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We prove that any 4-regular connected Cayley graph on a finite abelian group can be decomposed into two hamiltonian cycles. This answers a partial case of Alspach's conjecture concerning hamiltonian decompositions of $2k$-regular connected Cayley graphs. As a corollary we obtain the hamiltonian decomposition of 2-jump circulant graphs, also called double loops.

1. INTRODUCTION

This paper is motivated by a problem, posed by Bermond, Illiades, and Peyrat in [4], on double loops. Let a, b, n be three integers with $0 < a \neq b < n/2$; the vertices of the *double loop graph* DLG(a, b) are the integers modulo n, the vertex i being joined to the four vertices $i \pm a$, $i \pm b$. These graphs are also known in the literature as 2-jump circulant graphs (see, for example, the survey of Boesch and Tindell [5] on circulant graphs). DLG(a, b) is connected if and only if the greatest common divisor $\gcd(a, b, n)$ is 1. The problem posed in [4] was to prove that if DLG(a, b) is connected, then it is decomposable into two hamiltonian cycles (two loops). Such a property is interesting because routings on a hamiltonian cycle are easy; these graphs are good fault-tolerant networks because in case of edge failure one can always find a simple routing. This problem was also posed by Liang Sun [10]. It also appears as a special case of a conjecture of Alspach [1] concerning Cayley graphs.

Let G be a finite group (the operation being denoted by $+$) and S a symmetric subset of G (i.e., $s \in S \Rightarrow -s \in S$). The vertices of the *Cayley graph* Cay(S, G) are the elements of G and there is an edge between x and y if and only if $x - y \in S$. Note that Cay(S, G) is connected if and only if S generates the group G. It is known (although this was quoted as a problem in [5]) that any connected Cayley graph on an abelian group is hamiltonian (for the problem of the existence of a hamiltonian cycle in a general Cayley graph, see the survey of Witte and Gallian [11]). In [1]

Alspach posed the following problem: can the edges of every connected $2k$-regular Cayley graph on a finite abelian group be partitioned into k hamiltonian cycles? The problem of [4] corresponds to the particular case where G is the cyclic group Z_n of the integers modulo n and $S=\{\pm a, \pm b\}$. In that case the graph $\mathrm{Cay}(S, G)$ is regular of degree 4. The aim of this paper is to settle Alspach's conjecture for degree 4 and therefore the original problem. Partial results were obtained by Yixiu Huang [8], Loo Saoping [9], and Liang Sun [10].

MAIN THEOREM. *Every 4-regular connected Cayley graph on a finite abelian group can be decomposed into two hamiltonian cycles.*

2. NOTATION

Let the set $S=\{\pm a, \pm b\}$ be a generating set of the finite abelian group G and let us suppose $a\neq 0$ and $b\neq 0$. The associated Cayley graph Γ is connected. It is a simple 4-regular graph if $2a\neq 0$, $2b\neq 0$ and $a\neq \pm b$.

Remark. The case $a=\pm b$ is not interesting; indeed the graph consists of a double cycle, which can be viewed as two hamiltonian cycles. In the case $2a=0$, there exist double edges; one can consider the associated Cayley multigraph, but it is not always true that it can be decomposed into two hamiltonian cycles (see Remark 2 at the end of Sect. 3). We will therefore suppose in what follows that Γ is a simple graph.

Let us call an *a-edge* (resp. *b-edge*) an edge of Γ joining two vertices x and $x+a$ (resp. x and $x+b$). Then the partial graph generated by the a-edges (resp. b-edges) is a disjoint union of cycles called *a-cycles* (resp. *b-cycles*) each of length k_a(resp k_b), the order of a (resp. the order of b). Note that the a-edges, b-edges, a-cycles, and b-cycles have a natural orientation. *Let α be the number of a-cycles and β the number of b-cycles.* Then the number n of vertices satisfies: $n=\alpha k_a=\beta k_b$.

