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# Triangulations in CGAL - To non-Euclidean spaces and beyond!

Monique Teillaud

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Monique Teillaud. Triangulations in CGAL - To non-Euclidean spaces and beyond!. EuroCG 2020 - 36th European Workshop on Computational Geometry, Mar 2020, Würzburg, Germany. hal-02510046

**HAL Id: hal-02510046**

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Submitted on 17 Mar 2020

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# Triangulations in



Monique Teillaud

# Triangulations in

CGAL

To non-Euclidean spaces and beyond!



By Edo-biscuit - Own work, CC BY-SA 4.0

Monique Teillaud

# History of the CGAL project

**1995 - Start**

Academic project





# History of the project

**1995 - Start**

Academic project

**January 2003 - Creation of  Geometry Factory**

as an INRIA startup, by Andreas Fabri

sells commercial licenses, support, customized developments

**November 2003 - Open Source project**

new contributors

**November 2019 - CGAL 5.0**

C++14

# Goals

## Gather efforts

PlaGeo, SpaGeo (Utrecht)

XYZ GeoBench (ETH Zürich)

LEDA (MPII Saarbrücken)

C++GAL

(INRIA Sophia Antipolis)

...



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# Goals

Gather efforts

Promote research in CG

*“ make the large body of geometric algorithms developed in the field of computational geometry available for industrial applications ”*

⇒ high quality  
review process, . . .

⇒ robust code

# Goals

Gather efforts

Promote research in CG

Reward structure for implementations in academia

⇒ high quality  
review process, . . .

⇒ robust code

# Technical choices

C++

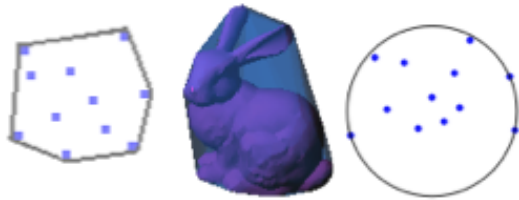
Genericity/flexibility through templates

Exact Geometric Computation [Yap]

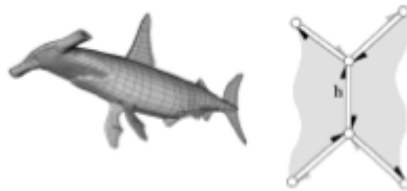
Exact predicates  $\rightsquigarrow$  exact decisions

# Contents

> 80 chapters in the manual



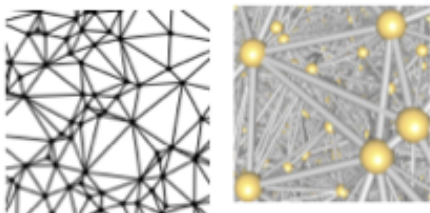
Bounding Volumes



Polyhedral Surface



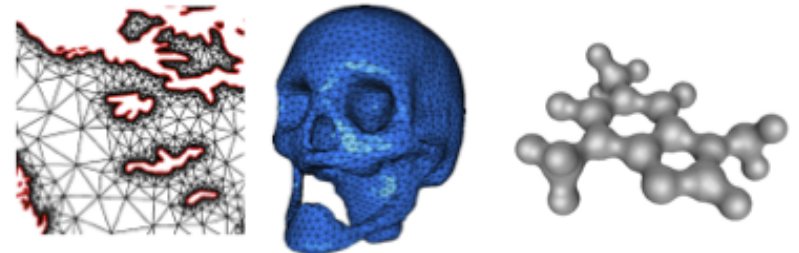
BooleanOperations



Triangulations



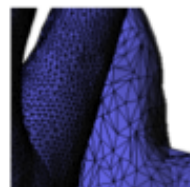
Voronoi Diagrams



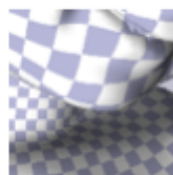
Mesh Generation



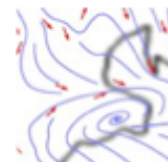
Subdivision



Simplification



Parameterization



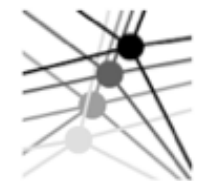
Streamlines



Ridge  
Detection



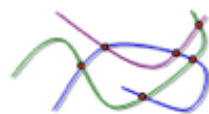
Neighbour  
Search



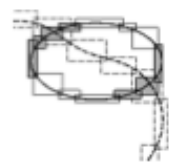
Kinetic  
Data structures



Lower Envelope



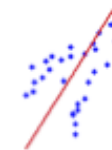
Arrangement



Intersection  
Detection



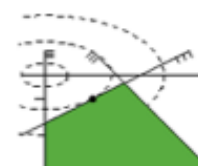
Minkowski  
Sum



PCA



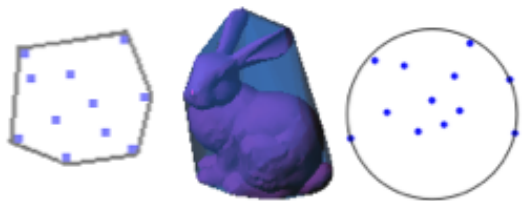
Polytope  
distance



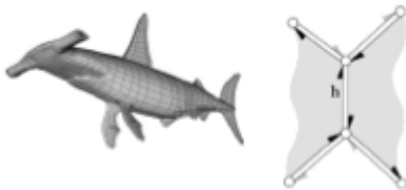
QP Solver

# Contents

> 80 chapters in the manual



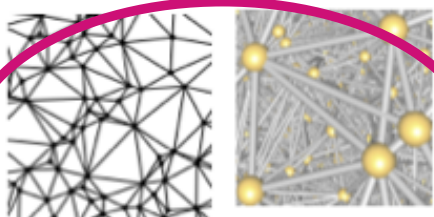
Bounding Volumes



Polyhedral Surface



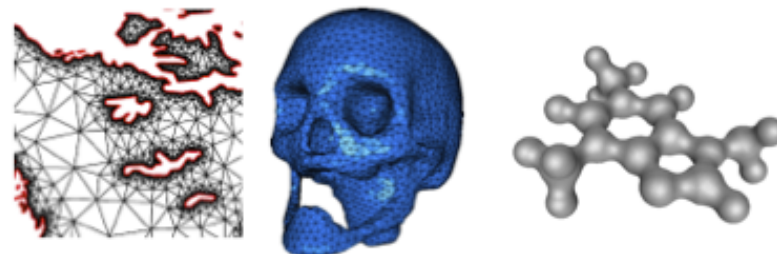
BooleanOperations



Triangulations



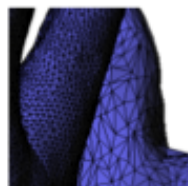
Voronoi Diagrams



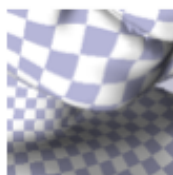
Mesh Generation



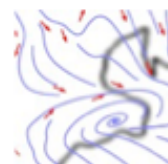
Subdivision



Simplification



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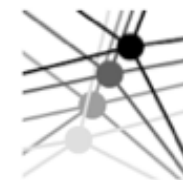
Streamlines



Ridge  
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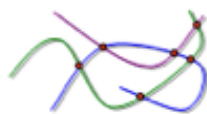
Neighbour  
Search



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Data structures



Lower Envelope



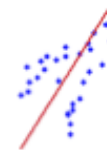
Arrangement



Intersection  
Detection



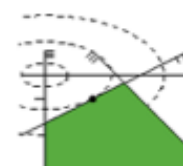
Minkowski  
Sum



PCA



Polytope  
distance



QP Solver

# Triangulations in

2D Triangulations (1997)

3D Triangulations (2000) and meshes (2009)

dD Triangulations (2015)

2D Periodic Triangulations (2013)

3D Periodic Triangulations (2009) and meshes (2018)

2D Hyperbolic Triangulations (2019)

2D Periodic Hyperbolic Triangulations (2019)



# General design

Triangulation < Geom\_traits, TDS >

# General design

Triangulation < **Geom\_traits** **TDS** >

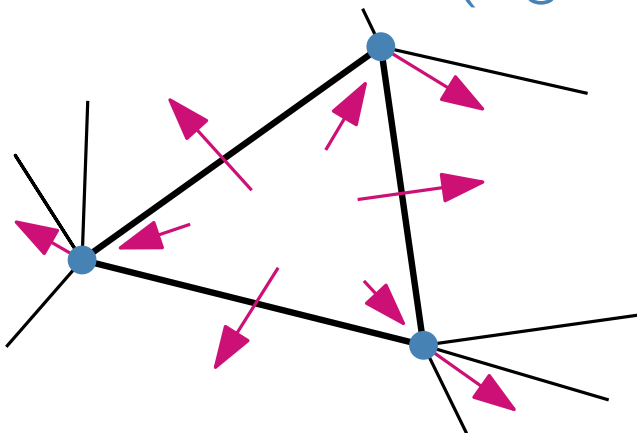
geometry

data structure

objects (points, etc)  
predicates (e.g., in\_circle)  
constructions (e.g., circumcenter)

Cell =  $d$ -simplex  
→  $d$  vertices  
→  $d$  adjacent cells

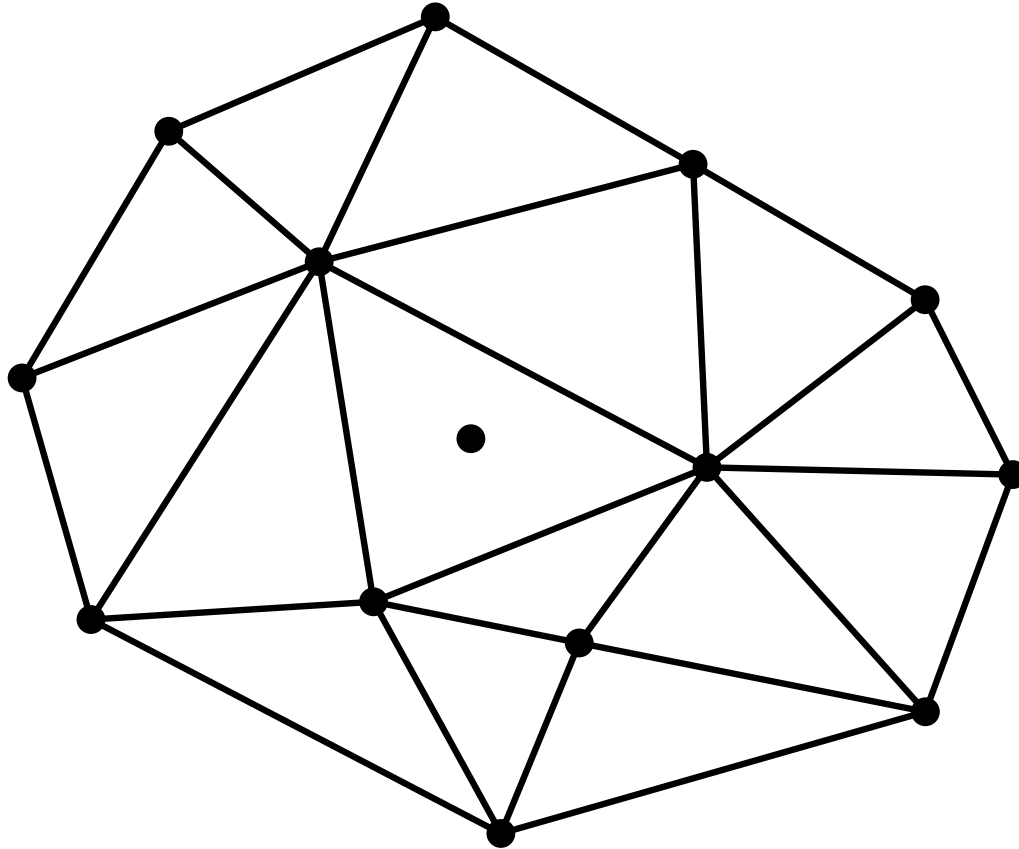
Vertex  
point  
→ one incident cell



# Euclidean Delaunay Triangulations

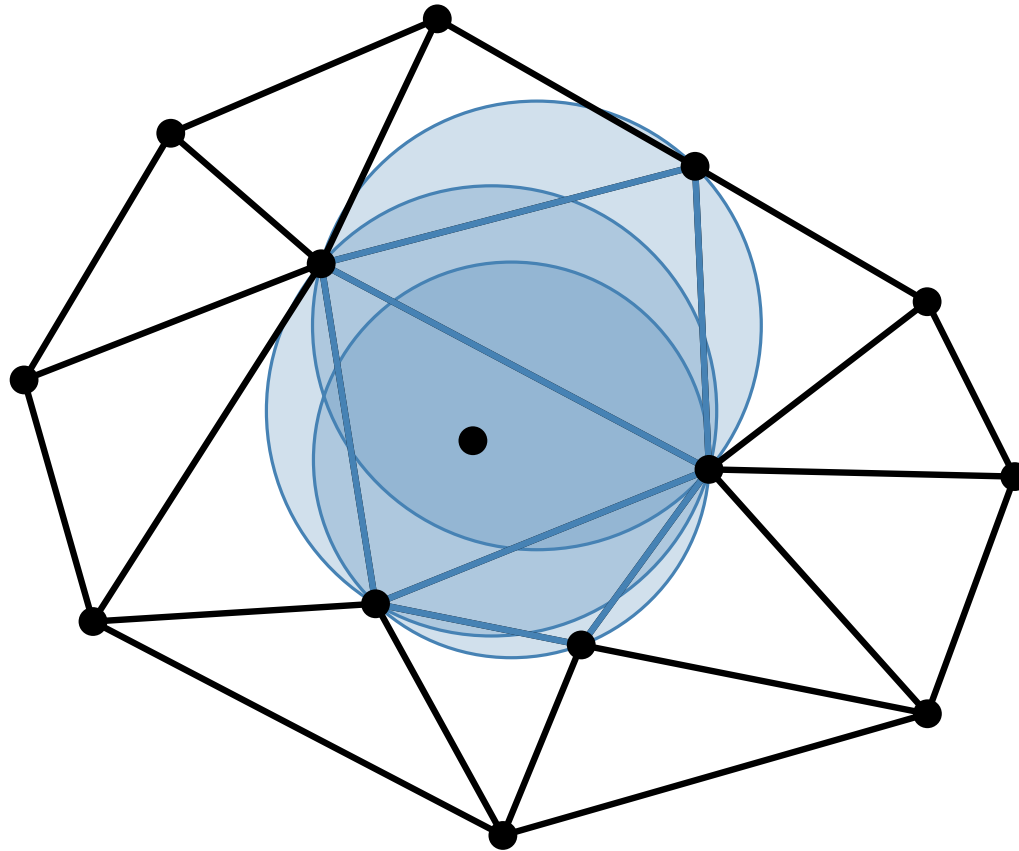
# Euclidean Delaunay Triangulations

Bowyer's incremental algorithm



# Euclidean Delaunay Triangulations

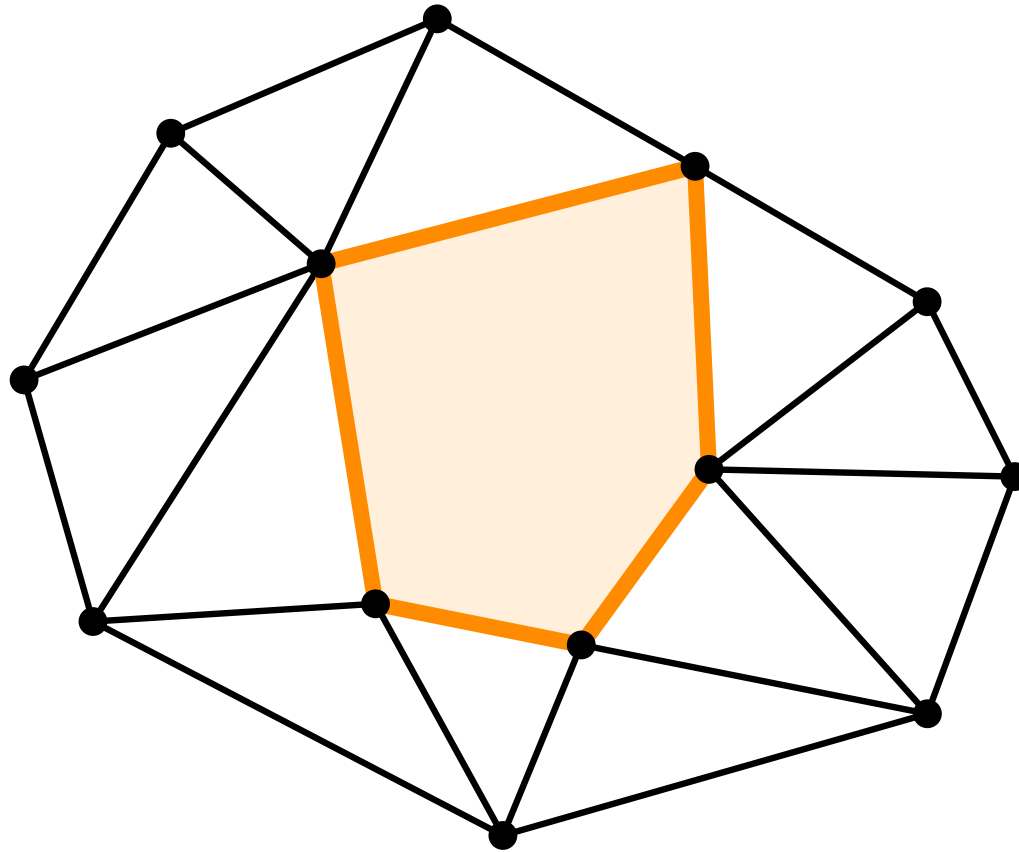
Bowyer's incremental algorithm



find simplices in conflict  
Triangulation  
using `Geom_traits`

# Euclidean Delaunay Triangulations

Bowyer's incremental algorithm

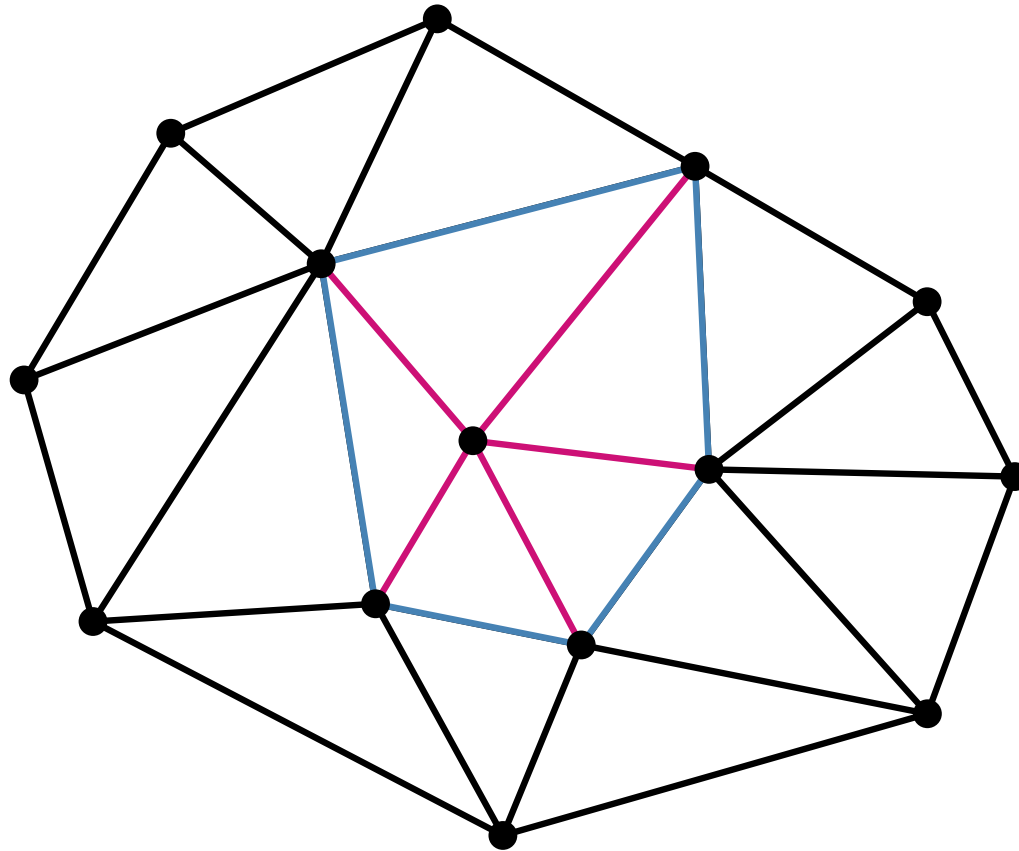


find simplices in conflict  
Triangulation  
using `Geom_traits`

**the conflict region forms a topological ball**

# Euclidean Delaunay Triangulations

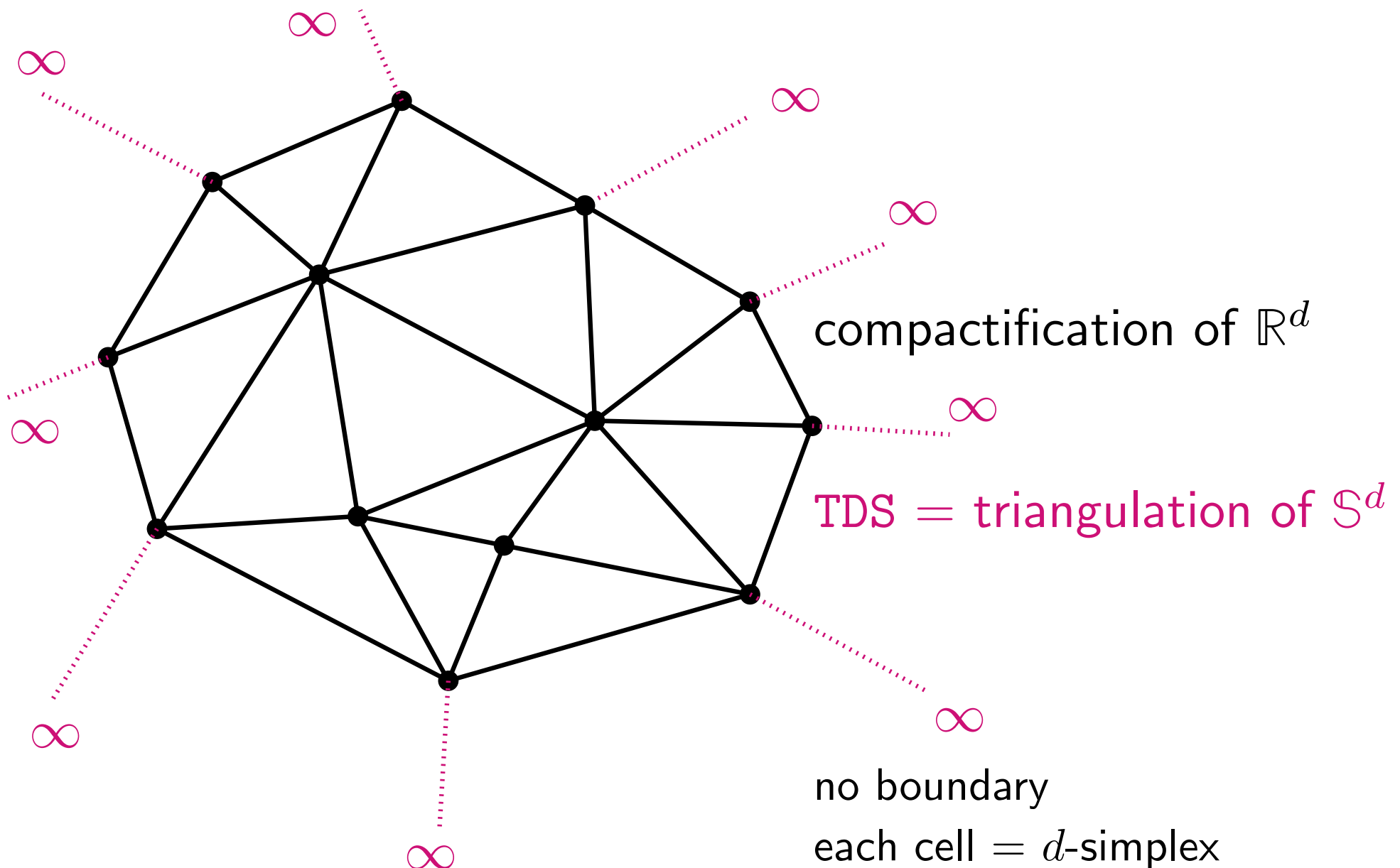
Bowyer's incremental algorithm



find simplices in conflict  
Triangulation  
using `Geom_traits`

create new simplices  
TDS

# Euclidean Delaunay Triangulations





# Euclidean Delaunay Triangulations

Robustness

Arithmetic issues

Exact Geometric Computation paradigm [Yap]

exact predicates  $\neq$  exact arithmetics

**Filtering**

easy cases are more frequent

$\implies$  cost  $\simeq$  cost of approximate (double) computation

# Euclidean Delaunay Triangulations

Robustness

Arithmetic issues

Approximate evaluation  $P^a(x)$   
+ Error  $\epsilon$

Filtering

$$P^a(x) > \epsilon$$

?

