Numerical Optimization for Frictional Contact Problems

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Introduction and motivation

The aim of this report is to show how one may reformulate the contact problems as an optimization programming and using numerical methods to solve the problems. We consider the problem of simulating the dynamics of mechanical systems involving unilateral contact and the Coulomb friction. The fact that these problems are difficult to numerically solve and even the mathematical properties of convergence of algorithms are required some strong assumptions. There are many research on numerically solving such systems we can see in [6, 7].

On examining of frictional contact problems (or incremental problems), the answer for existence a solution require strong assumptions, but in most cases in practice, they are not satisfied. And even to prove the convergence of an algorithm (in case of existence solution) is difficult. It makes confused when we determine the failure occurred in practice comes from the algorithm or because of no existence solution of the problem.

In [2, 3], V. Acary and Florent Cadoux used convex-analysis on reformulating the incremental problem, and give a proof of the existence of the solution to the problem by using fixed-point theory.

In this report, we will focus on solving optimization formulation by using Interior point method (IPM) on the large sets of examples coming from the simulation of a wide range of mechanical systems. The interior point method we use in this paper is base on solving KKT condition and using line-search methods to find the descend direction for the solution. Adding some modifications to be fitted with our problem, we also combine the fixed-point and projection algorithms to get the solution with the high accuracy.

This report is organized as follows. In the first chapter, we recall the incremental problem in the original form and conic complementarity formulation. We also introduce the optimization problem which is associated with the incremental problem. Especially, we study the properties of the new proposal form, in the last part of this chapter we introduce a new bi-level programming (also can see under the form of Mathematical Programming of Equilibrium Constraints) formulation which idea comes from professor Patrick Ballard. In chapter two, we recall some parts of Interior-point method base on paper of M. Friedlander and D. Orban in [5], using fixed-point theory in [2, 3] and projection theory in [4] to build the algorithm. The last chapter is the numerical experiences by using the algorithm introducing in the previous chapter, this part we also mention on performance profile of the solver to make some comparing between them. Finally, these are conclusions and some future works related to the report.

Chapter 1

Description of frictional contact problem

1.1 Incremental problem

In this section, following [2, 3], we present the frictional contact problem in finite dimension after a time–discretization. Let $n_c \in \mathbb{N}$ be the number of contacts and $n \in \mathbb{N}$ be the number of degrees of freedom. The problem can be formulated as a complementarity problem over second-order cones:

$$\begin{cases} Mv = f + H^T r \\ u = Hv + w \\ K_i^* \ni u_i + \mu_i ||u_{T,i}|| N \perp r^i \in K_i, \end{cases} \tag{1.1}$$

with $m = 3n_c$ is the number of unknown variables at contact , $M \in \mathbb{R}^{n\times n}_{++}$, $\mathbb{R}^{n\times n}_{++} = \{M \in \mathbb{R}^{n\times n} : x^T Mx > 0, \forall x \in \mathbb{R}^n\}$ and $H \in \mathbb{R}^{n\times m}$, $w \in \mathbb{R}^m$ $f \in \mathbb{R}^m$, $\mu \in \mathbb{R}^{n_c}$ are given. $N = (1, 0, 0)^T$. One note that the matrix H is usually rank-deficient.

The variables are the velocity vector $v := (v_1, v_2, ..., v_n) \in \mathbb{R}^n$ and the impulse vector $r := (r_1, r_2, ..., r_{n_c}) \in \mathbb{R}^m, r_i := (r_{N,i}, r_{T_1,i}, r_{T_2,i})^T \in \mathbb{R}^3$.

The set K_i is the so-called second-order cone defined by

$$K_i = \{r_i \in \mathbb{R}^3 : r_{N,i} \geq 0 \text{ and } ||r_{T,i}|| \leq \mu r_{N,i}\},$$

and its dual cone K_i^*

$$K_i^* = \{u_i \in \mathbb{R}^3 : u_{N,i} \geq 0 \text{ and } ||u_{T,i}|| \leq \frac{1}{\mu} u_{N,i}\}.$$

The first equation is known as the discretized dynamical equation, and the second one is the kinematic relation. The last condition is the cone complementarity condition.

If we set $\tilde{u} = u + g(u)$, the function $g(u) : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$g(u) = [(\mu_i ||u_{T,i}||, 0, 0)^T, i = 1...n_c],$$

we can formulate (1.1) as

$$\begin{cases} Mv = f + H^T r \\ \tilde{u} = Hv + w + g(s) \\ K_i^* \ni \tilde{u}_i \perp r_i \in K_i \\ s^i = ||\tilde{u}_{T,i}||. \end{cases} \tag{1.2}$$

For a fixed value of s, we build the conic linear complementarity problem over second-order cone:

$$\begin{cases} Mv = f + H^T r \\ \tilde{u} = Hv + w + g(s) \\ K_i^* \ni \tilde{u}_i \perp r_i \in K_i. \end{cases} \tag{1.3}$$

1.2 Fixed-point formulation

From the conic linear complementarity problem (1.3), addition the condition $M \in S^n_{++}$, with $S^n_{++} = \{M \in \mathbb{R}^{n\times n} : M = M^T \text{and } x^T Mx > 0, \ \forall x \in \mathbb{R}^n\}$ is the set of symmetrix positive-definite matrices, we can reformulate it as a Quadratic programming (QP) over second-order cone:

$$\begin{cases} \min_v \frac{1}{2} v^T Mv - v^T f \\ s.t \ \ Hv + w + g(s) \in K^*, \end{cases} \tag{1.4}$$

with $K^* = \prod_{i=1}^{n_c} K_i^*$.

Remark 1.1. *With the assumption of matrix $M \in S^n_{++}$, The problem* (1.4)*is strictly convex.*

We can also consider the problem on the variables u and r by computing

$$\tilde{u} = HM^{-1}f + HM^{-1}H^T r + w + g(s) = HM^{-1}H^T r + q(s),$$

with

$$q(s) = HM^{-1}f + w + g(s).$$

Let us define the Delassus matrix $W = HM^{-1}H^T$. The incremental problem can be rewritten as:

$$\begin{cases} \tilde{u} = Wr + q(s) \\ K^* \ni \tilde{u} \perp r \in K, \end{cases} \tag{1.5}$$

$K = \prod_{i=1}^{n_c} K_i$.

Remark 1.2. *The Delassus matrix is positive semi-definite. In most of case W is symmetric.*

The problem (1.5) is equivalent to the minimization problem

$$\begin{cases} \min_r \dfrac{1}{2} r^T W r + r^T q(s) \\ s.t \;\; r \in K. \end{cases} \tag{1.6}$$

Remark 1.3. *The problem (1.6) is convex since W is positive semi-definite.*

By using the typical property of second-order cone mentioned on [12], we reformulate (1.6) as:

$$\begin{cases} \min_r \dfrac{1}{2} r^T W r + r^T q(s) \\ s.t \;\; r_i^T Q_i r_i \geq 0, \;\; i = 1, ..., n_c \\ \qquad r_{i,N} \geq 0, \quad i = 1, ..., n_c. \end{cases}, \quad Q_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{-1}{\mu_i^2} & 0 \\ 0 & 0 & \dfrac{-1}{\mu_i^2} \end{pmatrix} \tag{1.7}$$

In the next chapter we use interior-point method for solving (1.7).

Remark 1.4.
- *It is not mandatory to reformulate the second-order cone constraint as quadratic constraints. The formulation of problems depends on the method we use to solve it, we try to perform it in the best way to use the solver. Example, if we want to use IPM we can reformulate as (1.7). If we want to solve by proximal-point technique, we include the cone constraint into the cost function by using indicator operator, or if we want to use ADMM we can add one linear constraint and include the cone constraint into cost function.*

1.3 Optimization problems over the second order cone

In this section, we introduce the optimization problem in the form

$$\begin{cases} \min_r \langle r, Wr+q\rangle + \sum_{i=1}^{n_c} \mu_i r_{N,i} ||(Wr+q)_{T,i}|| \\ s.t \qquad (Wr+q)_{N,i} \geq 0, \qquad i = 1..n_c \\ \qquad\quad r_i \in K_i, \qquad i = 1..n_c. \end{cases} \tag{1.8}$$

based on the fact that $\tilde{u} \in K^*, r \in K$. Along with examining the properties of the problem under this form, we will find the conditions to ensure the equivalence between formulations of (1.8) and (1.3). Hence, by using the idea of Least-square problem in [10, 11], we add some new slack variables s changing the problem to the new form

$$\begin{cases} \min_{r,s} \langle r, Wr+q\rangle + \sum_{i=1}^{n_c} \mu_i r_{N,i} s_i \\ s.t \qquad ||(Wr+q)_{T,i}|| \leq s_i, \qquad i = 1..n_c \\ \qquad\quad (Wr+q)_{N,i} \geq 0, \qquad i = 1..n_c \\ \qquad\quad r_i \in K_i, \qquad i = 1..n_c. \end{cases} \tag{1.9}$$

1.3.1 Equivalence between the Linear complementarity Problem (LCP) over cone and the QP.

We following consider the equivalence between LCP and QP in general cases. We know that LCP is equivalent with QP over the positive orthant $\mathbb{R}_+$ which is mentioned on [13, 14], so what about if we consider over the arbitrary cone?

Proposition 1.1. *Let W is a semi-definite positive matrix, not necessarily symmetric and K is a cone. Then, the following quadratic problem*

$$\begin{cases} \min_r \langle r, Wr+q\rangle \\ s.t \;\; Wr+q \in K^* \\ \qquad r \in K, \end{cases} \tag{1.10}$$

is equivalent to the following LCP

$$K^* \ni Wr + q \perp r \in K. \tag{1.11}$$

Proof. We assume that LCP has solution (i.e. feasible).

The equivalence in case that the solution of LCP is solution of QP and conversely.

" $\Leftarrow$ "

The Lagrangian of (1.10) is as follows

$$L(r, \lambda_1, \lambda_2) = \langle r, Wr + q\rangle - \lambda_1^T(Wr + q) - \lambda_2^T r.$$

The subdifferential of the Lagrangian with respect to r is

$$\partial_r(L(r, \lambda_1, \lambda_2)) = (W + W^T)r + q - W^T\lambda_1 - \lambda_2.$$

The KKT conditions

$$\begin{cases} 0 \in \partial_r(L(r, \lambda_1, \lambda_2)) \\ -\lambda_1 \in \partial\Psi_{K^*}(Wr + q) \\ -\lambda_2 \in \partial\Psi_K(r), \end{cases} \tag{1.12}$$

and can be formulated as follows

$$\begin{cases} W^T(r - \lambda_1) + u - \lambda_2 = 0 \\ u = Wr + q \\ -\lambda_1 \in \partial\Psi_{K^*}(u) \\ -\lambda_2 \in \partial\Psi_K(r). \end{cases} \tag{1.13}$$

Let us do the inner product of the first line in (1.12) with $r - \lambda_1$

$$\begin{aligned} 0 &= \langle r - \lambda_1, W^T(r - \lambda_1) + u - \lambda_2\rangle \\ &= \langle r - \lambda_1, W^T(r - \lambda_1)\rangle + \langle r - \lambda_1, u\rangle + \langle r - \lambda_1, -\lambda_2\rangle. \end{aligned}$$

Due to the complementarity between λ_1 and u and λ_2 and r, we can simplify as

$$0 = \langle r - \lambda_1, W^T(r - \lambda_1)\rangle + \langle r, u\rangle + \langle \lambda_1, \lambda_2\rangle. \tag{1.14}$$

Since W is semi–definite positive and the fact that $\lambda_1 \in K$ and $\lambda_2 \in K^*$, we can conclude that

$$\langle r, u\rangle \leq 0.$$

Since $r \in K$ and $u \in K^*$, we conclude that $\langle r, u\rangle = 0$.

Conversely, let us assume that r^* is the solution of LCP, then we have:

$$\begin{cases} \langle r^*, Wr^* + q\rangle = 0 \\ Wr^* + q \in K^* \\ r^* \in K. \end{cases}$$

From the constraints $r \in K,\ \ Wr + q \in K^* \Rightarrow \langle r, Wr + q\rangle \geq 0$, it mean that the cost function of QP is below bounded by $0 \Rightarrow r^*$ is solution of QP.

$\square$

Remark 1.5.

- *We note that the QP does not always have a solution. The existence of solution depends on feasiblity of the problem.*

- *Example, we consider the case the cone is the positive orthant, with the precise value of W, q as follows*

$$W = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, q = \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix}.$$

We see that W is satisfied the positive semi-definite condition. The feasible set is

$$F = \{r \in \mathbb{R}^3 : r \geq 0 \ \ and \ \ Wr + q \geq 0\},$$

But this set is empty, since $r \in F \Leftrightarrow$

$$\begin{pmatrix} r_1 + r_2 + q_1 \\ -r_1 - r_2 + q_2 \\ r_3 + q_3 \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} r_1 + r_2 \geq -q_1 \\ r_1 + r_2 \leq q_2 \\ r_3 \geq -q_3 \end{cases} \Leftrightarrow \begin{cases} r_1 + r_2 \geq 5 \\ r_1 + r_2 \leq 1 \\ r_3 \geq -q_3 \end{cases}$$

It is infeasible. So for the general case of cone, we do not have existence of solution for this problem for all W, q.

Remark 1.6.

- *It is the case for linear complementarity problem. We obtain the equivalence since the cost function of QP is convex, hence the solution of QP is global.*

- *Since it is for the arbitrary case, so it is true for the case of second-order cone (the frictional cone for contact problems).*

1.3.2 Equivalence between the frictional contact problem and the minimization problem

In this section, we examine the equivalence between the Frictional Contact Nonlinear Complementarity Problem (FCNCP):

$$\begin{cases} K^* \ni \hat{u} \perp r \in K \\ u = Wr + q \\ \hat{u} = u + g(u) \\ g(u) = (\mu^\alpha ||u_T^\alpha||, 0, 0)^T, \alpha = 1, ..., n_c. \end{cases} \tag{1.15}$$

and the Frictional Contact Optimization Problem (FCOP) of the form :

$$\begin{cases} \min_r \langle r, Wr + q\rangle + \langle r, (\mu||(Wr + q)_T||, 0, 0)^T)\rangle \\ s.t \;\; Wr + q + (\mu||(Wr + q)_T||, 0, 0)^T \in K^* \\ \qquad r \in K. \end{cases} \tag{1.16}$$

Proposition 1.2. *If the problem FCNCP* (1.15) *has a solution $r^\star, \hat{u}^\star$, then $r^\star$ is a optimal point of FCOP and the optimal cost is zero. Conversely, if $r^\star$ is a solution of FCOP with a optimal cost that vanishes, then $r^\star, \hat{u}^\star = Wr^\star + q + g(Wr^\star + q)$ is a solution of FCNCP* (1.15)

Proof. Let us introduce the following variables

$$u = Wr + q \quad \text{ and } \tilde{u} = u + g(u), \tag{1.17}$$

The minimization problem FCOP is then reformulated as

$$\begin{cases} \min_r \langle r, \tilde{u}\rangle \\ s.t \;\; \tilde{u} = Wr + q + g(Wr + q) \\ \qquad \tilde{u} \in K^* \\ \qquad r \in K. \end{cases} \tag{1.18}$$

Since $r \in K$ and $\tilde{u} \in K^*$, we have $\langle r, \tilde{u}\rangle \geq 0$. If the FCNCP has a solution, then $\langle r, \tilde{u}\rangle = 0$. □

Lemma 1.1. *For n_c contacts case, if r is solution of FCOP, then*

$$-\langle r-\lambda_1, G(\xi)\rangle \geq \langle r, \tilde{u}\rangle, \tag{1.19}$$

$$G(\xi) = (\mu_1(r-\lambda_1)_{N,1}\xi_1, 0, 0, \mu_2(r-\lambda_1)_{N,2}\xi_2, 0, 0, ..., \mu_{n_c}(r-\lambda_1)_{N,n_c}\xi_{n_c})^T.$$

Proof. The Lagrangian of FCOP

$$L(r;\lambda_1,\lambda_2) = \langle r, Wr+q\rangle + \langle r, g(Wr+q)\rangle - \lambda_1^T(Wr+q+g(Wr+q)) - \lambda_2^T r, \tag{1.20}$$

with $g(Wr+q) = (\mu_1||(Wr+q)_{T,1}||, 0, 0, \mu_2||(Wr+q)_{T,2}||, 0, 0, ..., \mu_{n_c}||(Wr+q)_{T,n_c}||, 0, 0)^T$,
$\Rightarrow \partial_r g(Wr+q) = (\mu_1\partial_r||(Wr+q)_{T,1}||, 0, 0, \mu_2\partial_r||(Wr+q)_{T,2}||, 0, 0, ..., \mu_{n_c}\partial_r||(Wr+q)_{T,n_c}||, 0, 0)^T$.
The subdifferential of the Lagrangian respect to r

$$\partial_r L = (W+W^T)r + q + g(Wr+q) + r^T\partial_r g(Wr+q) - W^T\lambda_1 - \lambda_1^T\partial_r g(Wr+q) - \lambda_2.$$

We set $-\xi_i \in \partial_r||(Wr+q)_{T,i}||, \xi = (\xi_1, 0, 0, \xi_2, 0, 0, ..., \xi_{n_c}, 0, 0)^T$,
$\Rightarrow \partial_r L = (W+W^T)r + q + g(Wr+q) - W^T\lambda_1 - \lambda_2$
$+ (\mu_1(r-\lambda_1)_{N,1}\xi_1, 0, 0, \mu_2(r-\lambda_1)_{N,2}\xi_2, 0, 0, ..., \mu_{n_c}(r-\lambda_1)_{N,n_c}\xi_{n_c})^T$
$= (W+W^T)r + q + g(Wr+q) - W^T\lambda_1 - \lambda_2 + G(\xi)$.
The KKT conditions are

$$\begin{cases} 0 \in \partial_r(L(r;\lambda_1,\lambda_2)) \\ -\lambda_1 \in \partial\Psi_{K^*}(Wr+q+g(Wr+q)) \\ -\lambda_2 \in \partial\Psi_K(r). \end{cases} \tag{1.21}$$

Let us do the same inner product with proposition (1.1), we obtain

$$-\langle r-\lambda_1, G(\xi)\rangle \geq \langle r, \tilde{u}\rangle.$$

□

The goal of computing the upper bound for inner product $\langle r, \tilde{u}\rangle$ in (1.19) is that we want to determine the conditions of frictional contact μ or the Delassus matrix W to be sure about the equivalence of FCNCP and FCOP.