In what follows we will use a particular representation of Γ. First note, as Γ is connected, that $0, b, \ldots, ib, \ldots, (\alpha-1)b$ belong to different a-cycles denoted $C_0, C_1, \ldots, C_i, \ldots, C_{\alpha-1}$. Indeed suppose that the vertex rb belongs to C_0, for some r, such that $0<r<\alpha$. That means that $rb=pa$ for some integer p and therefore every vertex of the graph can be written $x=\lambda a+\mu b=\lambda' a+\mu' b$ with $0\leqslant \mu'<r$ and the vertices of the graph will be included in at most r a-cycles with $r<\alpha$, a contradiction. Similarly one can show that αb is in C_0. Let c be defined by $\alpha b=ca$ with $0\leqslant c<k_a$.

Furthermore every vertex x of Γ can be uniquely written $x=ib+ja$ with $0\leqslant i<\alpha$ and $0\leqslant j<k_a$. Indeed if $x=\mu b+\lambda a$ then $x=ib+ja$ with

$$i\equiv \mu \ (\mathrm{mod.}\ \alpha) \qquad \text{and} \qquad j\equiv \lambda + c\cdot(\mu-i)/\alpha \ (\mathrm{mod.}\ k_a).$$

In fact this representation follows from a result of finite group theory: G is isomorphic to Z^2/K, where K is the kernel of the homomorphism Ψ from Z^2 to G defined by $\Psi(\mu, \lambda) = \lambda a + \mu b$. This kernel is a subgroup of G with basis $(0, k_a)$ and $(\alpha, -c)$. We will use the uniqueness of the representation to label the vertices (i, j) with $0 \leqslant i < \alpha$ and $0 \leqslant j < k_a$ where the first coordinate indicates the label of the cycle C_i containing the vertex and the second coordinate indicates the position of the vertex on the cycle.

DEFINITION. $\Gamma(\alpha, \beta)$. Let $\Gamma(\alpha, \beta)$ denote the class of simple graphs such that:

— there exist integers α, k, c with $\alpha \geqslant 1$, $k \geqslant 3$, $0 \leqslant c < k$, and $\beta = \gcd(k, c)$.

— the αk vertices of the graph can be labelled (i, j) with i taken modulo α and j taken modulo k.

— the edges are all the pairs:

first kind: $\{(i, j), (i, j+1)\}$

second kind: $\{(i, j), (i+1, j)\}$ for $0 \leqslant i \leqslant \alpha - 2$, and $\{(\alpha - 1, j), (0, j+c)\}$.

Note that the edges of the first kind form α cycles each of length k. Let us denote by β' the number of cycles of the second kind and by h their common length. We will show that $\beta' = \beta = \gcd(k, c)$. Clearly $h = z\alpha$ where zc is equal to the least common multiple of k and c. Thus $z \cdot \gcd(k, c) = k$ and $\beta' = \alpha k/h = \gcd(k, c)$.

One can have a pictorial representation of a graph of $\Gamma(\alpha, \beta)$ as α vertical disjoint cycles C_i, $0 \leqslant i < \alpha$, with a natural orientation and $\alpha - 1$ horizontal "parallel matchings" between the cycles C_i and C_{i+1} for $0 \leqslant i < \alpha - 1$ and a particular "parallel matching" between $C_{\alpha-1}$ and C_0 (drawn in heavy lines in the figures).

Remark. In the case $\alpha = 2$, a graph of $\Gamma(2, \beta)$ consists in fact of two cycles plus two perfect matchings between them (see Fig. 1). In the case $\alpha = 1$, a graph of $\Gamma(1, \beta)$ consists in fact of a cycle plus the chords joining $(0, j)$ to $(0, j+c)$ (see Fig. 2).

