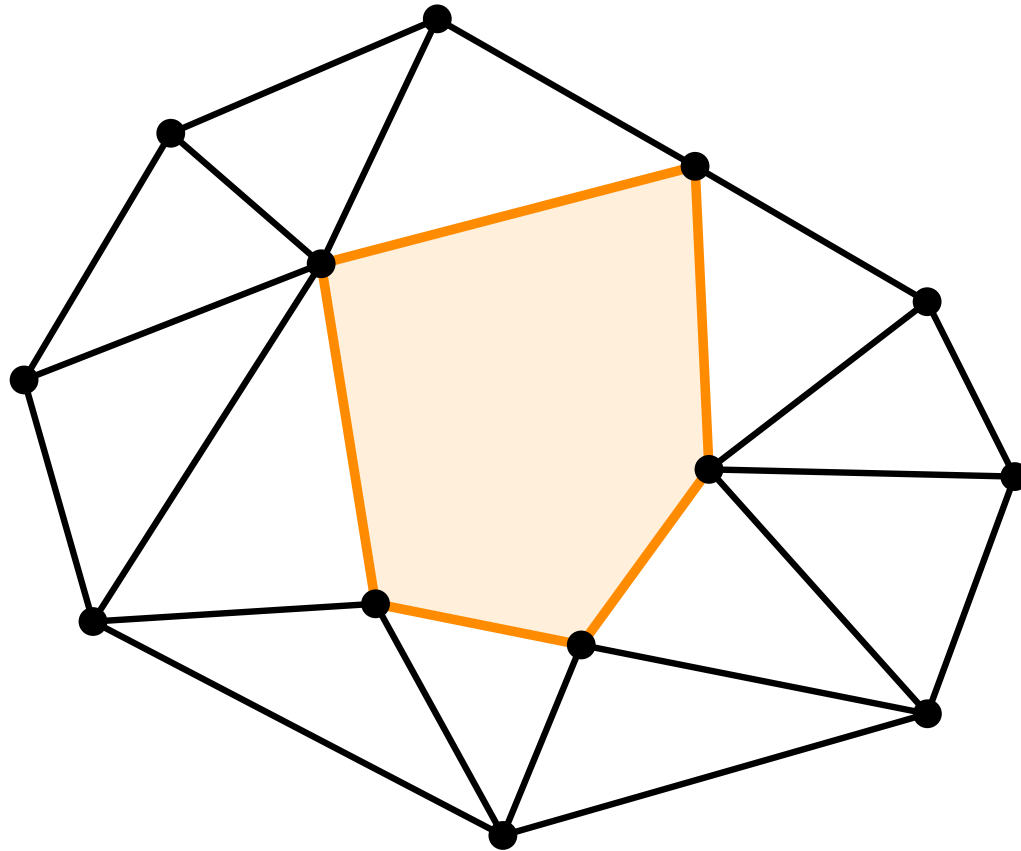
$$\mathbb{T}^3 \sim \mathbb{R}^3 / G$$

$$G = \langle t_x, t_y, t_z \rangle$$



Periodic Delaunay triangulations

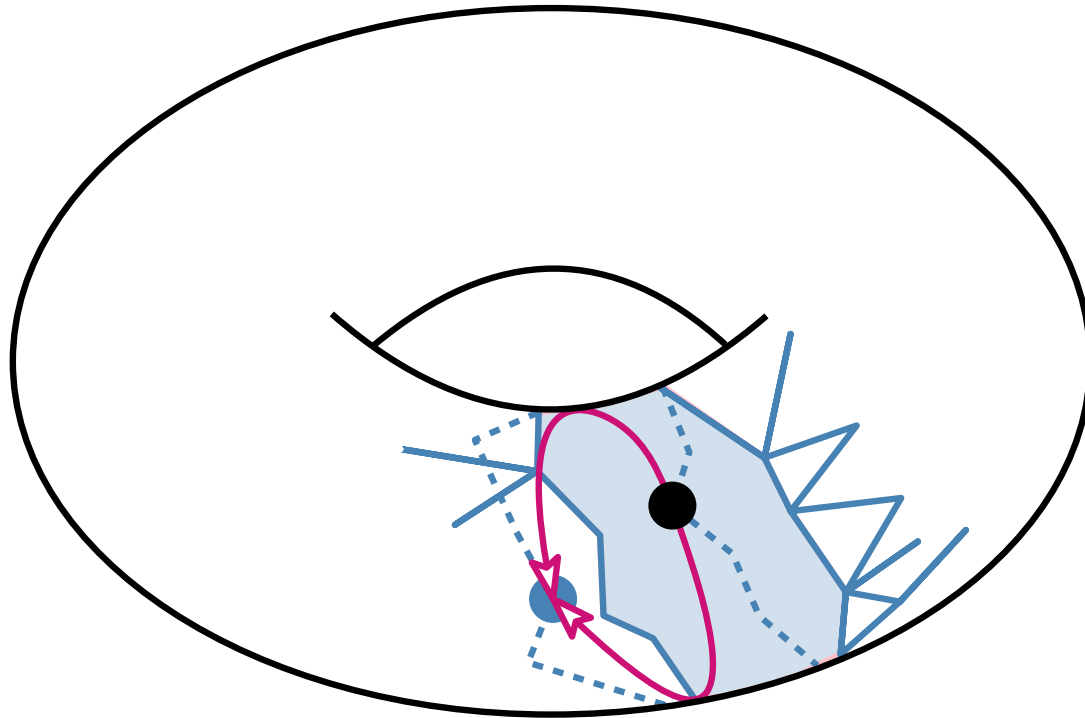
Bowyer's incremental algorithm



the conflict region forms a topological ball

Periodic Delaunay triangulations

Bowyer's incremental algorithm



the conflict region **does not** form a topological ball

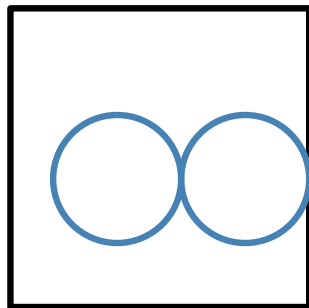
Periodic Delaunay triangulations

add a few dummy points (e.g., 36 in 3D)
and remove them asap

or

compute in a covering space (e.g., 27-sheeted in 3D)
and switch back to 1 sheet asap

so that the triangulation is always a **simplicial complex**



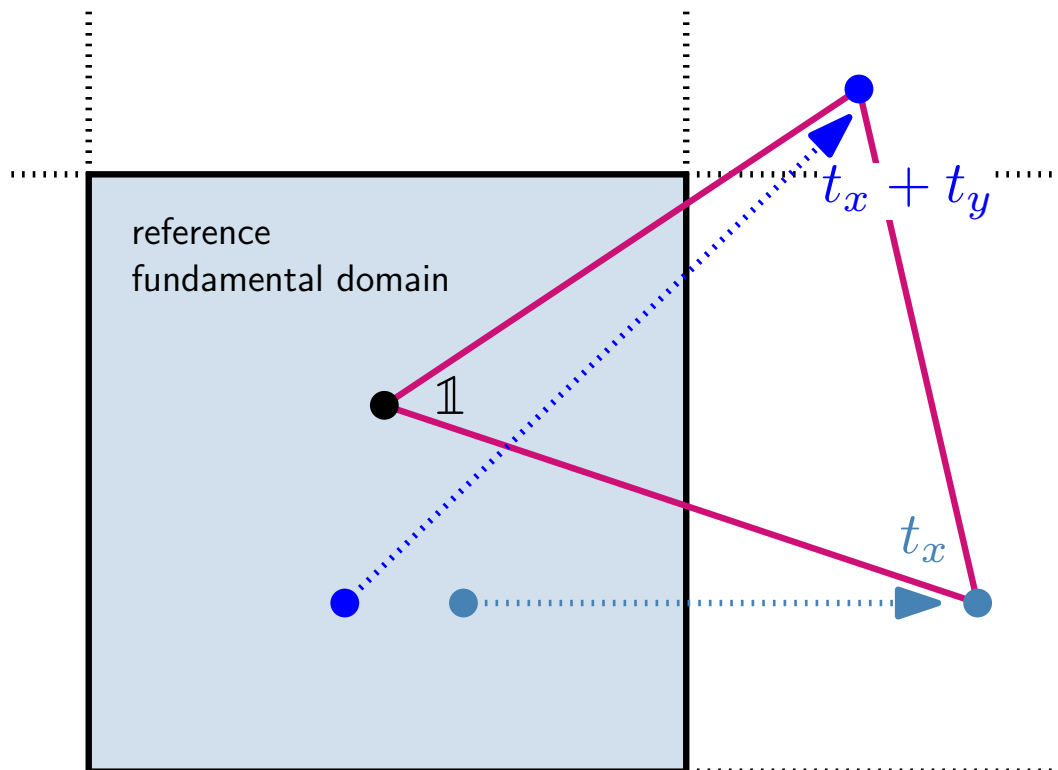
[Caroli, T. ESA'09]

$\Phi(\text{largest empty disk}) < \text{systole}/2$

Periodic Delaunay triangulations

Periodic_Delaunay_triangulation < TDS, Geom_traits >

combinatorial triangulation of \mathbb{S}^3 reused for \mathbb{T}^3



Cell = d -simplex

→ d vertices

→ d adjacent cells

d translations

Vertex

canonical point

→ one incident cell

Periodic Delaunay triangulations

[Caroli, T. CGAL'09]

2D [Kruithof CGAL'13]

also
weighted
alpha-shapes

periodic meshes [Bogdanov, Pellé, Rouxel-Labbé, T. CGAL'18]

Periodic Delaunay triangulations

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Periodic_3_Delaunay_triangulation_traits_3.h>
#include <CGAL/Periodic_3_Delaunay_triangulation_3.h>
#include <vector>

typedef CGAL::Exact_predicates_inexact_constructions_kernel      K;
typedef CGAL::Periodic_3_Delaunay_triangulation_traits_3<K>     Gt;
typedef CGAL::Periodic_3_Delaunay_triangulation_3<Gt>          P3DT3;
typedef P3DT3::Point      Point;
typedef P3DT3::Iso_cuboid Iso_cuboid;

int main(int, char**)
{
    Iso_cuboid domain(-1,-1,-1,2,2,2); // the fundamental domain

    std::vector<Point> pts;
    // pts is filled in some way...