Proposition 1.3. *If the problem is frictionless, i.e. $\mu = 0$, then FCNCP is equivalent to FCOP.*

Proof. Since $\mu = 0$, the nonlinear part $(\mu||u_T||, 0, 0)^T$ of $\tilde{u}$ is vanished, we come back to the quadratic form, using proposition (1.1) we have the equivalence.

Proposition 1.4. *If the value of Lagrange multiplier λ_1 corresponding to the constraint $\tilde{u} \in K^*$ is equal to r, then we obtain the equivalence.*

If $\lambda_1 = r \Rightarrow r - \lambda_1 = 0$

$$\Rightarrow \langle r - \lambda_1, G(\xi) \rangle = 0 \Rightarrow \rangle r, \tilde{u} \rangle \leq 0,$$

combine with the condition $r \in K, \tilde{u} \in K^* \Rightarrow \langle r, \tilde{u} \rangle = 0$, we obtain the equivalence. □

Remark 1.7. *It is difficult to give a mechanical interpretation of this condition, but it can be the object of future research.*

Monotony assumption and saddle-point problem : In the proposition (1.4), we mentioned on equality between r and Lagrange multiplier λ_1, this part we will make the precisely conditions for this proposition.

Definition 1.1. *(Saddle point in [11]) A point $(\hat{r}, \hat{\lambda})$ with $\hat{r} \geq 0$ and $\hat{\lambda} \geq 0$ is said to be a saddle point of the Lagrange function $L(r; \lambda)$ if:*

$$L(\hat{r}; \lambda) \leq L(\hat{r}; \hat{\lambda}) \leq L(r; \hat{\lambda}),$$

for all $r \geq 0, \lambda \geq 0$.

Here r, λ are variable and Lagrange multiplier, $\hat{r}$, $\hat{\lambda}$ are optimal solution and optimal mutiplier.

Proposition 1.5. *Let r^* is the solution of FCOP (1.16). If the point $(r^*; r^*, \tilde{u}(r^*))$, is the saddle point of the Lagrange function defined in (2.2), then the problem FCNCP is equivalent to FCOP.*

Proof. Firstly, If the point $(r^*; r^*, \tilde{u}(r^*))$ is the saddle point of the Lagrangian, then we have: r^* is the optimal solution of FCOP, and the pairs $(r^*, \tilde{u}(r^*))$ is the optimal Lagrangian multiplier (or they are solution of the dual problem of FCOP).
From the necessary condition, $(r^*; r^*, \tilde{u}(r^*))$ must be satisfied KKT condition, we can consider $r^* \backsim \lambda_1, \tilde{u}(r^*) \backsim \lambda_2$ in the above KKT condition.
Using proposition (1.4) we obtain the proof. □

Actually, we can obtain $(r^*, r^*, \tilde{u}(r^*))$ is the saddle point with the assumption of monotonicity of the function $\tilde{u}(r)$.

Proposition 1.6. *If $\hat{r}$ is the optimal solution of OP then $(\hat{r}, \hat{r}, \tilde{u}(\hat{r}))$ is the saddle point of Lagrange function, with the assumption $\tilde{u}$ is monotone.*

Proof. Let us denote by $\lambda = (\lambda_1, \lambda_2)$ the Lagrange multipliers.

Firstly, we prove $L(\hat{r};\lambda) \leq L(\hat{r};\hat{\lambda})$,

$$L(\hat{r};\lambda) \leq L(\hat{r};\hat{\lambda}) \Leftrightarrow \hat{r}\tilde{u}(\hat{r}) - \lambda_1\tilde{u}(\hat{r}) - \lambda_2\hat{r} \leq \hat{r}\tilde{u}(\hat{r}) - \hat{r}\tilde{u}(\hat{r}) - \hat{r}\tilde{u}(\hat{r})$$

$$\Leftrightarrow 2\hat{r}\tilde{u}(\hat{r}) - \lambda_1\tilde{u}(\hat{r}) - \lambda_2\hat{r} \leq 0.$$

It holds true since $\hat{r}$ is optimal solution of OP.

Hence, we continue proving $L(\hat{r},\hat{\lambda}) \leq L(r,\hat{\lambda})$, indeed

$$L(\hat{r},\hat{\lambda}) \leq L(r,\hat{\lambda}) \Leftrightarrow r\tilde{u}(r) - \hat{r}\tilde{u}(r) - \tilde{u}(\hat{r})r + \hat{r}\tilde{u}(\hat{r}) \geq 0$$

$$\Leftrightarrow r(\tilde{u}(r) - \tilde{u}(\hat{r})) + \hat{r}(\tilde{u}(\hat{r}) - \tilde{u}(r))$$

$$\Leftrightarrow (\tilde{u}(r) - \tilde{u}(\hat{r}))(r - \hat{r}) \geq 0.$$

If $\tilde{u}$ is monotone, $(\hat{r},\hat{\lambda})$ is saddle point of Lagrangian. □

Corollary 1.1. *If $\tilde{u}$ is monotone, then NCP is equivalent with OP.*

This result is from proposition (1.6) and proposition (1.1). Actually, we need to assume that FCOP has solution and the strong duality satisfied here.

Remark 1.8. *With the assumption of small value of friction coefficients μ and Delassus matrix is positive definite, we can obtain the monotony of $\tilde{u}$.*

Lemma 1.2. *(Asymptotic condition) With the value of $\mu \leq \mu_0$, $\mu_0 = \dfrac{1}{3n_c \max_{i,j}|w_{i,j}| + 10^K}$, $K >$ 0, $w_{i,j} \in W^T E_T^T$, with $E_T : \mathbb{R}^{3n_c} \to \mathbb{R}^{2n_c}$ such that $x_T = E_T x$, then we obtain $\mu||\xi|| \to 0$.*

Proof. Let us introduce the projector $E_T : \mathbb{R}^{3n_c} \to \mathbb{R}^{2n_c}$ such that $x_T = E_T x$. Using the chain rule, we get

$$\partial_r||(Wr+q)_T|| = \partial_r||E_T(Wr+q)|| = W^T E_T^T \partial_y||y|| \text{ with } y = E_T(Wr+q),$$

$\Rightarrow \xi = W^T E_T^T \partial_y ||y||$,

with

$$\partial_y||y|| = \begin{cases} \frac{y}{||y||}, & y \neq 0 \\ B[0,1], & y = 0. \end{cases} \tag{1.22}$$

$\Rightarrow ||\xi|| \leq ||W^T E_T^T||$.

We define the norm of matrix $W^T E_T^T$ as $||.||_1$ or $||.||_\infty$

$$\Rightarrow \mu||\xi|| \leq \mu \max_j \sum_i |w_{i,j}| << \mu 3n_c \max_{i,j} |w_{i,j}|, \; w_{i,j} \in W^T E_T^T.$$

Choosing $\mu_0 = \dfrac{1}{3n_c \max_{i,j}|w_{i,j}| + 10^K}$, $K > 0$, $w_{i,j} \in W^T E_T^T$, then $\forall\ \mu \leq \mu_0$ we obtain $\mu||\xi|| \to 0$. $\square$

Corollary 1.2. *With the condition of μ as in lemma (1.2) we obtain the equivalence of NCP and OP.*

1.3.3 A counter-example to the equivalence.

Proposition 1.7. *With the arbitrary value of μ, the solutions of the optimality conditions in (1.17) are not equivalent to the nonlinear complementarity problem (1.14).*

Proof. Let us proof this result by exhibiting a counter example.

Let us consider the case: $r_T = \mu r_N > 0, u_N > 0, u_T > 0$, with the setup value as follows:

$$\mu = 5, W = \begin{pmatrix} 6 & -2 \\ -2 & 1 \end{pmatrix}, q = \begin{pmatrix} 7 \\ -2.9 \end{pmatrix}$$

$$\Rightarrow u = \begin{pmatrix} 3 \\ 0.1 \end{pmatrix}, \tilde{u} = \begin{pmatrix} 3.5 \\ 0.1 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 9.5 \\ -1.9 \end{pmatrix}.$$

We check again all the equations of KKT condition for MP:

$$W^T r + \tilde{u} + \mu r_N \xi - \lambda_2 = \begin{pmatrix} 6 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 3.5 \\ 0.1 \end{pmatrix} - 5\begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 9.5 \\ -1.9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\tilde{u} = Wr + q + \begin{pmatrix} \mu(Wr+q)_T \\ 0 \end{pmatrix}.$$

$$\xi = -\begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow -\xi = 1 \times \begin{pmatrix} b \\ d \end{pmatrix} \Rightarrow -\xi \in \partial_r (Wr+q)_T,$$

$$\lambda_1 = 0 \Rightarrow -\lambda_1 \in N_{K^*}(\tilde{u}).$$

$$\lambda_2 = \begin{pmatrix} 9.5 \\ -1.9 \end{pmatrix} = 5\begin{pmatrix} 5 \\ -1 \end{pmatrix} \Rightarrow -\lambda_2 \in N_K(r) = -K^* \cap r^{\perp},\ \ r = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

We see that the value of $\langle r, \tilde{u}\rangle = 4 > 0$.

For this case, if we take the large value of μ such that $\mu > \dfrac{a}{-b}$, $b < 0$ then we get the counter-example.

$\square$

Figure (1.1) illustrates the function $G(r_N) = \langle r, \tilde{u} \rangle$.

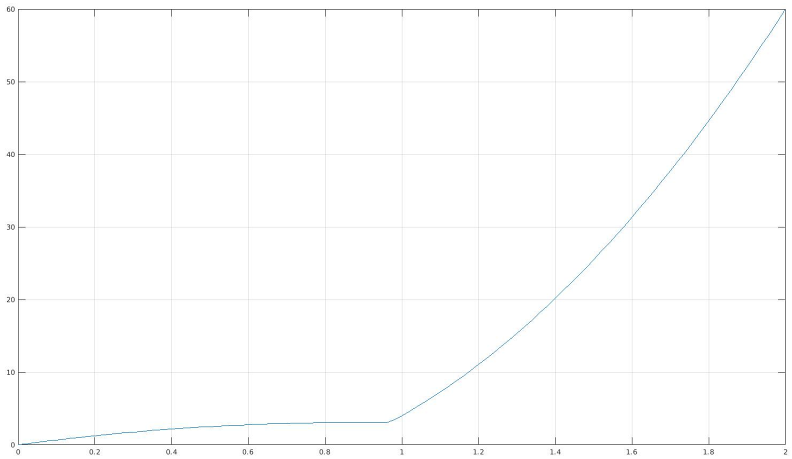


Figure 1.1: Illustration of counter example.

By using AMPL, we check the optimal solution (consider over interval $[0.5, +\infty)$ since we do not consider the take off case) reach when $r_N = 0.966667$, and the value of cost function is $G(r) = 3.02889$.

In this case, we consider the simple example: dimension of 2, and we already know the relation $r_T = \mu r_N$, both the cost function and constraints are one variables function (nonlinear function). So using AMPL here can give us the good solution.
One question is that the solution of AMPL is global? Actually by using solver suitable to the task we want (example nonlinear global: LGO, BARON, or ILP: CPLEX, GUROBI..) we can obtain the correct solution (in case the error is very small).

Remark 1.9.

- *In this simple example (in the space of dimension 2 with one contact), we see that at least the value of μ must be less than $-\dfrac{a}{b}$ in case $b < 0$.*
- *If $r = 0$ or $r_N = 0$ (in take off case) is the solution of MP, we always have the equivalence between FCOP and FCNCP (cost function reach at $0 \quad \forall \mu$).*
- *In the above example, there is a local minima are reached so we can not guarantee that the cost function equal zero, hence we have no equivalence.*
- *From the optimization point of view, FCOP is difficult to solve since it is nonsmooth and nonconvex cost function.*

We can see in the counter-example, u_T and r_T have the same direction, u_N and r_N is not in complementarity, this thing comes from the fact that if the system is satisfied the Signorini-Coulomb model, since we took a look at all the cases, there are no counter-examples. (Appendix)

1.3.4 Problem with adding slack variable s_i

The equivalence between two formulations

By using the idea of least square problem, we reformulate (1.8) as

$$\begin{cases} \min_{r,s}\langle r, Wr+q\rangle + \sum_{i=1}^{n_c}\mu_i r_{N,i}s_i \\ s.t \qquad ||(Wr+q)_{T,i}|| \le s_i, \qquad i=1..n_c \\ \qquad\quad (Wr+q)_{N,i} \ge 0, \qquad i=1..n_c \\ \qquad\quad r_i \in K_i, \qquad i=1..n_c. \end{cases} \tag{1.23}$$

Proposition 1.8. *The problem (1.8) is equivalent to the problem (1.23).*

Proof. We consider the equivalence between (1.8) and (1.23) in case that solution of (1.8) is solution of (1.23) and conversely.

"$\Rightarrow$"

We define the feasible set of (1.8) as:

$$F_1 = \{r \in \mathbb{R}^{3n_c} : (Wr+q)_{N,i} \ge 0 \text{ and } \ r_i \in K_i, i=1,..,n_c\}.$$

The feasible set of (1.23):

$$F_2 = \{(r,s) \in \mathbb{R}^{3n_c+n_c} : ||(Wr+q)_{T,i}|| \le s_i, (Wr+q)_{N,i} \ge 0 \text{ and } r_i \in K_i, i=1..n_c\}.$$

Assume that (1.8) has a solution and this solution is $r^* \in F_1$, we have:

$$\langle r^*, Wr^*+q\rangle + \sum_{i=1}^{n_c}\mu_i r^*_{N,i}||(Wr^*+q)_{T,i}|| \le \langle r, Wr+q\rangle + \sum_{i=1}^{n_c}\mu_i r_{N,i}||(Wr+q)_{T,i}||,$$

If we call $s^* = (s^*_1, s^*_2, ..., s^*_{n_c})$, $\ s^*_i = ||(Wr^*+q)_{T,i}||$, $\ i=1..n_c$, we will show that

(r^*, s^*) is solution of (1.23). Indeed,

$$\begin{aligned}\langle r^*, Wr^* + q\rangle + \sum_{i=1}^{n_c} \mu_i r_{N,i} s_i^* &= \langle r^*, Wr^* + q\rangle + \sum_{i=1}^{n_c} \mu_i r^*_{N,i} ||(Wr^* + q)_{T,i}|| \\ &\leq \langle r, Wr + q\rangle + \sum_{i=1}^{n_c} \mu_i r_{N,i} ||(Wr + q)_{T,i}|| \\ &\leq \langle r, Wr + q\rangle + \sum_{i=1}^{n_c} \mu_i r_{N,i} s_i,\end{aligned}$$

and we also have $(r^*, s^*) \in F_2$, so this is the solution of (1.23).

"$\Leftarrow$"

We consider the optimal problem (1.23): from the second constraint we see that $s_i \geq 0, \forall i = 1..n_c$.