PROPOSITION. *The class $\Gamma(\alpha, \beta)$ consists of the 4-regular connected Cayley graphs on a finite abelian group with a generating set $\{a, b\}$, where α is the number of a-cycles and β the number of b-cycles.*

Proof. The labelling of the Cayley graph associated to a group G given above has exactly the required properties in the definition of the class

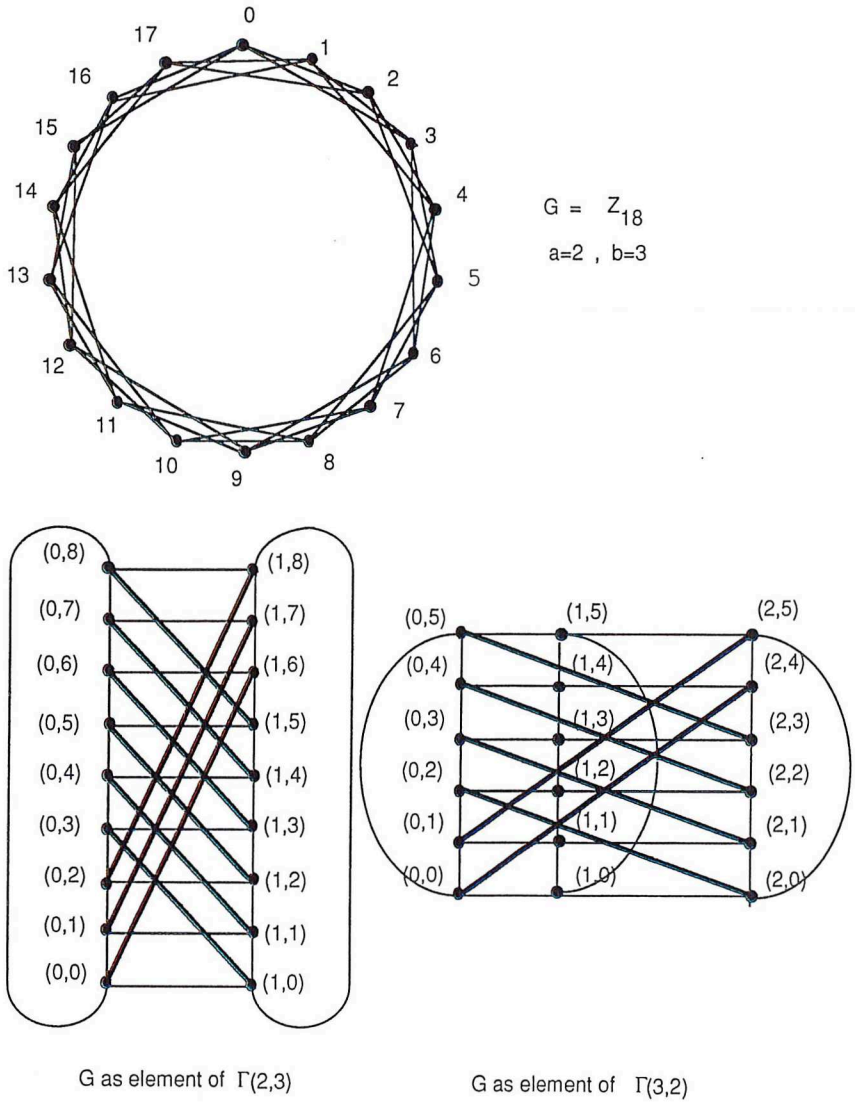

FIGURE 1

$\Gamma(\alpha, \beta)$; k is the order k_a of a and c is defined by $\alpha b = ca$, with $0 \leqslant c < k_a$. The a-edges (resp. b-edges) are those of the first kind (resp. second kind). Conversely a graph of $\Gamma(\alpha, \beta)$ can be considered as the Cayley graph on the abelian group isomorphic to Z^2/K where K is the subgroup with basis $(0, k)$, $(\alpha, -c)$ and where a and b are the images of $(0, 1)$ and $(1, 0)$ in the isomorphism $Z^2/K \to G$. It is shown furthermore in [6] that the Cayley graph determines the group G and the generators a and b up to a group isomorphism stabilizing $\{\pm a, \pm b\}$ with only one family of exceptions: $G = Z_{4n}$, $a = 1$, $b = 2n - 1$ whose Cayley graph is isomorphic to that of $G = Z_{2n} \times Z_2$, $a = (1, 0)$ and $b = (1, 1)$.