Yes

No

$$\text{Sign}(P(x)) = \text{Sign}(P^a(x))$$

Exact computation

# Euclidean Delaunay Triangulations

Robustness

Degenerate cases

symbolic perturbation

only perturbs the `in_sphere` predicate

[Devillers, T. SODA'03, CGTA'11]

# Euclidean Delaunay Triangulations

Tricks to improve efficiency

EuroCG'12 invited talk by Olivier Devillers

<https://hal.inria.fr/hal-00850561>

# Euclidean Delaunay Triangulations

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Delaunay_triangulation_3>

typedef CGAL::Exact_predicates_inexact_constructions_kernel K;
typedef CGAL::Delaunay_triangulation_3<K> Delaunay;
typedef Delaunay::Point Point;

int main()
{
    Delaunay T;

    T.insert(Point(0,0,0));
    T.insert(Point(1,0,0));
    T.insert(Point(0,1,0));
    T.insert(Point(0,0,1));
    T.insert(Point(2,2,2));
    T.insert(Point(-1,0,1));

    return 0;
}
```

predefined Geom\_traits

default TDS

# Euclidean Delaunay Triangulations

Flexibility

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Projection_traits_xy_3.h>
#include <CGAL/Delaunay_triangulation_2.h>
#include <fstream>
```

```
typedef CGAL::Exact_predicates_inexact_constructions_kernel K;
typedef CGAL::Projection_traits_xy_3<K> Gt;
typedef CGAL::Delaunay_triangulation_2<Gt> Terrain;
typedef K::Point_3 Point;
```

```
int main()
{
    Terrain T;

    T.insert(Point(0,0,0));
    T.insert(Point(1,0,0));
    T.insert(Point(0,1,0));
    // etc

    return 0;
}
```

Points : 3D

Predicates :

on their 2D projections

# Euclidean Delaunay Triangulations

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Delaunay_triangulation_3.h>
#include <CGAL/Delaunay_triangulation_cell_base_3.h>
#include <CGAL/Triangulation_vertex_base_with_info_3.h>
#include <CGAL/IO/Color.h>

typedef CGAL::Exact_predicates_inexact_constructions_kernel K;
typedef CGAL::Triangulation_vertex_base_with_info_3<CGAL::Color, K> Vb;
typedef CGAL::Delaunay_triangulation_cell_base_3<K> Cb;
typedef CGAL::Triangulation_data_structure_3<Vb, Cb> Tds;
typedef CGAL::Delaunay_triangulation_3<K, Tds> Delaunay;
typedef Delaunay::Point Point;