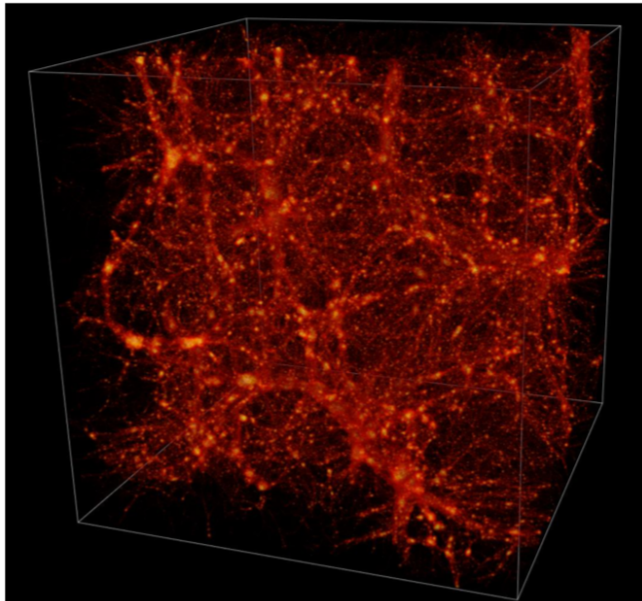
    P3DT3 T(pts.begin(), pts.end(), domain);