We also have assumption that $\mu_i \geq 0,\ \ r_{N,i} \geq 0,\ \ i = 1..n_c$. So the second part of cost function $\sum_{i=1}^{n_c} \mu_i r_{N,i} s_i \geq 0$.

$$\Rightarrow \min_{r,s} \langle r, Wr + q\rangle + \sum_{i=1}^{n_c} \mu_i r_{N,i} s_i = \min_{r} \langle r, Wr + q\rangle + \min_{r,s} \sum_{i=1}^{n_c} \mu_i r_{N,i} s_i, \tag{1.24}$$

with $||(Wr + q)_{T,i}|| \leq s_i, \forall i = 1..n_c$

$$\Rightarrow \min_{r,s} \sum_{i=1}^{n_c} \mu_i r_{N,i} s_i = \min_{r} \sum_{i=1}^{n_c} \mu_i r_{N,i} ||(Wr + q)_{T,i}||.$$

So cost function of (1.24) equivalent to:

$$\min_{r} \langle r, Wr + q\rangle + \sum_{i=1}^{n_c} \mu_i r_{N,i} ||(Wr + q)_{T,i}||,$$

this is the cost function of (1.8). $\square$

Properties of the problem (2.2)

Smooth objective function From adding the slack variable s_i in (1.8) we add n_c more constraints but we can obtain a smooth objective function

$$f(r, s) = \langle r, Wr + q\rangle + \sum_{i=1}^{n_c} \mu_i r_{N,i} s_i, \tag{1.25}$$

$f(r, s) \in C^\infty$. The gradient of $f(r, s)$ is:

$$\nabla f = \begin{pmatrix} \partial_r f \\ \partial_s f \end{pmatrix} = \begin{pmatrix} (W + W^T)r + q + (\mu_1 s_1, 0, 0, \mu_2 s_2, 0, 0, ..., \mu_{n_c} s_{n_c}, 0, 0)^T_{1\times 3n_c} \\ (\mu_1 r_{N,1}, \mu_2 r_{N,2}, ..., \mu_{n_c} r_{N,n_c})^T_{1\times n_c} \end{pmatrix}$$

By denoting $z = (r, s)^T$ we reformulate the objective function of MP as a quadratic form:

$$f(z) = z^T H_1 z + z^T h, \tag{1.26}$$

with

$$H_1 = \begin{pmatrix} (W + W^T)_{3n_c \times 3n_c} & \frac{1}{2}(\bar{\mu})^T_{3n_c \times n_c} \\ \frac{1}{2}(\bar{\mu})_{n_c \times 3n_c} & 0_{n_c \times n_c} \end{pmatrix}, \quad h = \begin{pmatrix} q \\ 0 \end{pmatrix}_{4n_c \times 1}. \tag{1.27}$$

It is clear that the cost function is smooth.

Properties of the Hessian The Hessian can be compactly written as:

$$H = H_1 + H_1^T = \begin{pmatrix} W + W^T & \bar{\mu}^T \\ \bar{\mu} & 0 \end{pmatrix}, \tag{1.28}$$

with the matrix $\bar{\mu}$ is as follows

$$\bar{\mu} = (\mu_{i,j})_{n_c \times 3n_c}, \mu_{i,j} = \begin{cases} \mu_i, & if \;\; j = 3i - 2, \\ 0, & otherwise. \end{cases}$$

To make simple the notation, from now we define the matrix has the same structure with $\tilde{\mu}$ as $\tilde{\mu} = (\mu^i_{(i,3i-2)})_{n_c \times 3n_c}$.

Remark 1.10. *From the form of the Hessian matrix, we see that if exist one value of $\mu_i = 0$ then the Hessian matrix is singular. In this case, we can reduce the size of H by eliminate a row (in $\bar{\mu}$) and a collum (in $\bar{\mu}^T$) corresponding to the value of $\mu = 0$ to avoid the singular property of Hessian matrix.*

Remark 1.11. *The Hessian matrix H is copositive.*

Proposition 1.9. *In general case, H is indefinite, even with the small value of μ.*

Proof. Firstly, we show that $\exists (r, s) \in \mathbb{R}^{4n_c}$ such that $(r, s)H(r, s)^T \geq 0$.
Example,$(r, s) : r_{N,i} \in \mathbb{R}_+, r_{T,i} \in \mathbb{R}^2, s_i \in \mathbb{R}_+$ we always obtain $(r, s)H(r, s)^T \geq 0$.
Secondly, we will demonstrate that even with the very small value of $0 \neq \mu << 1$, we still can choose the vector $(r, s) : (r, s)H(r, s)^T < 0$.
Indeed,

$$\Rightarrow (r, s)H(r, s)^T = r^T (W + W^T) r + \mu r^T A s, \;\; A = (1_{3i-2,i})_{3n_c \times n_c}, i = 1, ..., n_c.$$

If we choose $s = -\alpha r$ with $\alpha > \dfrac{||r||^2_{(W+W^T)}}{\mu||r||^2_A}$ then we always have the negative definite property of H.

In case $\mu = 0$ (no friction in contact) we have the SDP property of H. $\square$

Remark 1.12. *Since the Hessian matrix is either SDP or negative definite, so we can not ensure about the convexity of cost function. It leads to the difficulty in solving our problems.*

Remark 1.13. *One question is that is the objective function convex over the cone* $K \times \mathbb{R}^{n_c}_+$*. Normally, if we have the linear constraints then the cone here becomes linear subspace, so the objective function is convex. But in our case the cone is not linear subspace, it is second-order cone, can we talk about the convexity of cost function? In general the answer is "No".*

Existence of solutions In ([2], [3], [4]) have an examining of conditions to make the existence of solution of incremental problems. With the following assumptions the problem has a solution

- The mass matrix $M \in S^n_{++}$.
- $\exists v \in \mathbb{R}^m \;\; s.t \;\; u = Hv + w = (u_1, u_2, .., u_m), u_i \in intK^*_i, \;\; \forall i.$

We note that if the matrix H has full rank, then the second assumption is satisfied, so in this case, incremental problem has a solution.

In case $w = 0$, we also obtain the existing solution of problem.

By reformulating the incremental problems as fixed-point formulation we can see that if the Delassus matrix is positive-definite we get the existence of the problem.

1.3.5 A new optimization formulation thanks to Patrick Ballard.

In this part of report we propose the new reformulation of MP. The idea comes from the following lemma.

Lemma 1.3.

$$||u_T|| = \max_{\alpha \in \mathbb{B}[0,1]} \alpha^T u_T, \;\; u_T, \alpha \in \mathbb{R}^2.$$

Proof. Firstly, consider the case $n = 1$, we have:

- If $u_T < 0 \Rightarrow \max_{\alpha \in [-1,1]} \alpha u_T = -u_T = |u_T|$ (reach when $\alpha = -1$).

- If $u_T \leq 0 \Rightarrow \max_{\alpha \in [-1,1]} \alpha u_T = u_T = |u_T|$ (reach when $\alpha = 1$).

Next we consider the case $n = 2$. To make it more convenient, we denote:

$$\alpha = (\alpha_1, \alpha_2)^T, u_T = (u_{T,1}, u_{T,2})^T$$

$$\Rightarrow (\alpha^T u_T)^2 = (\alpha_1 u_{t,1} + \alpha_2 u_{T,2})^2.$$

By using Cauchy-Schwarz inequality, and the fact that $\alpha \in \mathbb{B}[0,1]$ we obtain:

$$\Rightarrow (\alpha^T u_T)^2 \leq (\alpha_1^2 + \alpha_2^2)(u_{T,1}^2 + u_{T,2}^2) \leq (u_{T,1}^2 + u_{T,2}^2) = ||u_T||^2$$

$$\Rightarrow (\alpha^T u_T)^2 - ||u_T||^2 \leq 0 \Leftrightarrow (\alpha^T u_T - ||u_T||)(\alpha^T u_T + ||u_T||) \leq 0$$

$$\Leftrightarrow -||u_T|| \leq \alpha^T u_T \leq ||u_T||.$$

□

Apply lemma (1.3), we can reformulate MP as follows:

$$\begin{cases} \max \alpha \\ ||\alpha_i|| \leq 1 \\ \begin{cases} \min_r r^T W r + r^T q + \sum_{i=1}^{n_c} \mu_i r_{N,i} \alpha_i^T (Wr + q)_{T,i} \\ s.t (Wr + q)_N \geq 0 \\ r \in K \end{cases} \end{cases} \tag{1.29}$$

Remark 1.14. *In general, This problem is not QP and not convex.*

The problem can be seen as a bi-level programming if we reformulate as follows:

$$\begin{cases} \max_{\alpha,r} r^T W r + r^T q + \sum_{i=1}^{n_c} \mu_i r_{N,i} \alpha_i^T (Wr + q)_{T,i} \\ s.t \;\; r \in \;\; \mathrm{argmin}_{(\mathrm{Wr+q})_{\mathrm{N,i}} \geq 0, \mathrm{r} \in \mathrm{K}} \mathrm{r^T W r + r^T q} + \sum_{\mathrm{i}=1}^{\mathrm{n_c}} \mu_{\mathrm{i}} \mathrm{r_{N,i}} \alpha_{\mathrm{i}}^{\mathrm{T}} (\mathrm{Wr + q})_{\mathrm{T,i}}, \\ \qquad ||\alpha_i|| \leq 1 \end{cases} \tag{1.30}$$

By writing the KKT condition of sub-problem we can obtain the "Mathematical

Programming with Equilibrium Constraints" (MPEC).

$$\begin{cases} \max_{\alpha,r,\lambda_1,\lambda_2} r^T W r + r^T q + \sum_{i=1}^{n_c} \mu_i r_{N,i} \alpha_i^T (Wr+q)_{T,i} \\ s.t\ \ (W+W^T)r + q + A_f - A_{g_1}^T \lambda_1 - \lambda_2 = 0 \\ \qquad 0 \le -\lambda_1 \perp g_1 \ge 0 \\ \qquad -\lambda_2 \in \partial\psi_{K^*}(r), \\ \qquad ||\alpha_1|| \le 1, \end{cases} \tag{1.31}$$

with A_f is gradient of $f = \sum_{i=1}^{n_c} \mu_i r_{N,i} \alpha_i^T (Wr+q)_{T,i}$ respect to r, $g_1 = [(Wr+q)_{N,i}, i = 1...n_c]$, $A_{g_1} = \nabla_r g_1$.

Chapter 2

Numerical method for solving contact problems

In this part of report, we present IPM to solve optimization formulation of contact problems. We focus on solving two formulations which are the modified fixed-point problem and FCOP. IPM is well-known for solving linear and quadratic programming which have a large size of data. The method is based on solving the KKT condition system which can be written explicitly in our case tend to adapt IPM for solving fixed-point formulation.

2.1 Interior point method for solving fixed-point optimization problem

In this section we adapt interior point method base on [5, 8, 10, 11] to solve the modified fixed-point formulation (1.7)

$$\begin{cases} \min_r \dfrac{1}{2} r^T W r + r^T q(s) \\ s.t\ \ r_i^T Q_i r_i \geq 0,\ \ i = 1, ..., n_c, \\ \qquad r_{N,i} \geq 0, \quad i = 1, ..., n_c. \end{cases} \qquad Q_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{-1}{\mu_i^2} & 0 \\ 0 & 0 & \dfrac{-1}{\mu_i^2} \end{pmatrix}$$

The Lagrangian:

$$L(r, \lambda_1, \lambda_2) = \frac{1}{2} r^T W r + r^T q(s) - \lambda_1^T g_1(r) - \lambda_2^T g_2(r),$$

$g_1(r) = (g_1^1(r), g_1^2(r), ..., g_1^{n_c}(r)),\ \ g_1^i(r) = r_i^T Q_i r_i,\ \ g_2(r) = (g_2^1(r), g_2^2(r), ..., g_2^{n_c}(r)),\ \ g_2^i(r) = r_{N,i},\ \ i = 1, ..., n_c,$

and the lagrange multipliers $\lambda_1 = (\lambda_1^1, \lambda_1^2, ..., \lambda_1^{n_c}),\ \ \lambda_2 = (\lambda_2^1, \lambda_2^2, ..., \lambda_2^{n_c})\ \ i = 1, ..., n_c$.

$$\Rightarrow \nabla_r L(r, z) = Wr + q(s) - \nabla g_1^T \lambda_1 - \nabla g_2^T \lambda_2,$$

with

$$\nabla g_1 = \begin{pmatrix} 2(r_1^T Q_1)_{1\times 3} & [0] & [0] & ... & [0] \\ [0] & 2(r_2 Q_2)_{1\times 3} & [0] & ... & [0] \\ ... & ... & ... & & \\ [0] & [0] & [0] & .... & 2(r_{n_c} Q_{n_c})_{1\times 3} \end{pmatrix}_{n_c \times 3n_c}, \nabla g_2 = (1_{i,3i-2})_{n_c \times 3n_c}.$$

The KKT conditions are

$$\begin{cases} \nabla_r L(r,z) = 0 \\ \Lambda_1 G_1 e = 0 \\ \Lambda_2 G_2 e = 0 \\ r_i^T Q_i r_i \geq 0, \ \ i = 1, ..., n_c \\ r_{N,i} \geq 0, \ \ i = 1, ..., n_c \\ \lambda_1, \lambda_2 \geq 0, \end{cases} \tag{2.1}$$

$\Lambda_1 = diag(\lambda_1), \Lambda_2 = diag(\lambda_2), G_1 = diag(g_1), G_2 = diag(g_2), e = (1, 1, ..., 1)^T_{n_c \times 1}$.

With the perturbed KKT conditions are

$$\begin{cases} \nabla_r L(r,z) = 0 \\ \Lambda_1 G_1 e = \sigma_1 e \\ \Lambda_2 G_2 e = \sigma_2 e \\ r_i^T Q_i r_i \geq 0, \ \ i = 1, ..., n_c \\ r_{N,i} \geq 0, \ \ i = 1, ..., n_c \\ \lambda_1, \lambda_2 \geq 0, \end{cases} \tag{2.2}$$

with the small value of $\sigma_1 = \sigma_2 = 0.1$.

We introduce the residual at step k

$$r_k = \begin{pmatrix} Wr^k + q(s) - \nabla g_1^T(r^k)\lambda_1^k - \nabla g_2^T(r^k)\lambda_2^k \\ diag(\lambda_1^k) diag(g_1(r^k))e - \sigma_1 e \\ diag(\lambda_2^k) diag(g_2(r^k))e - \sigma_2 e \end{pmatrix}. \tag{2.3}$$

We use Newton's method to solve system:

$$\begin{pmatrix} W-C_1 & -A_{g_1} & -A_{g_2} \\ C_2 & G_1^k & [0]_{n_c\times n_c} \\ C_4 & [0]_{n_c\times n_c} & G_2^k \end{pmatrix} \begin{pmatrix} \triangle r \\ \triangle\lambda_1 \\ \triangle\lambda_2 \end{pmatrix} = - \begin{pmatrix} Wr^k + q(s) - \nabla g_1^T(r^k)\lambda_1^k - \nabla g_2^T(r^k)\lambda_2^k \\ diag(\lambda_1^k)diag(g_1(r^k))e - \sigma_1 e \\ diag(\lambda_2^k)diag(g_2(r^k))e - \sigma_2 e \end{pmatrix} \tag{2.4}$$

with

$$C_1 = 2\begin{pmatrix} \lambda_1^1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & -\lambda_1^1\frac{1}{\mu_1^2} & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & -\lambda_1^1\frac{1}{\mu_1^2} & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & & & \\ 0 & 0 & 0 & ... & \lambda_1^{n_c} & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & -\lambda_1^{n_c}\frac{1}{\mu_{n_c}^2} & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & -\lambda_1^{n_c}\frac{1}{\mu_{n_c}^2} \end{pmatrix}_{3n_c\times 3n_c}$$

$$A_{g_1} = 2\begin{pmatrix} Q_1r_1 & [0] & [0] & ... & [0] \\ [0] & Q_2r_2 & [0] & ... & [0] \\ ... & ... & ... & ... & \\ [0] & [0] & [0] & ... & Q_{n_c}r_{n_c} \end{pmatrix}_{3n_c\times n_c} , [0] = [0]_{3\times 1}, A_{g_2} = (1_{3i-2,i})_{3n_c\times n_c}$$

$$C_2 = 2\begin{pmatrix} \lambda_1^1 r_1^T Q_1 & [0] & ... & [0] \\ [0] & \lambda_1^2 r_2^T Q_2 & ... & [0] \\ [0] & [0] & ... & \lambda_1^{n_c} r_{n_c}^T Q_{n_c} \end{pmatrix}_{n_c\times 3n_c} , C4 = \begin{pmatrix} \lambda_2^1 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^2 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & & & & \\ 0 & 0 & 0 & 0 & ... & \lambda_2^{n_c} & 0 & 0 \end{pmatrix}_{n_c\times 3n_c}$$

We define a neighborhood $\mathcal{N}$ of the central path as follows:

$$\mathcal{N} = \{(r,\lambda_1,\lambda_2) : \lambda_1 > 0,\ \ \lambda_2 > 0,\ \ r_i^T Q_i r_i > 0,\ \ r_{i,N} > 0,\ \ i = 1,...,n_c\}.$$