int main()
{
    Delaunay T;
    T.insert(Point(0,0,0));
    // etc
    Delaunay::Finite_vertices_iterator vit;
    for (Delaunay::Vertex_handle v : T.finite_vertex_handles())
        if (T.degree(v) == 6)
            v->info() = CGAL::red();
    return 0;
}
```

Flexibility



# Euclidean Delaunay Triangulations

fully dynamic

fully robust

also weighted

**2D** (insertion with flips?)

$\simeq 10$  M points / second

[Yvinec CGAL'97]

also constrained

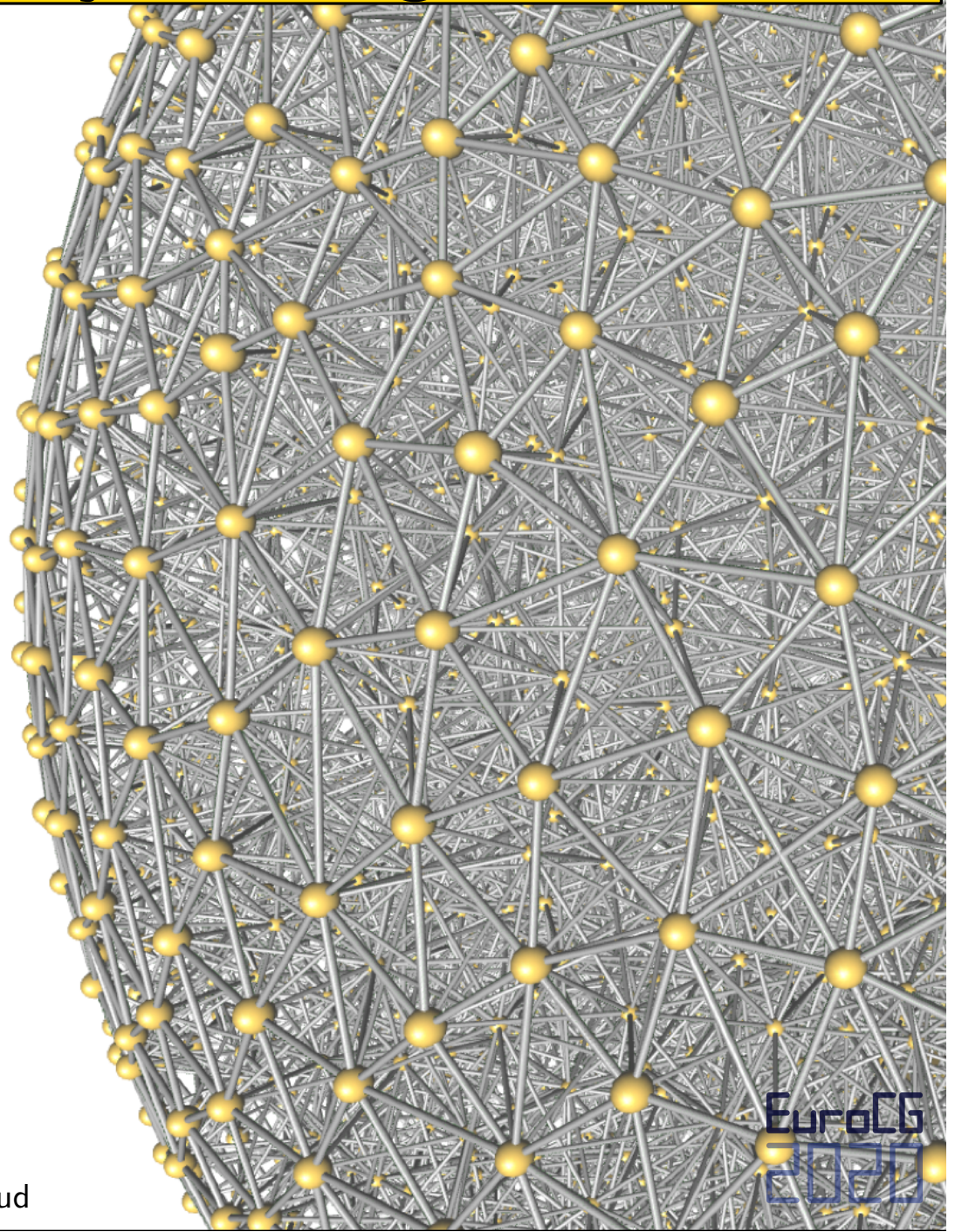
**3D**

$\simeq 1$  M points / second

[T. CGAL'00] [Pion, T. CGAL'01\*]

multicore version

[Jamin CGAL'14]





# Hyperbolic triangulations

# Hyperbolic triangulations

[Boissonnat 1988]

Once upon a time...

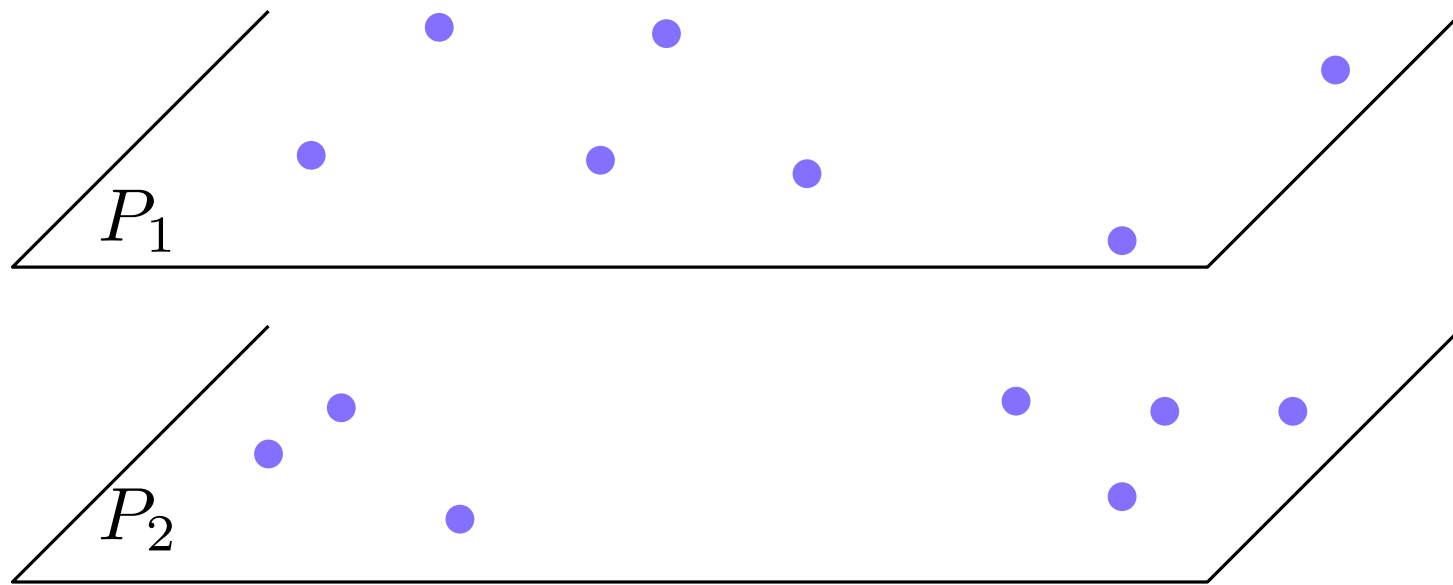
Compute the 3D Delaunay triangulation

of a set of points

lying in two parallel planes

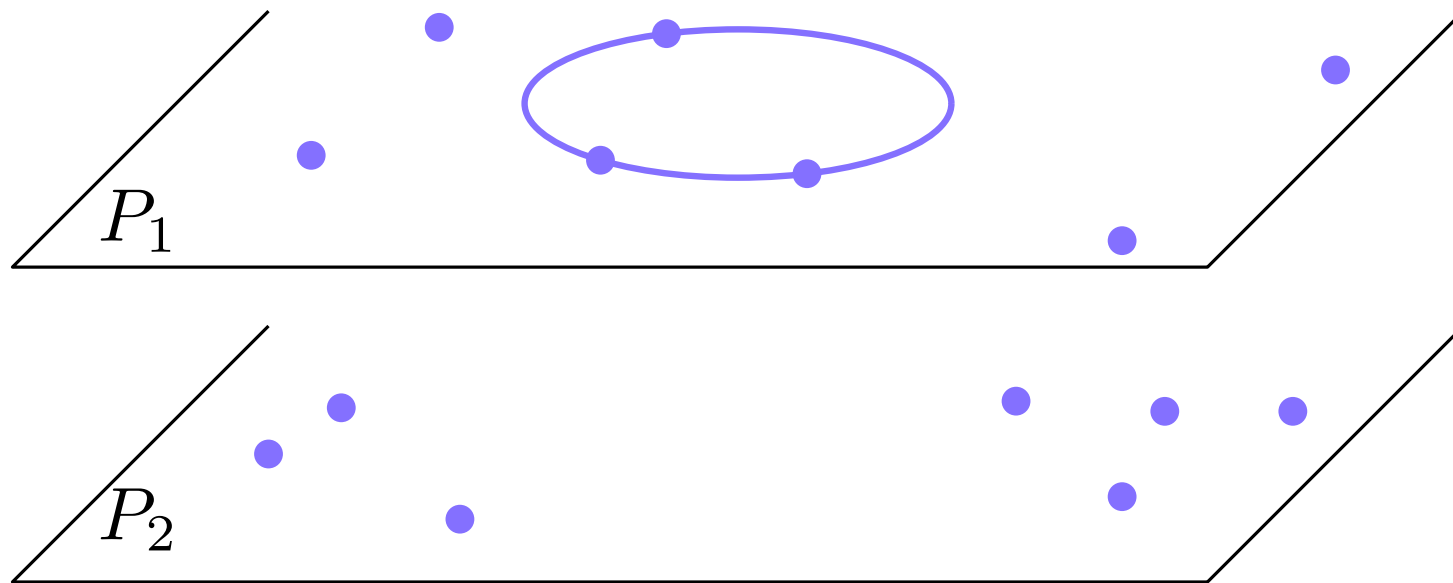
# Hyperbolic triangulations

[Boissonnat 1988]



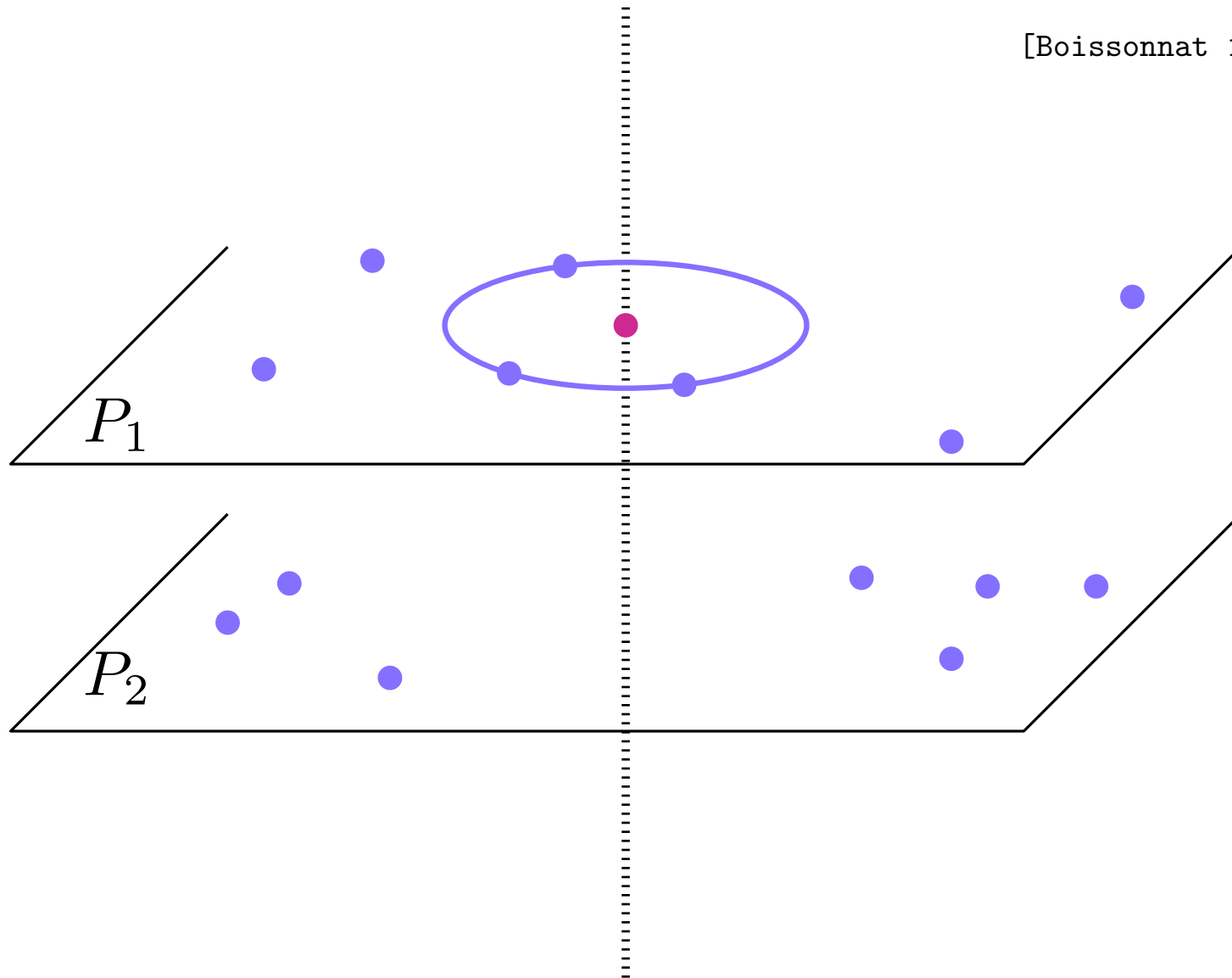
# Hyperbolic triangulations

[Boissonnat 1988]

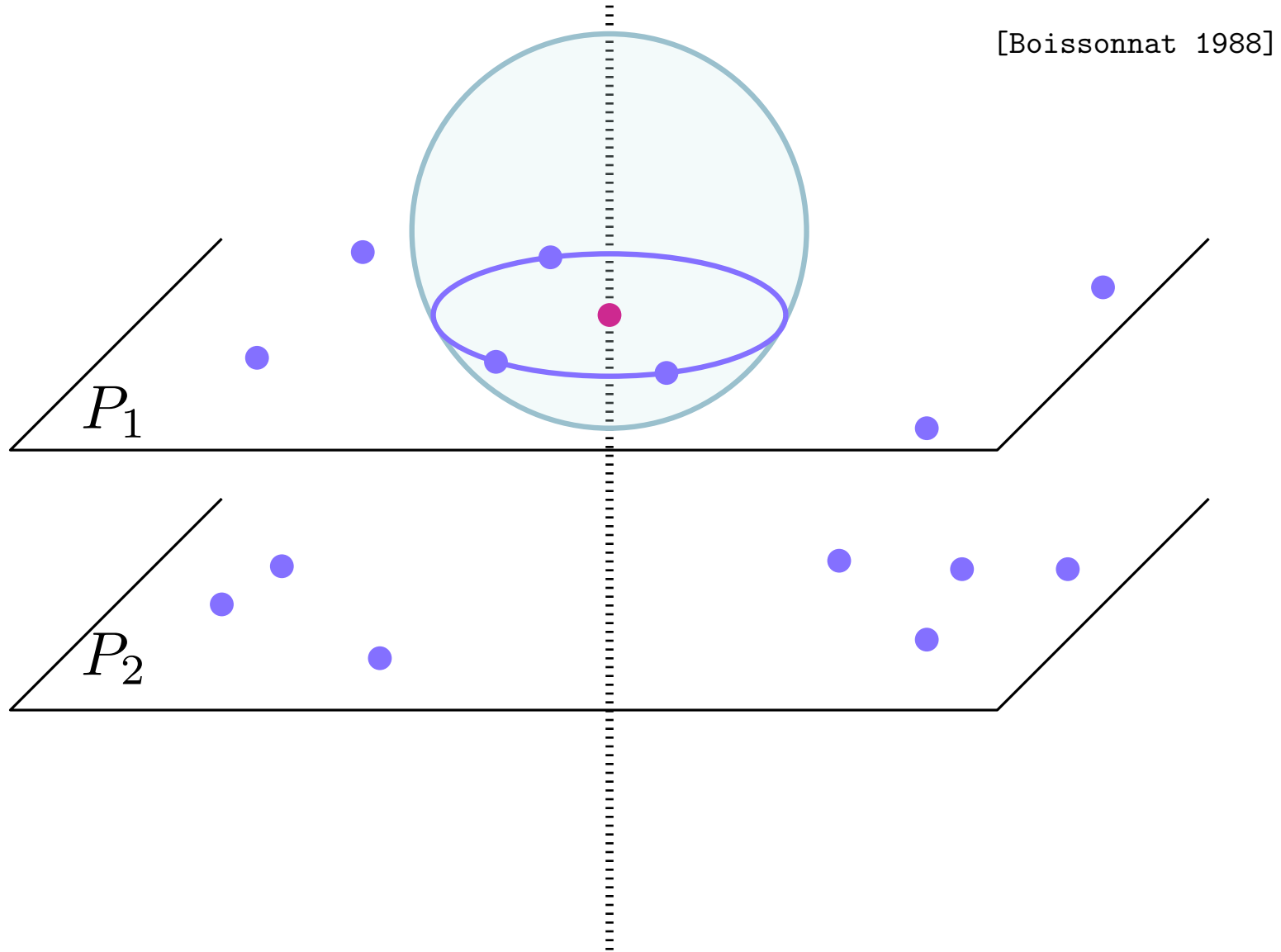


# Hyperbolic triangulations

[Boissonnat 1988]

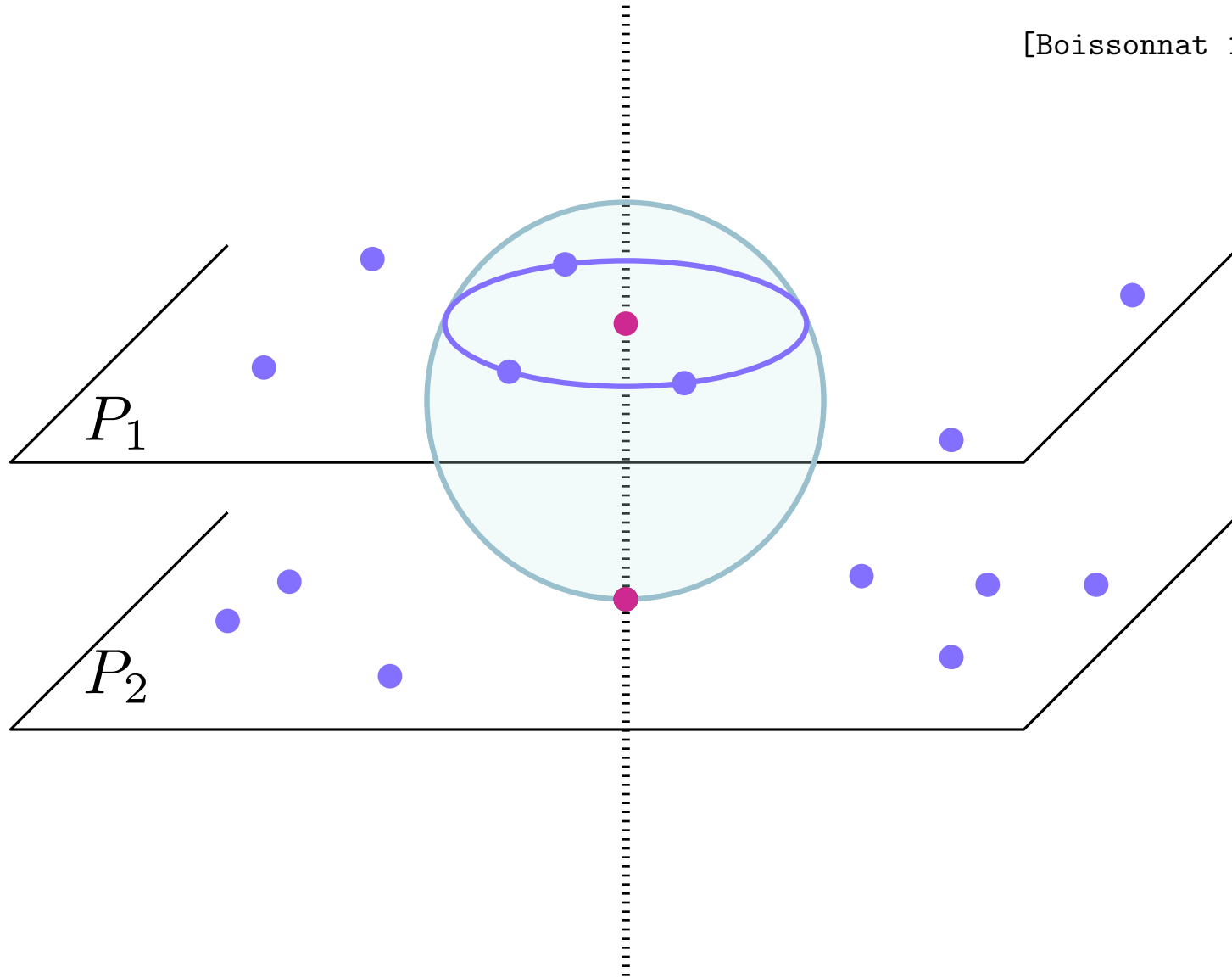


# Hyperbolic triangulations



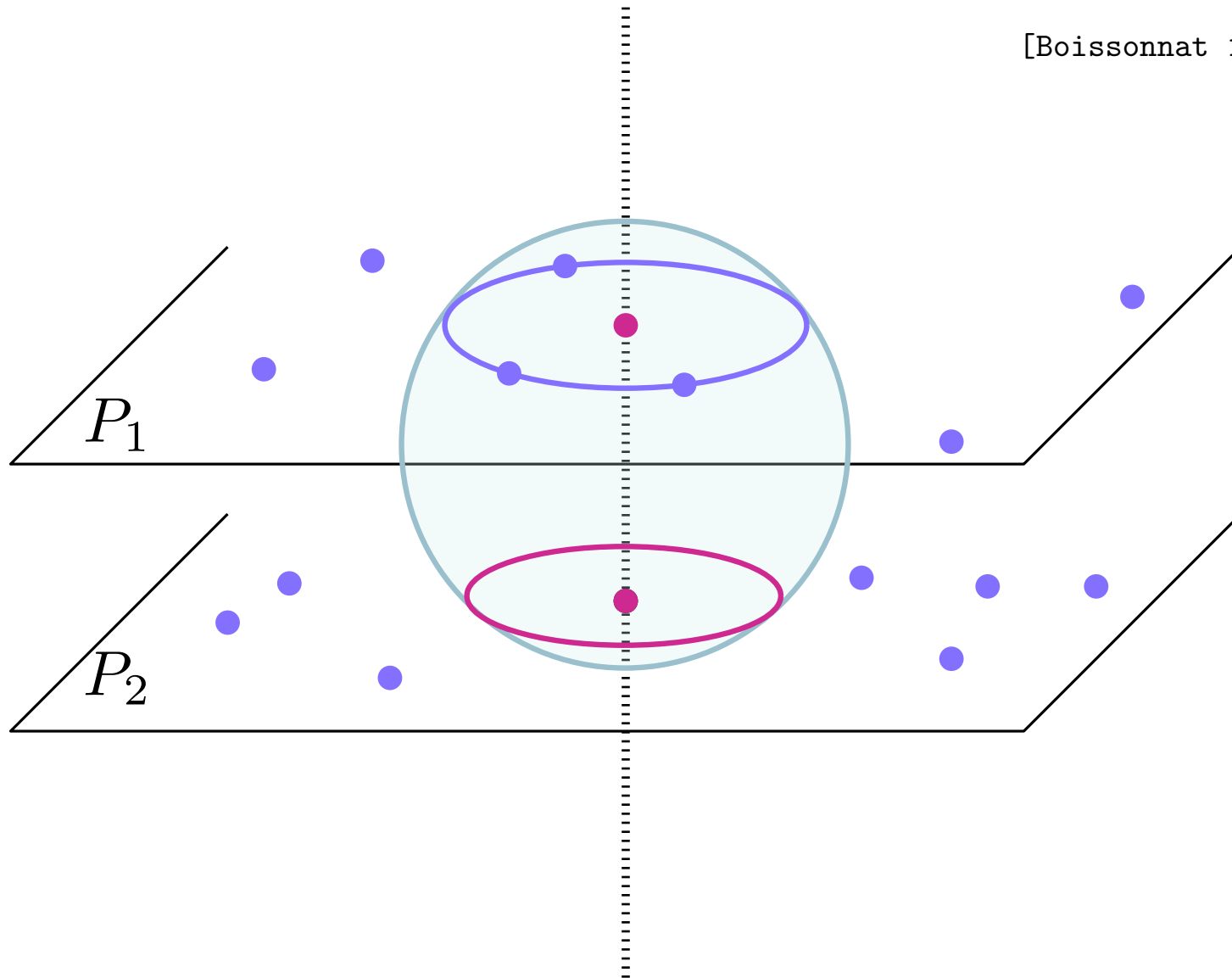
# Hyperbolic triangulations

[Boissonnat 1988]



# Hyperbolic triangulations

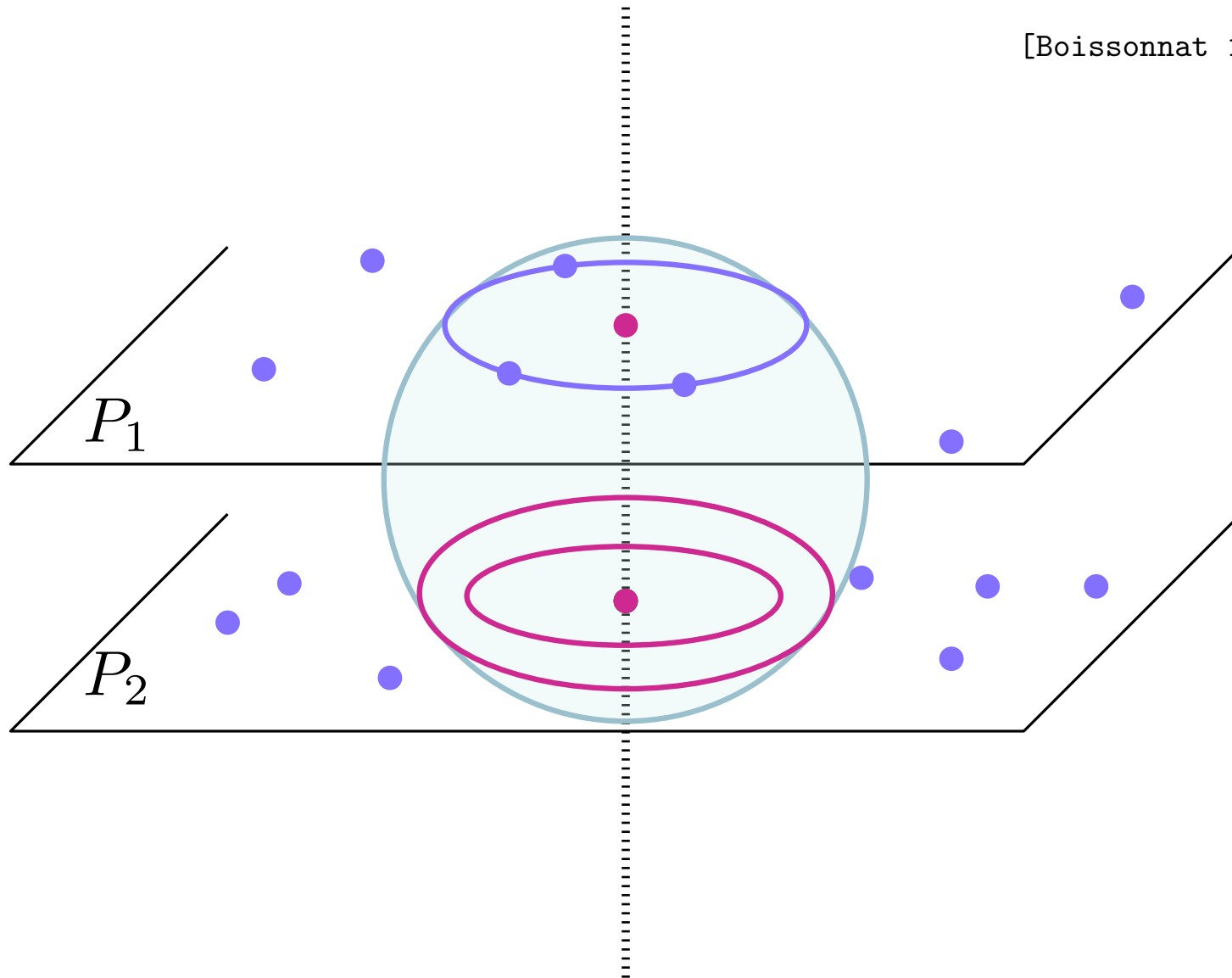
[Boissonnat 1988]





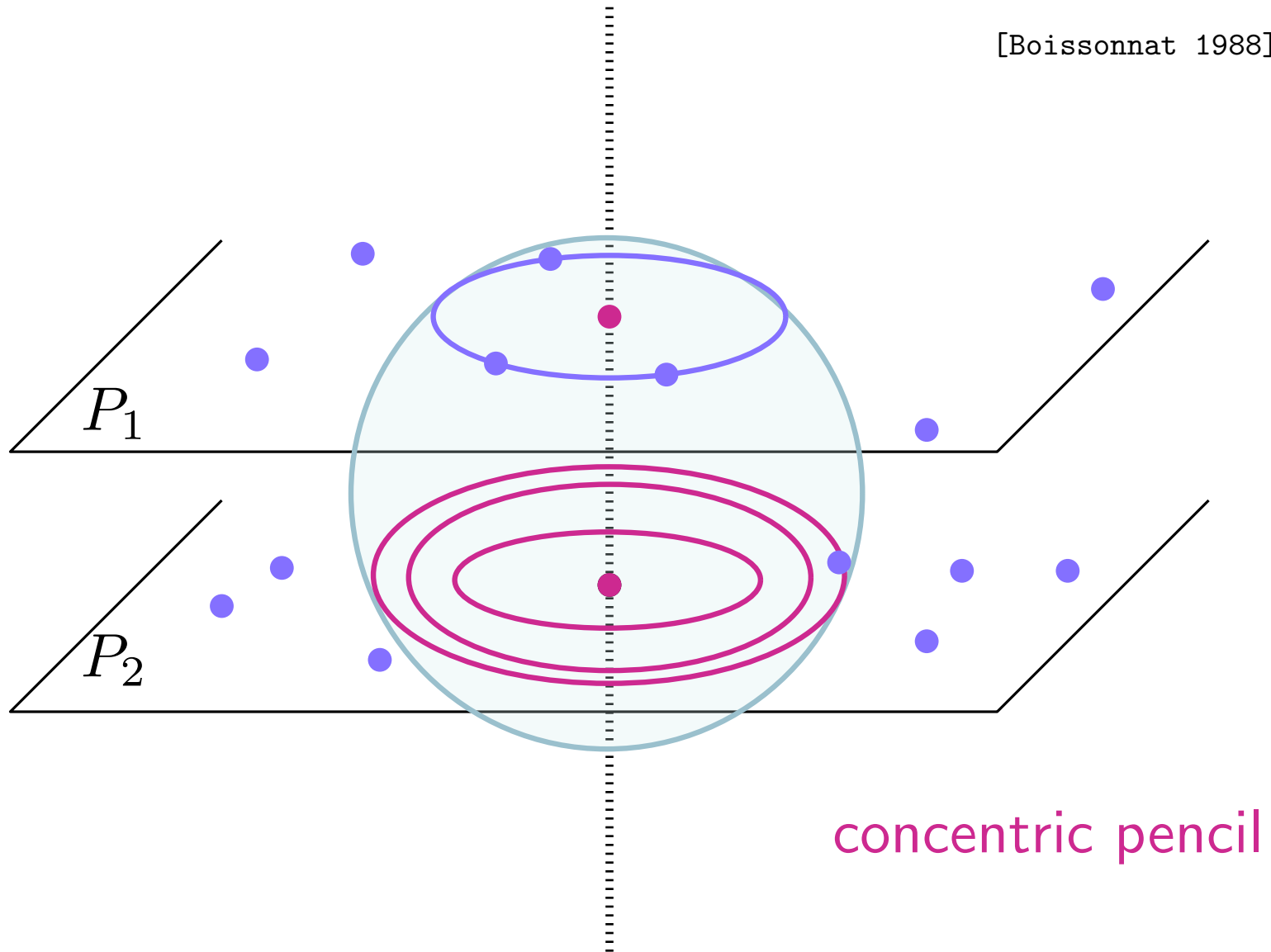
# Hyperbolic triangulations

[Boissonnat 1988]



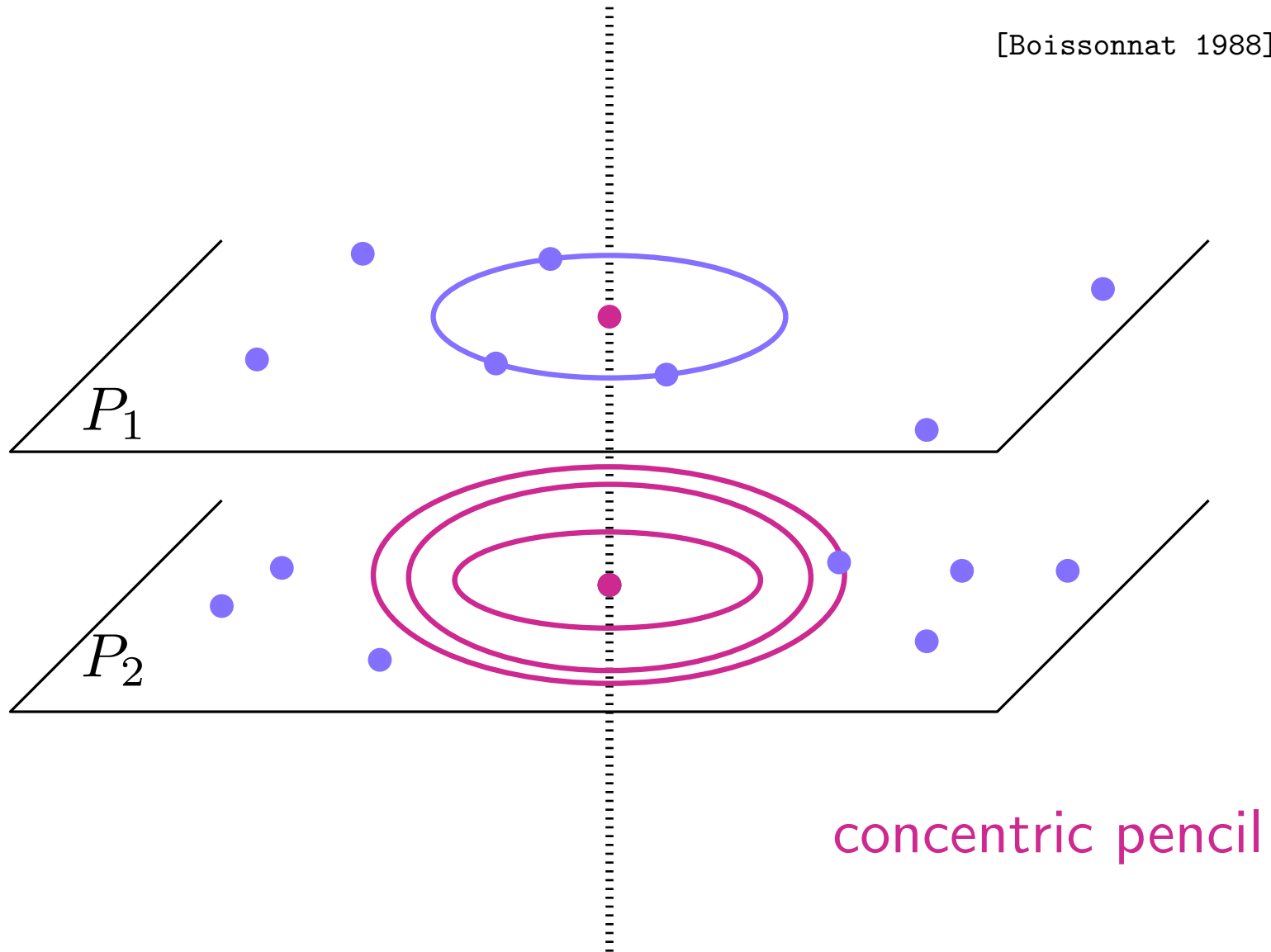
# Hyperbolic triangulations

[Boissonnat 1988]



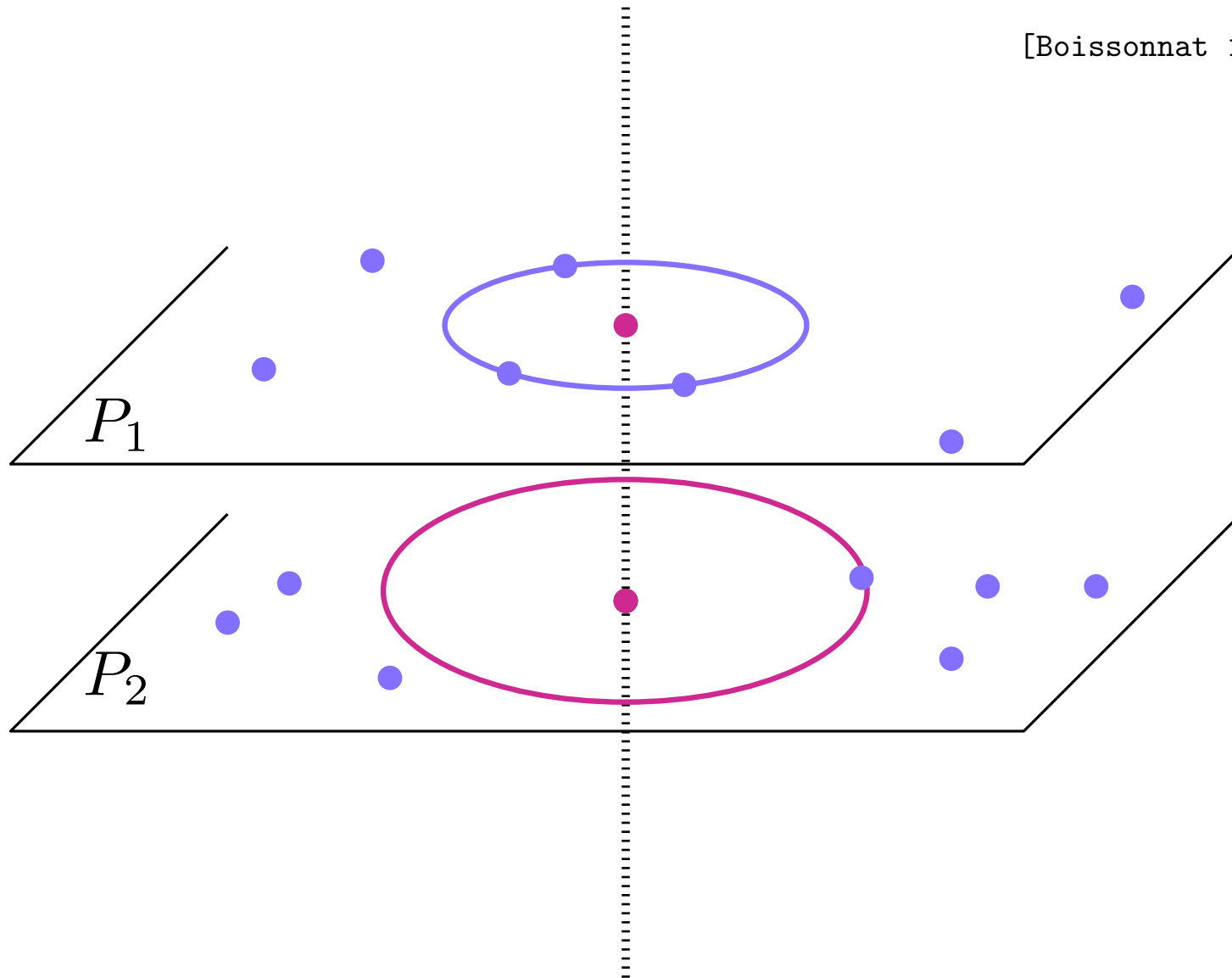
# Hyperbolic triangulations

[Boissonnat 1988]



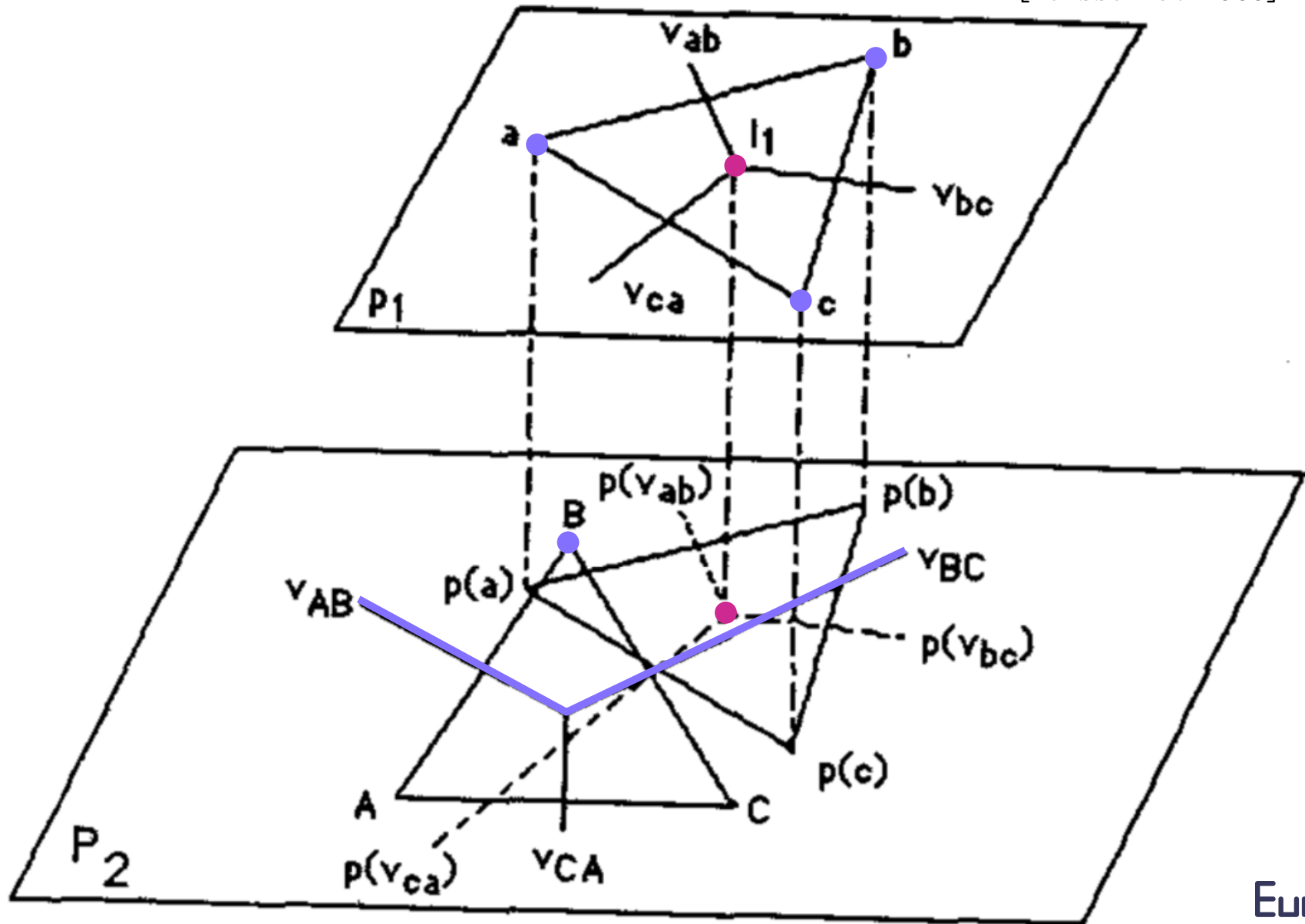
# Hyperbolic triangulations

[Boissonnat 1988]



# Hyperbolic triangulations

[Boissonnat 1988]



# Hyperbolic triangulations

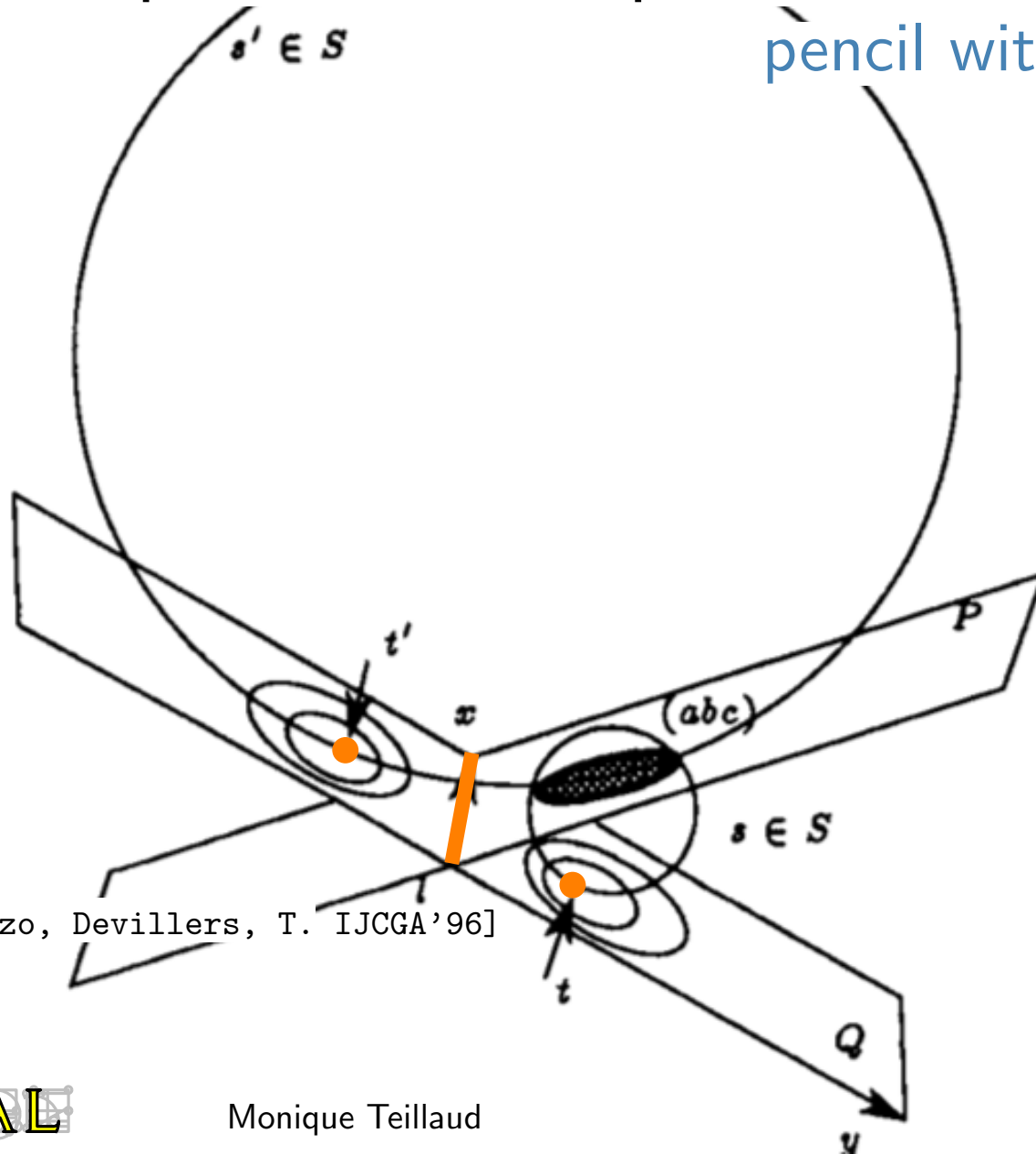
What if the planes are **not** parallel?

[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