    return 0;
}
```

Periodic Delaunay triangulations

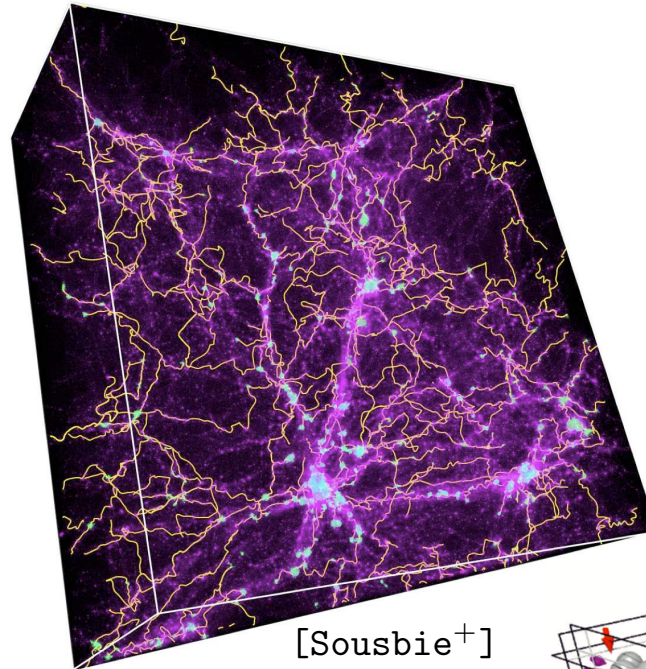
Users

Astrophysics



[v.d. Weijgaert⁺]

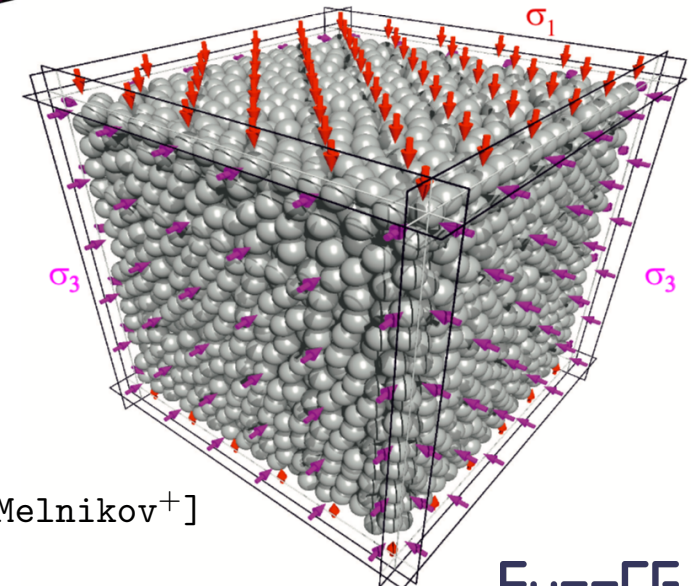
[SoCG'12 video]



[Sousbie⁺]

Particle physics
Nanostructures
etc

Granular materials



[Melnikov⁺]

Periodic Delaunay triangulations

Users are always asking for more...

non-cubic case

dD closed flat manifolds [Caroli, T. SoCG'11 DCG'16]

uses covering spaces

in practice: dummy points In progress

$\Phi(\text{largest empty disk}) < \text{systole}/2$

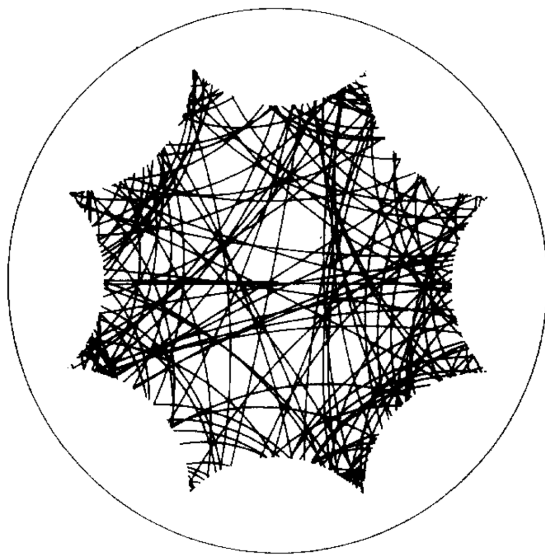
Periodic hyperbolic triangulations

Delaunay triangulation of the Bolza surface

Periodic hyperbolic triangulations

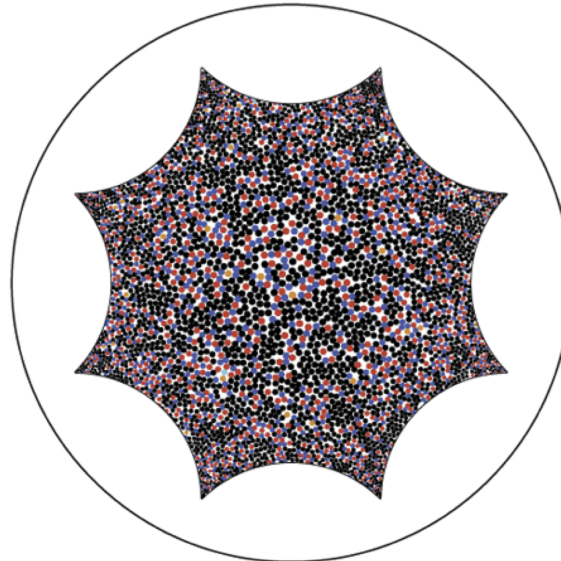
Delaunay triangulation of the Bolza surface

Motivation: e.g., in mathematical physics



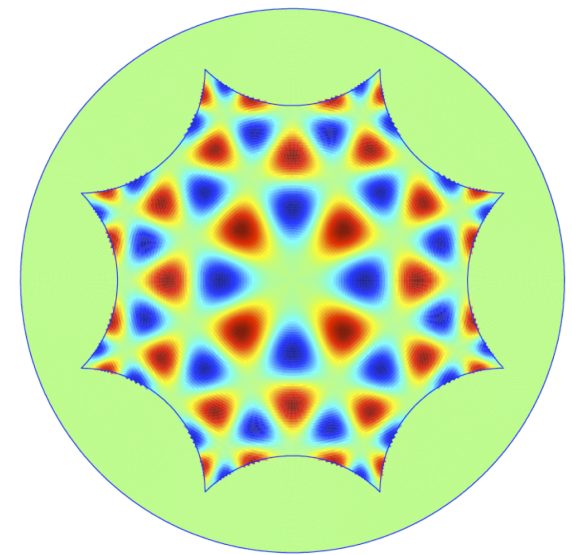
[Balazs, Voros '86]

Chaotic motion



[Sausset, Tarjus, Viot '08]

Glass-forming liquid



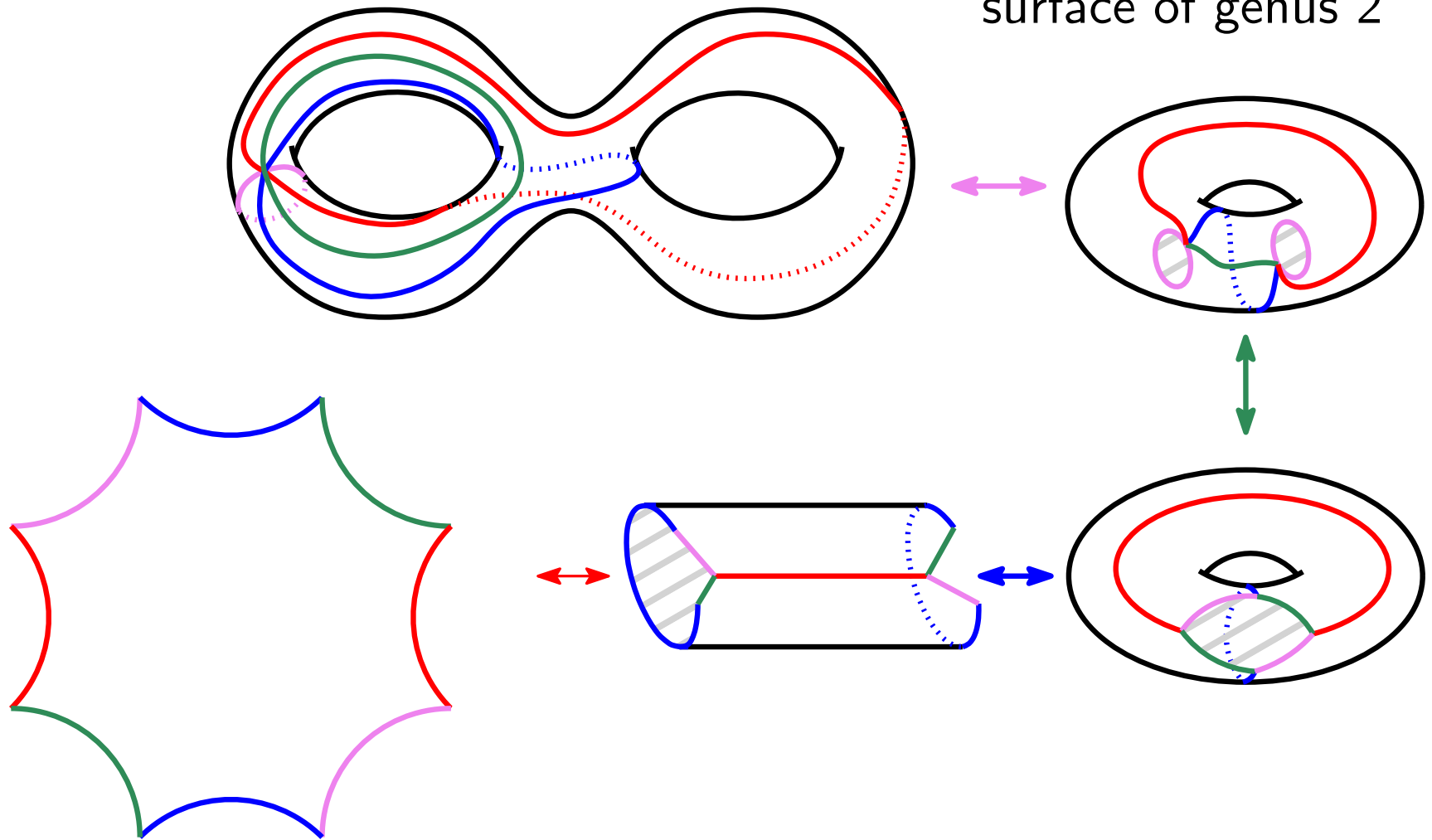
[Chossat, Faye, Faugeras '11]

Visual perception
of textures

Periodic hyperbolic triangulations

Delaunay triangulation of the Bolza surface

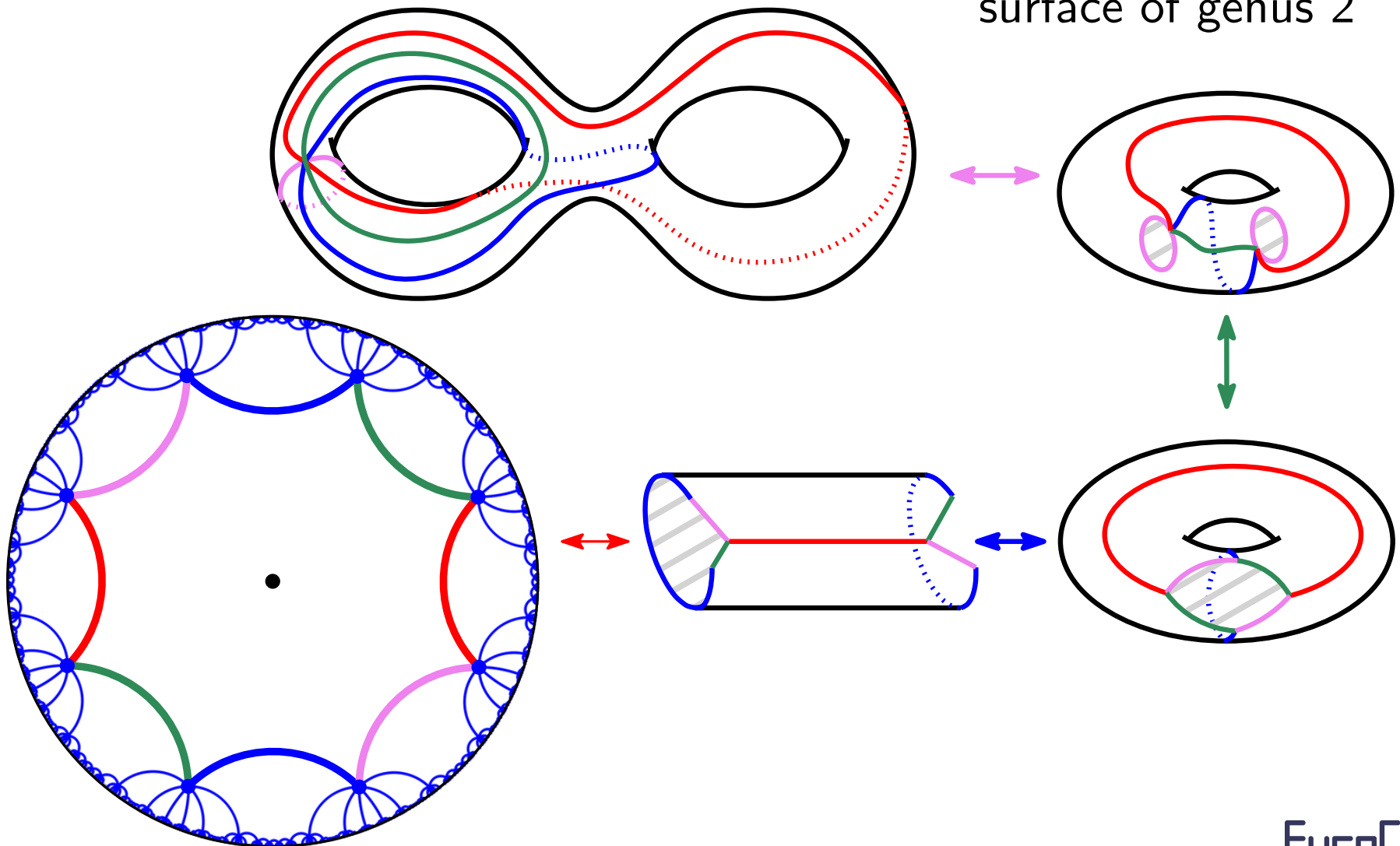