After we compute the direction $(\triangle r, \triangle\lambda_1, \triangle\lambda_2)$ we compute the step lenght $(\alpha_r^k, \alpha_{\lambda_1}^k, \alpha_{\lambda_2}^k)$:

$$(r^k + \alpha_r^k\triangle r^k, \lambda_1^k + \alpha_{\lambda_1}^k\triangle\lambda_1^k, \lambda_2^k + \alpha_{\lambda_2}^k\triangle\lambda_2^k) \in \mathcal{N}.$$

We choose as follows:

According to Armijo rule

- Compute the largest $\alpha_{\lambda_1} : \lambda_1^{k+1} > 0 \Leftrightarrow \lambda_1^k + \alpha_{\lambda_1}^k \triangle\lambda_1^k > 0$, that is:

$$\alpha_{\lambda_1} = 0.9 \min\{1, \min_{i:\triangle\lambda_1^i<0} \frac{-\lambda_1^{ik}}{\triangle\lambda_1^{ik}}\}.$$

- Compute the largest $\alpha_{\lambda_2} : \lambda_2^{k+1} > 0 \Leftrightarrow \lambda_2^k + \alpha_{\lambda_2}^k \triangle\lambda_2^k > 0$, that is:

$$\alpha_{\lambda_2} = 0.9 \min\{1, \min_{i:\triangle\lambda_2^i<0} \frac{-\lambda_2^{ik}}{\triangle\lambda_2^{ik}}\}.$$

- Compute the largest $\alpha_{rN} : r_{i,N}^{k+1} > 0 \ \ \forall i = 1, .., n_c$ or $r_{i,N}^k + \alpha_{r_i} \triangle r_{i,N}^k > 0$, that is:

$$\alpha_{r_{iN}} = 0.9 \min\{1, \min_{i:\triangle r_{i,N}^k<0} \frac{-r_{i,N}^k}{\triangle r_{i,N}^k}\}.$$

$$\alpha_{rN} = rand(1) \min\{\alpha_{r_{iN}}\}.$$

- Compute the largest $\alpha_{rT} : r_i^T Q_i r_i \geq 0 \ \ \forall i = 1, .., n_c$, that is

$$\alpha_{rT} = \min\{\alpha_{r_iT}\},$$

α_{r_iT} is the largest value of α such that $(r_i + \alpha_k \triangle r^i)^T Q_i (r_i + \alpha_k \triangle r^i) \geq 0$ (We use Armijo rule in this step:
while $(r_i + \alpha_k \triangle r^i)^T Q_i (r_i + \alpha_k \triangle r^i) \leq 0$
$\alpha_{r_iT} = \dfrac{\alpha_{r_iT}}{2}$.
We choose

$$\alpha = \min\{\alpha_{rN}, rand(1)\alpha_{rT}\}, .$$

According to backtracking strategy in [10]

- $\alpha_{\lambda_1}, \alpha_{\lambda_2}$ is computted the same.

- To compute α_r^k, we start at α_{r_N} as Armijo rule, we substitute value of α_{r_N} to the quadratic constraints, and reduce the value of α_{r_N} by multiply with the random value $\gamma \in (0, 1)$ until we get the positive of constraints.

Update

After finding the direction $\triangle r, \ \triangle\lambda_1, \ \triangle\lambda_2$ and the step length $\alpha_r, \ \alpha_{\lambda_1}, \ \alpha_{\lambda_2}$, we update the value of r, z in the next iteration as follows:

$$r \leftarrow r + \alpha_r \triangle r, \ \ \lambda_1 \leftarrow \lambda_1 + \alpha_{\lambda_1} \triangle\lambda_1, \ \ \lambda_2 \leftarrow \lambda_2 + \alpha_{\lambda_2} \triangle\lambda_2. \tag{2.5}$$

Stop criterion

Firstly, to stop iterations inside the loop of IPM, we normally use

$$dgap \le \epsilon_{IPM}, \ \ dgap = \frac{(g_1(r), g_2(r))^T(\lambda_1, \lambda_2)}{n_c}, \ \ \epsilon_{IPM} \backsim 1e-6. \tag{2.6}$$

Since we are considering s as a parameter inside the loop of IPM, to find the solution of the problem we need to change value of s outside the interation of the Interior point algorithm.

In this study, we use as an error criterion the natural map, mentioned on [4], which is defined as follows

$$F_\rho^{nat}(r) = r - \prod_K (r - \rho(Wr + q(s))), \rho > 0. \tag{2.7}$$

where $\prod_K$ is the Euclidean projector on the second-order cone K of $\mathbb{R}^3$ which is defined as

$$\prod_K(r) = \begin{cases} r & \text{if } r \in K \\ 0 & \text{if} - r \in K^* \\ \dfrac{1}{1+\mu^2}(r_N + \mu||r_T||)\begin{pmatrix} 1 \\ \mu\dfrac{r_T}{||r_T||}\end{pmatrix} & \text{if } r \notin K \text{and} - r \notin K^* \end{cases} \tag{2.8}$$

in our case K is second-order cone, r is solution of incremental problem if and only if $F_\rho^{nat}(r) = 0$.

From this, we stop when

$$err = \frac{||r - \prod_K (r - \rho(Wr + q(s)))||}{||q||}, \ \ ||q|| > 1e-16. \tag{2.9}$$

Algorithm 1 Friedlander-Orban-Armijo

Input: W, q, μ.

Output: Optimal solution of MP.

Require: $r^0 \in \mathbb{R}^{3n_c \times 1}, \lambda_1^0 \in \mathbb{R}_{++}^{n_c \times 1}, \lambda_2^0 \in \mathbb{R}_{++}^{n_c \times 1}$, $\sigma_1, \sigma_2 \in [\sigma_{min}, \sigma_{max}], 0 < \sigma_{min} < \sigma_{max} < 1, \beta_1, \beta_2, \beta_3, \beta_4 \in (0,1)$, $tol = 1e-8$,, itemax, and set $k = 0$.

```
procedure FRIEDLANDER-ORBAN-A(W, q, μ, tol, s) → ( r, u)
    for k = 0, 1, ... do
        dgap_k := (g_1(r)^k, g_2(r^k))^T (λ_1^k, λ_2^k) / n_c.        ▷ Compute the dual gap
        if dgap_k ≤ tol then
            fprintf('Optimal found')
            break, end
        else
            k = k + 1
        Solve system (2.4)                                          ▷ Searching for direction
        Using Armijo rule to compute α_r^k, α_{λ1}^k, α_{λ2}^k such that      ▷ Searching for step-length
        (r^{k+1}, λ_1^{k+1}, λ_2^{k+1}) = (r^k, λ_1^k, λ_2^k) + (α_r^k △r^k, α_{λ1}^k △λ_1^k, α_{λ2}^k △λ_2^k) ∈ N.
        Update
        r^{k+1} ← r^k α_r^k △r^k, λ_1^{k+1} ← λ_1^k + α_{λ1}^k △λ_1^k, λ_2^{k+1} ← λ_2^k + α_{λ2}^k △λ_2^k, k ← k + 1.
        u^{k+1} = W r^{k+1} + q
    end if
    end for
```

Algorithm 2 Friedlander-Orban-Backtracking

Same procedure with Friedlander-Orban-Armijo, but using backtracking strategy to find the value of step-length.

Algorithm 3 Mehrotra-Armijo

Input: W, q, μ.

Output: Optimal solution of MP.

Require: $r^0 \in \mathbb{R}^{3n_c\times 1}, \lambda_1^0 \in \mathbb{R}_{++}^{n_c\times 1}, \lambda_2^0 \in \mathbb{R}_{++}^{n_c\times 1}$, $\sigma_1, \sigma_2 \in [\sigma_{min}, \sigma_{max}], 0 < \sigma_{min} < \sigma_{max} < 1, \beta_1, \beta_2, \beta_3, \beta_4 \in (0,1), \tau = 0.99$, $tol = 1e-8$,, itemax, and set $k = 0$.

procedure MEHROTRA-A$(W, q, \mu, tol, s) \to (\ r, u)$
    for $k = 0, 1, ...$ **do**
        $dgap_k := \dfrac{(g_1(r)^k, g_2(r^k))^T(\lambda_1^k, \lambda_2^k)}{n_c}$. ▷ Compute the dual gap
        if $dgap_k \leq tol$ **then**
            fprintf('Optimal found')
            break, end
        else
            $k = k + 1$
        Solve system (2.4). ▷ Searching for predictor direction
        Using Armijo rule to compute $\alpha_r^k, \alpha_{\lambda_1}^k, \alpha_{\lambda_2}^k$ such that ▷ Searching for predictor step-length
        $(r^{k+1}, \lambda_1^{k+1}, \lambda_2^{k+1}) = (r^k, \lambda_1^k, \lambda_2^k) + (\alpha_r^k \triangle r^k, \alpha_{\lambda_1}^k \triangle \lambda_1^k, \alpha_{\lambda_2}^k \triangle \lambda_2^k) \in \mathcal{N}$.
        Predictor Update $r^{k+1} \leftarrow r^k \alpha_r^k \triangle r^k, \lambda_1^{k+1} \leftarrow \lambda_1^k + \alpha_{\lambda_1}^k \triangle \lambda_1^k, \lambda_2^{k+1} \leftarrow \lambda_2^k + \alpha_{\lambda_2}^k \triangle \lambda_2^k, k \leftarrow k + 1$.
end if
        Centering parameter
        $dgap_* = \dfrac{(g_1(r^{k+1}), g_2(r^{k+1}))^T(\lambda_1, \lambda_2)}{n}$,
        Set the centering parameter $\sigma = rand(1)0.9(\dfrac{dgap_*}{dgap})^3$.
        Solve system (2.4) with $\sigma_1 = \sigma_2 = \sigma$ ▷ Searching for corrector direction.
        Using Armijo rule to compute $\alpha_r^k, \alpha_{\lambda_1}^k, \alpha_{\lambda_2}^k$ such that ▷ Searching for corrector step-length
        $(r^{k+1}, \lambda_1^{k+1}, \lambda_2^{k+1}) = (r^k, \lambda_1^k, \lambda_2^k) + (\alpha_r^k \triangle r^k, \alpha_{\lambda_1}^k \triangle \lambda_1^k, \alpha_{\lambda_2}^k \triangle \lambda_2^k) \in \mathcal{N}$
        Corrector Update
        $r^{k+1} \leftarrow r^k \alpha_r^k \triangle r^k, \lambda_1^{k+1} \leftarrow \lambda_1^k + \alpha_{\lambda_1}^k \triangle \lambda_1^k, \lambda_2^{k+1} \leftarrow \lambda_2^k + \alpha_{\lambda_2}^k \triangle \lambda_2^k, k \leftarrow k + 1, u^{k+1} = Wr^{k+1} + q$.
    end for

Algorithm 4 Mehrotra-Backtracking

Same procedure with Mehrotra-Armijo, but using backtracking strategy to find the value of step-length.

After run algorithm, we obtain: $IPM(W, q, \mu, s) \rightarrow (r, u)$.

Run outside the loop

Algorithm 5 Outside the loop

Require: $\epsilon_{out} = 1e-8$, the initial value of s, we call $s_0 > 0$. The starting value of $\rho = 1$, $v \in (0,1)$ is a random value. Set up erro $err = 1$.

1: **while** $err > \epsilon_{out}$ **do**
2: $[r, u] = IPM(W, q, \mu, s)$
3: $s = ||u_T||$
4: $v = rand(1)$
5: $\rho = v\ \rho$
6: $err = \dfrac{||r - \prod_K (r - \rho(Wr + q(s)))||}{||q||}$

2.2 Interior-point method for solving NOP

We recall the formulation NOP (2.2)

$$\begin{cases} \min_{r,s} \langle r, Wr + q\rangle + \sum_{i=1}^{n_c} \mu_i r_{N,i} s_i \\ s.t \qquad ||(Wr+q)_{T,i}|| \le s_i, \qquad i = 1..n_c \\ \qquad (Wr+q)_{N,i} \ge 0, \qquad i = 1..n_c \\ \qquad r_i \in K_i, \qquad i = 1..n_c. \end{cases}$$

We use the same idea with the fixed-point formulation to reformulate NOP as

$$\begin{cases} \min_{r,s} \quad (r^T, s^T) H(r,s) + (r^T, s^T)(q, [0]_{n_c \times 1}) \\ s.t \qquad (r^T, s^T) A_i (r,s)^T + (r^T, s^T) B_i + B_i^T (r,s) + q_{T,i}^T q_{T,i} \le 0, \quad i = 1..n_c \\ \qquad W(3i-2,:) r + q(3i-2) \ge 0, \qquad i = 1..n_c \\ \qquad r_i^T Q_i r_i \ge 0, \qquad i = 1..n_c \\ \qquad r_{N,i} \ge 0, \qquad i = 1..n_c \\ \qquad s_i \ge 0, \qquad i = 1..n_c, \end{cases}$$

with

$$A_i = \begin{pmatrix} A_{1,i}^T A_{1,i} & [0] \\ [0] & [-1_{i,i}]_{n_c \times n_c} \end{pmatrix}, \ A_{1,i} = \begin{pmatrix} W(3i-1,:) \\ W(3i,:) \end{pmatrix}, \ B_i = \begin{pmatrix} A_{1,i}^T q_{T,i} \\ [0] \end{pmatrix}, \ q_{T,i} = \begin{pmatrix} q(3i-1) \\ q(3i) \end{pmatrix}$$

We denote

$$
\begin{aligned}
&g_1 = (g_{1,i})_{n_c\times 1}, g_{1,i} = (r^T, s^T)A_i(r,s)^T + (r^T, s^T)B_i + B_i^T(r,s) + q_{T,i}^T q_{T,i},\\
&g_2 = (g_{2,i})_{n_c\times 1}, g_{2,i} = W(3i-2,:)r + q(3i-2),\\
&g_3 = (g_{3,i})_{n_c\times 1}, g_{3,i} = r_i^T Q_i r_i,\\
&g_4 = (g_{4,i})_{n_c\times 1}, g_{4,i} = r_{N,i},\\
&g_5 = s.
\end{aligned} \tag{2.10}
$$

The Lagragian is as follows

$$L(r,s,\lambda) = (r^T, s^T)H(r,s) + (r^T, s^T)(q,[0]) + \lambda_1^T g_1 - \lambda_2^T g_2 - \lambda_3^T g_3 - \lambda_4^T g_4 - \lambda_5^T g_5.$$

$$\Rightarrow \nabla_{r,s}L = (H+H^T)(r,s)^T + (q,[0]) + \nabla_{r,s}g_1^T\lambda_1 - \nabla_{r,s}g_2^T\lambda_2 - \nabla_{r,s}g_3^T\lambda_3 - \nabla_{r,s}g_4^T\lambda_4 - \nabla_{r,s}g_5^T\lambda_5,$$

with

$$\nabla_{r,s}g_1^T = \begin{pmatrix} 2A_1(r,s)^T + 2B_1 & \dots & 2A_{n_c}(r,s)^T + 2B_{n_c} \end{pmatrix}_{4n_c\times n_c}, \nabla_{r,s}g_2^T = \begin{pmatrix} [W(3i-2)^T]_{3n_c\times n_c} \\ [0]_{n_c\times n_c} \end{pmatrix}$$

$$\triangle_{r,s}g_3^T = \begin{pmatrix} diag(2Q_i r_i)_{3n_c\times n_c} \\ [0]_{n_c\times n_c} \end{pmatrix}, \nabla_{r,s}g_4^T = (1_{3i-2,i})_{4n_c\times n_c}, \nabla_{r,s}g_5^T = \begin{pmatrix} [0]_{3n_c\times n_c} \\ I_{n_c\times n_c} \end{pmatrix}$$

KKT condition:

$$
\begin{cases}
\nabla_{r,s}L & = 0\\
\Lambda G e & = 0\\
\lambda & \geq 0\\
g & \geq 0.
\end{cases}
\Leftrightarrow
\begin{cases}
(H+H^T)(r,s)^T + (q,[0]) - \nabla_{r,s}g^T\lambda & = 0\\
\Lambda G e & = 0\\
\lambda & \geq 0\\
g & \geq 0,
\end{cases} \tag{2.11}
$$

with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^T, g = (-g_1, g_2, g_3, g_4, g_5), \Lambda = diag(\lambda), G = diag(g)$.

KKT perturb a small value of $\sigma \in (0,1)$ in the complementarity condition:

$$
\begin{cases}
(H+H^T)(r,s)^T + (q,[0]) - \nabla_{r,s}g^T\lambda & = 0\\
\Lambda G e & = \sigma e, \quad e = ones(5n_c, 1)\\
\lambda & \geq 0\\
g & \geq 0.
\end{cases} \tag{2.12}
$$

.