# Hyperbolic triangulations

What if the planes are **not** parallel?

pencil with limit points

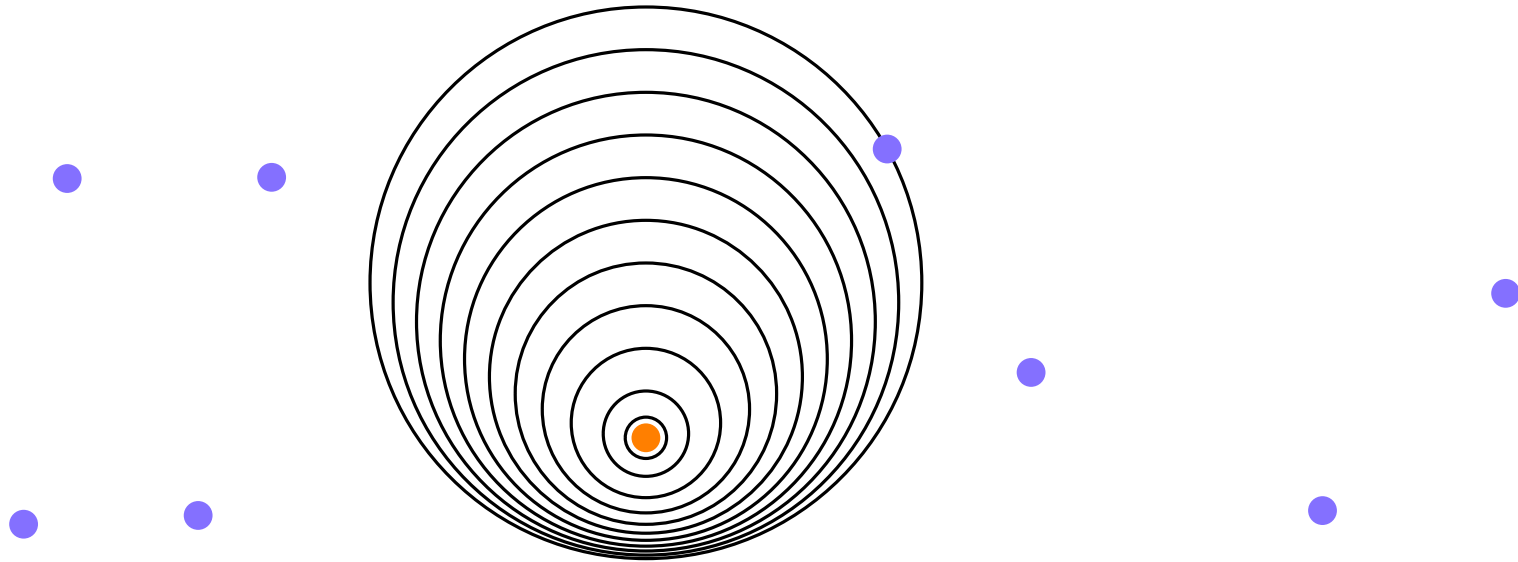


[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

# Hyperbolic triangulations

What if the planes are **not** parallel?

- pencil with limit points



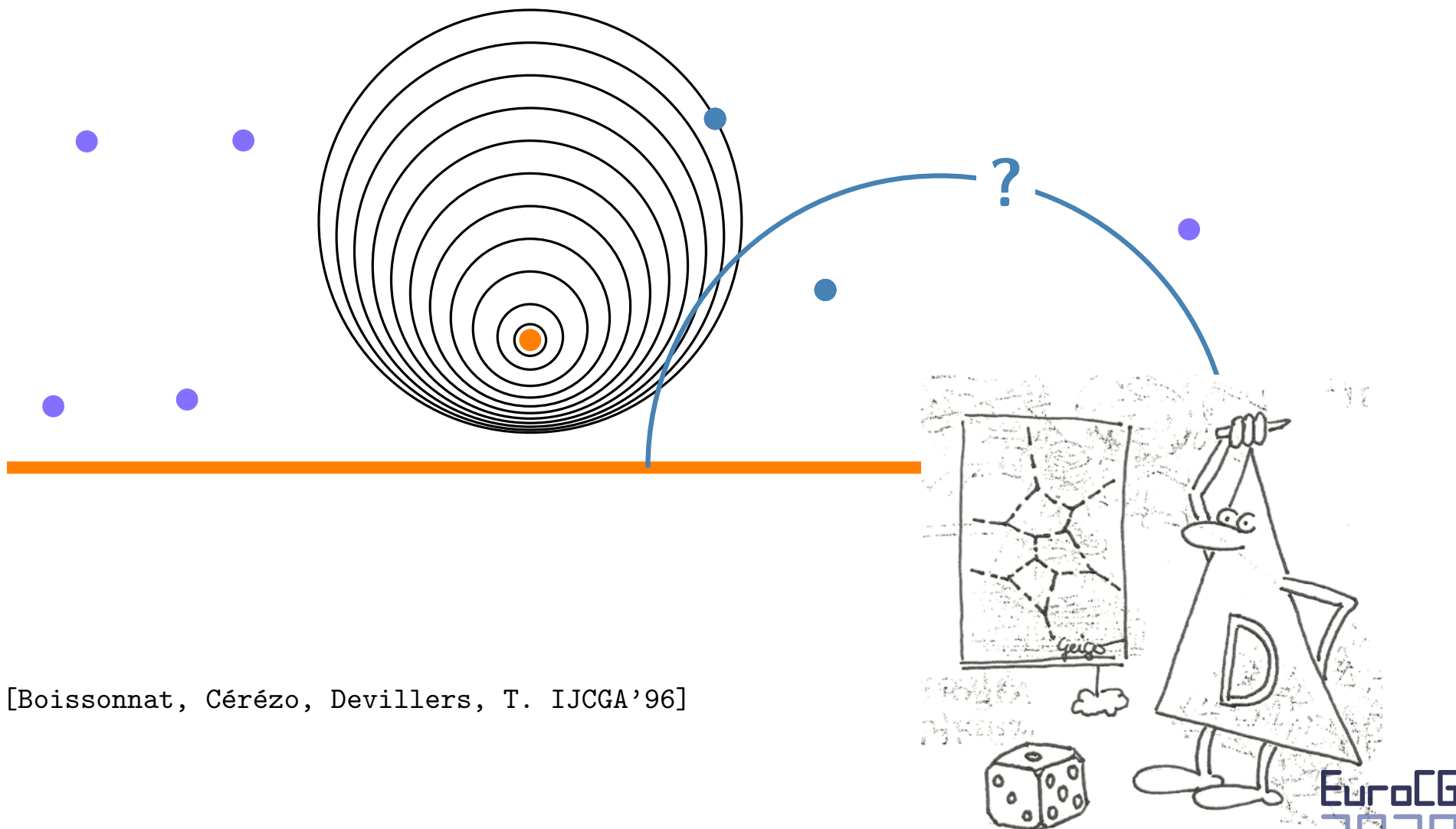
[Boissonnat, Cérézo, Devillers, T. IJCGA'96]



# Hyperbolic triangulations

What if the planes are **not** parallel?

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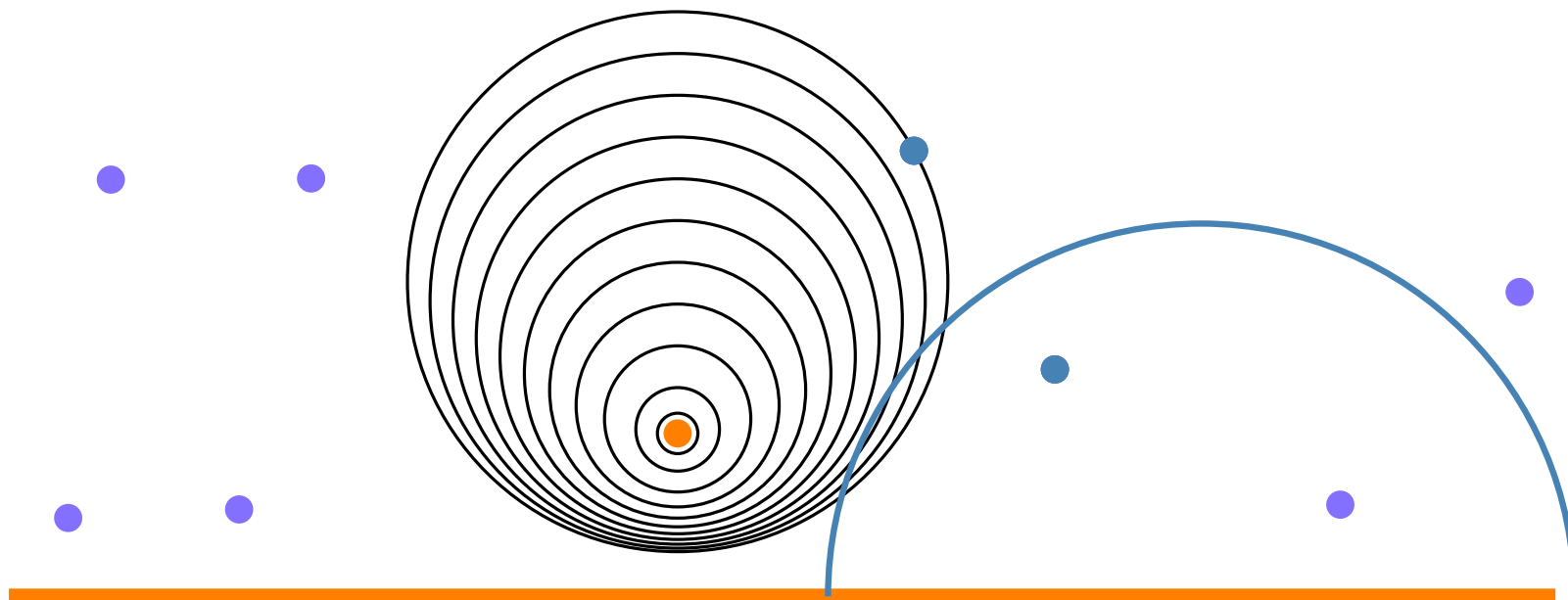


[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

# Hyperbolic triangulations

What if the planes are **not** parallel?

- pencil with limit points



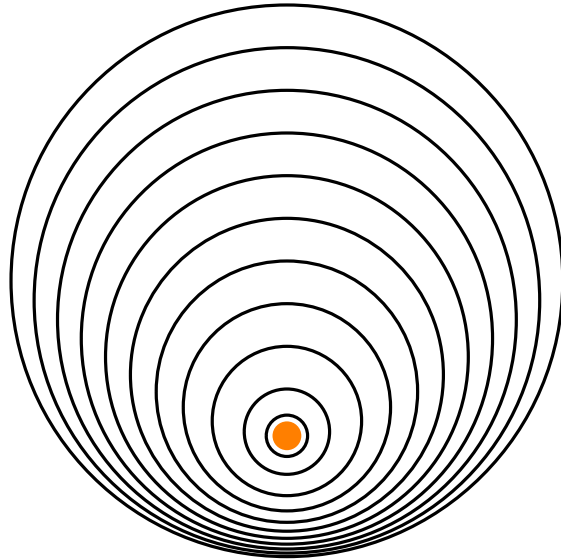
hyperbolic line  
in the Poincaré half-plane

[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

# Hyperbolic triangulations

What if the planes are **not** parallel?

pencil with limit points



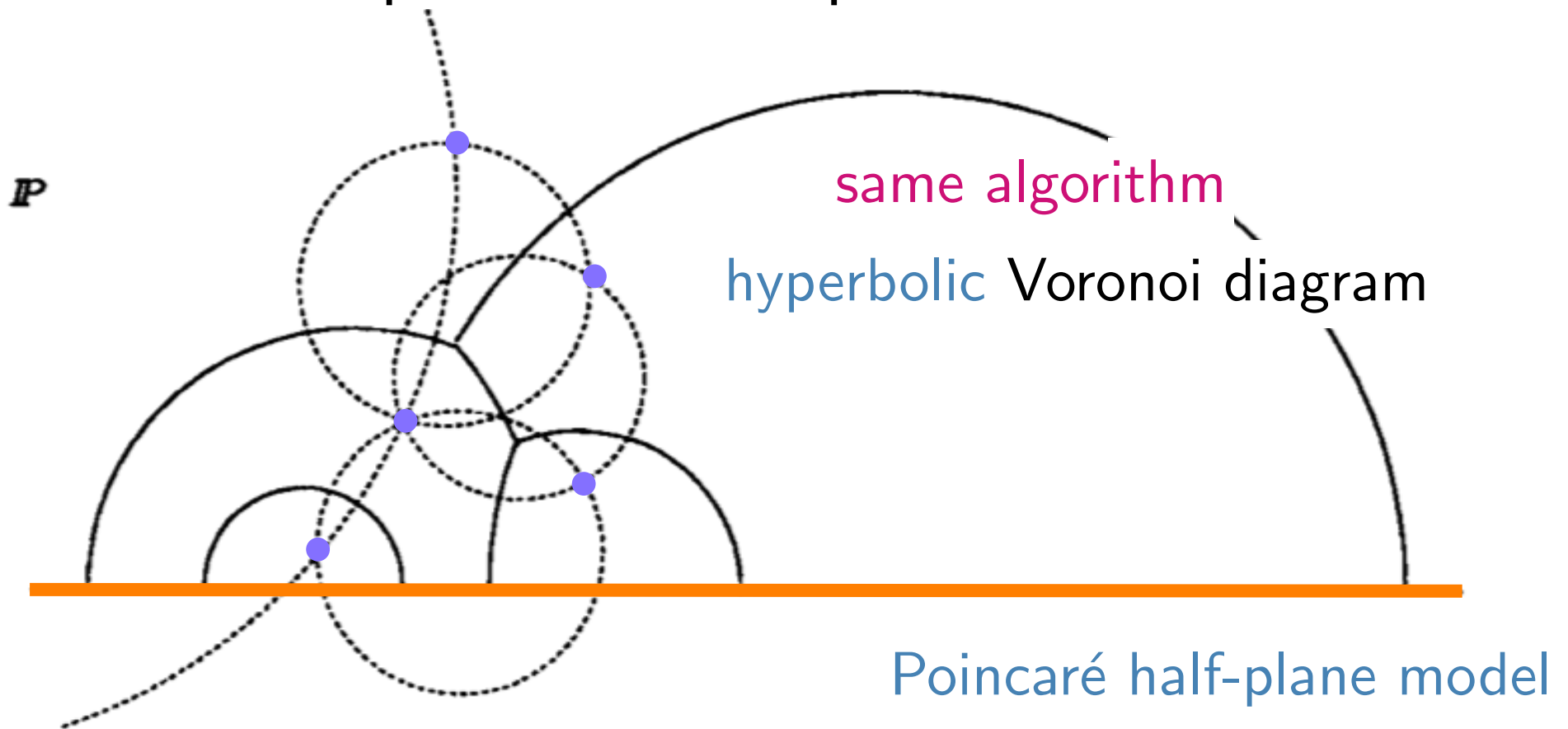
---

concentric pencil  
in the Poincaré half-plane

[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

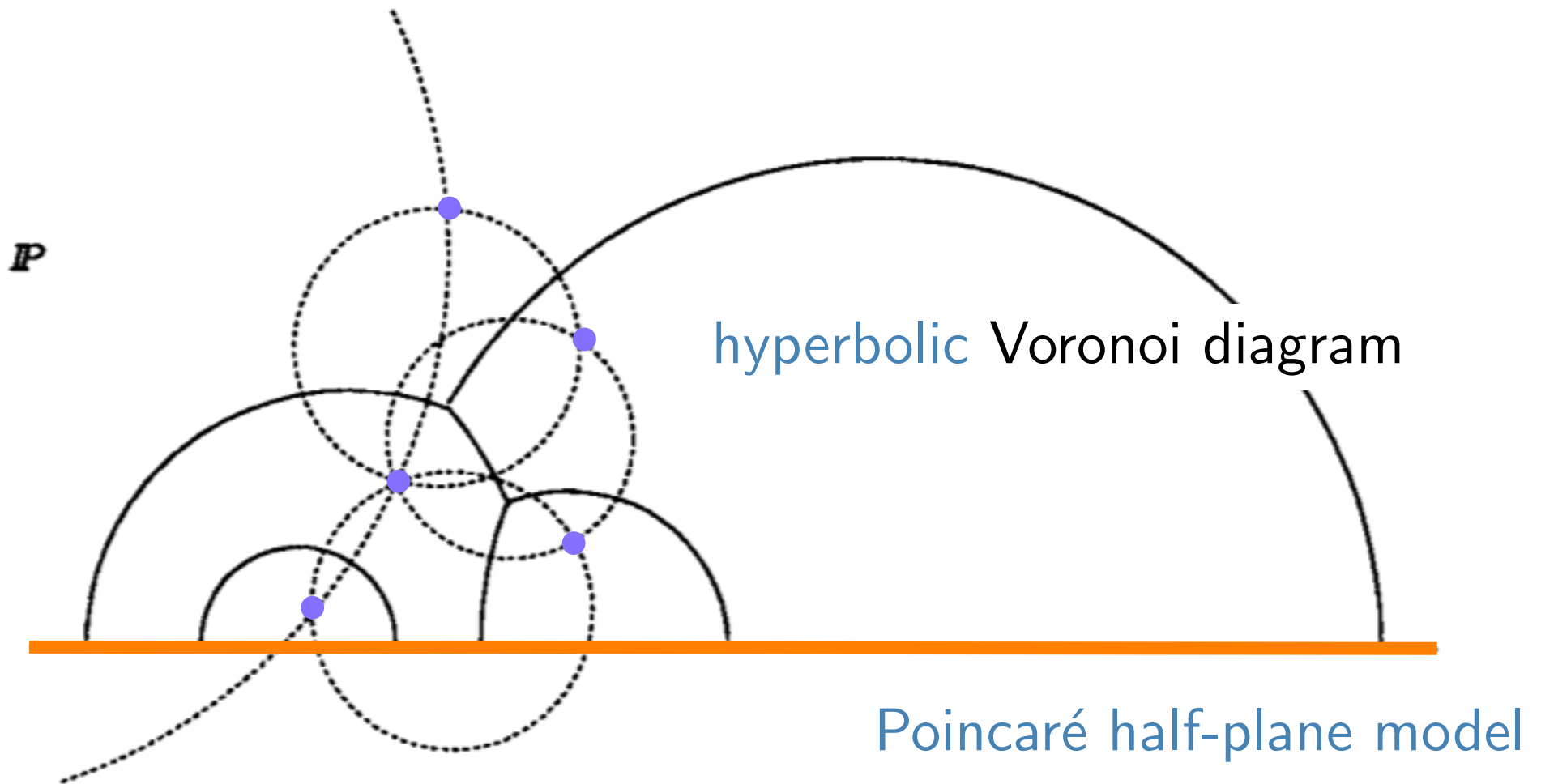
# Hyperbolic triangulations

What if the planes are **not** parallel?



[Boissonnat, Cérézo, Devillers, T. IJCGA'96]

# Hyperbolic triangulations

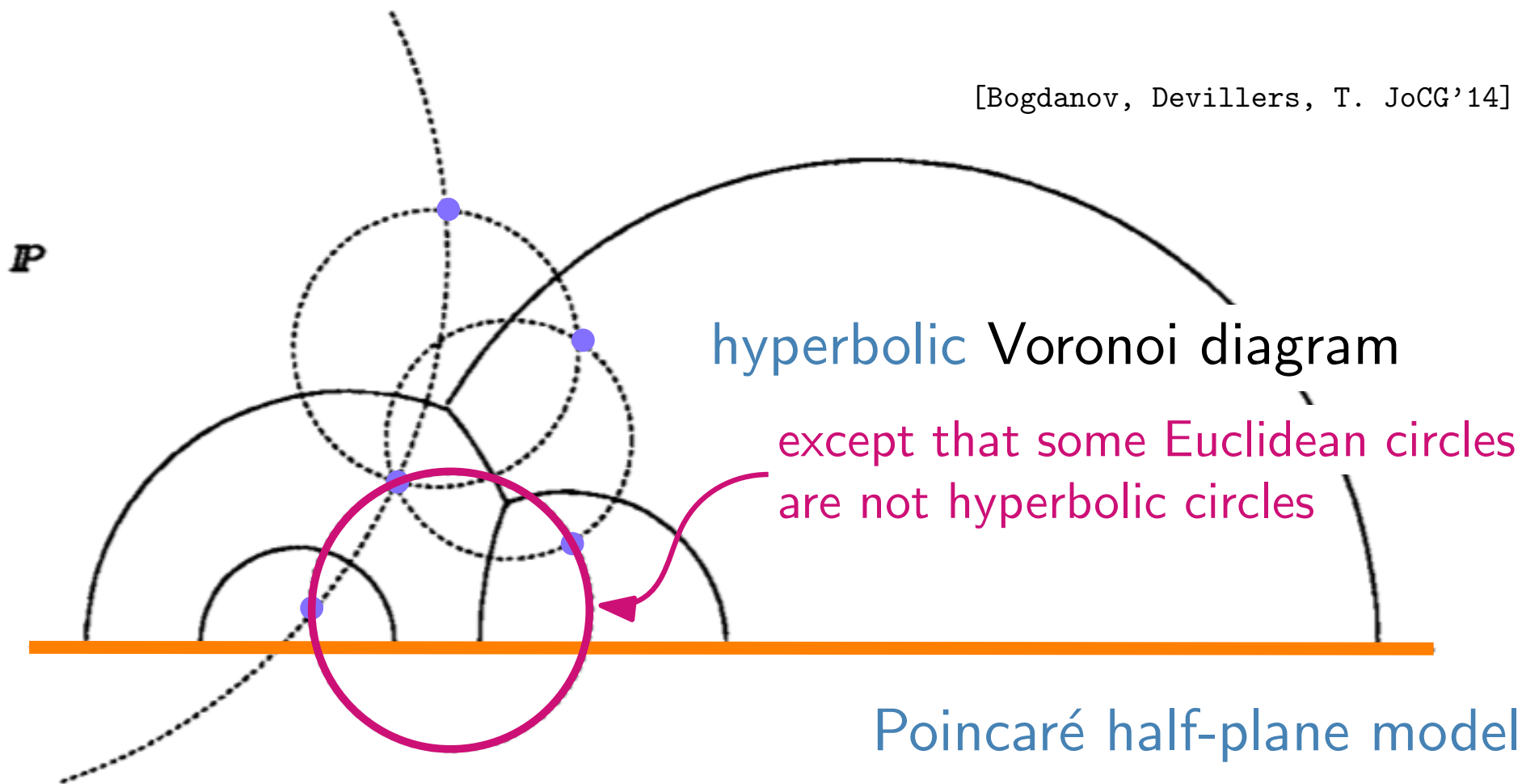


hyperbolic circles are Euclidean circles

↳ same combinatorics as Euclidean Voronoi diagram

# Hyperbolic triangulations

[Bogdanov, Devillers, T. JoCG'14]



hyperbolic circles are Euclidean circles

↳ same combinatorics as Euclidean Voronoi diagram

# Hyperbolic triangulations

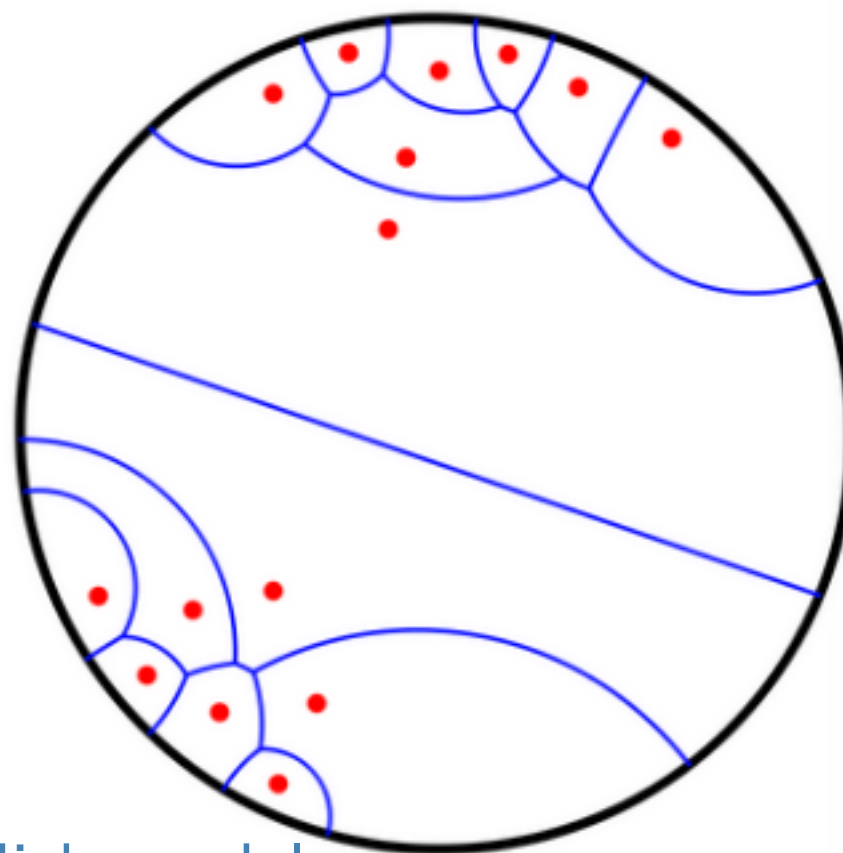
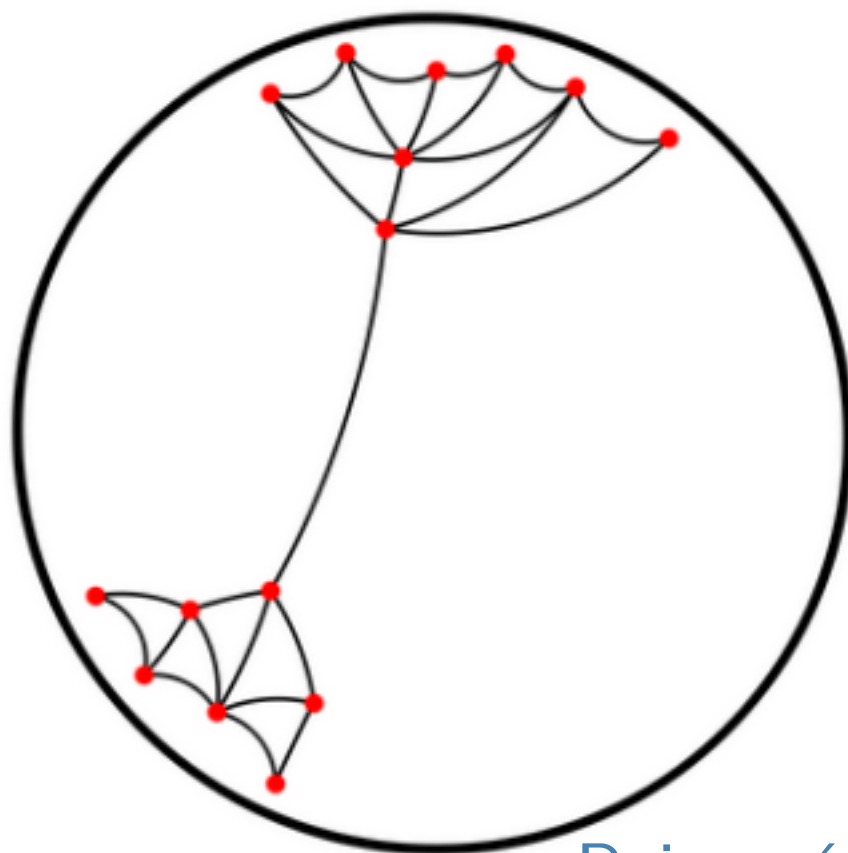
Hyperbolic\_Delaunay\_triangulation\_2

inherits from

Delaunay\_triangulation\_2

[Bogdanov, Devillers, T. JoCG'14]

[Bogdanov, Iordanov, T. CGAL'19]



Poincaré disk model

# Hyperbolic triangulations