surface of genus 2



Periodic hyperbolic triangulations

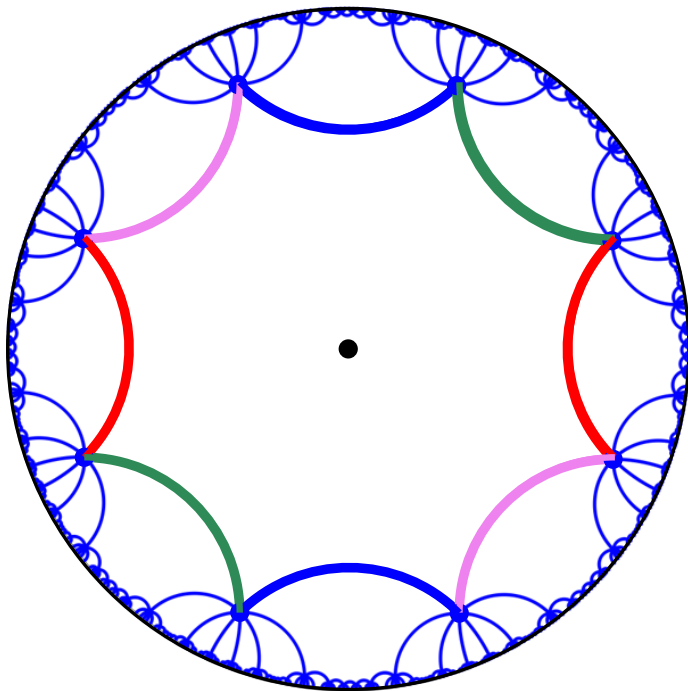
Delaunay triangulation of the Bolza surface

surface of genus 2



Periodic hyperbolic triangulations

Delaunay triangulation of the Bolza surface



the regular octagon tiles the
hyperbolic plane = covering space

Periodic hyperbolic triangulations

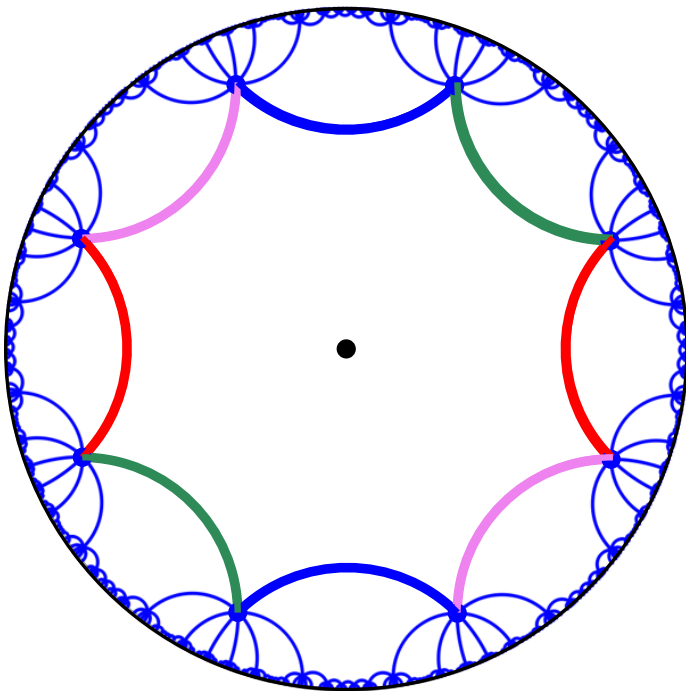
Delaunay triangulation of the Bolza surface

$$\mathbb{M}_2 \sim \mathbb{H}^2 / G$$

$$G = \langle a, b, c, d \mid \text{relation} \rangle$$

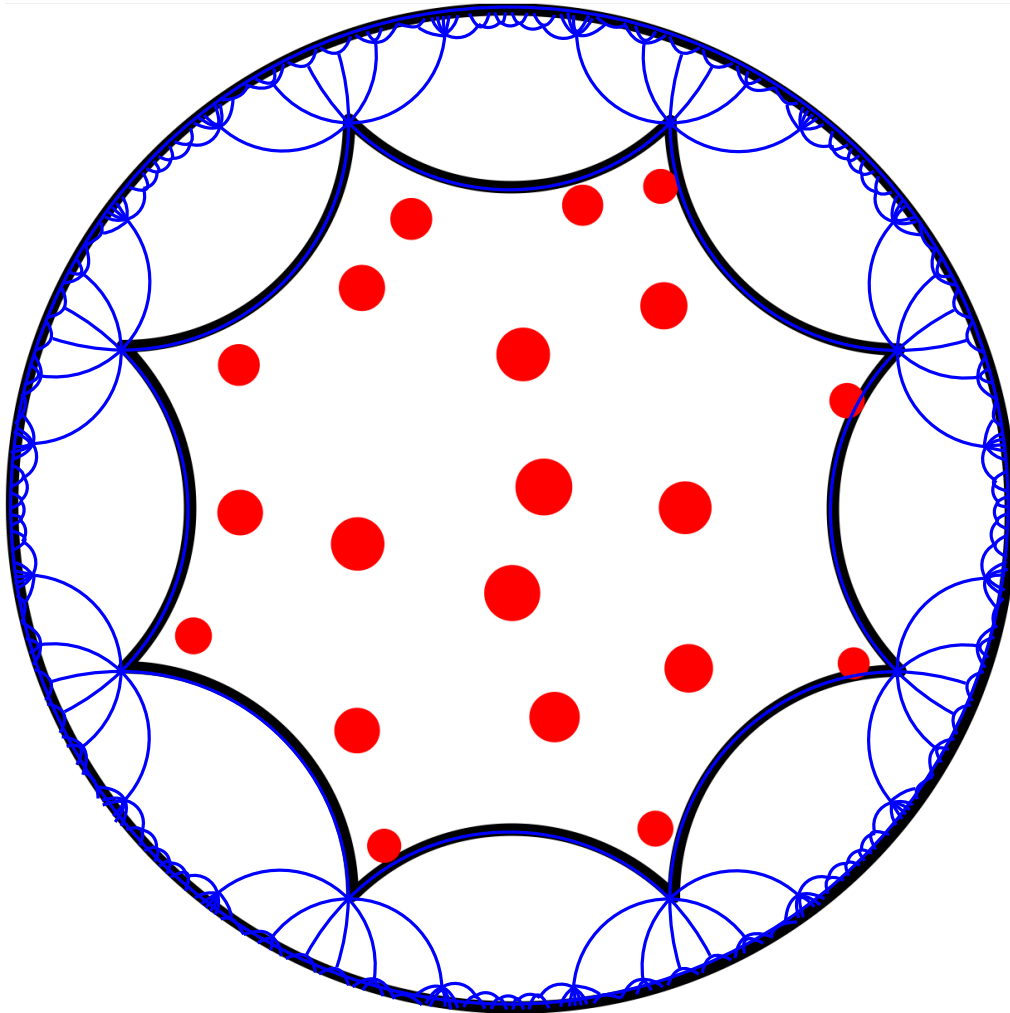
a, b, c, d hyperbolic translations

do not commute



Periodic hyperbolic triangulations

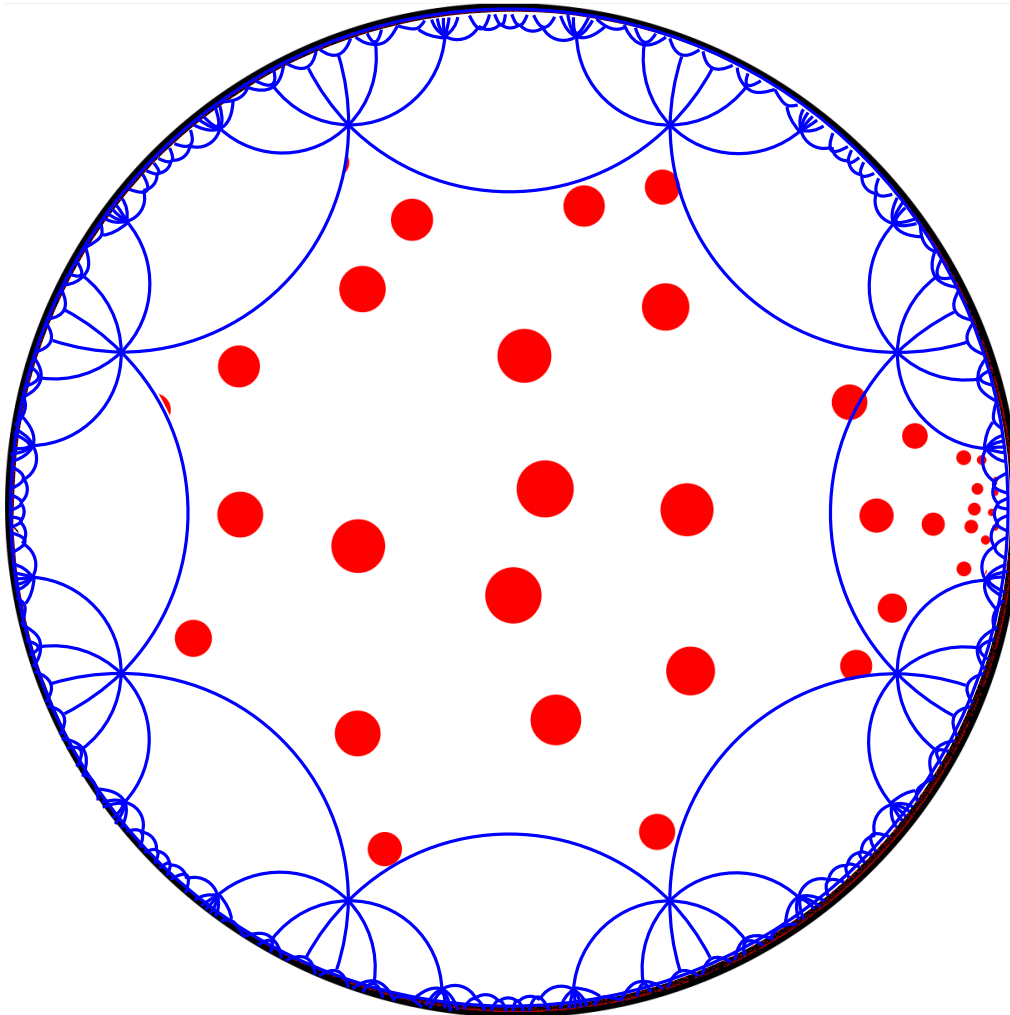
Delaunay triangulation of the Bolza surface



points on the surface

Periodic hyperbolic triangulations

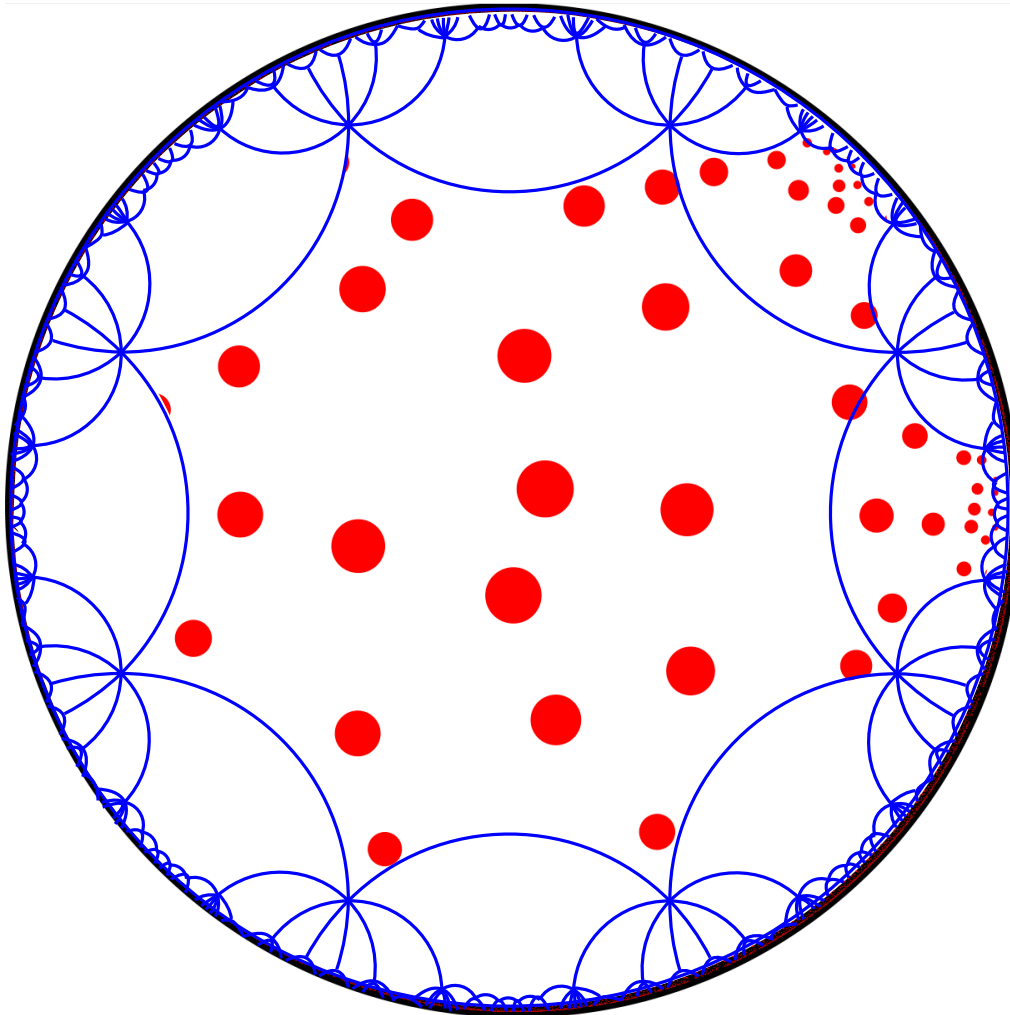
Delaunay triangulation of the Bolza surface



translated images

Periodic hyperbolic triangulations

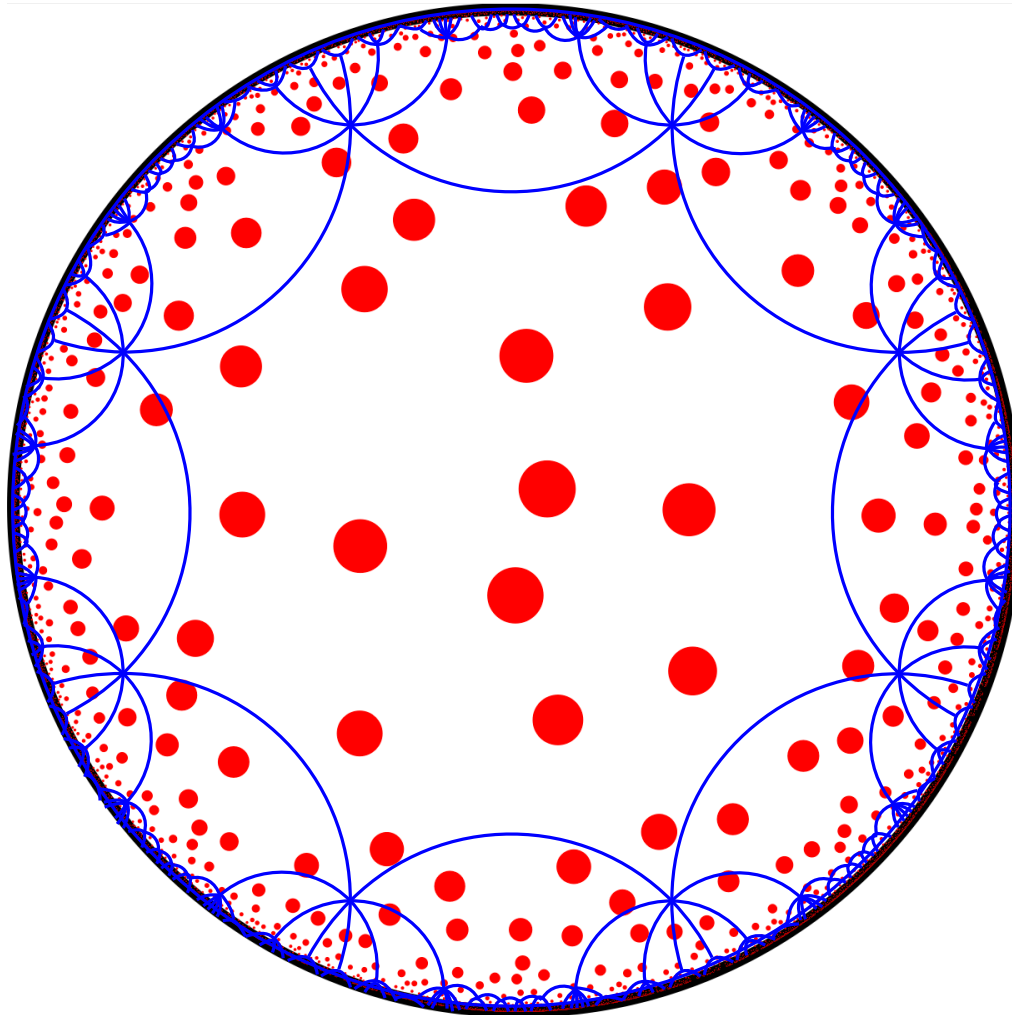
Delaunay triangulation of the Bolza surface



translated images

Periodic hyperbolic triangulations

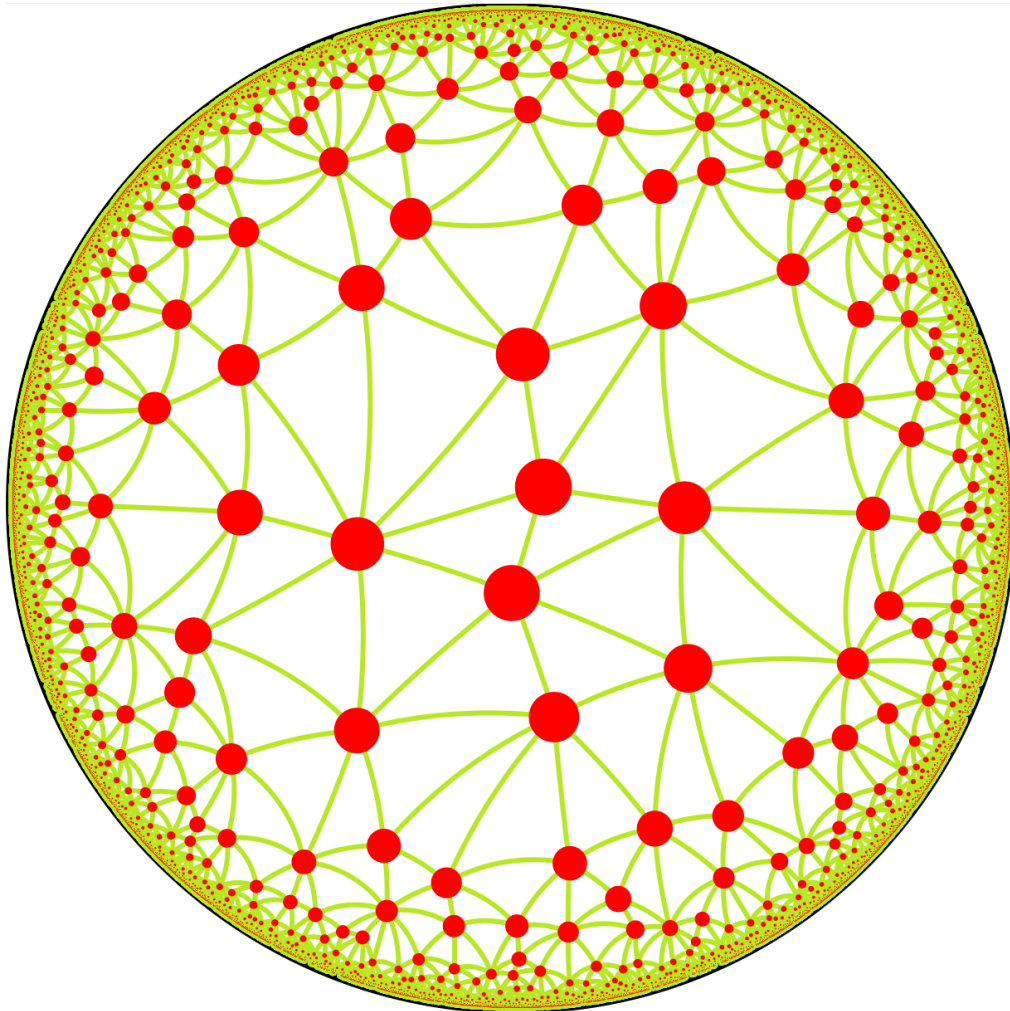
Delaunay triangulation of the Bolza surface