Using Newton's method to solve system:

$$\begin{pmatrix} (H + H^T) - D & -\nabla g_{5n_c \times 4n_c} \\ \Lambda \nabla g & G \end{pmatrix},$$

with

$$D = D_1 - D_2, D_1 = \begin{pmatrix} diag(Q_i \lambda_i^3) & [0]_{3n_c \times n_c} \\ [0]_{n_c \times 3n_c} & [0]_{n_c \times n_c} \end{pmatrix}, D_2 = (d_{i,j})_{4n_c \times 4n_c}, d_{i,j} = \sum_{k=1}^{n_c} \lambda_k^1 A_k(i,j).$$

We use the Friedlander-Orban and Mehrotra algorithm above for solving this formulation.

2.3 Relax s

To simplify the problem and compare with our algorithm, we consider $s = 0$, it mean that the problem is completely convex. We solve only IPM inside the loop.

Chapter 3

Numerical experience

In this chapter, we adapt interior point method for solving the problems with precise data. After that are some comparison framework.

3.1 Performance profiles

This section, we base on [[4]] for introducing the concept of performance profiles. For the set of n_p problems $P = \{p_1, p_2, ..., p_{n_p}\}$, and the set of n_s solvers $S = \{s_1, s_2, ..., s_{n_s}\}$, we define a performance criterion for a solver s, a problem p and a required precision tol by

$$t_{p,s} = computing\ time\ required\ for\ s\ to\ solve\ p\ at\ precision\ tol, \tag{3.1}$$

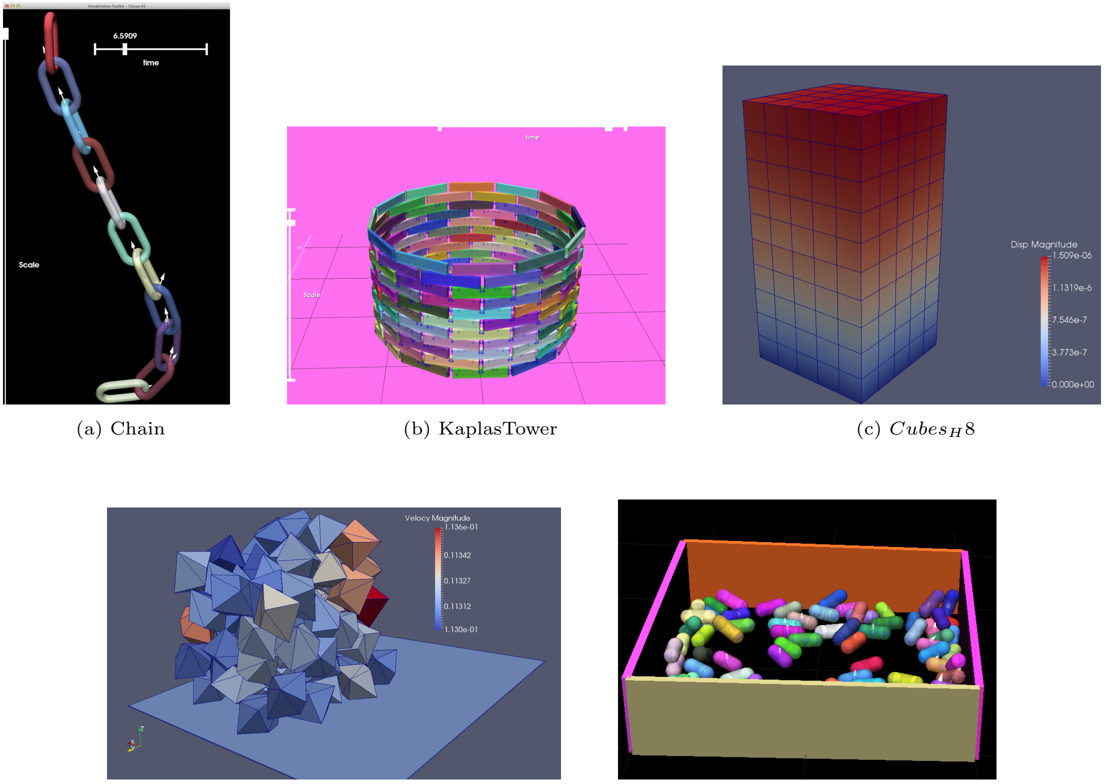
A performance ratio over all the solvers is defined by (fix problem p)

$$r_{p,s} = \frac{t_{p,s}}{\min\{t_{p,s}, s \in S\}} \geq 1. \tag{3.2}$$

For $\tau \geq 1$, we define a distribution function ρ_s for the performance ratio for a solver s as (fix solver)

$$\rho_s(\tau) = \frac{1}{n_p} card\{p \in P, r_{p,s} \leq \tau\} \leq 1. \tag{3.3}$$

This distribution computes the number of problems p that are solved with a perpormance ratio below a given threshold τ. If the value of $\rho_s(\tau)$ is near 1, it mean that the solver is good and can solve almost of problems. The performance profiles denotes a graph of the function $\rho_s(\tau), \tau > 1$. The computational time is use to measure performance in the algorithms.



(a) Chain (b) KaplasTower (c) $Cubes_H8$

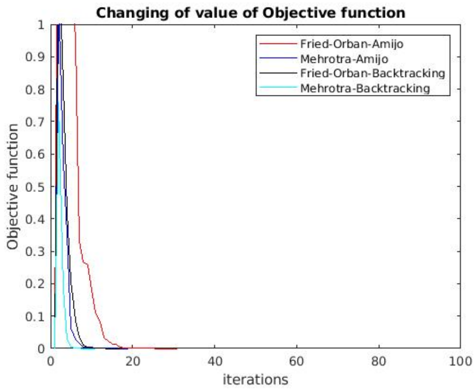
(a) $100_PR_PerioBox$ (b) Capsules

Figure 3.1: Test sets

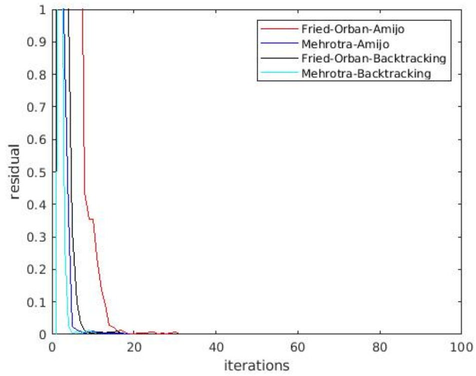
3.2 Result

We use the collection of problems in FClib (hdf5 format) which is described in ([9]). The following table is the details of data sets for each example.

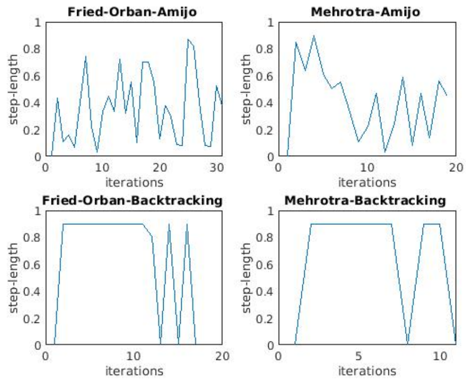
Test set	μ	#contacts	C_W	$[\lambda_{min}, \lambda_{max}]$	$\#\lambda = 0$
Chain-i93-120	0.3	12	3.0171e+16	[-2.97e-16, 8.6261]	0
KaplasTower-i4-10	0.3	136	7.7987e+20	[2.2e -15-9.2e-15i,3.9e+3]	0
Cubes-H8-20	0.3	369	4.65e+04	[3.755e-8,9.288e-4]	0
PR-PeroBox-14	[0.3;0.5]	14	2.2847e+18	[1.17e-7,5.14e-5]	19
PR-PeroBox-60	[0.3;0.5]	60	1.3696e+21	[1.05e-7,1.05e-4]	108
PR-PeroBox-143	[0.3;0.5]	143	2.017e+20	[2.143e-8,1.18e-4]	267
PR-PeroBox-254	[0.3;0.5]	254	1.8322e+21	[2.85e-9,1.17e-4]	486
Capsules-i2007-12	0.7	17	8.7861e+18	[-1.55e-15,9.54]	1



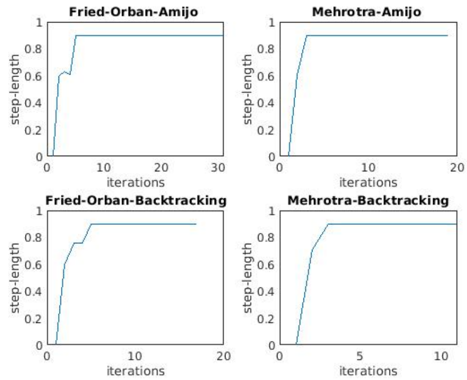
(a) Objective function Chain-12



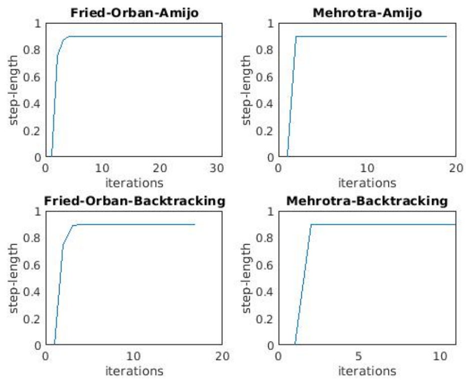
(b) Residual Chain-12



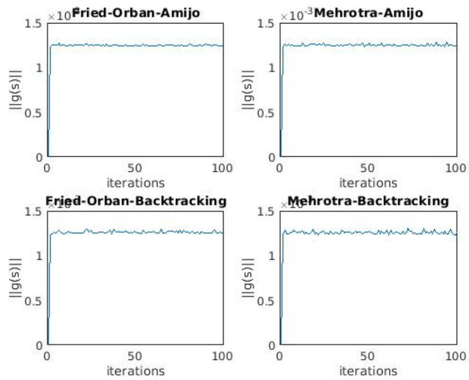
(c) Alphar Chain-12



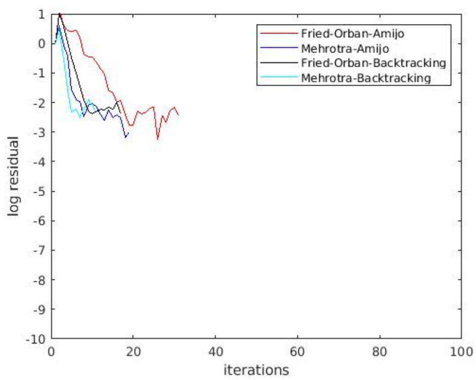
(d) Alphaz1 Chain-12



(e) Alphaz2 Chain-12

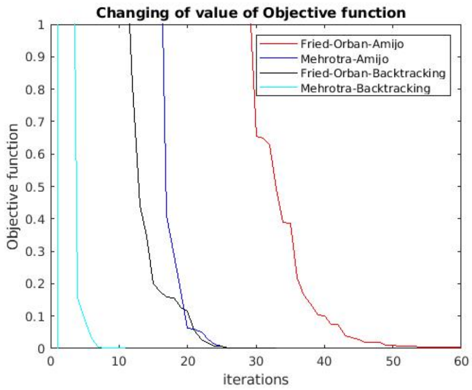


(f) norm(g(Wr+q)) Chain-12

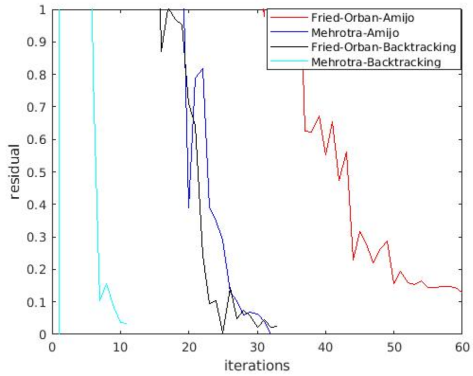


(g) log residual Chain-12

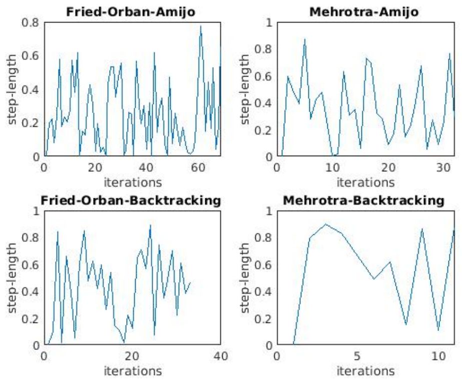
Figure 3.2: Chain-12 test set result.



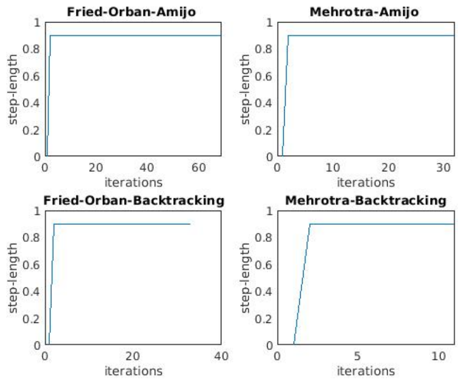
(a) Objective function KaplasTower



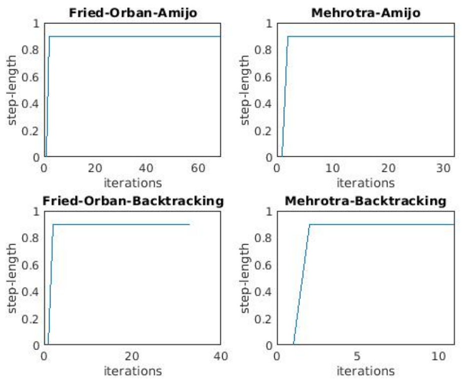
(b) Residual KaplasTower



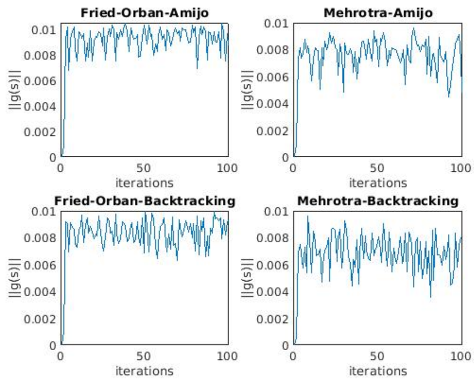
(c) Alphar KaplasTower



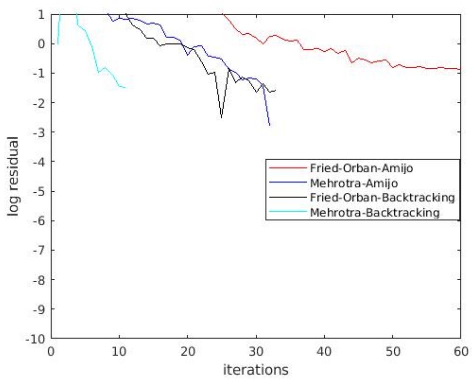
(d) Alphaz1 KaplasTower



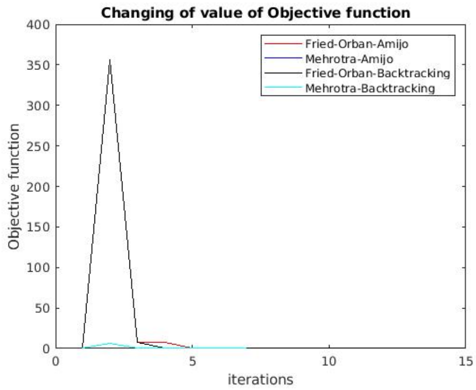
(e) Alphaz2 KaplasTower



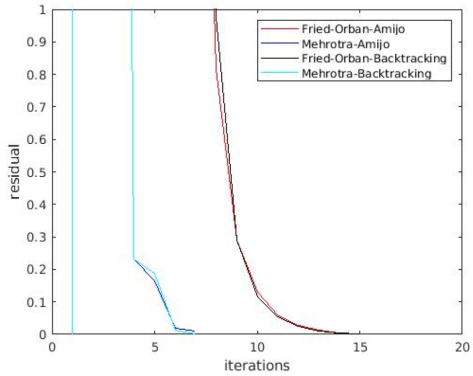
(f) norm(g(Wr+q)) KaplasTower



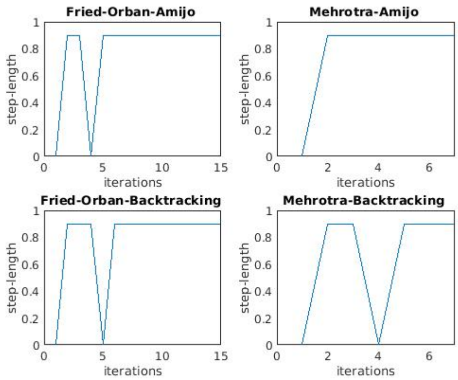
(g) log residual KaplasTower



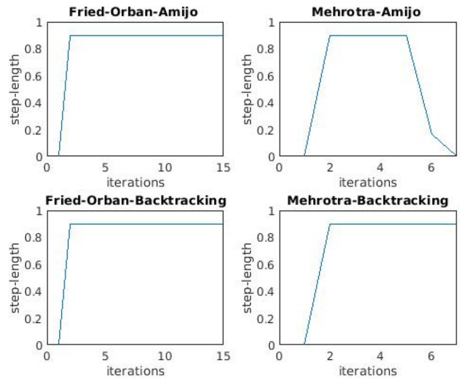
(a) Objective function LMGC-Cubes-H8 369