Remarks. (1) A graph of $\Gamma(\alpha, \beta)$ can also be considered as a graph of $\Gamma(\beta, \alpha)$.

(2) The cartesian sum of two cycles is a special case. It corresponds to $G = Z_\alpha \times Z_\beta$, $a = (0, 1)$, $b = (1, 0)$, that is $c = 0$, $\alpha = k_b$, $\beta = k_a$.

EXAMPLE 1. Figure 1 shows the Cayley graph associated to $G = Z_{18}$ with $a = 2$ and $b = 3$. Here $\alpha = 2$ and $\beta = 3$. A first representation shows it as an element of $\Gamma(2, 3)$; the vertex (i, j) corresponds to the vertex $3i + 2j$. The vertex $(0, j)$ is joined to the vertex $(1, j)$ and the vertex $(1, j)$ to $(0, j + 3)$ (edges in heavy lines). Here $c = 3$, $k = 9$. A second representation shows it as an element of $\Gamma(3, 2)$; the vertex (i, j) corresponding to the vertex $2i + 3j$. The vertex $(0, j)$ is joined to $(1, j)$, $(1, j)$ to $(2, j)$, and $(2, j)$ to $(0, j + 2)$ (these last edges are drawn in heavy lines). Here $c = 2$, $k = 6$.

EXAMPLE 2. Figure 2 shows the Cayley graph associated to $G = Z_6$ with $a = 1$, $b = 2$. Here $\alpha = 1$, $\beta = 2$. We give its representation as an element of $\Gamma(1, 2)$.

3. PROOF OF THE THEOREM

We will prove that *any graph of* $\Gamma(\alpha, \beta)$ *can be decomposed into* 2 *hamiltonian cycles.*

The idea of the proof is to build one hamiltonian cycle from the α-cycles by deleting some of their edges and adding some edges of the parallel matchings. This must be done in such a way that the remaining edges form a hamiltonian cycle. One can see in Fig. 3a (on the right) what happens in Example 1 considered as an element of $\Gamma(3, 2)$ and (on the left) what happens in Example 2. In Example 1 we delete from the cycles of the first kind the edges $\{(0, 0), (0, 1)\}$, $\{(0, 2), (0, 3)\}$, $\{(1, 0), (1, 1)\}$, $\{(1, 5), (1, 0)\}$, $\{(2, 0), (2. 1)\}$, $\{(2, 5), (2, 0)\}$, and add $\{(2, 0), (0, 2)\}$, $\{(2, 1), (0, 3)\}$, $\{(0, 0), (1, 0)\}$, $\{(0, 1), (1, 1)\}$, $\{(1, 0), (2, 0)\}$, $\{(1, 5), (2, 5)\}$. Figure 3b represents the second hamiltonian cycle on the same representation. Figure