translated images

Periodic hyperbolic triangulations

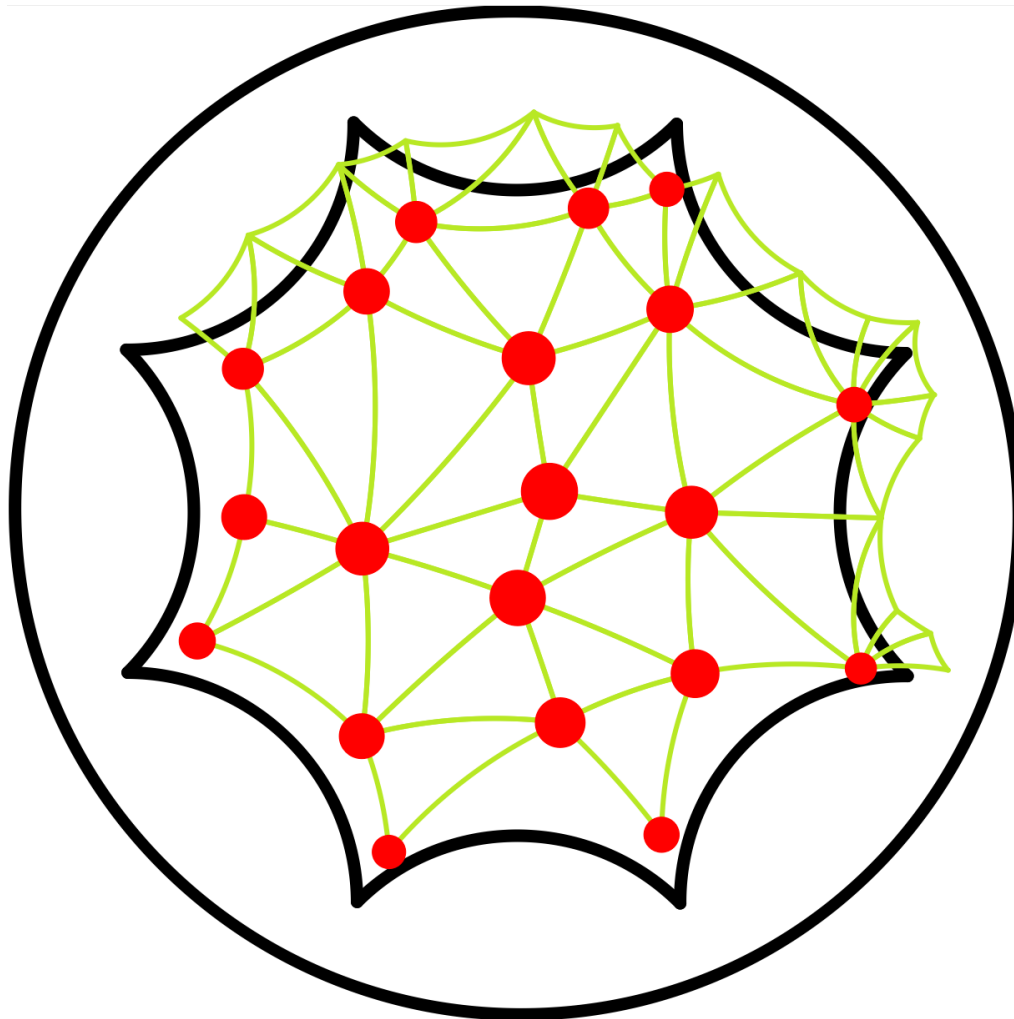
Delaunay triangulation of the Bolza surface



Delaunay triangulation
of the infinite point set

Periodic hyperbolic triangulations

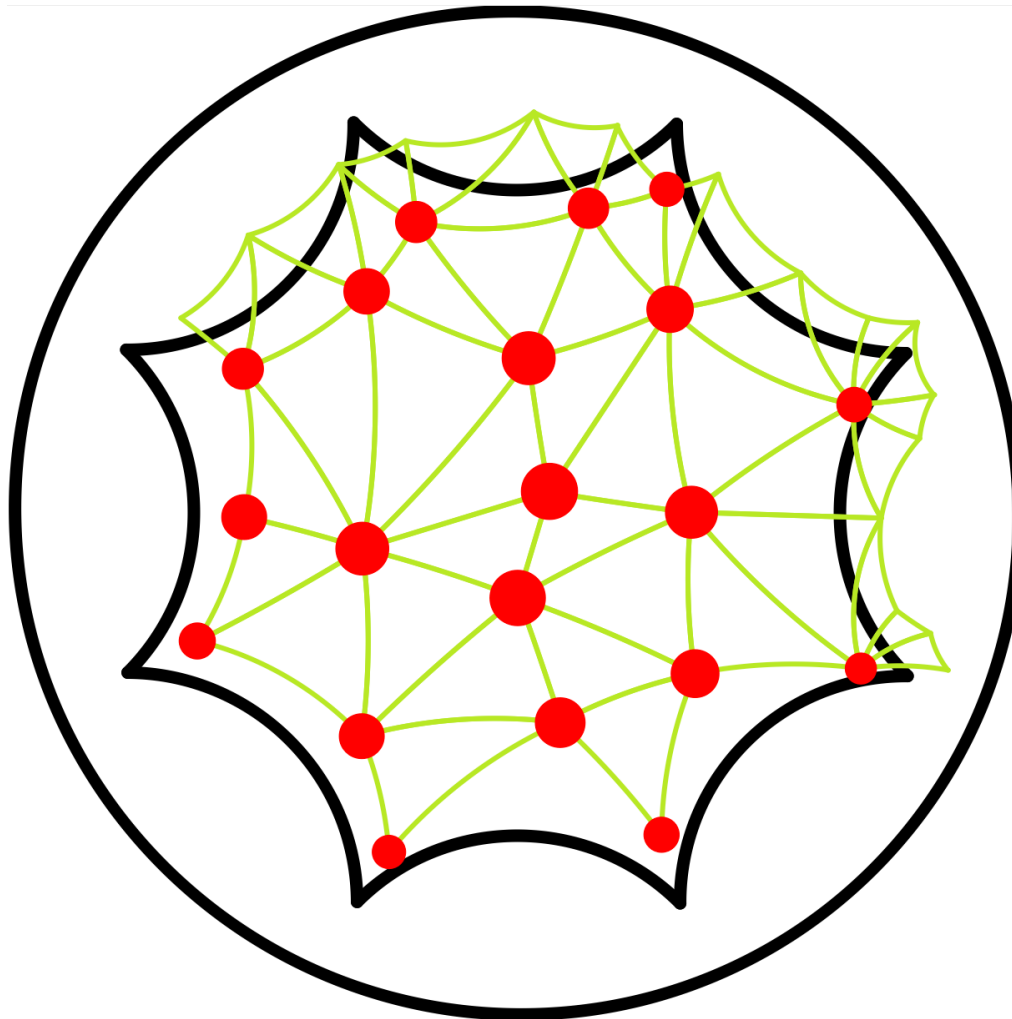
Delaunay triangulation of the Bolza surface



Delaunay triangulation
of the surface

Periodic hyperbolic triangulations

Delaunay triangulation of the Bolza surface



Delaunay triangulation
of the surface

(14 "dummy" points)

$\Phi(\text{largest empty disk}) < \text{systole}/2$

[Bogdanov, T., Vegter SoCG'16]

[Iordanov, T. SoCG'17]

[Iordanov, T. CGAL'19]

The CGAL project

Becoming a user

<http://www.cgal.org>

Many platforms

Linux, MacOS, Windows
g++, VC++, clang, . . .

Becoming a user

<http://www.cgal.org>

Many platforms

Linux, MacOS, Windows
g++, VC++, clang, . . .

Download from `github`

or use

Linux distribution (Debian, . . .)

Mac distribution (macports, brew, . . .)

Euclidean 2D and 3D triangulations

- in Matlab
- in Python through CGAL bindings

Becoming a user

<http://www.cgal.org>

Follow “Getting started”

easier than ever!