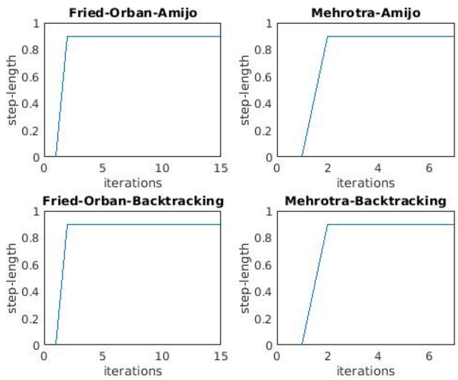
(b) Residual LMGC-Cubes-H8



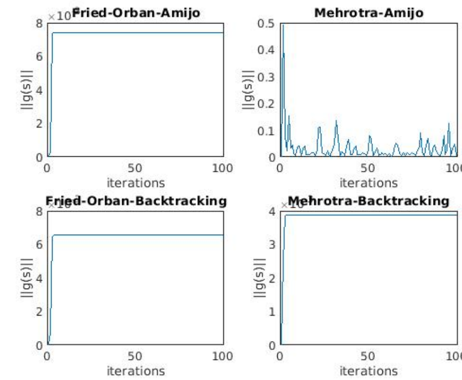
(c) Alphar LMGC-Cubes-H8



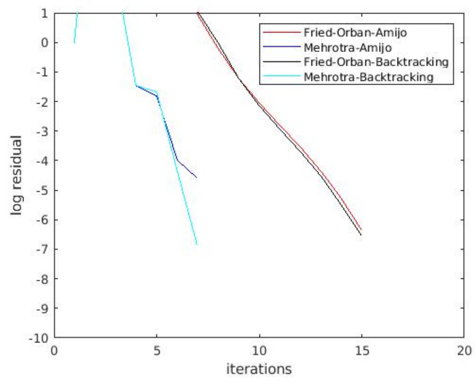
(d) Alphaz1 LMGC-Cubes-H8



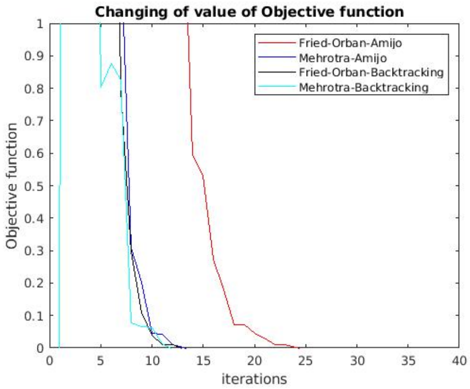
(e) Alphaz2 LMGC-Cubes-H8



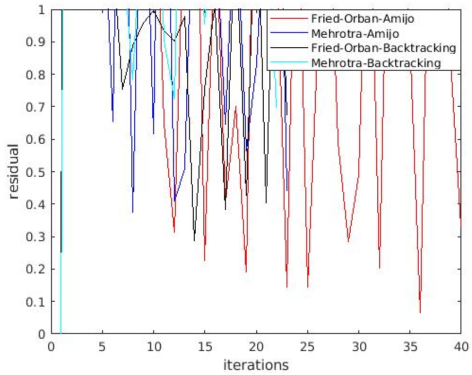
(f) norm(g(Wr+q)) LMGC-Cubes-H8



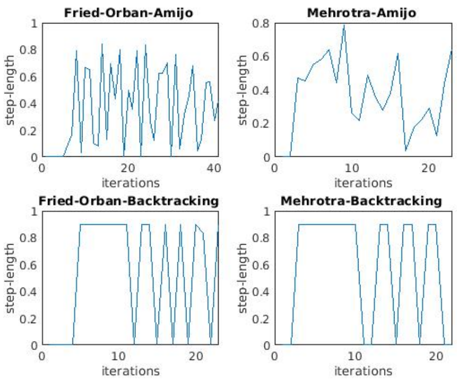
(g) log residual LMGC-Cubes-H8



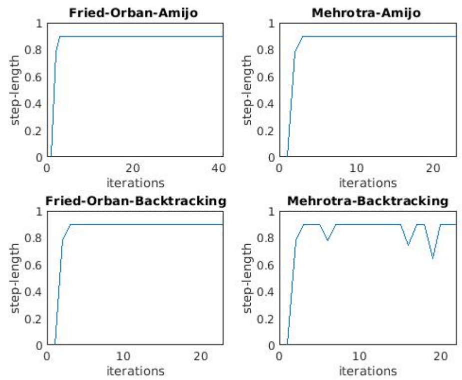
(a) Objective function Capsules



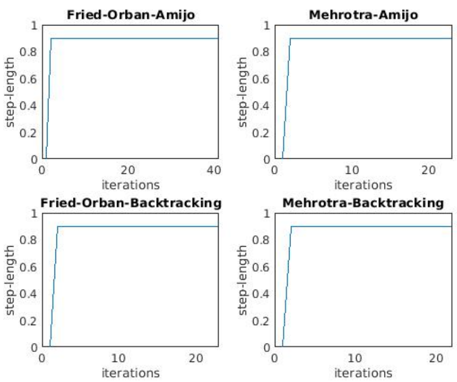
(b) Residual Capsules



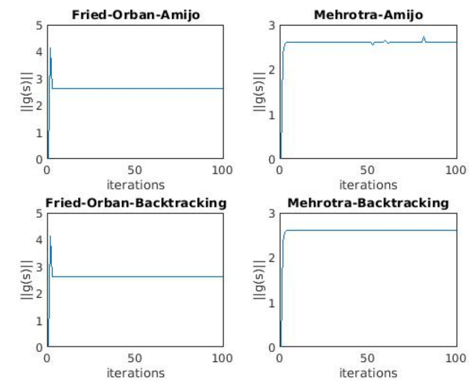
(c) Alphar Capsules



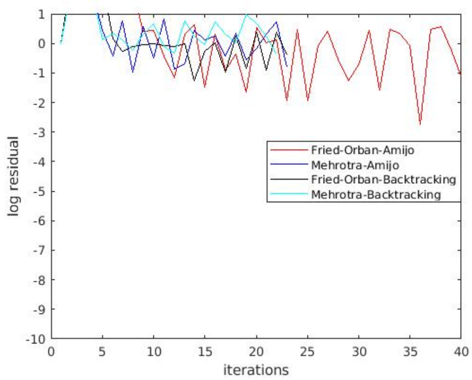
(d) Alphaz1 Capsules



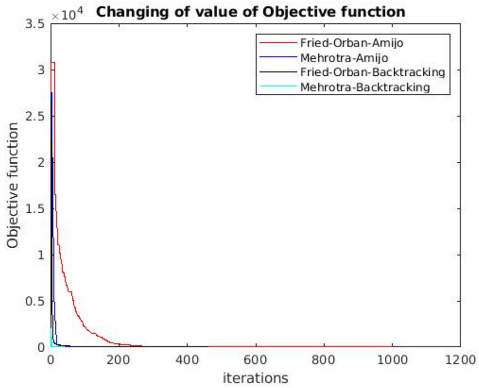
(e) Alphaz2 Capsules



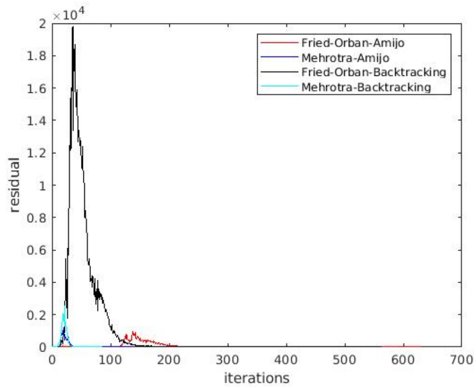
(f) Norm($g(Wr + q)$) Capsules



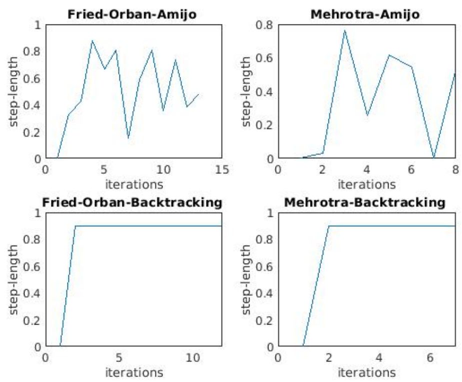
(g) Log residual Capsules



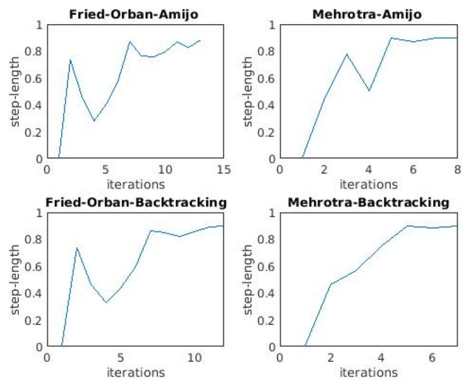
(a) Objective function PerioBox-14



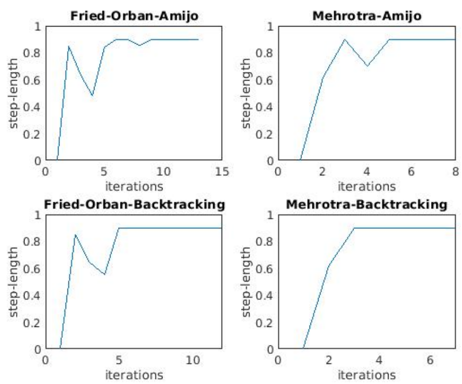
(b) Residual PerioBox-14



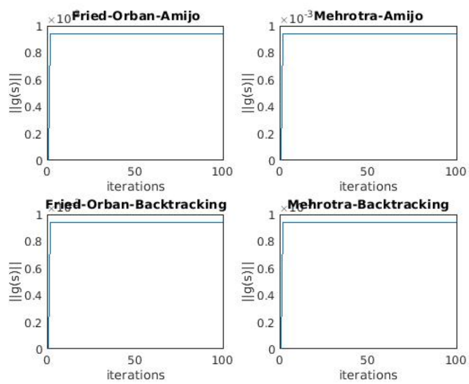
(c) Alphar PerioBox-14



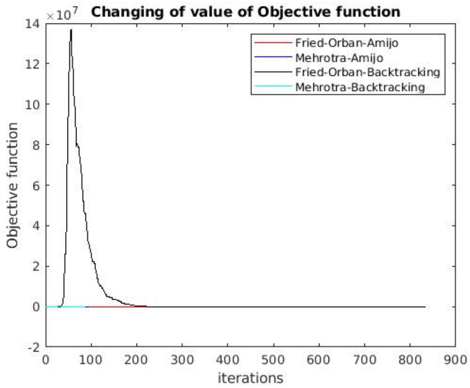
(d) Alphaz1 PerioBox-14



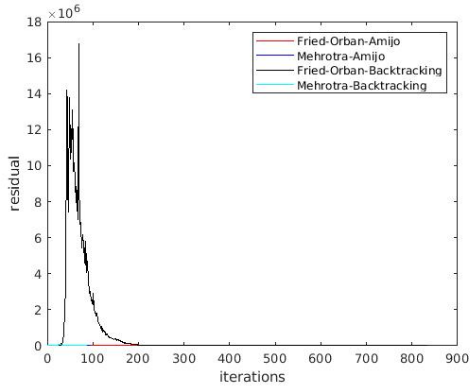
(e) Alphaz2 PerioBox-14



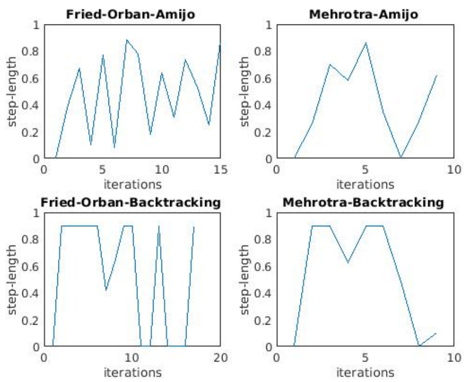
(f) norm(g(Wr+q)) PerioBox-14



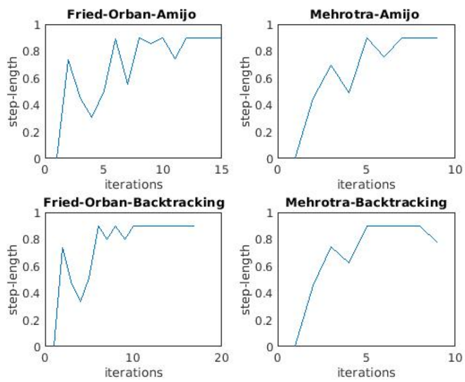
(a) Objective function PerioBox-60



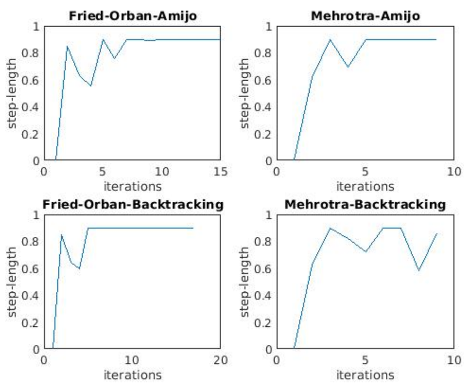
(b) Residual PerioBox-60



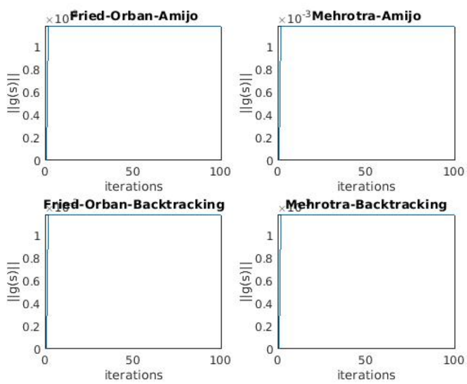
(c) Alphar PerioBox-60



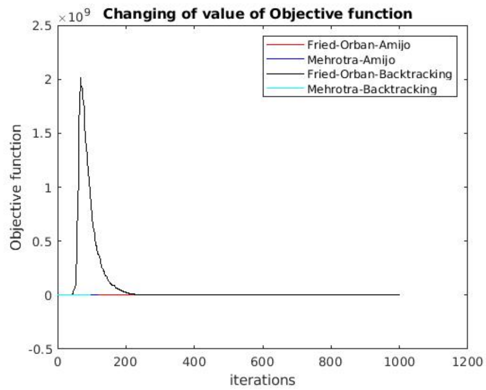
(d) Alphaz1 PerioBox-60



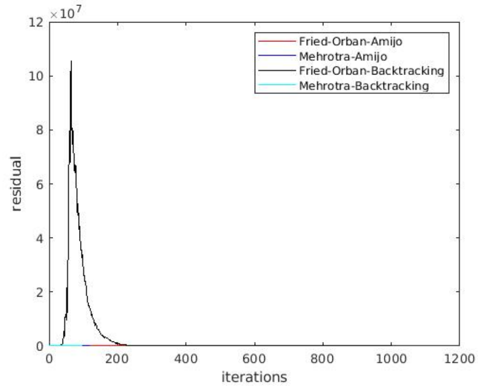
(e) Alphaz2 PerioBox-60



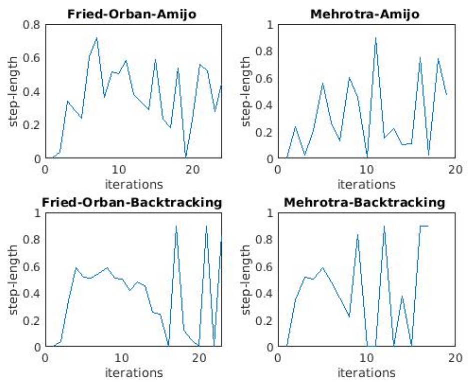
(f) norm(g(Wr+q)) PerioBox-60



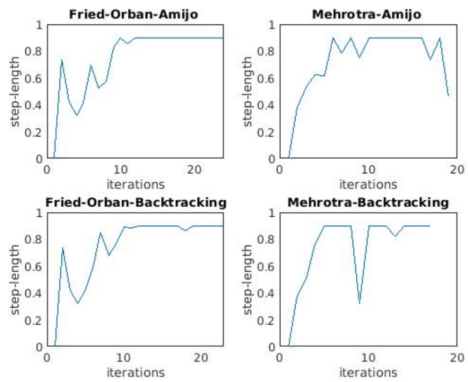
(a) Objective function PerioBox-254



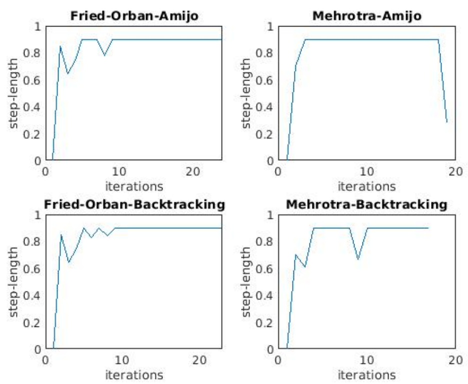
(b) Residual PerioBox-254



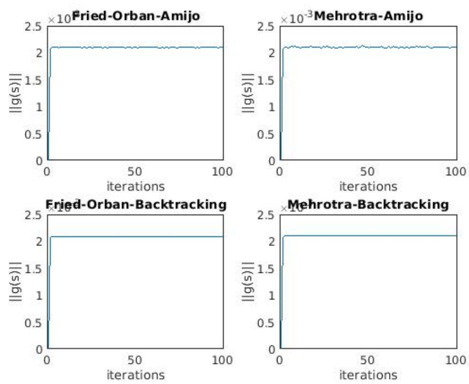
(c) Alphar PerioBox-254



(d) Alphaz1 PerioBox-254



(e) Alphaz2 PerioBox-254



(f) norm(g(Wr+q)) PerioBox-254

	Fried-Orban-A	Mehrotra-A	Fried-Orban-B	Mehrotra-B
Chain 12	0.233	0.0758	0.1923	0.0916
Kaplas 136	18.978	0.5527	20.8413	0.3002
Cubes 369	5.519	0.3224	28.5546	0.3382
PerioBox 14	0.031	0.0046	0.0484	0.0053
Perio 60	0.3489	0.0821	0.514	0.0631
Perio 143	3.7064	0.4455	4.8333	0.3807
Perio 254	10.1878	2.3127	11.7256	2.9865
Capsules 17	0.4162	10.1317	0.5105	0.0384