```
#include <CGAL/Hyperbolic_Delaunay_triangulation_2.h>
#include <CGAL/Hyperbolic_Delaunay_triangulation_traits_2.h>
#include <vector>

typedef CGAL::Hyperbolic_Delaunay_triangulation_traits_2<> Gt;
typedef Gt::Point_2 Point_2;
typedef CGAL::Hyperbolic_Delaunay_triangulation_2<Gt> Dt;

int main(int argc, char** argv)
{
    std::vector<Point_2> pts;    \\ pts filled in some way

    Dt dt_during;    // hyperbolic filtering at each step:
    std::vector<Point_2>::iterator ip;
    for(ip = pts.begin(); ip != pts.end(); ++ip) dt_during.insert(*ip);

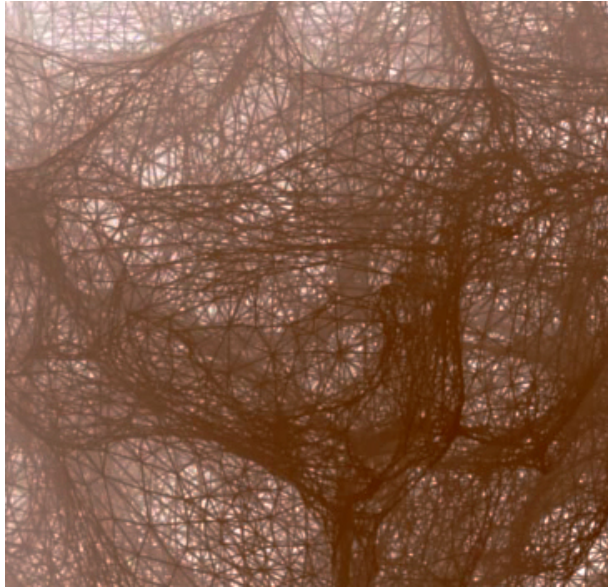
    Dt dt_end;    // hyperbolic filtering only once at the end:
    dt_end.insert(pts.begin(),pts.end());

    return 0;
}
```



# Periodic Delaunay triangulations

# Periodic Delaunay triangulations



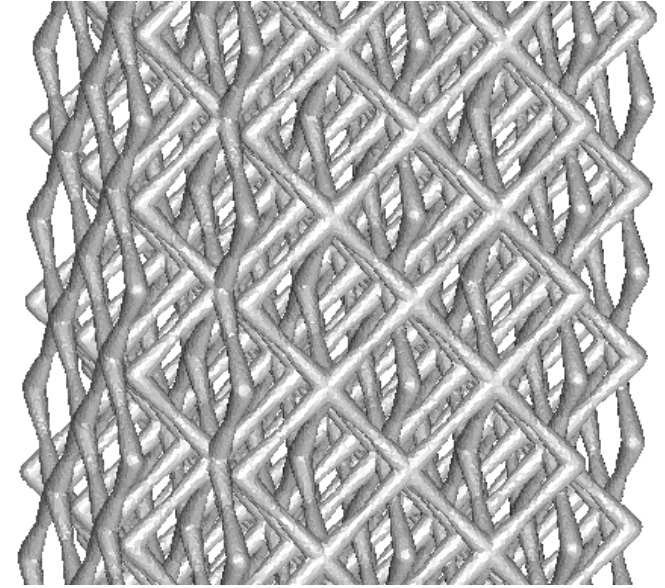
cosmic web

[v.d. Weijgaert, Groningen]

motivation:

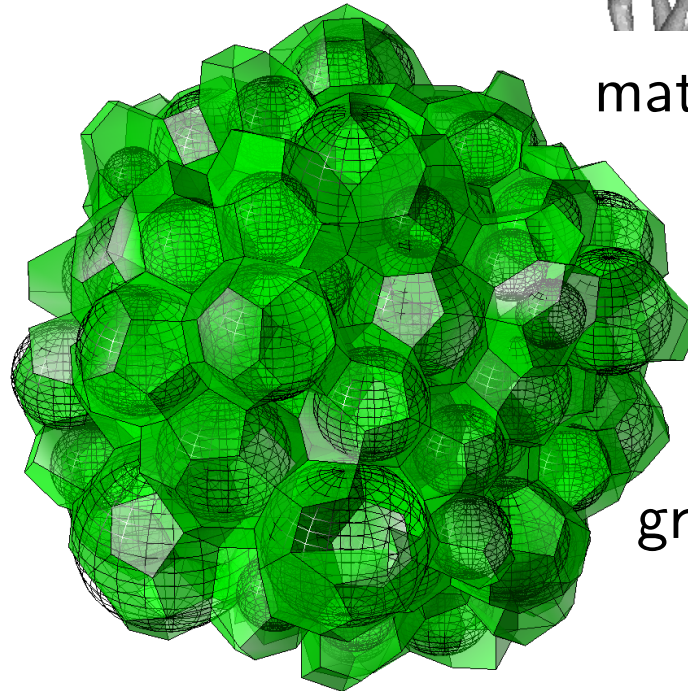


users



material for bone scaffold

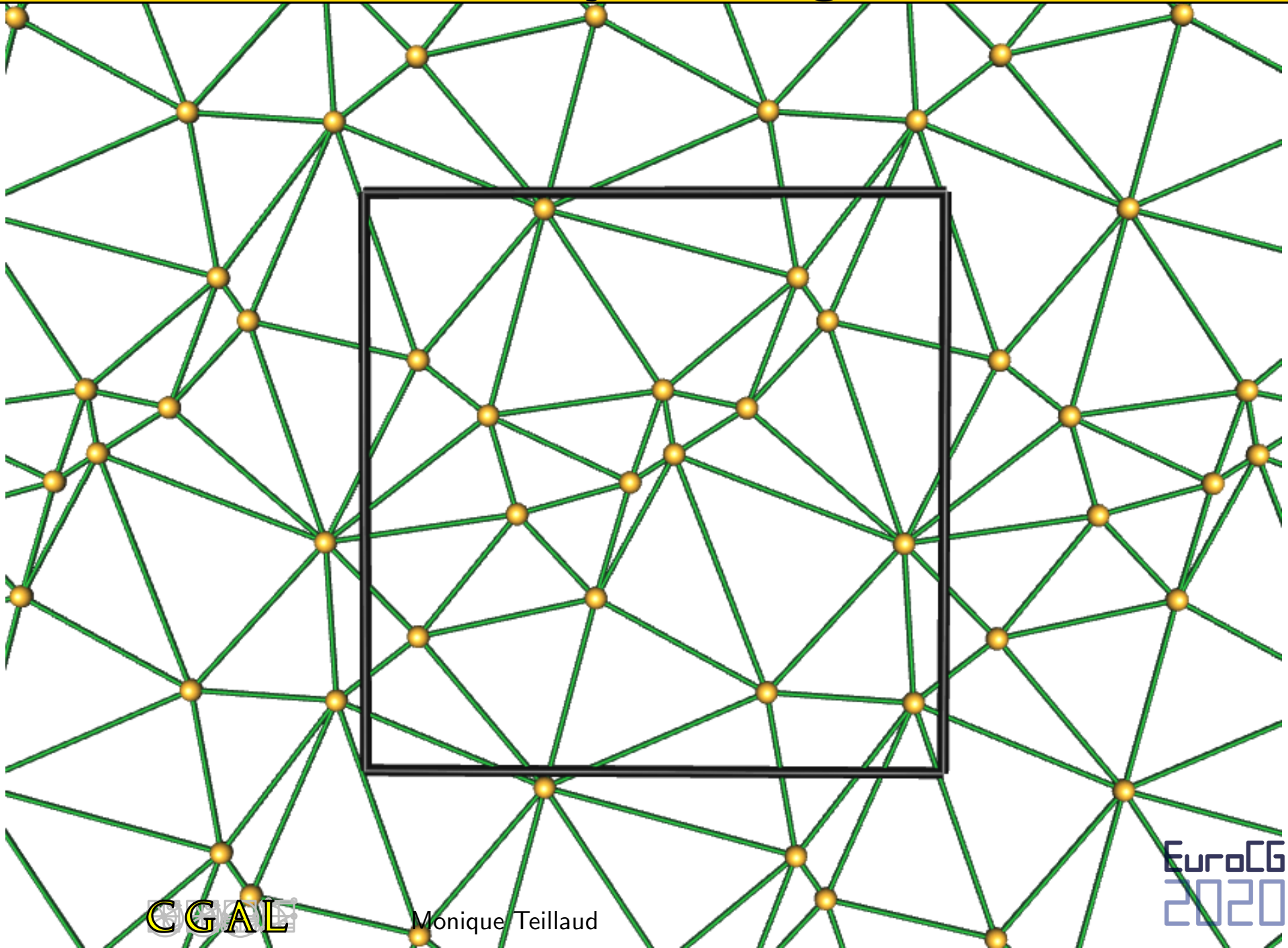
[Moesen<sup>+</sup>, Leuven]



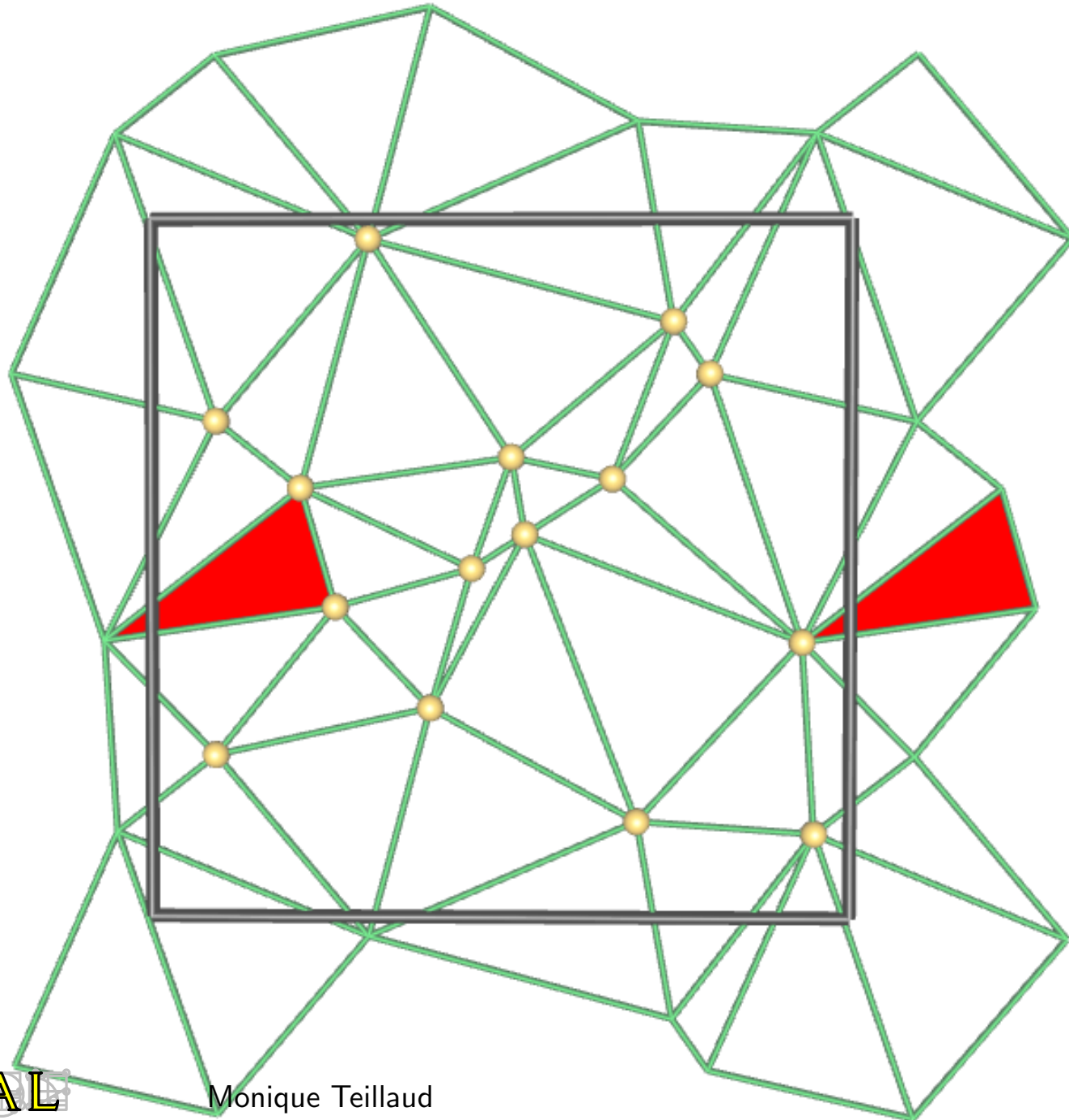
granular material

[Ludig<sup>+</sup>, Twente]

# Periodic Delaunay triangulations

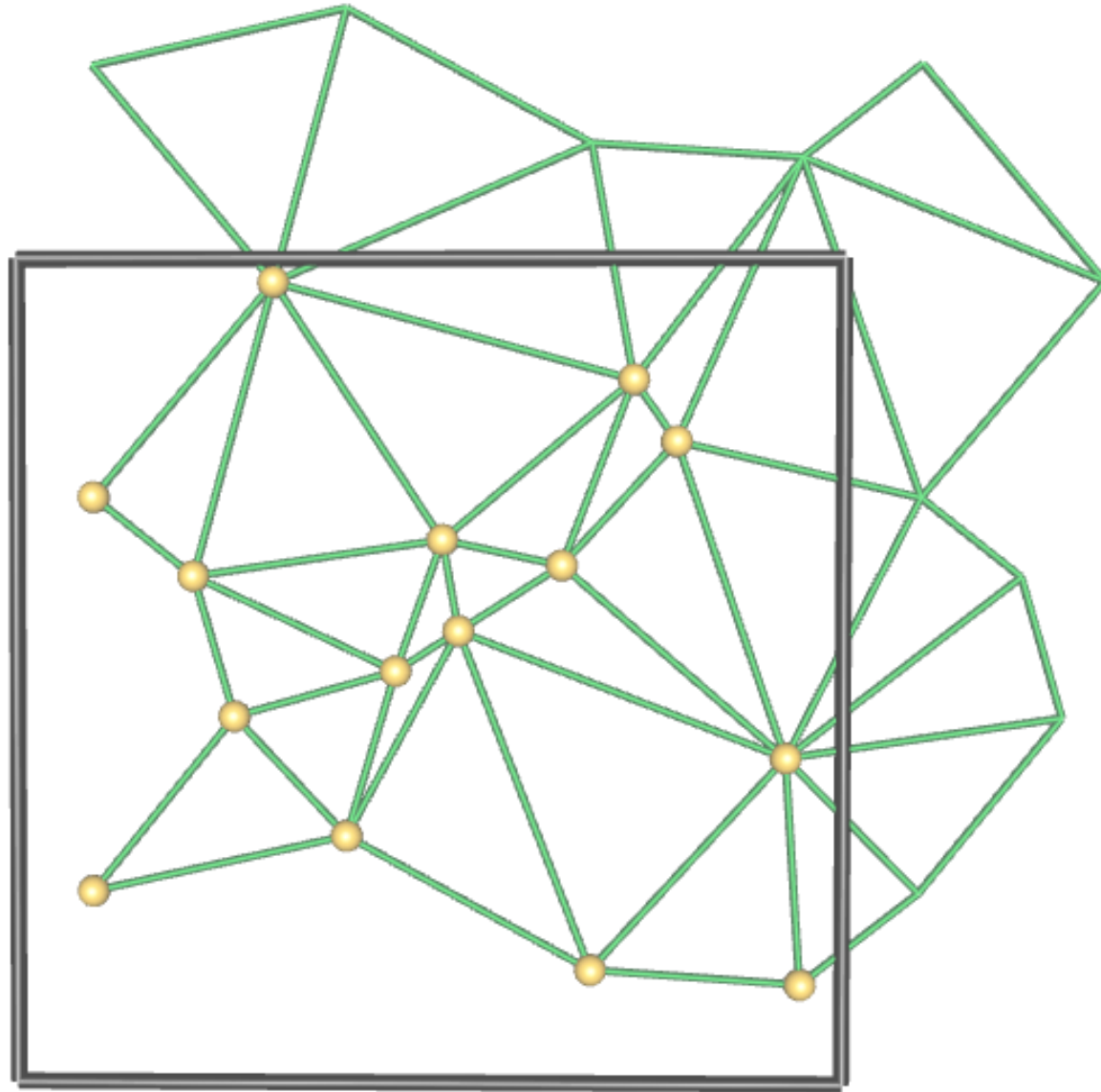


# Periodic Delaunay triangulations





# Periodic Delaunay triangulations



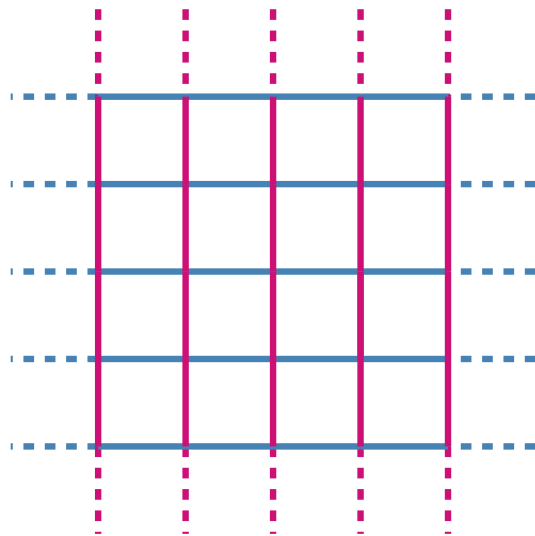
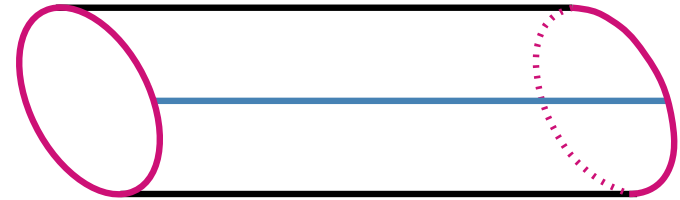
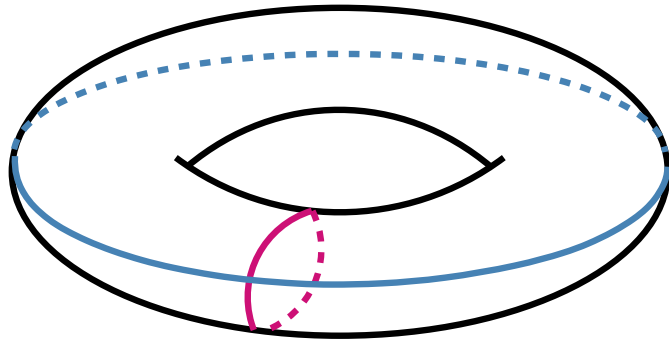
[Caroli, T. SoCG'08 video]

# Periodic Delaunay triangulations

flat torus

$$\mathbb{T}^2 \sim \mathbb{R}^2 / G$$

$$G = \langle t_x, t_y \rangle$$



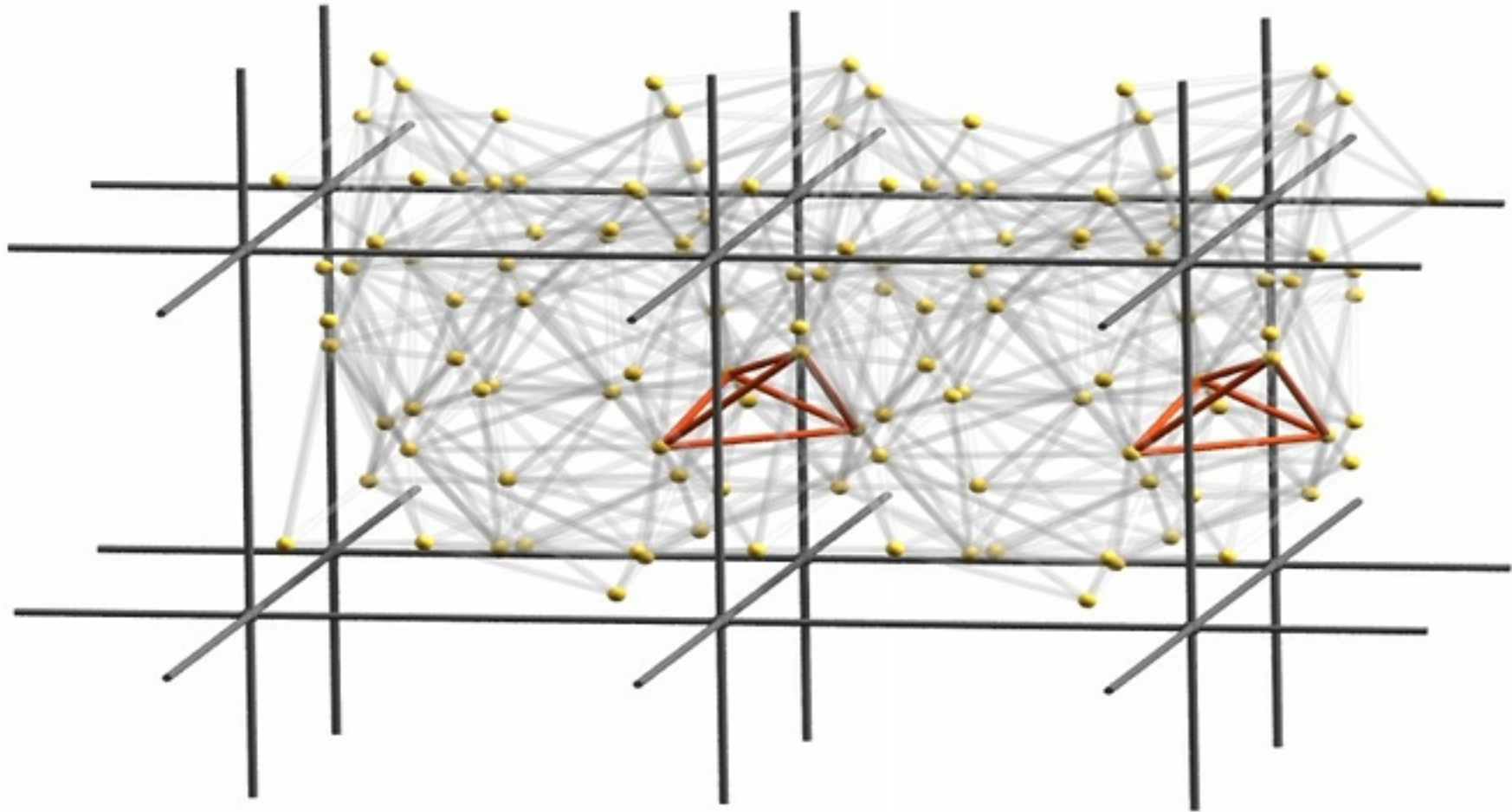
universal covering space

# Periodic Delaunay triangulations

flat torus

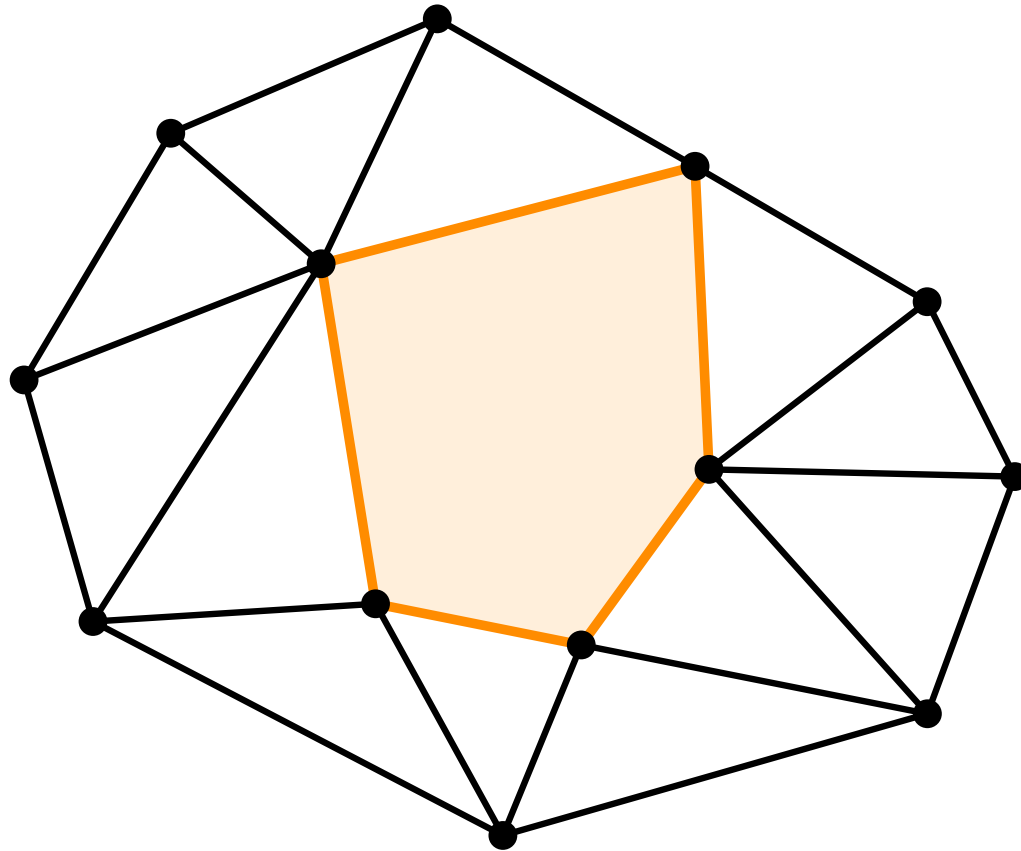
$$\mathbb{T}^3 \sim \mathbb{R}^3 / G$$

$$G = \langle t_x, t_y, t_z \rangle$$



# Periodic Delaunay triangulations

Bowyer's incremental algorithm

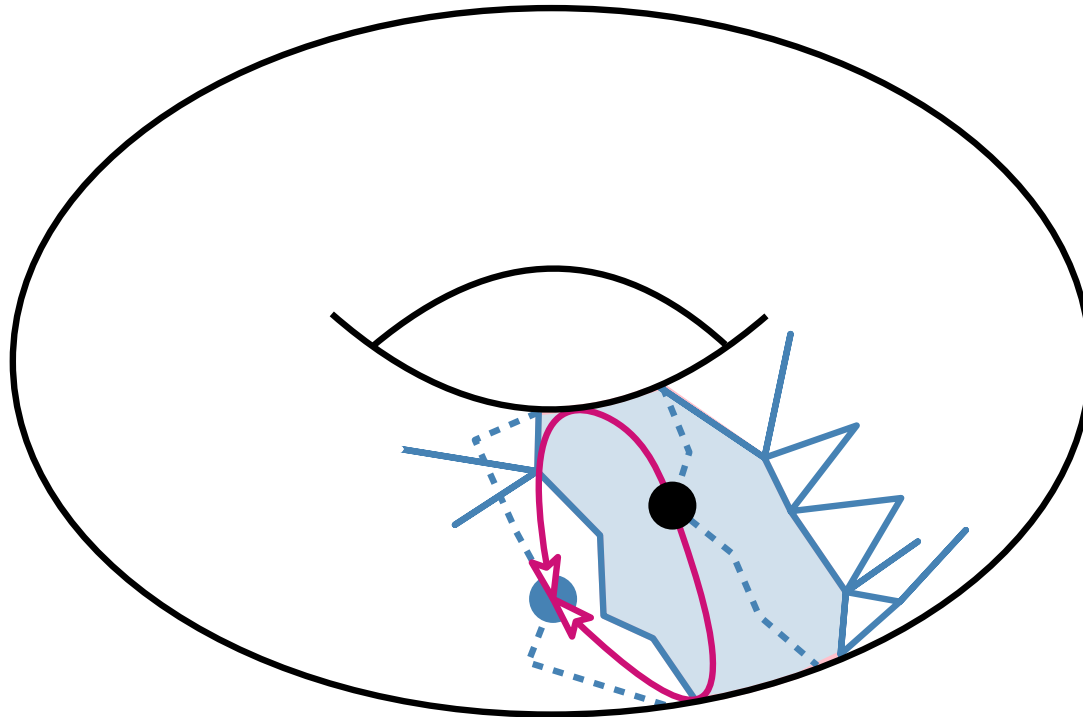


the conflict region forms a topological ball



# Periodic Delaunay triangulations

Bowyer's incremental algorithm



the conflict region **does not** form a topological ball

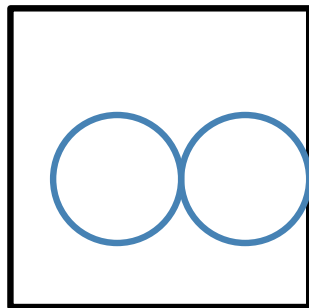
# Periodic Delaunay triangulations

add a few dummy points (e.g., 36 in 3D)  
and remove them asap

or

compute in a covering space (e.g., 27-sheeted in 3D)  
and switch back to 1 sheet asap

so that the triangulation is always a **simplicial complex**



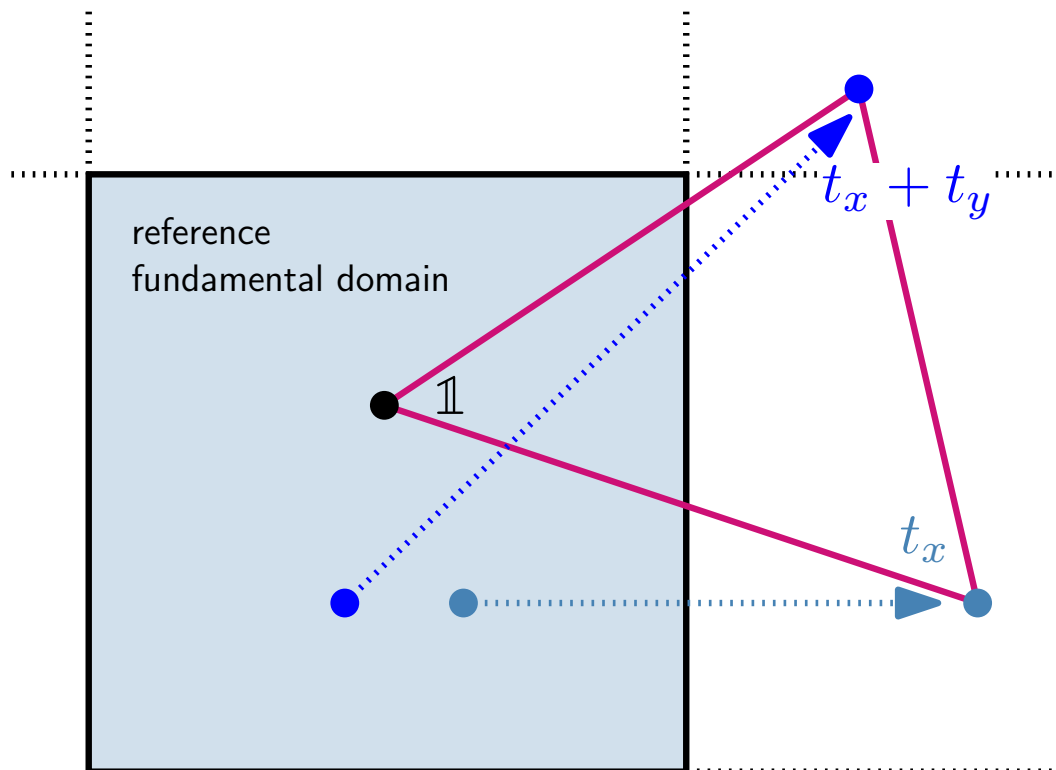
[Caroli, T. ESA'09]

$\Phi(\text{largest empty disk}) < \text{systole}/2$

# Periodic Delaunay triangulations

Periodic\_Delaunay\_triangulation < **TDS**, Geom\_traits >

combinatorial triangulation of  $\mathbb{S}^3$  reused for  $\mathbb{T}^3$



Cell =  $d$ -simplex

→  $d$  vertices

→  $d$  adjacent cells

$d$  translations

Vertex

canonical point

→ one incident cell



# Periodic Delaunay triangulations

[Caroli, T. CGAL'09]

2D [Kruithof CGAL'13]

also  
weighted  
alpha-shapes

periodic meshes [Bogdanov, Pellé, Rouxel-Labbé, T. CGAL'18]



# Periodic Delaunay triangulations

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Periodic_3_Delaunay_triangulation_traits_3.h>
#include <CGAL/Periodic_3_Delaunay_triangulation_3.h>
#include <vector>

typedef CGAL::Exact_predicates_inexact_constructions_kernel      K;
typedef CGAL::Periodic_3_Delaunay_triangulation_traits_3<K>      Gt;
typedef CGAL::Periodic_3_Delaunay_triangulation_3<Gt>           P3DT3;
typedef P3DT3::Point                                             Point;
typedef P3DT3::Iso_cuboid                                       Iso_cuboid;

int main(int, char**)
{
    Iso_cuboid domain(-1,-1,-1,2,2,2); // the fundamental domain

    std::vector<Point> pts;
    // pts is filled in some way...

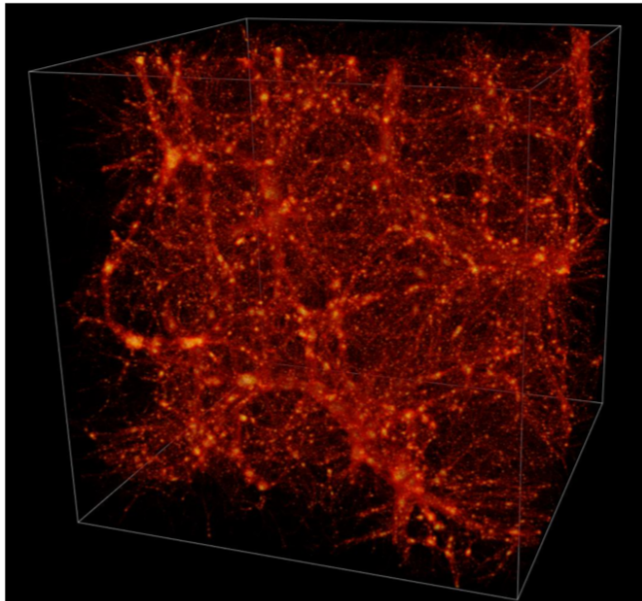
    P3DT3 T(pts.begin(), pts.end(), domain);

    return 0;
}
```