- “ Since CGAL version 5.0, CGAL is header-only by default, which means that there is no need to build CGAL before it can be used. ”
- Simple viewer

Becoming a user

<http://www.cgal.org>

Follow “Getting started”

Read the **User manual** of your favorite package

Read, compile and run **examples**

Becoming a user

<http://www.cgal.org>

Follow “Getting started”

Read the **User manual** of your favorite package

Read, compile and run **examples**

Write your own code!

looking at the **Reference manual**

Obey license GPLv3+ (or buy a commercial license)

Becoming a user

<http://www.cgal.org>

Molecular Modeling

Particle Physics, Fluid Dynamics, Microstructures

Medical Modeling and Biophysics

Geographic Information Systems

Games

Motion Planning

Sensor Networks

Architecture, Buildings Modeling, Urban Modeling

Astronomy

2D and 3D Modelers

Mesh Generation and Surface Reconstruction

Geometry Processing

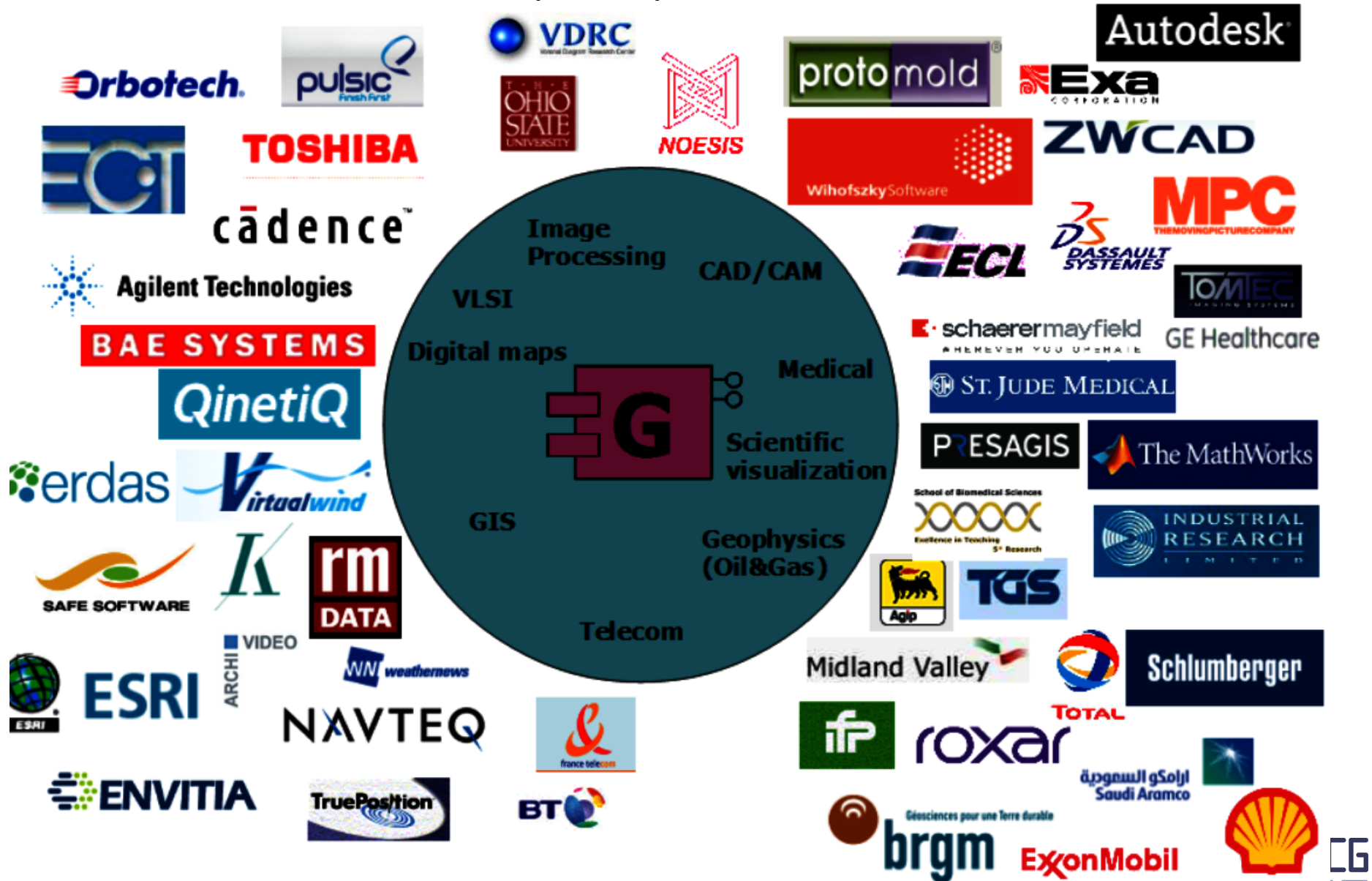
Computer Vision, Image Processing, Photogrammetry

Computational Topology and Shape Matching

Computational Geometry and Geometric Computing

Becoming a CGAL user

some commercial users (2012)



Becoming a CGAL user

Ipelets!

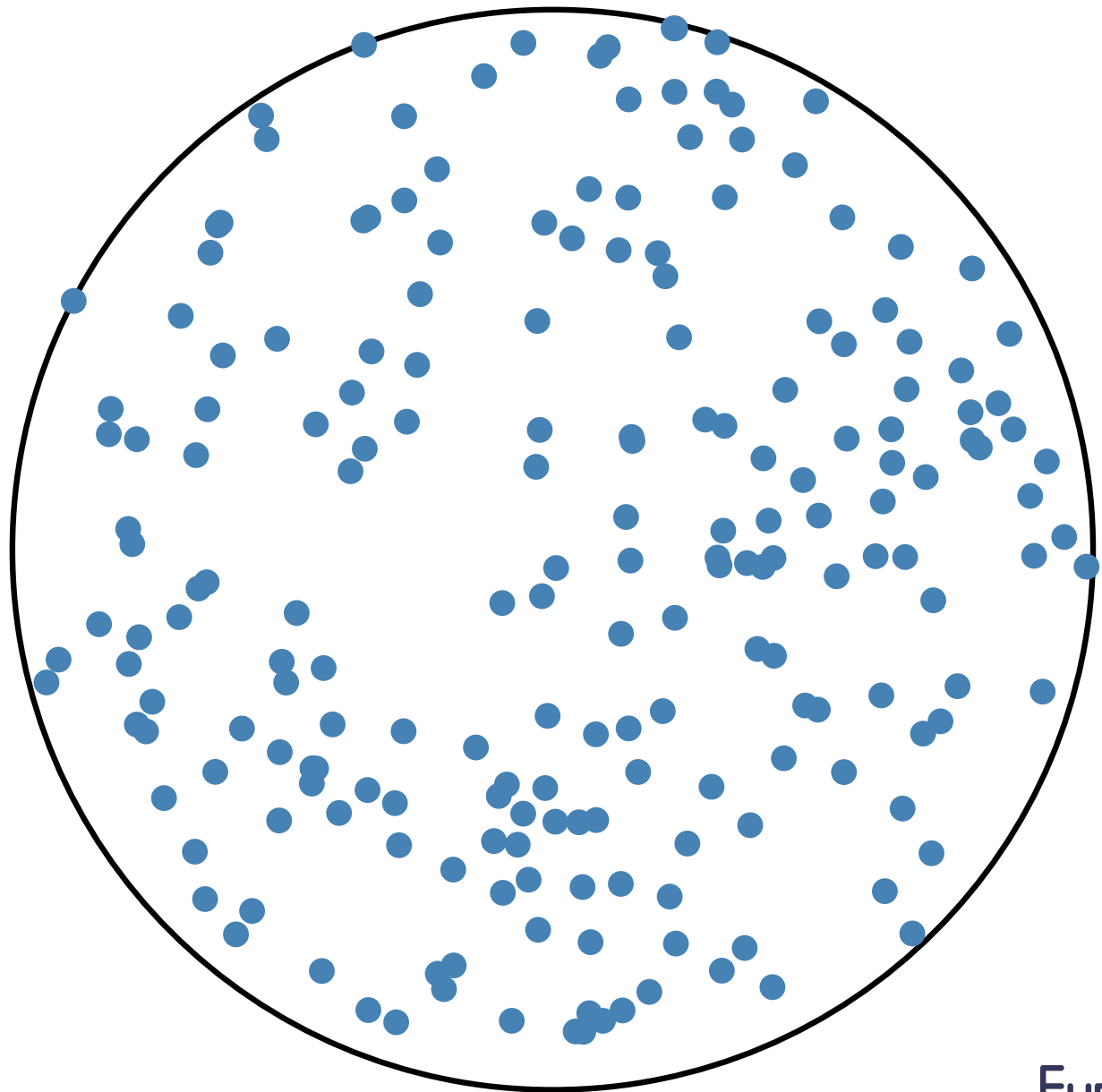