Table 3.1: Table of CPU time running

Performance frofiles

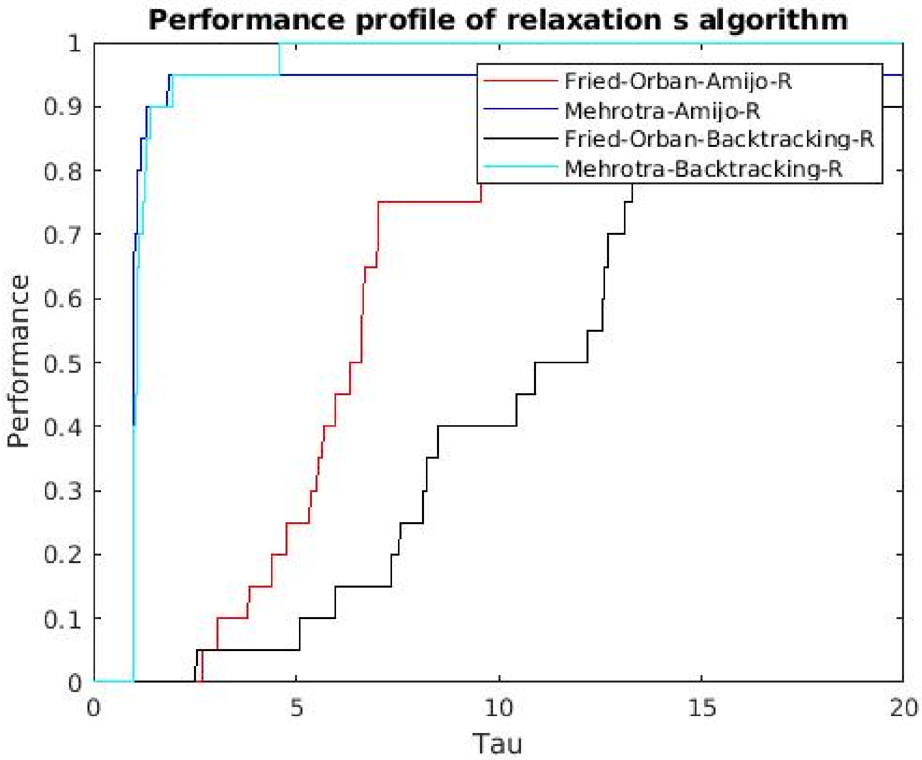


Figure 3.3: Performance profiles of relaxation s with above fix-problems.

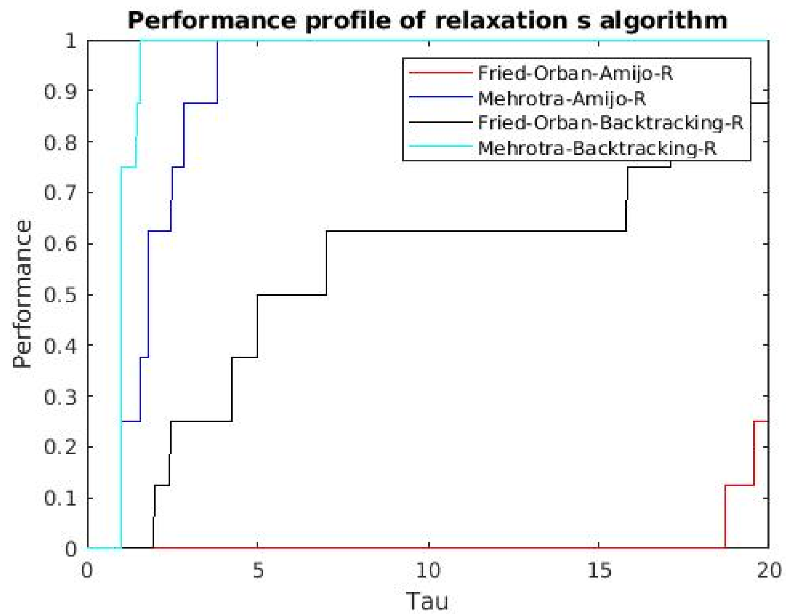


Figure 3.4: Chain problems.

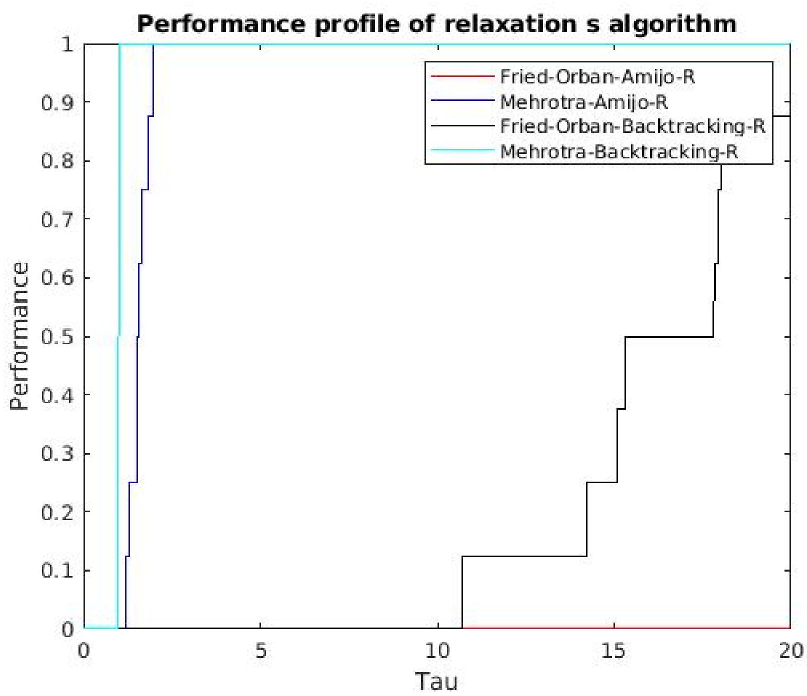


Figure 3.5: Kaplas problems.

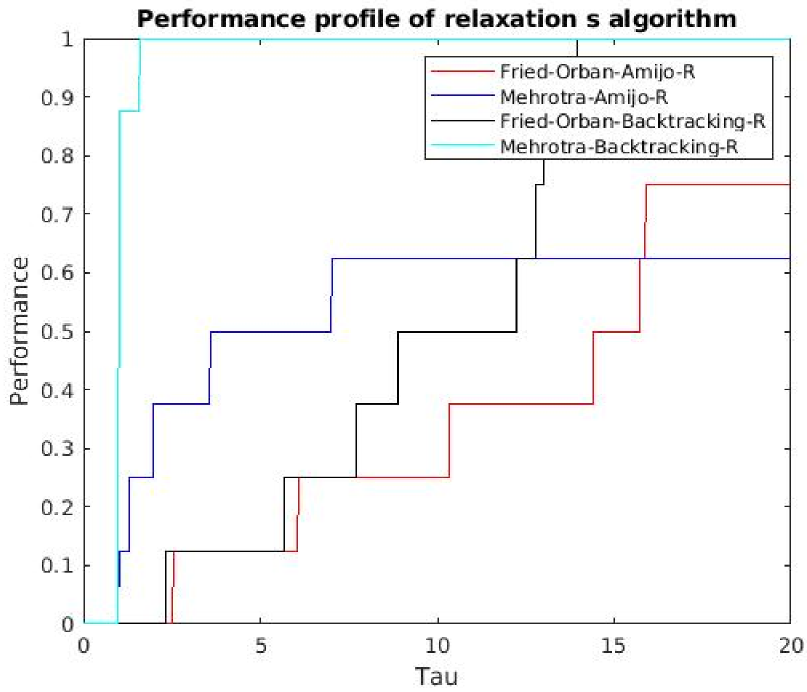


Figure 3.6: PerioBox problems.

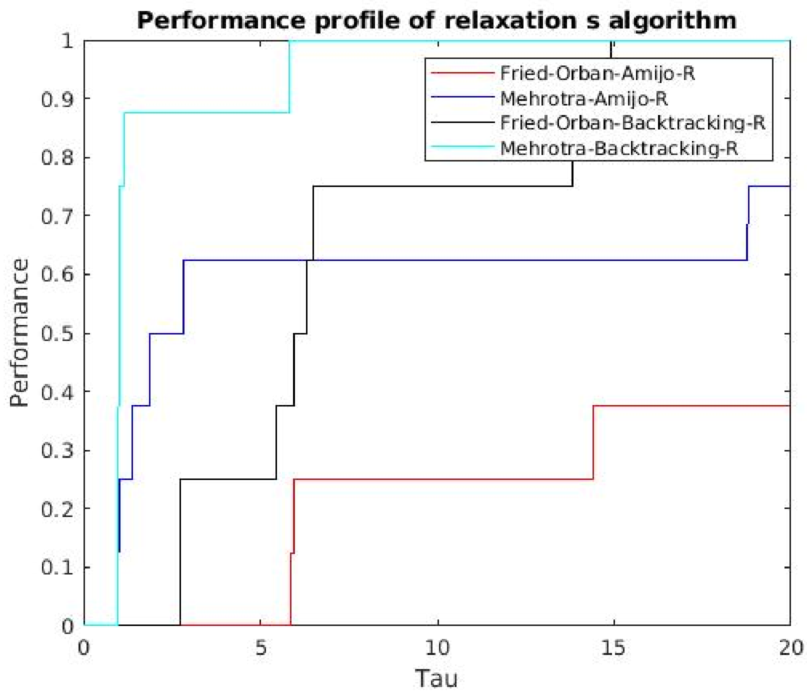


Figure 3.7: Cubes H8 problems.

Comments

We have some following comments

- The first observation is that the value objective function and residual tends to 0.
- About the number of iterations: in most cases, Mehrotra-backtracking has the smallest number of iteration, then to Mehrotra- Armijo, Friedlander-Orban-Backtracking, and the last one is Friedlander-Orban- Armijo. On the comparison of the number of iteration, Mehrotra method gives us the better result.
- About the norm of $g(s)$, for the set of small value of friction coefficients $\mu \backsim 0.3-0.5$ we get the norm of $g(s)$ is very small. Example, set of Chain $||g(s)|| \backsim 1e-3$, KaplasTower $||g(s)|| \backsim 1e-4 \rightarrow 1e-2$, Cubes $||g(s)|| \backsim 1e-3 \rightarrow 1e-1$, PerioBox $||g(s)|| \backsim 1e-3$. In case the values of μ is large (Capsules $\mu = 0.8$), the value of $||g(s)|| \backsim 3$. it tends us to think about using relaxation s to solve the contacts problems with the small value of μ.
- Performance profile: For the test sets and using relaxation, we see that Mehrotra-backtracking is the best method which reached 1 just for the value of $\tau \backsim 1.5$, next method is Friedlander-Orban-Backtracking which reached 1 when $\tau \backsim 16$, Mehrotra-Amijo and Friedlander-Orban-Amijo depict the worst behaviour.
- When we consider the separated set:
  1. In case of small values of $\mu = 0.3$ and the number of contacts $n_c \leq 200$ such as Chain and Kaplas sets of test, we get Mehrotra-backtracking and Mehrotra-Amijo suffers the robustness while Friedlander-Orban-Armijo and Friedlander-Orban-Backtracking show the worse behaviour.
  2. In case of small value of $\mu = 0.3$ but with the number of contacts is large > 300 such as Cubes H8, we get Mehrotra-backtracking and Friedlander-Orban-

Backtracking suffers the robustness while Mehrotra-Armijo and Friedlander-Orban-Armijo show the worse behaviour.

3. In case of the larger value of $\mu \in [0.3, 0.5]$, the properties are the same as the above case.

- In case which has the big number of contacts or the big value of μ, one possible thing happened is that the solution is not satisfied the second-order cone constraints (example in the case of PerioBox 104 contacts or Cubes H8 361 contacts the objective function is less than 0 it can be the cause of using for iterate and roundoff errors on computing).

Conclusions

The thesis focus on studying the properties of new reformulation of incremental problems which are large systems of frictional contacts and implement IPM for solving the fixed-point optimization problem. There are a lot of proposal methods to solve the incremental problem under any type of formulation such as fixed point iteration (FP), fixed point extra gradient (FP-EG) for solving variational inequality, ADMM or proximal point method for solving second-order cone formulation. Here the author proposes IPM for the problem after a reformulation as the quadratic objective function, quadratic constraints and linear constraints.

We used the Armijo rule and backtracking strategy to pass the quadratic constraints problems. We also modified the way to determine the step length to fit the problem.

By using MATLAB, we implemented the method with the set of data, from that we have a look at the properties of IPM adapting to frictional contact problems which are commented on the previous chapter.

Future work: There are several interesting directions for future work.

Firstly, speed up IPM by choosing a strategy to find the optimal step length.

Secondly, by susing IPM for solving KKT system it tends to question can we use IPM for solving directly incremental problems.

Furthermore, as we mentioned in remark (1.7), examining about Lagrange multipliers can be interesting in mathematics point of view.

Besides, in practice we update the value of s after running internal loops of IPM, it takes a lot of time, one other direction is that study about conditions to make sure about the equivalence of FCNLCP and FOP where s is a variable and can update inside the loop.

In this report, we proposed a new formulation which is MPEC for frictional contact problem. It is also interesting if we can continue to study the MPEC formulation and propose an algorithm to solve it.

Appendix A

Counter-example

From (1.19) we have:

$$\langle r, \tilde{u} \rangle \;=\; -\left[\langle r-\lambda_1, W^T(r-\lambda_1)\rangle + \langle r-\lambda_1, \mu(r-\lambda_1)_N \xi\rangle + \langle \lambda_1, \lambda_2\rangle\right]$$

If $\langle r-\lambda_1, W^T(r-\lambda_1)\rangle + \langle r-\lambda_1, \mu(r-\lambda_1)_N \xi\rangle + \langle \lambda_1, \lambda_2\rangle$ is strictly negative, then $\langle r, \tilde{u}\rangle$ is strictly positive so it does not satisfy the complementarity between r and $\tilde{u}$.

Let us consider the case: $r_T = \mu r_N > 0, u_N > 0, u_T > 0$. Since $u_N > 0 \Rightarrow \tilde{u} \in \mathrm{intK}^*$. We obtain

$$-\lambda_1 \in \partial\psi_{K^*}(\tilde{u}) = N_{K^*}(\tilde{u}) \Rightarrow \lambda_1 = 0.$$

$$-\lambda_2 \in \partial\psi_K(r) = -K^* \cap r^{\perp}, r = (1,\mu)r_N \Rightarrow -\lambda_2 = \beta \begin{pmatrix} -\mu \\ 1 \end{pmatrix}, \beta \geq 0 \Rightarrow \lambda_2 = \beta \begin{pmatrix} \mu \\ -1 \end{pmatrix},$$

$$-\xi \in \partial_r ||E_T(Wr+q)|| \Rightarrow -\xi \in W^T E_T^T \partial_{u_T} ||u_T|| \Rightarrow \xi = -\gamma \begin{pmatrix} b \\ d \end{pmatrix}, \gamma = 1.$$

$$\begin{aligned}\Rightarrow \langle r, \tilde{u}\rangle &= -\{\langle r-\lambda_1, W^T(r-\lambda_1)\rangle + \langle r-\lambda_1, \mu(r-\lambda_1)_N \xi\rangle + \langle \lambda_1, \lambda_2\rangle\} \\ &= -\{\langle r, W^T r\rangle + \langle r, \mu r_N \xi\rangle\} \\ &= -(a + 2b\mu + \mu^2 d - \mu\gamma b - \mu^2 \gamma d) r_N^2.\end{aligned}$$

Let us define the function

$$f(\mu) = (d - \gamma d)\mu^2 + (2b - \gamma b)\mu + a.$$

The goal is to find condition on μ such that $f(\mu) < 0$. Since $\gamma = 1$, we obtain

$$f(\mu) = (a + b\mu) < 0 \Leftrightarrow a < -b\mu,$$

combine with the assumption on $W \in S_+^2$, it implies $a > 0 \Rightarrow b < 0$.