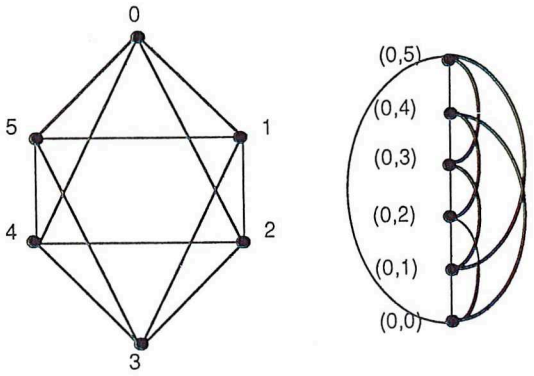

FIGURE 2

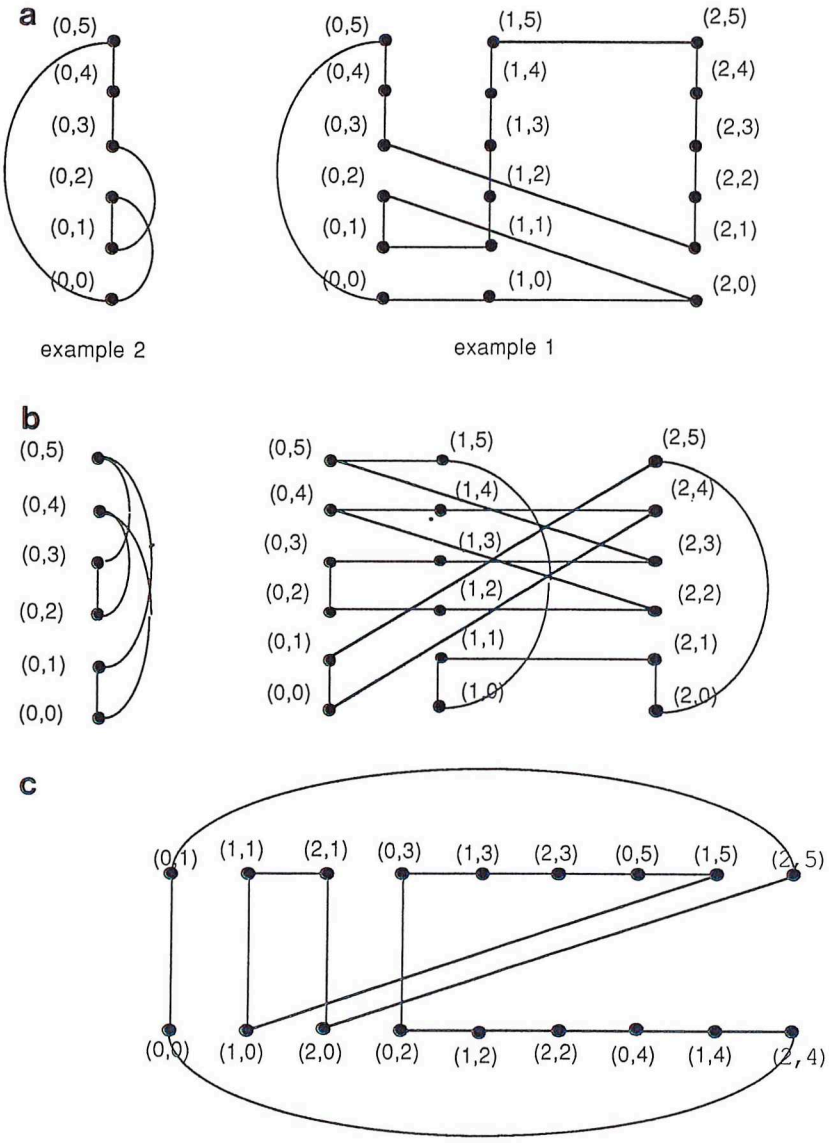

FIG. 3. Decompositions into hamiltonian cycles of Examples 1 and 2.

3c represents the second hamiltonian cycle of Example 1 on the representation with the cycles of the second kind.

The following lemma will reduce the problem to small values of α and β.

Let Γ be a graph of $\Gamma(\alpha+2, \beta)$ and let Γ' *be the reduced graph*, obtained from Γ by deleting the vertices of the cycles C_α and $C_{\alpha+1}$, that is, the vertices (α, j) and $(\alpha+1, j)$, and by adding the edges $(\alpha-1, j)$, $(0, j+c)$ joining $C_{\alpha-1}$ and C_0, where c is given in the definition of $\Gamma(\alpha+2, \beta)$. Γ' might in some cases (see the end of the proof) be a multigraph, but if it is a simple graph, it is an element of $\Gamma(\alpha, \beta)$ (k and c being unchanged). For example the graph of Fig. 2 is the reduced graph of that of Fig. 1 considered as an element of $\Gamma(3, 2)$.