Becoming a CGAL user

Ipelets!

Generators

e.g. 200 points in a disk

\mathbb{R}^2



Becoming a CGAL user

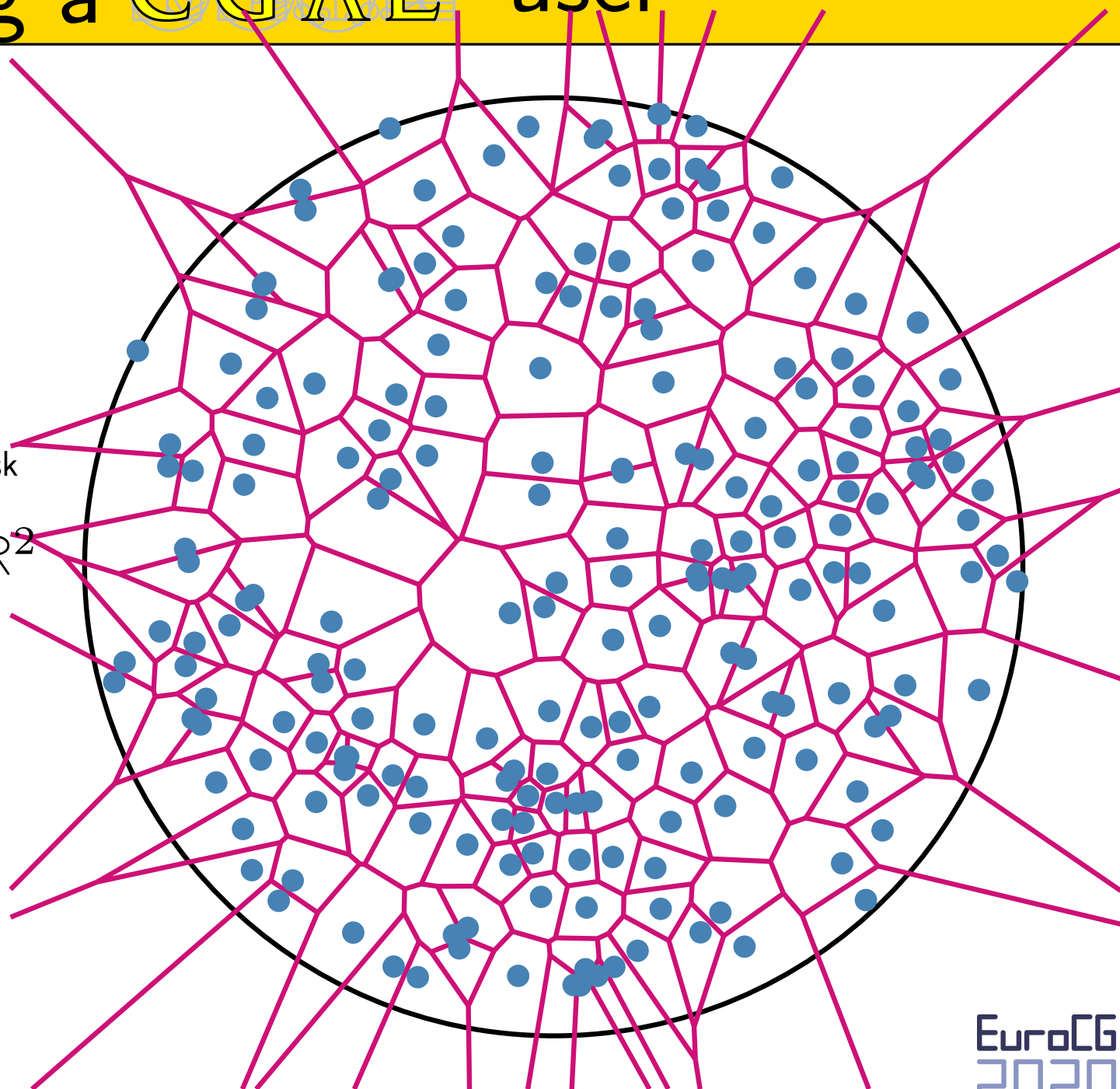
Ipelets!

Generators

e.g. 200 points in a disk

\mathbb{R}^2

“Diagrams”
→ Voronoi



Becoming a CGAL user

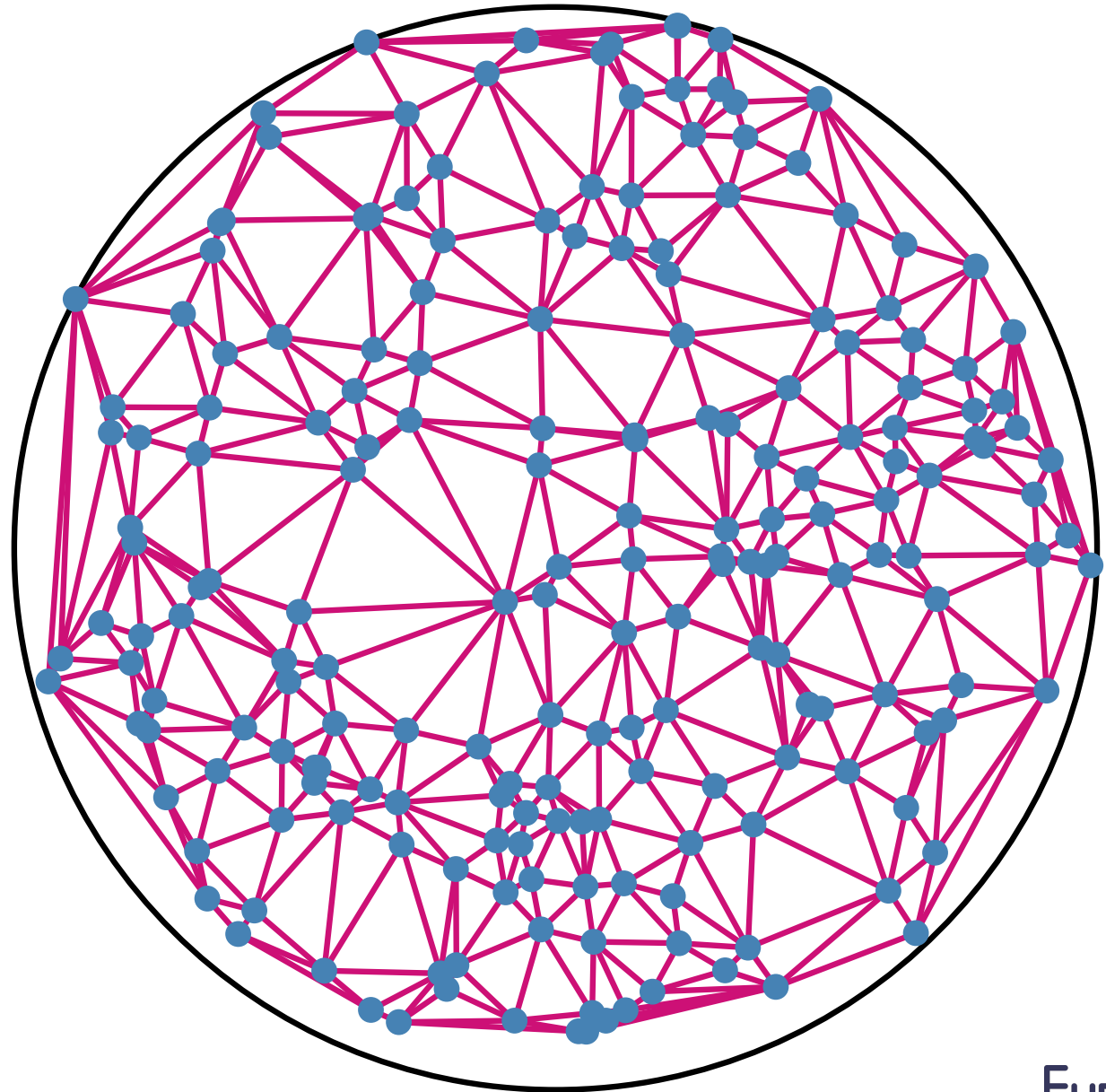
Ipelets!

Generators

e.g. 200 points in a disk

\mathbb{R}^2

“Triangulations”
→ Delaunay



Becoming a developer

Any idea for a new package?

Contact the editorial board!

even before starting to code

<http://www.cgal.org>

Design and coding style

Infrastructure

Review and integration process

Becoming a CGAL developer

Contributors are clearly acknowledged **Visibility**

- Appear as Authors in manual chapters

CGAL 5.0.1 - 3D Periodic Triangulations

CGAL 5.0.1 - 3D Periodic Triangulations

- ▶ User Manual
- ▶ Reference Manual
- Refinement Relationships
- Is Model Relationships
- Has Model Relationships
- Bibliography
- ▶ Class and Concept List
- ▶ Examples

User Manual

Authors
Manuel Caroli, Aymeric Pellé, Mael Rouxel-Labbé and Monique Teillaud

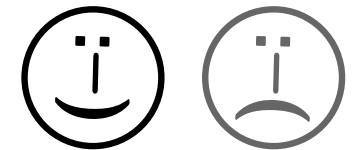
- Listed in the “People” web page

Becoming a CGAL developer