# Periodic Delaunay triangulations

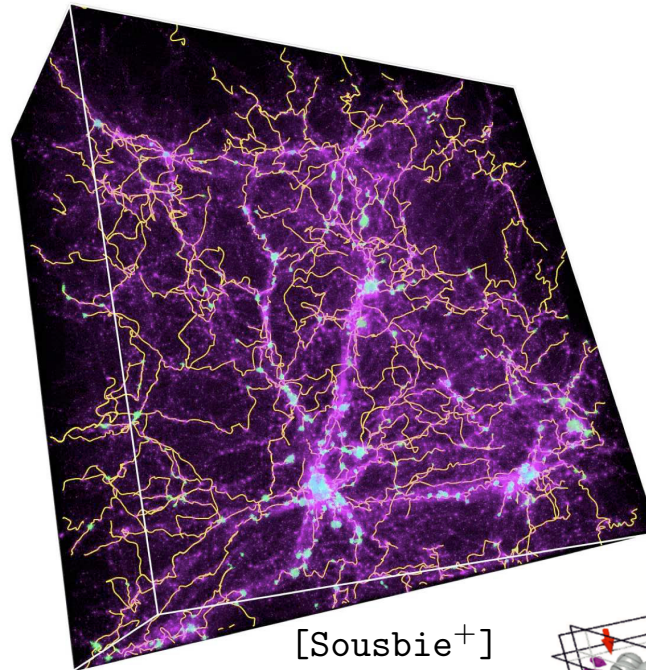
Users

Astrophysics



[v.d. Weijgaert<sup>+</sup>]

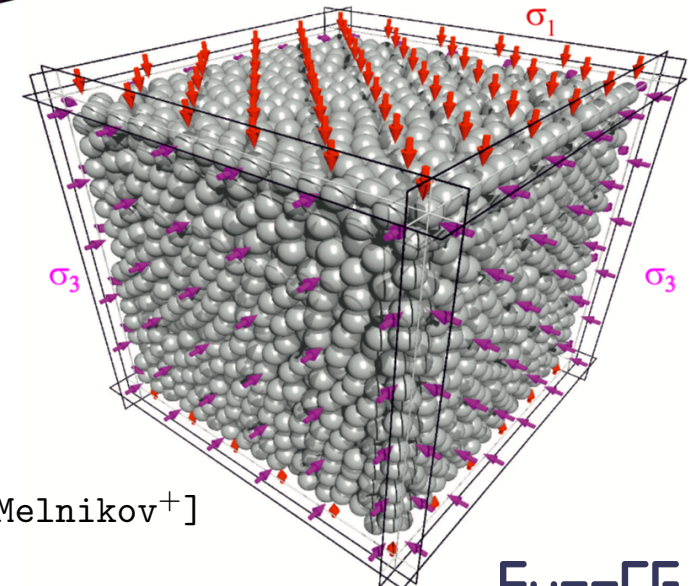
[SoCG'12 video]



[Sousbie<sup>+</sup>]

Particle physics  
Nanostructures  
etc

Granular materials



[Melnikov<sup>+</sup>]

# Periodic Delaunay triangulations

Users are always asking for more...

non-cubic case

$dD$  closed flat manifolds [Caroli, T. SoCG'11 DCG'16]

uses covering spaces

in practice: dummy points      In progress

$\Phi(\text{largest empty disk}) < \text{systole}/2$

# Periodic hyperbolic triangulations

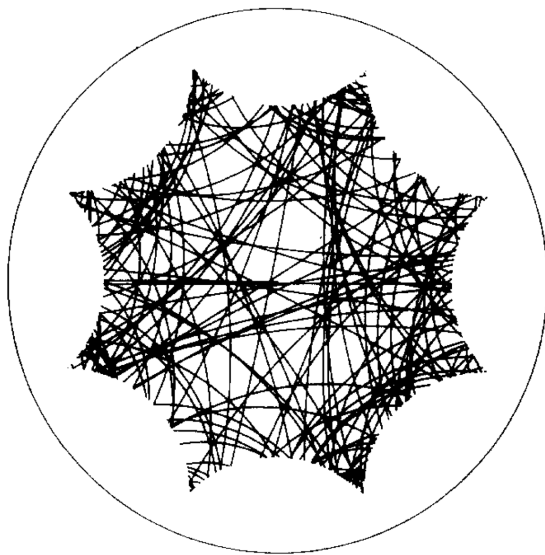
Delaunay triangulation of the Bolza surface



# Periodic hyperbolic triangulations

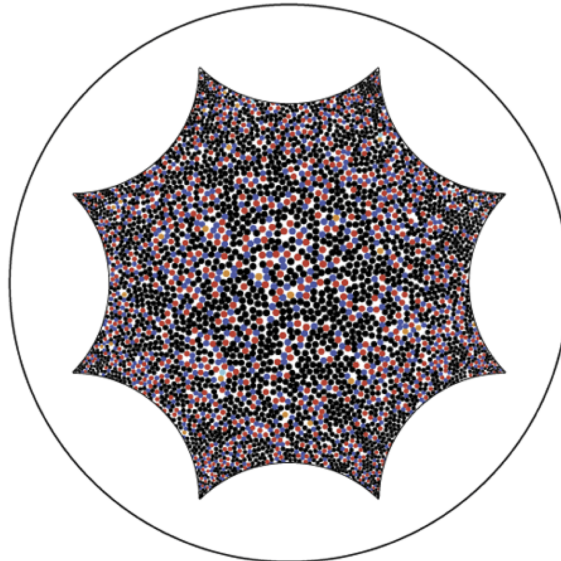
## Delaunay triangulation of the Bolza surface

Motivation: e.g., in mathematical physics



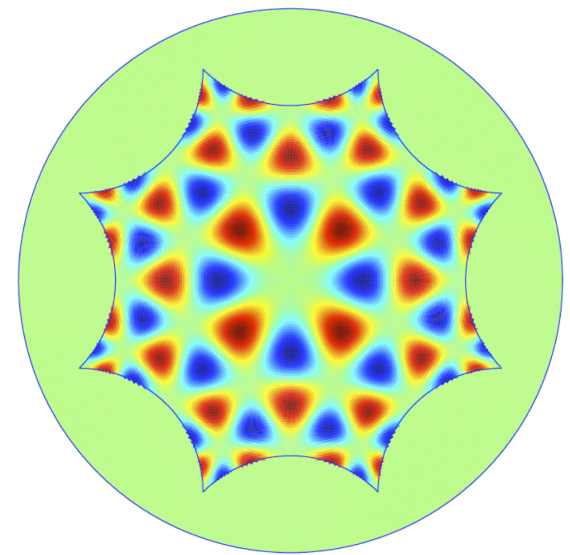
[Balazs, Voros '86]

Chaotic motion



[Sausset, Tarjus, Viot '08]

Glass-forming liquid



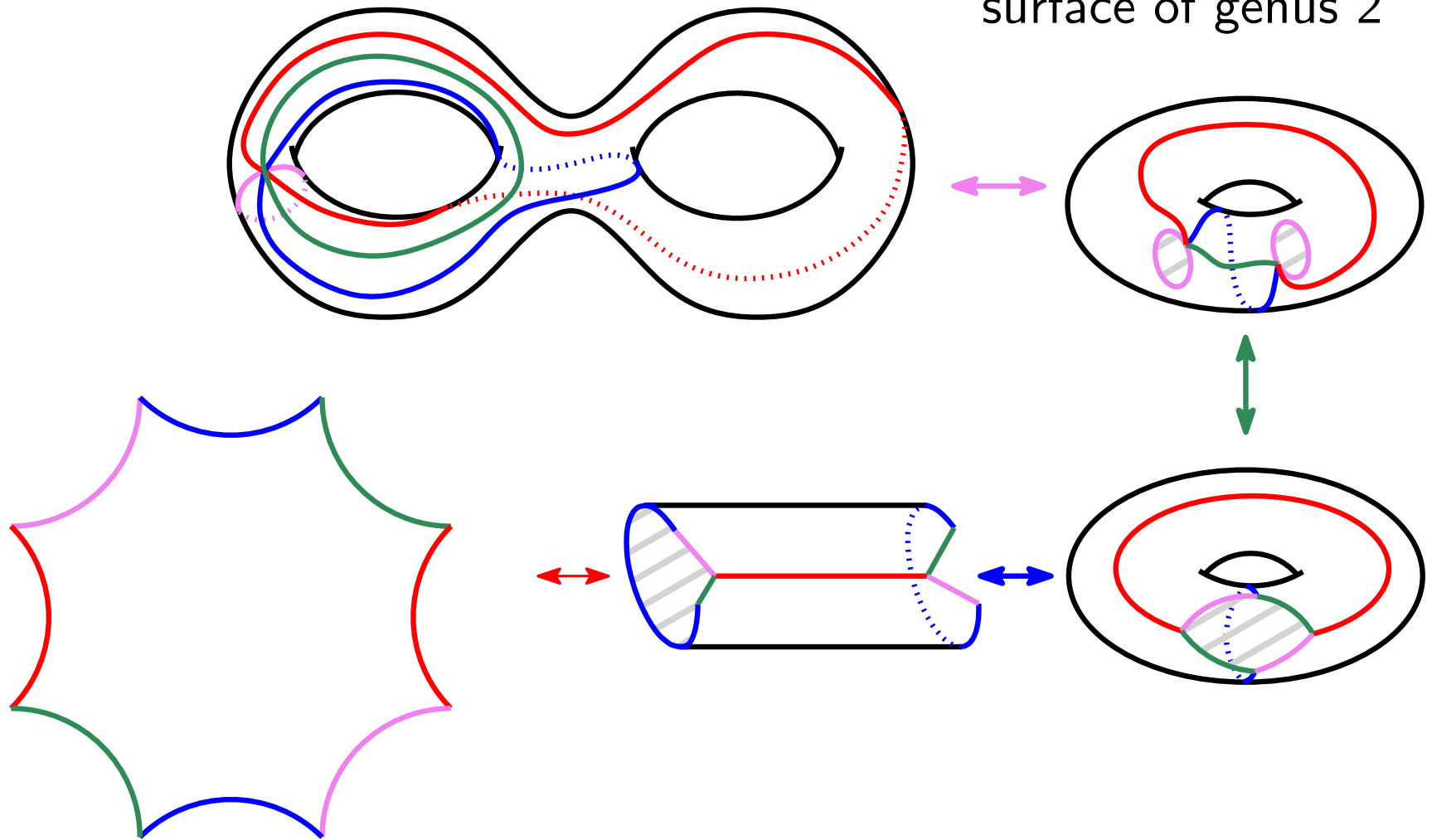
[Chossat, Faye, Faugeras '11]

Visual perception  
of textures

# Periodic hyperbolic triangulations

Delaunay triangulation of the Bolza surface

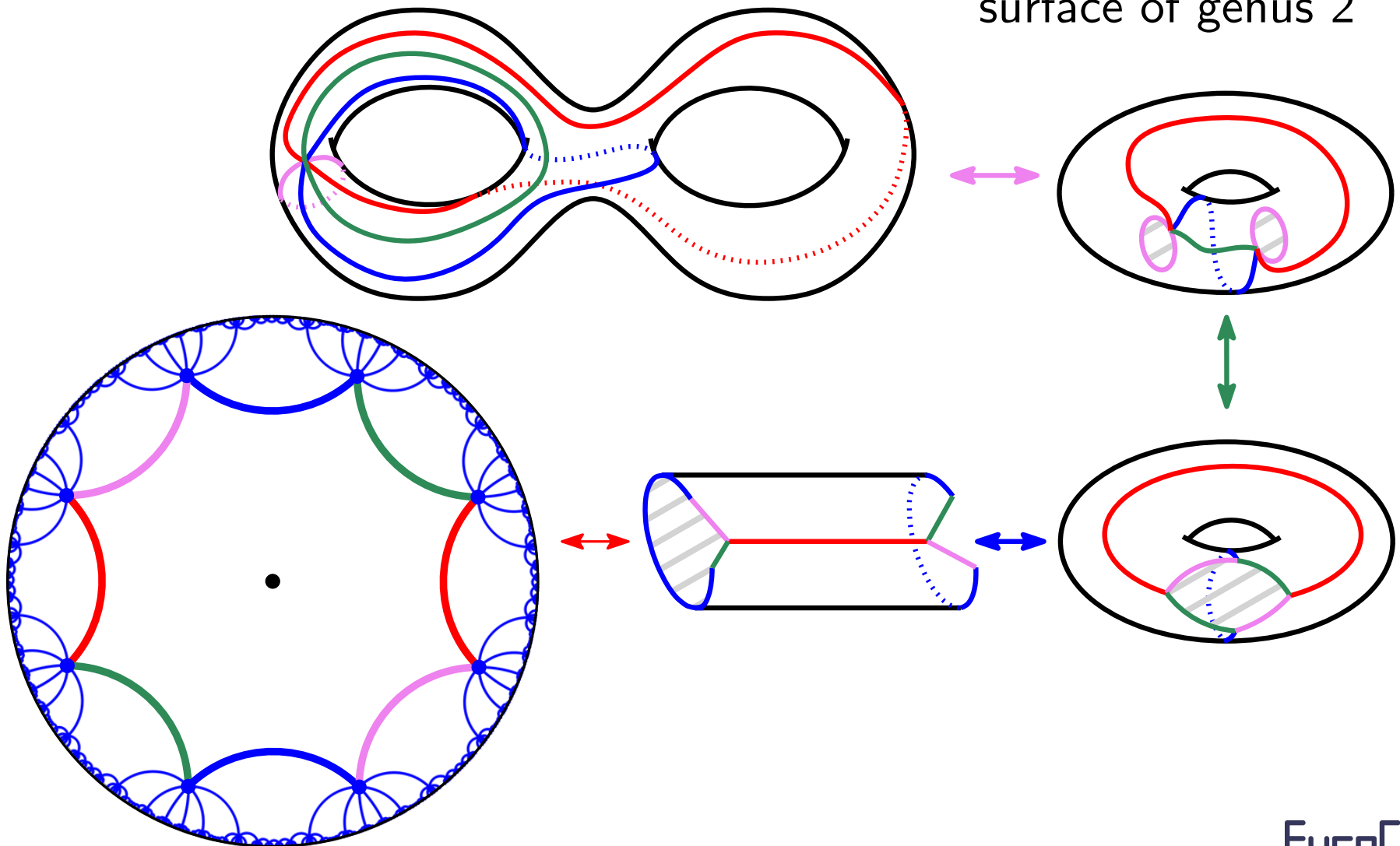
surface of genus 2



# Periodic hyperbolic triangulations

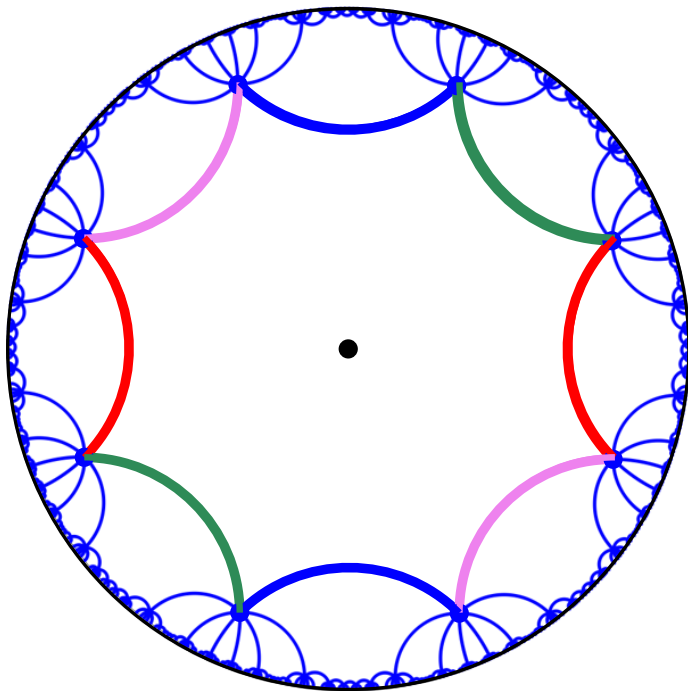
Delaunay triangulation of the Bolza surface