DEFINITION. We say that a hamiltonian decomposition of a graph belonging to $\Gamma(\alpha, \beta)$ has the *property P* between C_i and C_{i+1}, $0 \leqslant i \leqslant \alpha - 1$ (with $C_\alpha = C_0$), if both hamiltonian cycles use at least one edge of the matching between C_i and C_{i+1}.

If $\alpha = 2$, it is necessary to distinguish the matching between $C_{\alpha-1} = C_1$ and C_0 and that between C_0 and C_1.

LEMMA 1. *Let Γ be a graph of $\Gamma(\alpha+2, \beta)$. If the reduced graph Γ' admits a hamiltonian decomposition having the property P between $C_{\alpha-1}$ and C_0, then Γ admits a hamiltonian decomposition having the property P between $C_{\alpha+1}$ and C_0.*

Proof. The idea is similar to one given in the article of M. Foregger [7]. Let H'_1 and H'_2 be the two hamiltonian cycles of the decomposition of Γ'. Let the m_1 edges joining $C_{\alpha-1}$ and C_0 in H'_1 be $\{(\alpha-1, j_m), (0, j_m + c)\}$ where $0 \leqslant m \leqslant m_1 - 1$. By property P, $m_1 \geqslant 1$.

Then replace each of these edges by the path of length $2(j_{m+1} - j_m) + 1$: $(\alpha - 1, j_m)$, (α, j_m), $(\alpha, j_m + 1)$, ..., $(\alpha, j_{m+1} - 1)$, $(\alpha + 1, j_{m+1} - 1)$, $(\alpha + 1, j_{m+1} - 2)$, ... $(\alpha + 1, j_m)$, $(0, j_m + c)$ (see Fig. 4). Note that if $j_{m+1} = j_m + 1$ the path is simply: $(\alpha - 1, j_m)$, (α, j_m), $(\alpha + 1, j_m)$, $(0, j_m + c)$.

In that way one clearly obtains a hamiltonian cycle H_1 in Γ. The same construction with the hamiltonian cycle H'_2 of Γ' using the m_2 edges of H'_2 between $C_{\alpha-1}$ and C_0 gives rise to a hamiltonian cycle H_2 of Γ. We claim that H_1 and H_2 are edge-disjoint. Indeed an edge of the type $\{(\alpha-1, j), (\alpha, j)\}$ or $\{(\alpha+1, j), (0, j+c)\}$ belongs to H_1 (resp. H_2) if $\{(\alpha-1, j), (0, j+c)\}$ belongs to H'_1 (resp. H'_2). The edge $\{(\alpha, j), (\alpha+1, j)\}$ belongs to H_1 (resp. H_2) if $\{(\alpha-1, j+1), (0, j+c+1)\}$ belongs to H'_1 (resp. H'_2). The edges $\{(\alpha, j), (\alpha, j+1)\}$ and $\{(\alpha+1, j), (\alpha+1, j+1)\}$) belong to H_1(resp. H_2) if $\{(\alpha-1, j+1), (0, j+c+1)\}$ does not belong to H'_1

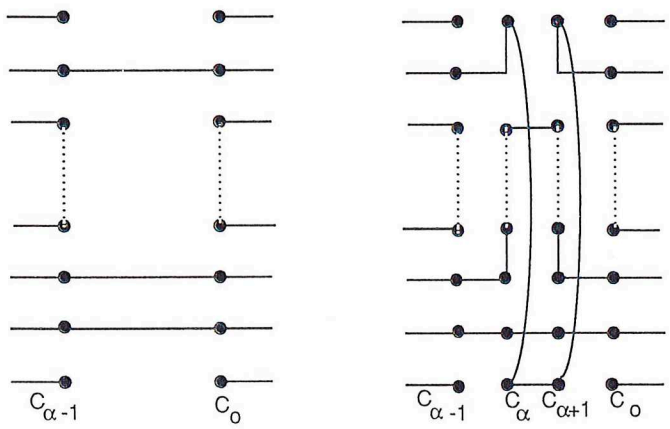

FIGURE 4

(resp. H'_2). The other edges of H_1(resp. H_2) are those of H'_1 (resp. H'_2). Therefore H_1 and H_2 form a hamiltonian decomposition of the original graph having property P between $C_{\alpha+1}$ and C_0. Indeed H_1 (resp. H_2) contains the edges $\{(\alpha+1, j), (0, j+c)\}$ such that $\{(\alpha-1, j), (0, j+c)\}$ belong to H'_1 (resp. H'_2).