Contributors are clearly acknowledged **Visibility**

appreciated by

companies (Google, Intel, Apple, ...)



research institutes (INRIA, ETHZ, ...?)

Becoming a developer

Contributors are clearly acknowledged **Visibility**

[The institution of the] authors keep the copyright

(must agree to distribute the code in CGAL under GPLv3+)

Becoming a developer

Key for success = Diversity of its members

gather many skills:

maths

algorithms

C++

development tools

...

Join the crowd!

Becoming a developer





Datum/Date: 10.09.2016 Zug/Train: 571 (12:33) Position: -1b 09.09.2016 15:27:10

Reserviert Réservé

Bitte Sitzplätze für Gruppen freihalten
Veuillez laisser les places libre pour les groupes
Per favore tenere liberi i posti per gruppi

CGAL-ETH Zürich (Salow) 20
Zürich HB (13:37) - Landquart (14:41)



September 2016

EuroCG
2020



September 2016

EuroCG
2020

Thanks to

past,
present,
future

developers
and

users
of



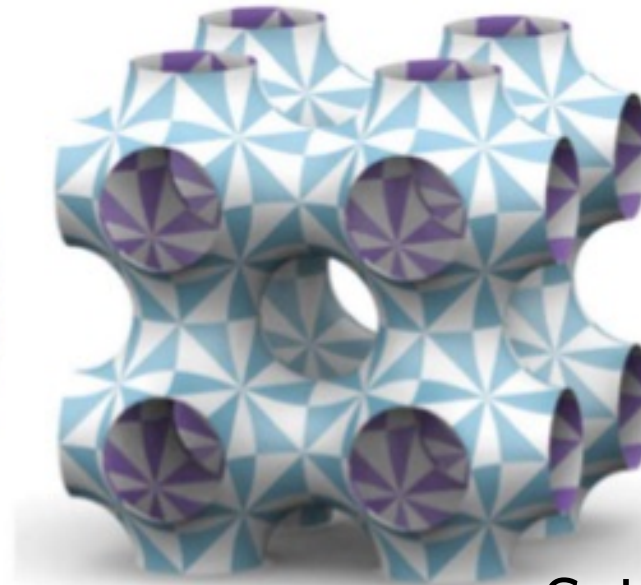
you all!

Otfried for Ipe

Triply periodic minimal surfaces



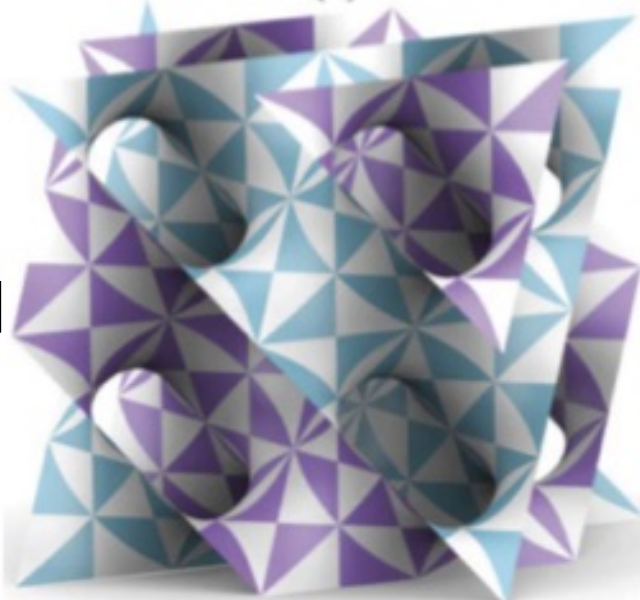
(a)



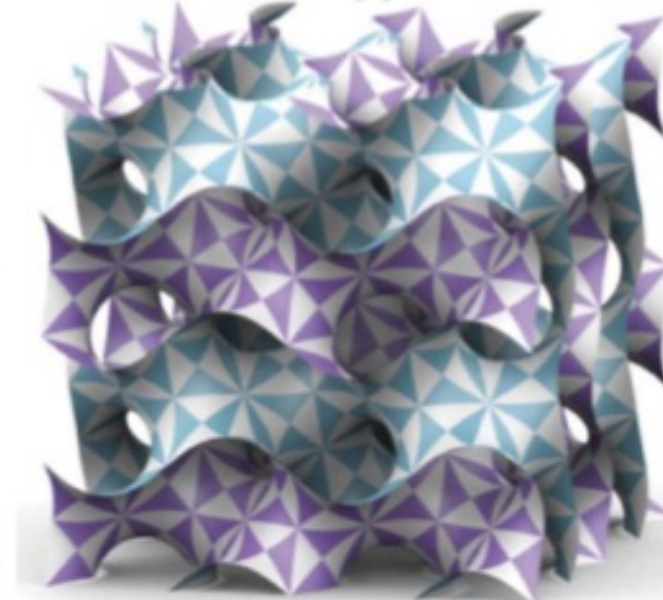
Schwartz P

(b)

Diamond



(c)



Gyroid

(d)