surface of genus 2



# Periodic hyperbolic triangulations

Delaunay triangulation of the Bolza surface



the regular octagon tiles the  
hyperbolic plane = covering space

# Periodic hyperbolic triangulations

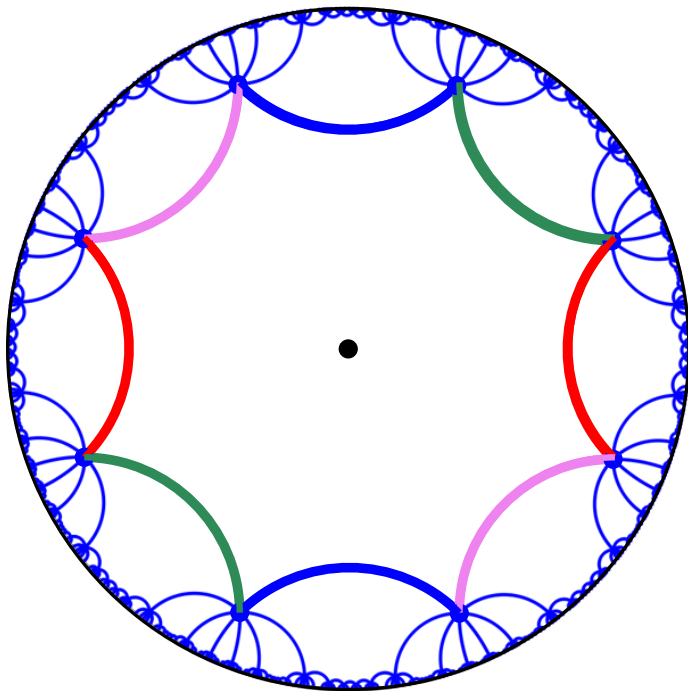
Delaunay triangulation of the Bolza surface

$$\mathbb{M}_2 \sim \mathbb{H}^2 / G$$

$$G = \langle a, b, c, d \mid \text{relation} \rangle$$

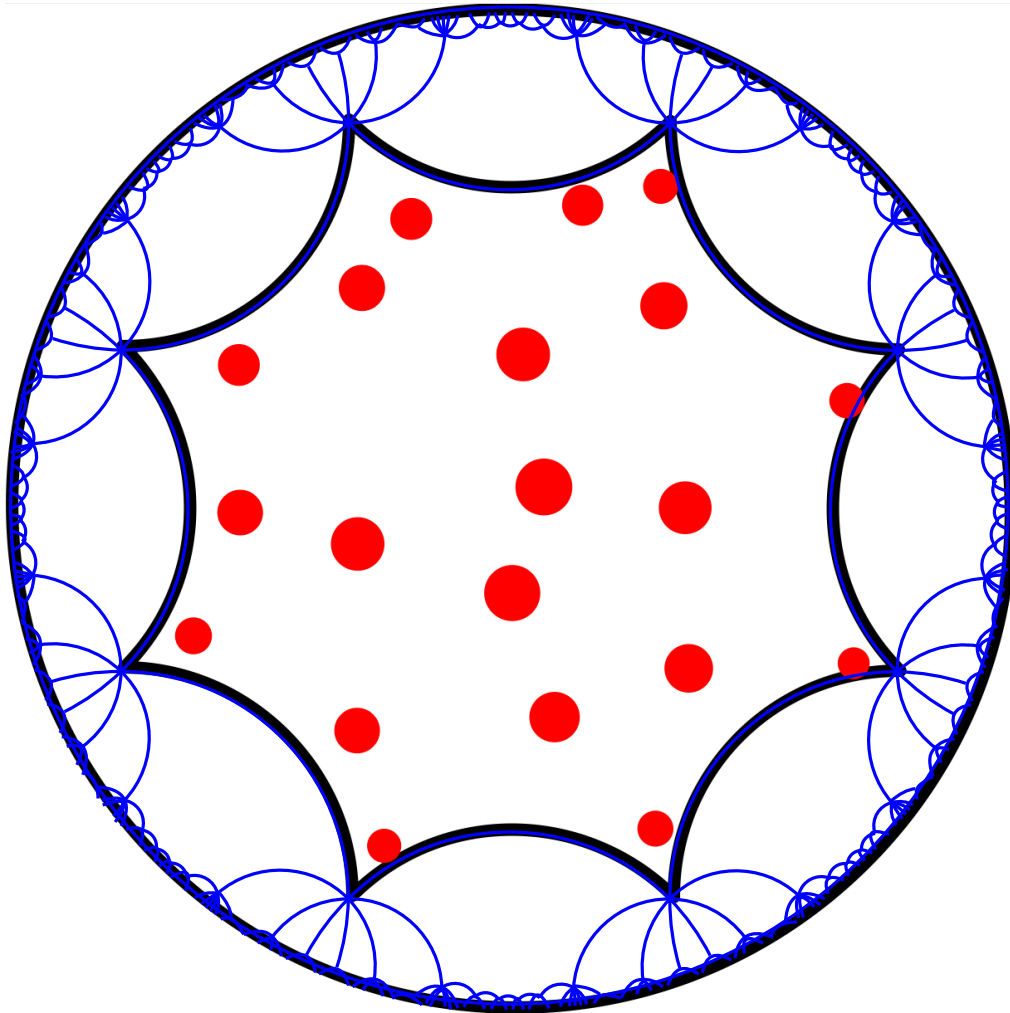
$a, b, c, d$  hyperbolic translations

do not commute



# Periodic hyperbolic triangulations

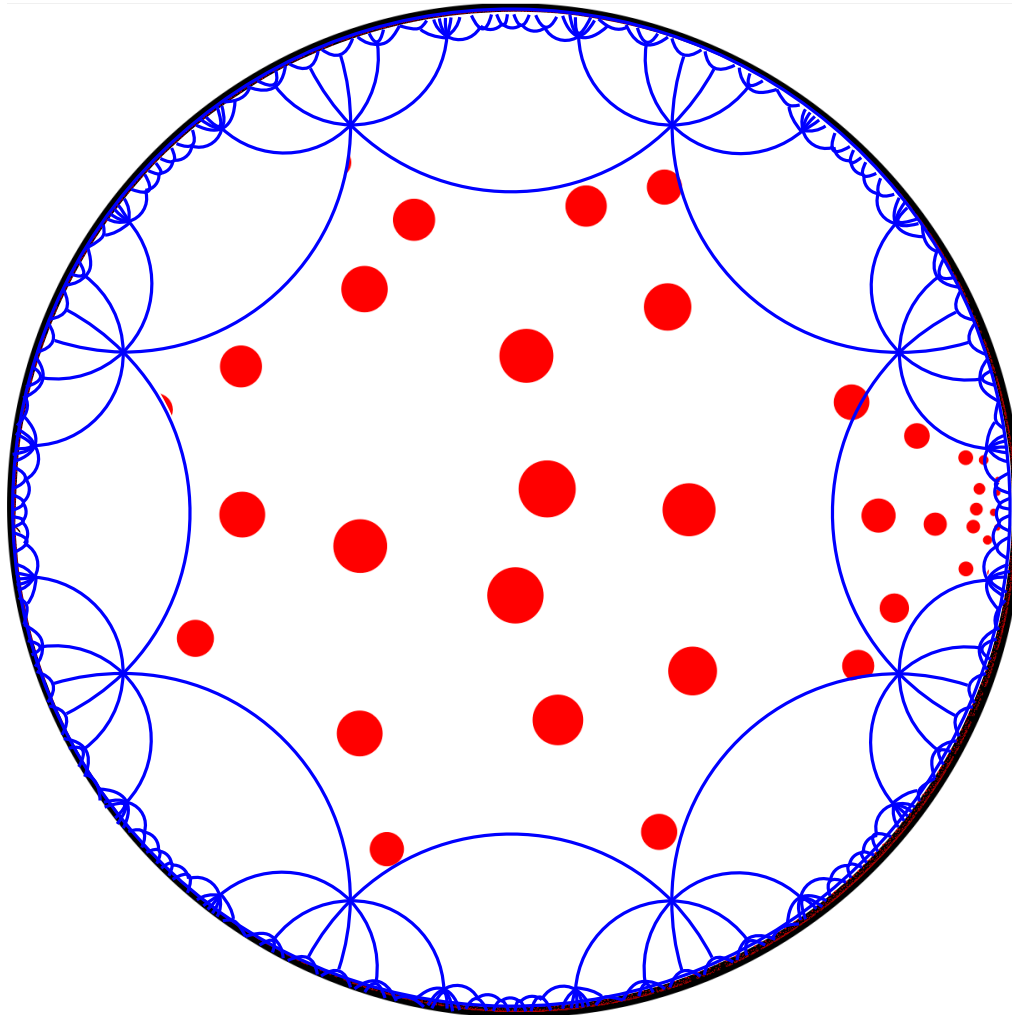
Delaunay triangulation of the Bolza surface



points on the surface

# Periodic hyperbolic triangulations

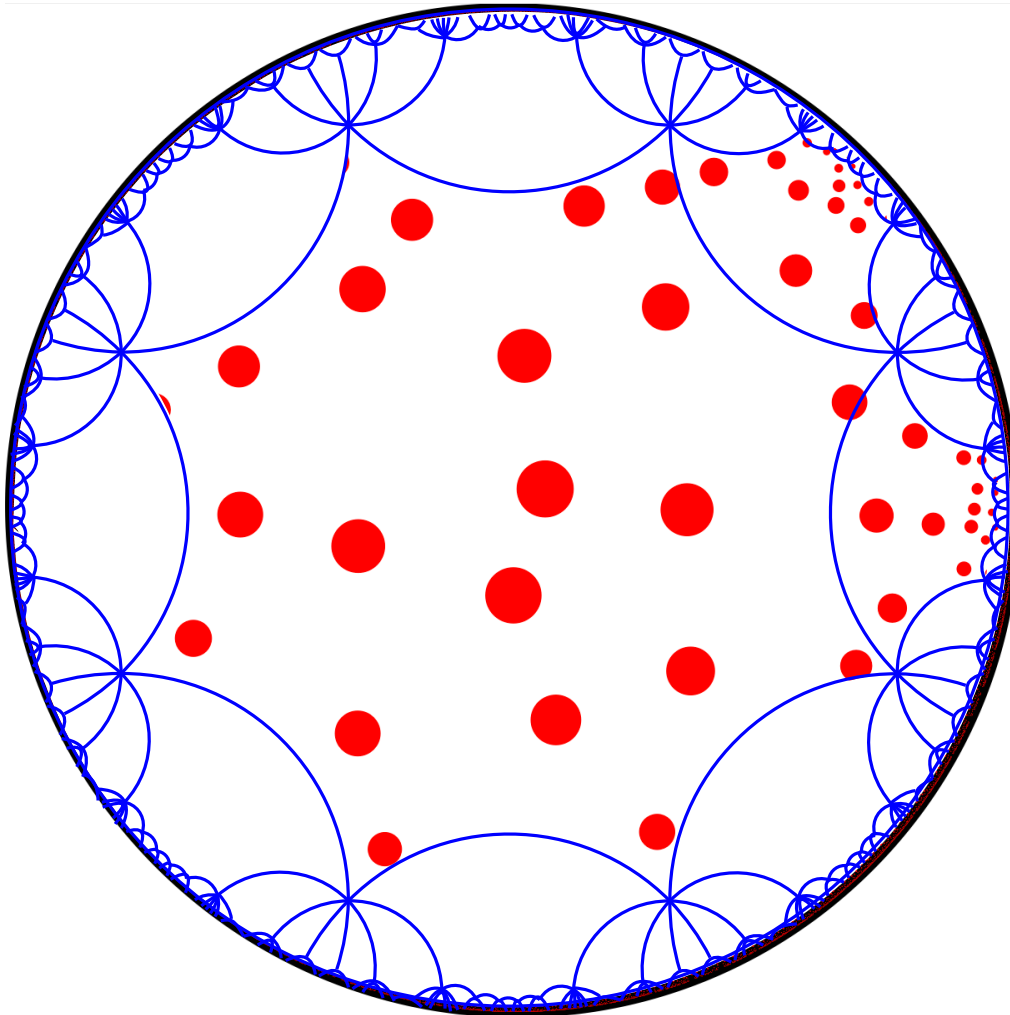
Delaunay triangulation of the Bolza surface



translated images

# Periodic hyperbolic triangulations

Delaunay triangulation of the Bolza surface

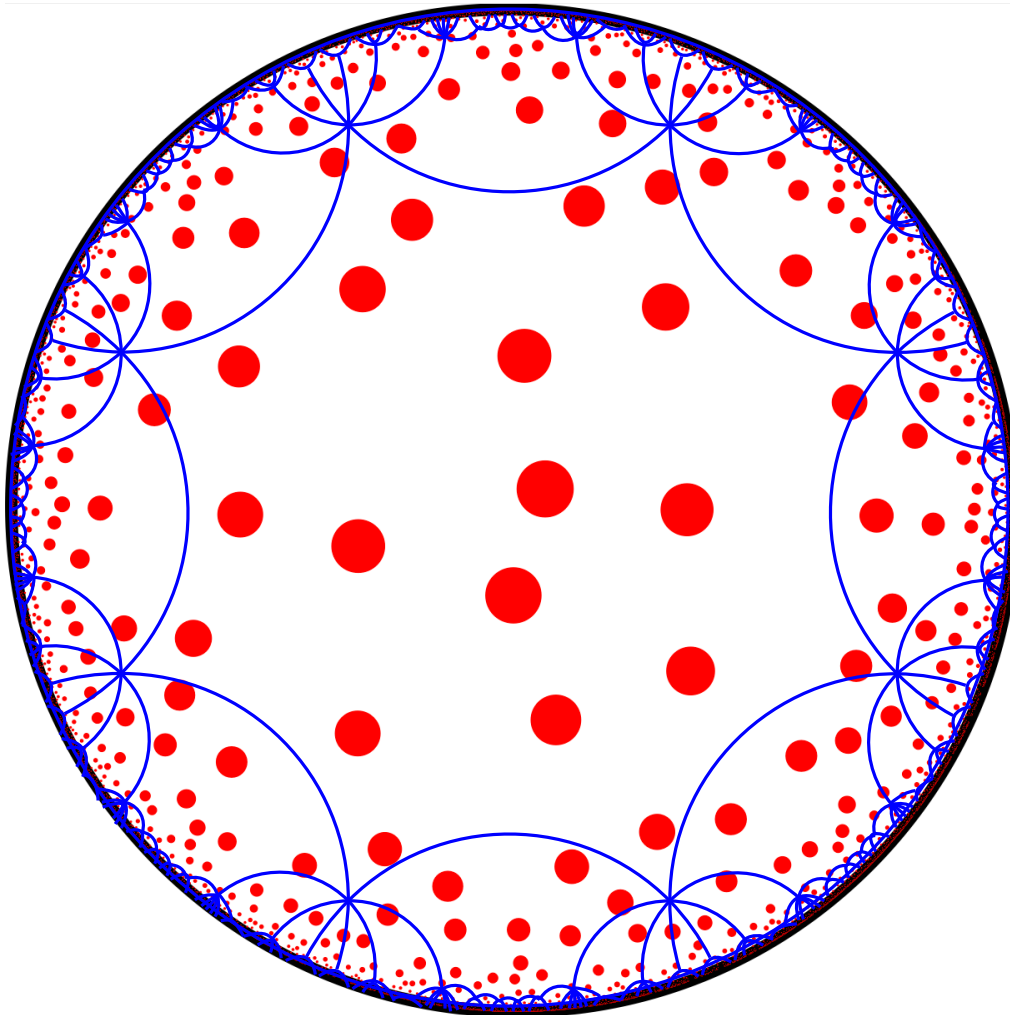


translated images



# Periodic hyperbolic triangulations

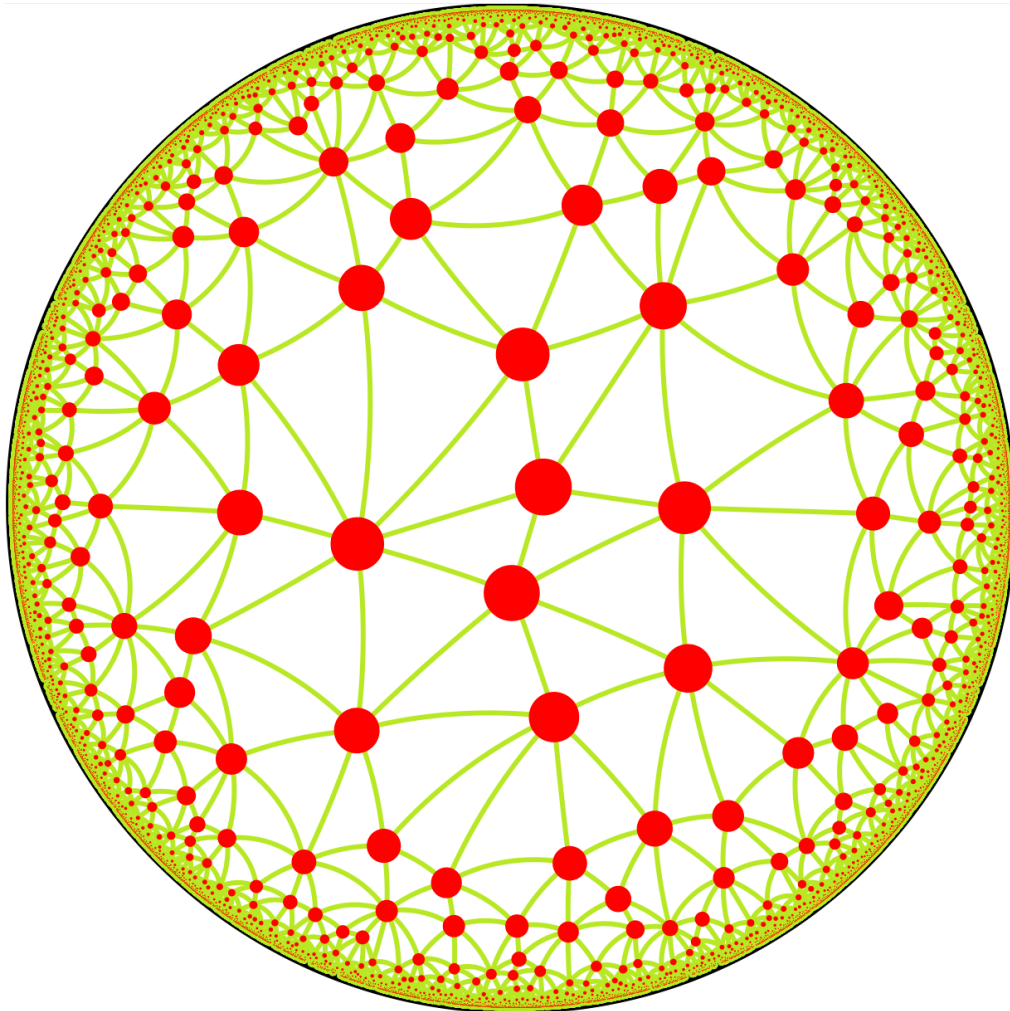
Delaunay triangulation of the Bolza surface



translated images

# Periodic hyperbolic triangulations

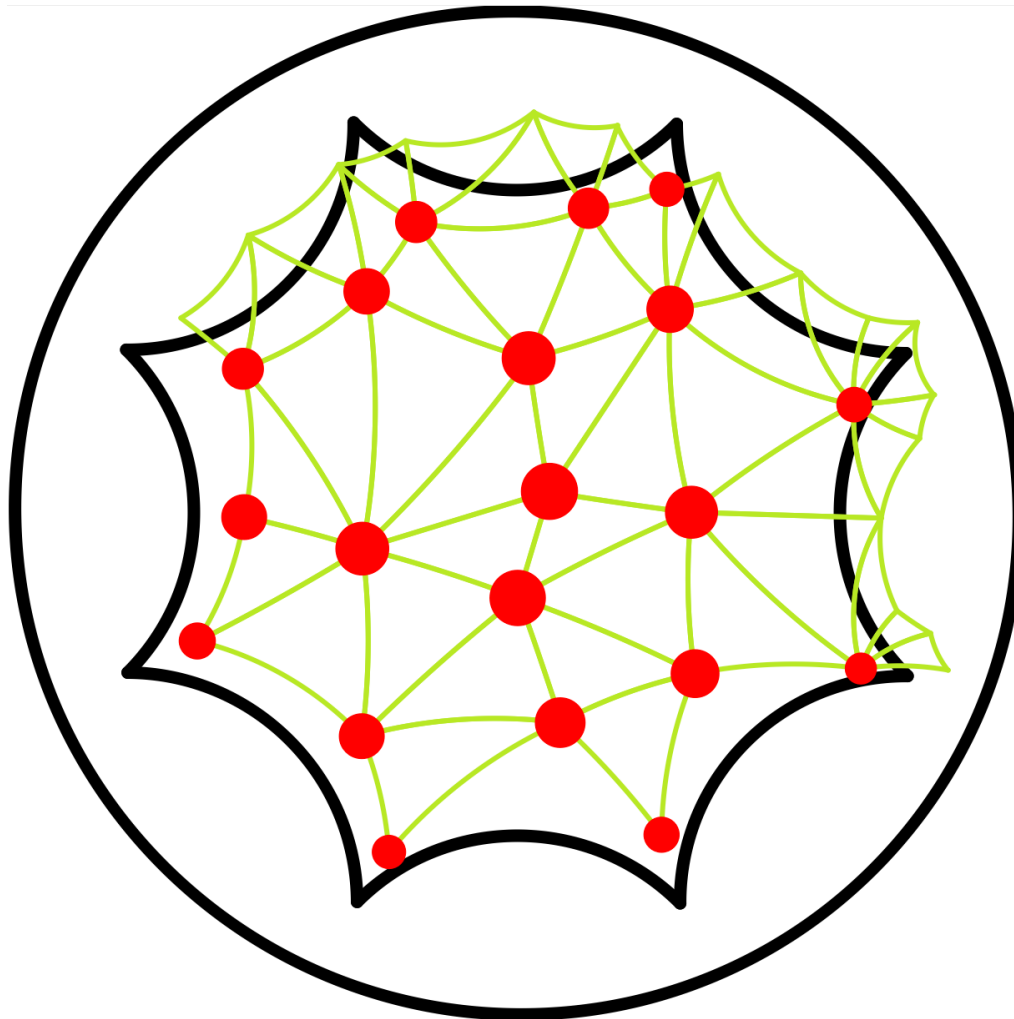
Delaunay triangulation of the Bolza surface



Delaunay triangulation  
of the infinite point set

# Periodic hyperbolic triangulations

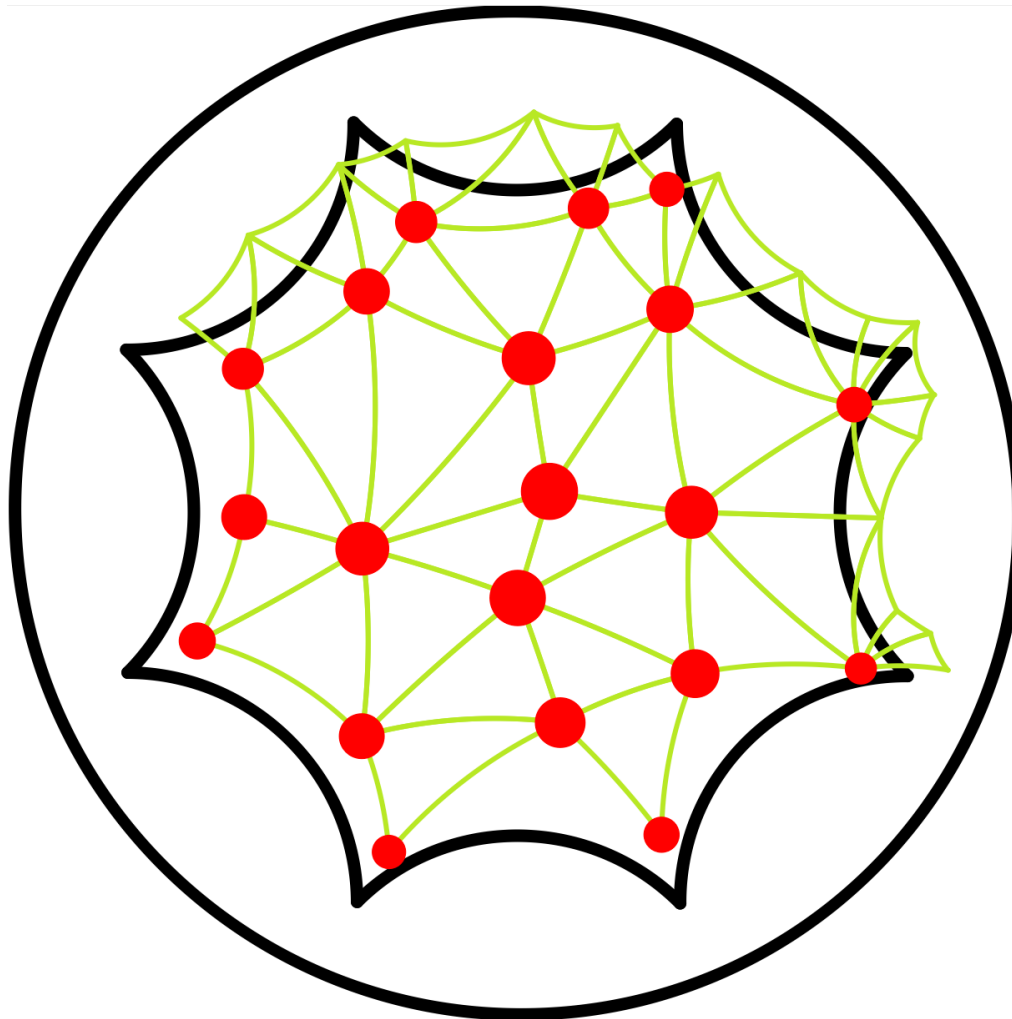
Delaunay triangulation of the Bolza surface



Delaunay triangulation  
of the surface

# Periodic hyperbolic triangulations

## Delaunay triangulation of the Bolza surface



Delaunay triangulation  
of the surface

(14 "dummy" points)

$\Phi(\text{largest empty disk}) < \text{systole}/2$

[Bogdanov, T., Vegter SoCG'16]

[Iordanov, T. SoCG'17]

[Iordanov, T. CGAL'19]

# The CGAL project

# Becoming a user

<http://www.cgal.org>

Many platforms

Linux, MacOS, Windows  
g++, VC++, clang, . . .