Choosing b and μ such that the inequality $a < -b\mu$ provides a counter example.

Let us choose $\mu = 5, b = -2, a = 6, d = 1$ this condition satisfies the strictly positive of $\langle r, \tilde{u} \rangle$ and $W \in S^2_+$. Now, let us find the condition on q such that all the KKT equations are satisfied. From KKT conditions, we have:

$$W^T r + \tilde{u} + \mu r_N \xi - \lambda_2 = 2Wr + Q - \mu \begin{pmatrix} b \\ d \end{pmatrix} - \lambda_2 = 0,$$

with

$$Q = q + \begin{pmatrix} \mu|u_T| \\ 0 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 5(-2+5+q_2) \\ 0 \end{pmatrix} = \begin{pmatrix} q_1 + 15 + 5q_2 \\ q_2 \end{pmatrix}$$

$$\Rightarrow \lambda_2 = 2Wr + Q - \mu \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} q_1 + 15 + 5q_2 \\ q_2 \end{pmatrix}.$$

We find the condition of q:

$$\alpha \begin{pmatrix} \mu \\ -1 \end{pmatrix} = \begin{pmatrix} q_1 + 15 + 5q_2 + 2 \\ q_2 + 1 \end{pmatrix} \Leftrightarrow \begin{cases} q_1 + 17 + 5q_2 & = 5\alpha, \\ q_2 + 1 & = -\alpha, \end{cases}$$

$$\Leftrightarrow \begin{cases} q_1 & = 10\alpha - 12, \\ q_2 & = -\alpha - 1, \end{cases}$$

combine to the condition

$$\begin{cases} u_N = a + b\mu + q_N > 0, \\ u_T = b + \mu d + q_T > 0, \end{cases} \Rightarrow \begin{cases} q_N > -(a + b\mu), \\ q_T > -(b + \mu d), \end{cases} \Rightarrow \begin{cases} q_1 > 4, \\ q_2 > -3, \end{cases}$$

$$\Rightarrow 1.6 < \alpha < 2,$$

we choose $\alpha = 1.9$, then we got the result as follows:

$$q = \begin{pmatrix} 7 \\ -2.9 \end{pmatrix}, \quad u = \begin{pmatrix} 3 \\ 0.1 \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} 3.5 \\ 0.1 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 9.5 \\ -1.9 \end{pmatrix}, \quad \langle r, \tilde{u} \rangle = 4.$$

Appendix B

Finding counter-example in another cases

We consider some cases as follow:

1. Take off case: $r = 0 \Rightarrow \langle r, \tilde{u} \rangle = 0, \forall \tilde{u} \in K^*$. So this case are not considered.

2. Sticking case: $u = 0 \Rightarrow \tilde{u} = 0 \Rightarrow \langle r, \tilde{u} \rangle = 0, \forall r \in K$. We do not consider this case.

3. Sliding case:

   - $r_T = \mu r_N > 0, u_T < 0$ (not consider)
   - $r_T = -\mu r_N < 0, u_T > 0$(not consider)
   - $r_T = \mu r_N > 0, u_T > 0, u_N = 0$: We characterize the Lagrange multipliers:

     $$-\lambda_1 \in \partial \psi_{K^*}(\tilde{u}) = -K \cap \tilde{u}^\perp \Rightarrow -\lambda_1 = \alpha \begin{pmatrix} -\frac{1}{\mu} \\ 1 \end{pmatrix}, \alpha \geq 0 \Rightarrow \lambda_1 = \alpha \begin{pmatrix} \frac{1}{\mu} \\ -1 \end{pmatrix},$$

     $$-\lambda_2 \in \partial \psi_K(r) = -K^* \cap r^\perp \Rightarrow -\lambda_2 = \beta \begin{pmatrix} -\mu \\ 1 \end{pmatrix}, \beta \geq 0 \Rightarrow \lambda_2 = \beta \begin{pmatrix} \mu \\ -1 \end{pmatrix},$$

     $$-\xi \in \partial_r ||E_T(Wr+q)|| \Rightarrow -\xi \in W^T E_T^T \partial_{Wr+q} ||E_T(Wr+q)|| \Rightarrow \xi = -\gamma \begin{pmatrix} b \\ d \end{pmatrix}.$$

     From the KKT condition, we have:

     $$\langle r, \tilde{u} \rangle = -\{\langle r - \lambda_1, W^T(r - \lambda_1) \rangle + \langle r - \lambda_1, \mu(r - \lambda_1)_N \xi \rangle + \langle \lambda_1, \lambda_2 \rangle\},$$

     with

     $$\langle r - \lambda_1, W^T(r - \lambda_1) \rangle = a(r_N - \frac{\alpha}{\mu})^2 + 2b\mu(r_N - \frac{\alpha}{\mu})(r_N + \frac{\alpha}{\mu}) + d\mu^2(r_N + \frac{\alpha}{\mu})^2,$$

     $$\langle \lambda_1, \lambda_2 \rangle = 2\alpha\beta,$$

     $$\langle r - \lambda_1, \mu(r - \lambda_1)_N \xi \rangle = -\gamma\mu b(r_N - \frac{\alpha}{\mu})^2 - \gamma\mu^2 d(r_N - \frac{\alpha}{\mu})(r_N + \frac{\alpha}{\mu}).$$

$\Rightarrow \langle r, \tilde{u}\rangle = -\{a(r_N - \frac{\alpha}{\mu})^2 + 2b\mu(r_N - \frac{\alpha}{\mu})(r_N + \frac{\alpha}{\mu}) + d\mu^2(r_N + \frac{\alpha}{\mu})^2 - \gamma\mu b(r_N - \frac{\alpha}{\mu})^2 - \gamma\mu^2 d(r_N - \frac{\alpha}{\mu})(r_N + \frac{\alpha}{\mu}) + 2\alpha\beta\} := f(\mu)$.
We have to find the conditions of μ such that $f(\mu)$ is strickly negative. We see that with the large value of μ then $\frac{\alpha}{\mu} \to_{\mu\to\infty} 0$ then $f(\mu) \geq 0$, so there will be no counter example in the case of having very large value of μ.
Let set:

$$\begin{cases} x &= r_N - \frac{\alpha}{\mu}, \\ y &= r_N + \frac{\alpha}{\mu}. \end{cases}$$

$$\Rightarrow f(\mu) = ax^2 + 2b\mu xy + d\mu^2 y^2 - \gamma\mu b x^2 - \gamma\mu^2 dxy + 2\alpha\beta,$$

with

$$f_1(\mu) = ax^2 + 2b\mu xy + d\mu^2 y^2 - \gamma\mu b x^2 - \gamma\mu^2 dxy$$

has quadratic form, we find the normal form of this by using Lagrange transformation:

$$f_1(\mu) = \frac{1}{a - \gamma\mu b} x_1^2 + (d\mu^2 - (b\mu - \gamma\frac{\mu^2 d}{2})^2))y_1^2$$

$$\Rightarrow f(\mu) = \frac{1}{a - \gamma\mu b} x_1^2 + (d\mu^2 - (b\mu - \gamma\frac{\mu^2 d}{2})^2))y_1^2 + z^2,$$

with

$$\begin{cases} x_1 &= (a - \gamma\mu b)x + (b\mu - \gamma\frac{\mu^2 d}{2}))y, \\ y_1 &= y, \\ z &= \sqrt{2\alpha\beta}. \end{cases}$$

We already know that f is strickly negative if all coefficients in normal form are negative. But here it is not impossible since we can get some cases as:

- $a - \gamma\mu b \leq 0,\ and\ d\mu^2 - (b\mu - \gamma\frac{\mu^2 d}{2})^2) \leq 0 \Rightarrow q = (-1, -1, 1).$
- $a - \gamma\mu b \leq 0,\ and\ d\mu^2 - (b\mu - \gamma\frac{\mu^2 d}{2})^2) \geq 0 \Rightarrow q = (-1, 1, 1).$
- $a - \gamma\mu b \geq 0,\ and\ d\mu^2 - (b\mu - \gamma\frac{\mu^2 d}{2})^2) \leq 0 \Rightarrow q = (1, -1, 1).$
- $a - \gamma\mu b \geq 0,\ and\ d\mu^2 - (b\mu - \gamma\frac{\mu^2 d}{2})^2) \geq 0 \Rightarrow q = (1, 1, 1).$

- $r_T = -\mu r_N < 0, u_T < 0$: The same result with the bellow case.

- $r_T = -\mu r_N < 0, u_T > 0, u_N > 0, r_N > 0 (\tilde{u} \in intK^*)$

$$\Rightarrow -\lambda_1 \in \partial\psi_{K^*}(\tilde{u}) = N_{K^*}(\tilde{u}) \Rightarrow \lambda_1 = 0$$

$$-\lambda_2 \in \partial\psi_K(r) = -K^* \cap r^\perp, r = (1, -\mu)r_N \Rightarrow -\lambda_2 = \beta \begin{pmatrix} \mu \\ 1 \end{pmatrix}, \beta \geq 0 \Rightarrow \lambda_2 = \beta \begin{pmatrix} -\mu \\ -1 \end{pmatrix},$$

$$-\xi \in \partial_r ||E_T(Wr+q)|| \Rightarrow -\xi \in W^T E_T^T \partial_{u_T} ||u_T|| \Rightarrow \xi = -\gamma \begin{pmatrix} b \\ d \end{pmatrix}, \gamma \geq 0.$$

$$\begin{aligned} \Rightarrow \langle r, \tilde{u} \rangle &= -\{\langle r-\lambda_1, W^T(r-\lambda_1)\rangle + \langle r-\lambda_1, \mu(r-\lambda_1)_N \xi\rangle + \langle \lambda_1, \lambda_2 \rangle\} \\ &= -\{\langle r, W^T r\rangle + \langle r, \mu r_N \xi \rangle\} \\ &= -(a - 2b\mu + \mu^2 d - \mu\gamma b + \mu^2 \gamma d) r_N^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow f(\mu) &= a - 2b\mu + \mu^2 d - \mu\gamma b + \mu^2\gamma d \\ &= (d+\gamma d)\mu^2 - (2b+\gamma b)\mu + a \\ \triangle &= (2b+\gamma b)^2 - 4(d+\gamma d)a \\ &= b^2\gamma^2 + 4(b^2 - da)\gamma + 4(b^2 - da) := g(\gamma) \end{aligned}$$

$$\triangle'_{g(\gamma)} = 4da(da - b^2)$$

$$\Rightarrow \gamma \in (\frac{2(da - b^2 + \sqrt{\triangle'_{g(\gamma)}})}{b^2}, +\infty).$$

$$\begin{cases} \mu_1 &= \dfrac{(\gamma b + 2b) - \sqrt{\triangle'_{g(\gamma)}}}{2d(1+\gamma)}, \\ \mu_2 &= \dfrac{(\gamma b + 2b) + \sqrt{\triangle'_{g(\gamma)}}}{2d(1+\gamma)}. \end{cases}$$

We check the positive property of m_1:

$$\mu_1 > 0 \Leftrightarrow (\gamma b + 2b) - \sqrt{\triangle_{g(\gamma)}}' > 0$$

$$\Leftrightarrow -4da(\gamma+1) > 0 \quad \Leftrightarrow \gamma < -1$$

It is impossible.

- $r_T = \mu r_N > 0, u_N > 0, u_T = 0 \Rightarrow \langle r, \tilde{u}\rangle = r_N u_N$

$$-\lambda_1 \in \partial\psi_{K^*}(\tilde{u}) = -K \cap \tilde{u}^{\perp}; \tilde{u} = (u_N, 0)^T = (1,0)u_N \Rightarrow -\lambda_1 = 0$$

$$-\lambda_2 \in \partial\psi_K(r) = -K^* \cap r^{\perp}, r = (1,\mu)r_N \Rightarrow -\lambda_2 = \beta \begin{pmatrix} -\mu \\ 1 \end{pmatrix}, \beta \geq 0 \Rightarrow \lambda_2 = \beta \begin{pmatrix} \mu \\ -1 \end{pmatrix},$$

$$-\xi \in \partial_r ||E_T(Wr+q)|| \Rightarrow -\xi \in W^T E_T^T \partial_{u_T} ||u_T|| \Rightarrow \xi = -\gamma \begin{pmatrix} b \\ d \end{pmatrix}, \gamma \in [0,1].$$

From the KKT condition, we have:

$$\langle r, \tilde{u}\rangle = -\{\langle r-\lambda_1, W^T(r-\lambda_1)\rangle + \langle r-\lambda_1, \mu(r-\lambda_1)_N\xi\rangle + \langle \lambda_1, \lambda_2\rangle\},$$

with

$\langle r-\lambda_1, W^T(r-\lambda_1)\rangle = \langle r, W^T r\rangle = ar_N^2 + 2\mu b r_N^2 + \mu^2 d r_N^2,$

$\langle \lambda_1, \lambda_2\rangle = 0,$

$\langle r-\lambda_1, \mu(r-\lambda_1)_N\xi\rangle = \langle r, \mu r_N \xi\rangle = -\gamma\mu b(r_N)^2 - \gamma\mu^2 d r_N^2.$

$$\Rightarrow \langle r, \tilde{u}\rangle = -(a + 2b\mu + \mu^2 d - \mu\gamma b - \mu^2\gamma d)r_N^2$$

$$\Rightarrow \langle r, \tilde{u}\rangle > 0 \Leftrightarrow (a + 2b\mu + \mu^2 d - \mu\gamma b - \mu^2\gamma d) < 0$$

We consider the left handside of inequality as a function of μ:

$$f(\mu) = (d - \gamma d)\mu^2 + (2b - \gamma b)\mu + a < 0$$

$$\triangle(\gamma) = (2b - \gamma b)^2 - 4a(d - \gamma d)$$

$$= b^2\gamma^2 + (4da - 4b^2)\gamma + 4(b^2 - da)$$

since $ad \geq b^2 \Rightarrow \triangle(\gamma)$ always has solution of γ, we find conditions of γ such that $\triangle(\gamma) > 0$:

$$\triangle^{'} = (2da - b^2)^2 - b^2 4(b^2 - da)$$

$$= 4d^2a^2 - 4dab^2 = 4da(da - b^2) \geq 0 \forall a, b, d \ satisfy \ W \in S_+.$$

(assume W is P-matrix).

We obtain the table of varying γ:

$-\infty$	0	$\dfrac{-2(da-b^2) + 2\sqrt{da(da-b^2)}}{b^2}$	1	$+\infty$
+	-	+	+	+

So with the value of $\gamma \in (\frac{-2(da-b^2)+2\sqrt{da(da-b^2)}}{b^2}, 1)$, we find two solution of $f(\mu)$:

$$\begin{cases} \mu_1 &= \frac{(\gamma b - 2b) - \sqrt{\triangle(\gamma)}}{2d(1-\gamma)}, \\ \mu_2 &= \frac{(\gamma b - 2b) + \sqrt{\triangle(\gamma)}}{2d(1-\gamma)}. \end{cases}$$

and the table varying of μ

-∞	μ_1	μ_2	$+\infty$
$+$	0 -	0$+$	$+$

So to obtain $f(\mu) < 0$ we have the condition of μ is: $\mu \in (\mu_1, \mu_2),\ \ with\ \ \gamma \in (\frac{-2(da-b^2)+2\sqrt{da(da-b^2)}}{b^2}, 1)$

But the thing is that both $\mu_1\ \ and\ \ \mu_2$ need to be positive. We check:

$$\mu_1 = \frac{(\gamma b - 2b) - \sqrt{\triangle(\gamma)}}{2d(1-\gamma)} > 0$$

$$\Leftrightarrow (\gamma b - 2b) - \sqrt{\triangle(\gamma)} > 0$$

$$\Leftrightarrow \sqrt{\triangle(\gamma)} > 2b - \gamma b$$

$$\Leftrightarrow 4da(\gamma - 1) > 0$$

It is impossible since $\gamma \in (0, 1)$, the same with μ_2.

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