# Becoming a user

<http://www.cgal.org>

Many platforms

Linux, MacOS, Windows  
g++, VC++, clang, . . .

Download from `github`

or use

Linux distribution (Debian, . . .)

Mac distribution (macports, brew, . . .)

Euclidean 2D and 3D triangulations

- in Matlab
- in Python through CGAL bindings

# Becoming a user

<http://www.cgal.org>

Follow “Getting started”

**easier than ever!**

- “ Since CGAL version 5.0, CGAL is header-only by default, which means that there is no need to build CGAL before it can be used. ”
- Simple viewer



# Becoming a user

<http://www.cgal.org>

Follow “Getting started”

Read the **User manual** of your favorite package

Read, compile and run **examples**

# Becoming a user

<http://www.cgal.org>

Follow “Getting started”

Read the **User manual** of your favorite package

Read, compile and run **examples**

**Write your own code!**

looking at the **Reference manual**

Obey license GPLv3+ (or buy a commercial license)

# Becoming a user

<http://www.cgal.org>

Molecular Modeling

Particle Physics, Fluid Dynamics, Microstructures

Medical Modeling and Biophysics

Geographic Information Systems

Games

Motion Planning

Sensor Networks

Architecture, Buildings Modeling, Urban Modeling

Astronomy

2D and 3D Modelers

Mesh Generation and Surface Reconstruction

Geometry Processing

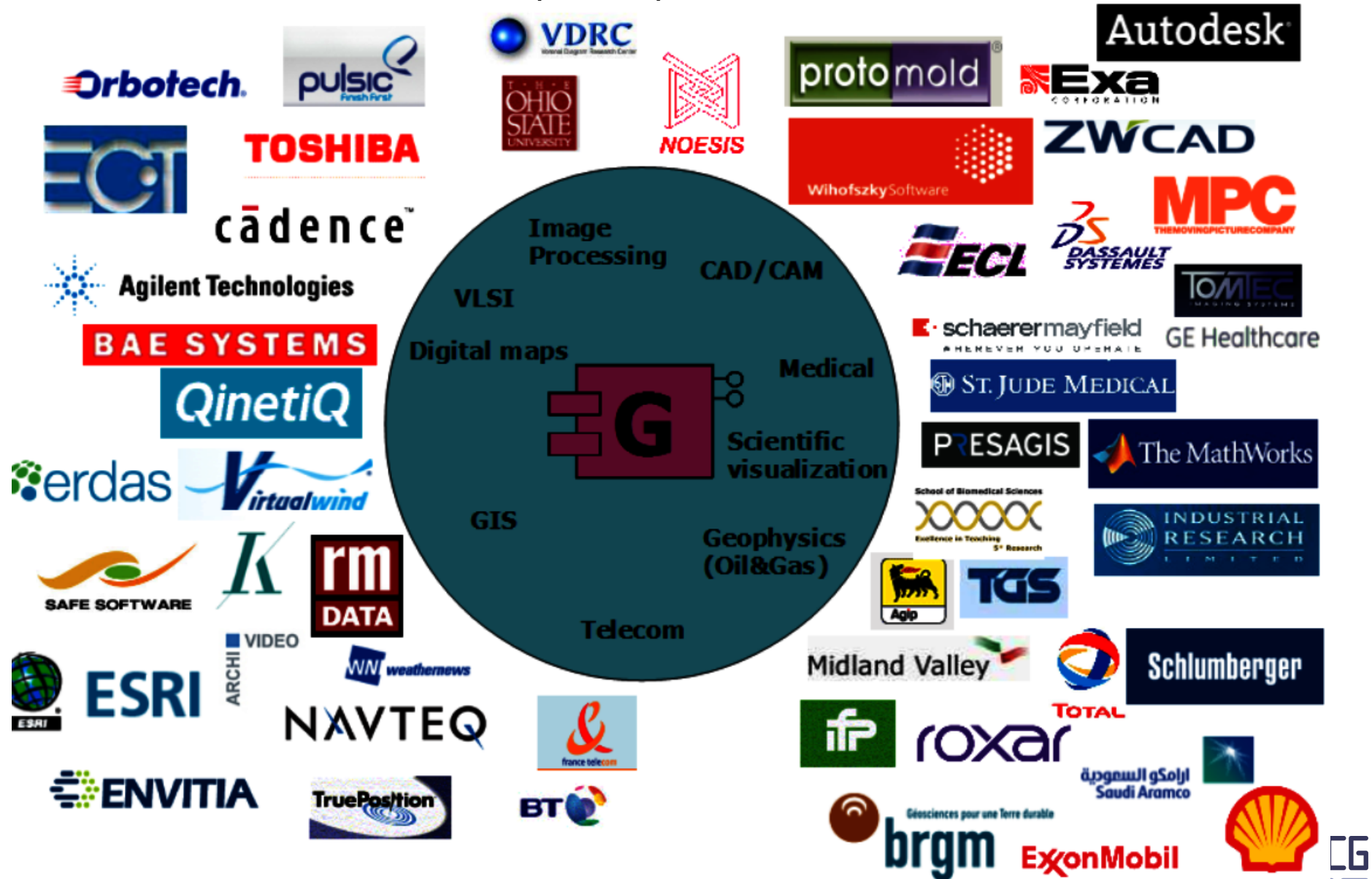
Computer Vision, Image Processing, Photogrammetry

Computational Topology and Shape Matching

Computational Geometry and Geometric Computing

# Becoming a CGAL user

some commercial users (2012)



# Becoming a CGAL user

Ipelets!

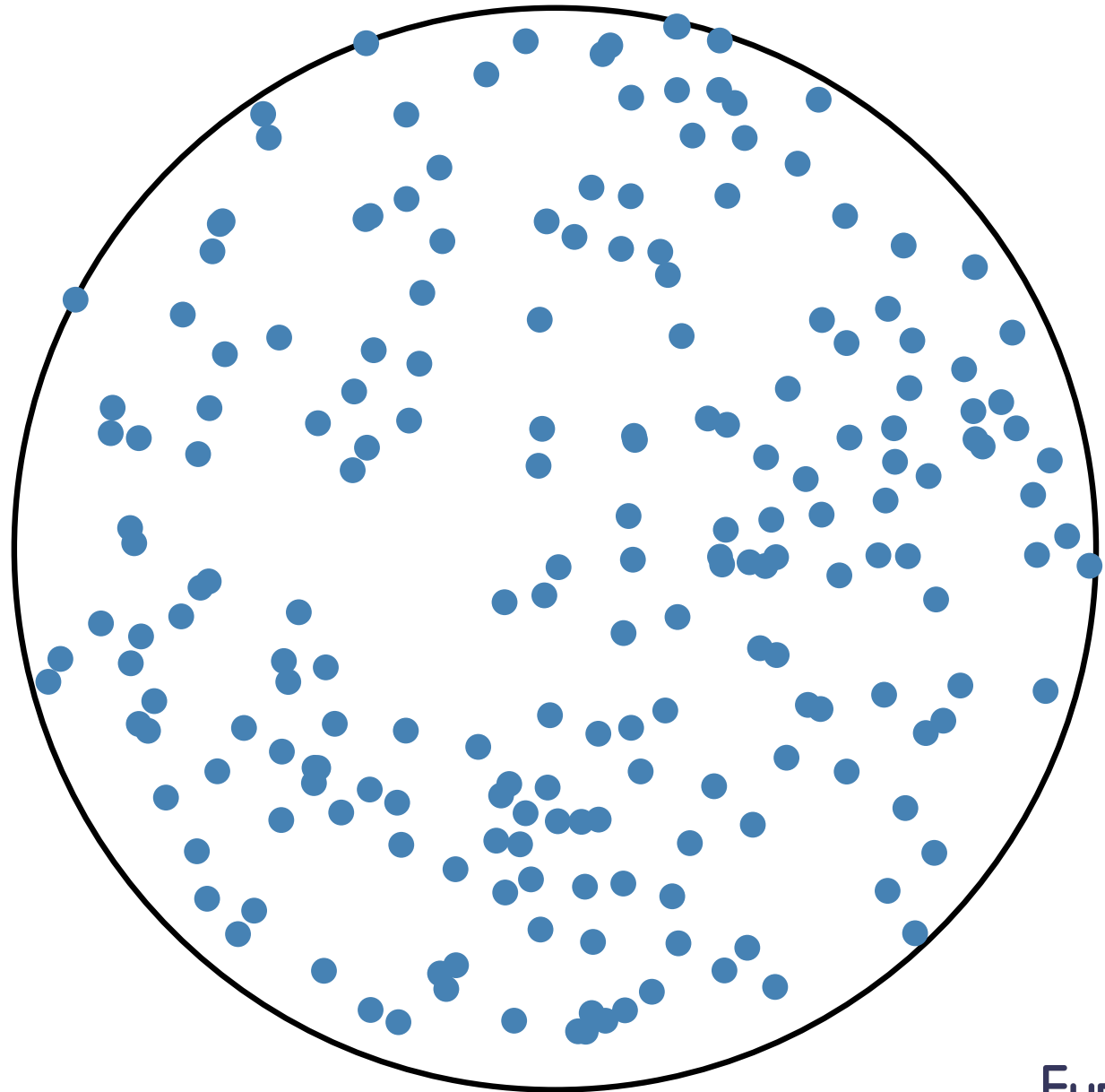
# Becoming a CGAL user

Ipelets!

Generators

e.g. 200 points in a disk

$\mathbb{R}^2$



# Becoming a CCGAL user

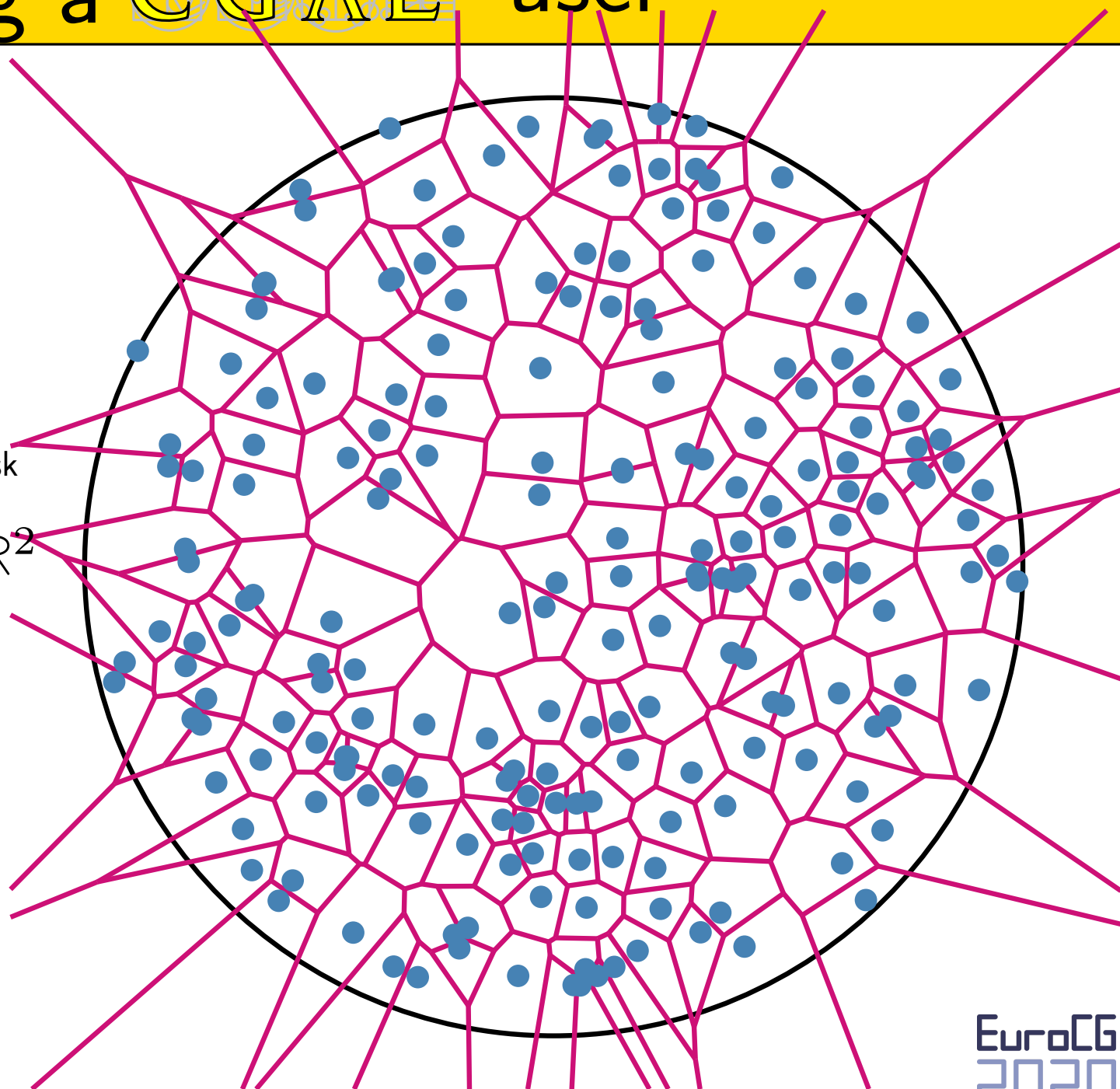
Ipelets!

Generators

e.g. 200 points in a disk

$\mathbb{R}^2$

“Diagrams”  
→ Voronoi



# Becoming a CGAL user

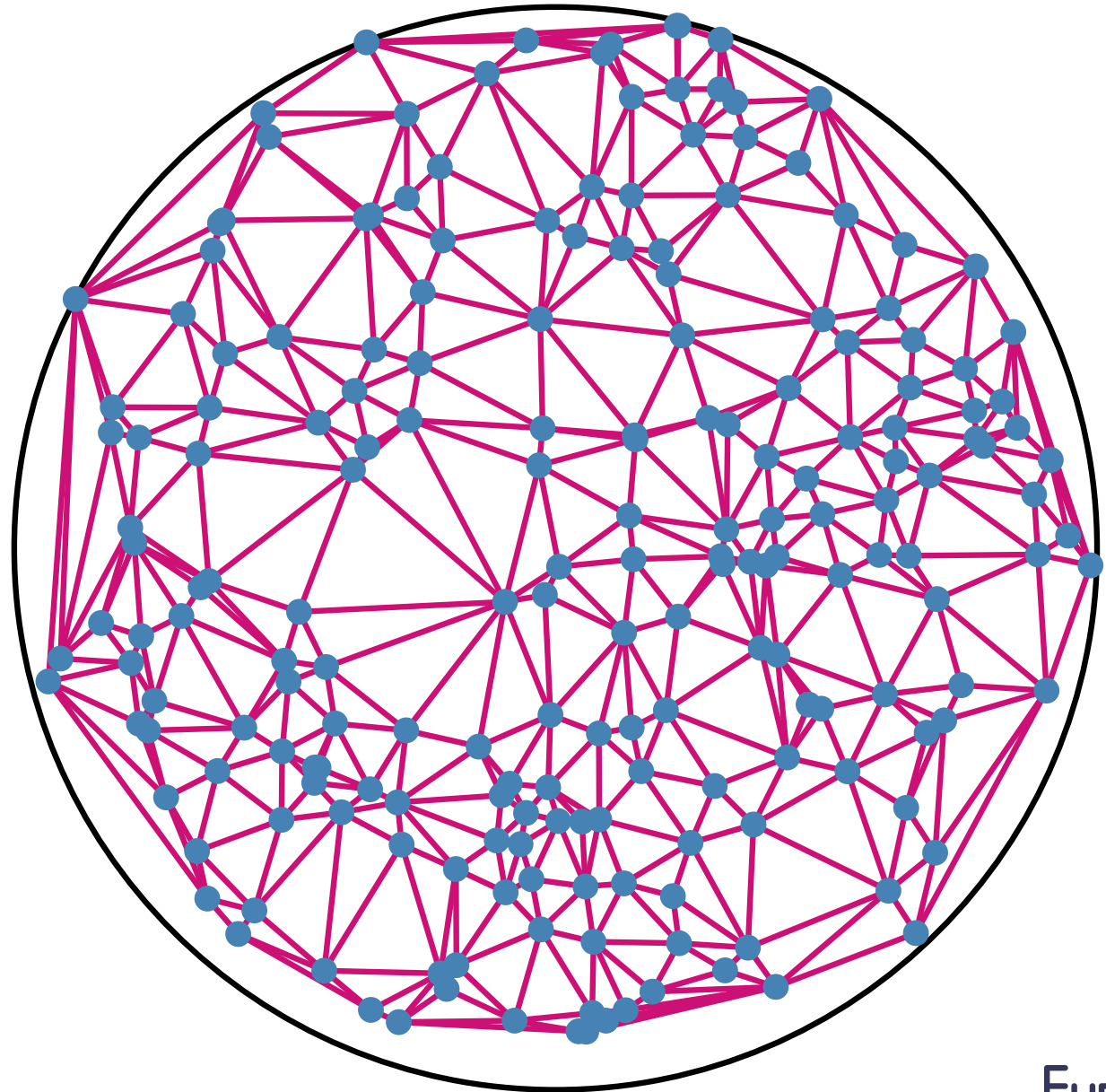
Ipelets!

Generators

e.g. 200 points in a disk

$\mathbb{R}^2$

“Triangulations”  
→ Delaunay





# Becoming a developer

Any idea for a new package?

**Contact the editorial board!**

even before starting to code

<http://www.cgal.org>

Design and coding style

Infrastructure

Review and integration process

# Becoming a CGAL developer

Contributors are clearly acknowledged **Visibility**

- Appear as Authors in manual chapters

## CGAL 5.0.1 - 3D Periodic Triangulations

CGAL 5.0.1 - 3D Periodic Triangulations

- ▶ User Manual
- ▶ Reference Manual
- Refinement Relationships
- Is Model Relationships
- Has Model Relationships
- Bibliography
- ▶ Class and Concept List
- ▶ Examples

### User Manual

**Authors**  
Manuel Caroli, Aymeric Pellé, Mael Rouxel-Labbé and Monique Teillaud

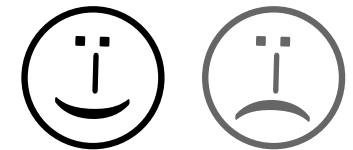
- Listed in the “People” web page

# Becoming a CGAL developer

Contributors are clearly acknowledged **Visibility**

appreciated by

companies (Google, Intel, Apple, ...)



research institutes (INRIA, ETHZ, ...?)

# Becoming a developer

Contributors are clearly acknowledged **Visibility**

[The institution of the] authors keep the copyright

( must agree to distribute the code in CGAL under GPLv3+ )

# Becoming a CGAL developer

**Key for success = Diversity of its members**

gather many skills:

maths

algorithms

C++

development tools

...

**Join the crowd!**



# Becoming a developer







Datum/Date: 10.09.2016 Zug/Train: 571 (12:33) Position: -1b 09.09.2016 15:27:10

# Reserviert Réservé

Bitte Sitzplätze für Gruppen freihalten  
Veuillez laisser les places libre pour les groupes  
Per favore tenere liberi i posti per gruppi

**CGAL-ETH Zürich (Salow) 20**  
Zürich HB (13:37) - Landquart (14:41)



September 2016

EuroCG  
2020







# Thanks to

past,  
present,  
future

developers  
and

users  
of



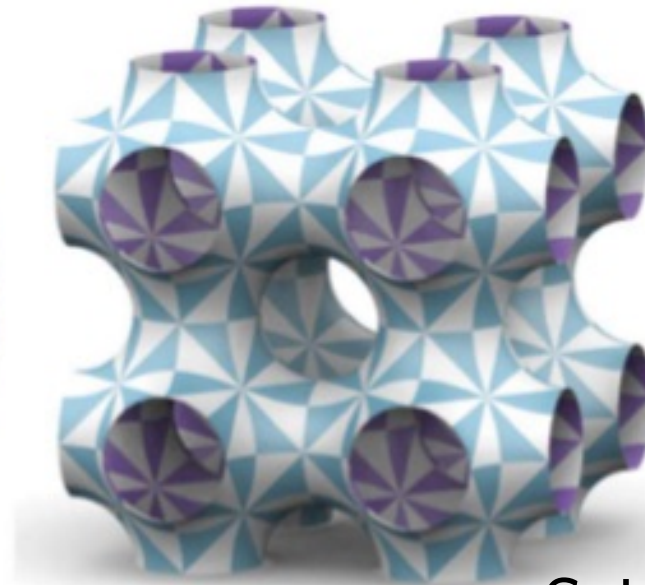
you all!

Otfried for Ipe

# Triply periodic minimal surfaces



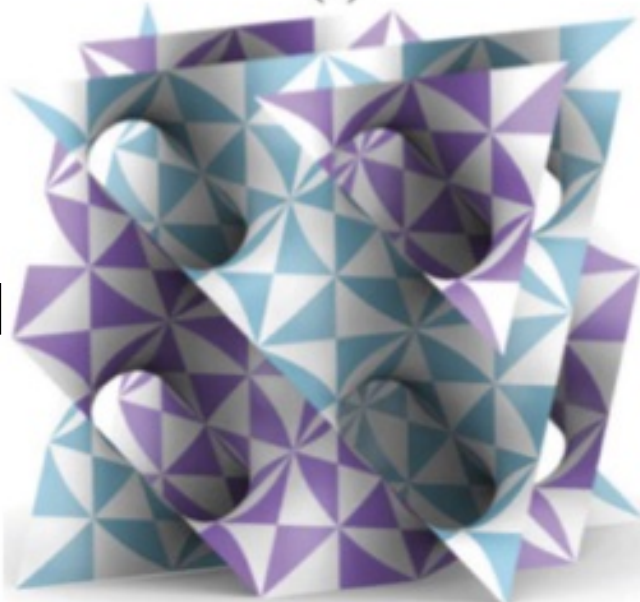
(a)



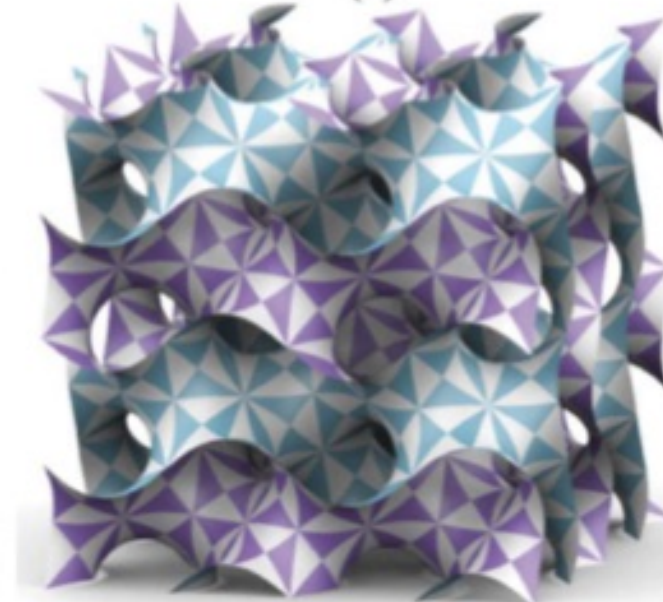
(b)

Schwarz P

Diamond



(c)



(d)

Gyroid