EXAMPLE. Figures 3a and 3b show the cycles H_1 and H_2 of Example 1 constructed from a hamiltonian decomposition H'_1 and H'_2 of the reduced graph of Example 2.

LEMMA 2. *For $1 \leqslant \alpha \leqslant \beta \leqslant 2$, there exists a hamiltonian decomposition, having the property P between two cycles of the first kind and also between two cycles of the second kind, of any graph of $\Gamma(\alpha, \beta)$.*

Proof. In this proof we consider the graph as a Cayley graph with two generators a and b (by the proposition). The first hamiltonian cycle H_1 is obtained from the partial graph generated by the unique cycle ($\alpha=1$) or the two cycles ($\alpha=2$) by deleting the edges $(0, a)$ and $(-b, a-b)$ and adding the edges $(-b, 0)$ and $(a-b, a)$. The remaining edges form H_2, which is obtained in a symmetric way from the partial graph generated by the b-cycles.

As the graph is simple, k is greater than or equal to 3 and H_1 and H_2 contain at least one edge of the particular matching between $C_{\alpha-1}$ and C_0.

End of the Proof of the Theorem

The proof is by induction. Let Γ be a graph of $\Gamma(\alpha, \beta)$; by successive reductions, we find a reduced graph in $\Gamma(\alpha-2\lambda, \beta-2\mu)$ such that $1 \leqslant \alpha-2\lambda \leqslant 2$ and $1 \leqslant \beta-2\mu \leqslant 2$. If this graph is simple, by Lemma 2 it has a hamiltonian decomposition satisfying property P between two cycles of the first kind and also of the second kind. By applying Lemma 1, λ times, we obtain a hamiltonian decomposition of the reduced graph in $\Gamma(\alpha, \beta-2\mu)$. The graph considered as an element of $\Gamma(\beta-2\mu, \alpha)$ has property P between two cycles of the first kind (previously cycles of the second kind). By applying Lemma 1, μ times, we have a hamiltonian decomposition of Γ.

Thus, we have to examine when the reduced graph is not simple (in this case, we have to stop the reduction one or two steps before). That happens only in three cases:

— $\alpha=3$ and $c=0$; in that case we obtain loops in Γ'.

— $\alpha=4$ and $c=0$; then the edges in Γ' between C_0 and C_1 and between C_1 and C_0 are the same.

— $\alpha=3$ and $c=k/2$; then the edges are again double edges.

Note that the two first cases correspond to the cartesian sum; in that case $c=0$ and $k=\beta$. If $\beta \geqslant 5$ we reduce according to β; therefore to start the induction, we have to decompose C_3+C_3, C_3+C_4, C_4+C_4, the only elements of $\Gamma(3,3)$, $\Gamma(3,4)$, $\Gamma(4,4)$ with $c=0$. These graphs can easily be decomposed in hamiltonian cycles (see for example [7]). Let us consider the last case $\alpha=3$, $c=k/2$. Then $\beta=c$; as Γ is simple, then $k \geqslant 3$ and $\beta \geqslant 2$. If $\beta \geqslant 4$ we first reduce according to β. Thus, to start the induction we have to consider the two cases $\beta=2$ and $\beta=3$. These cases contain only two graphs, one in $\Gamma(3,3)$ (Fig. 5) and one in $\Gamma(3,2)$ (Fig. 6); each of these graphs can be decomposed into hamiltonian cycles satisfying property P between two cycles of the first kind and second kind.

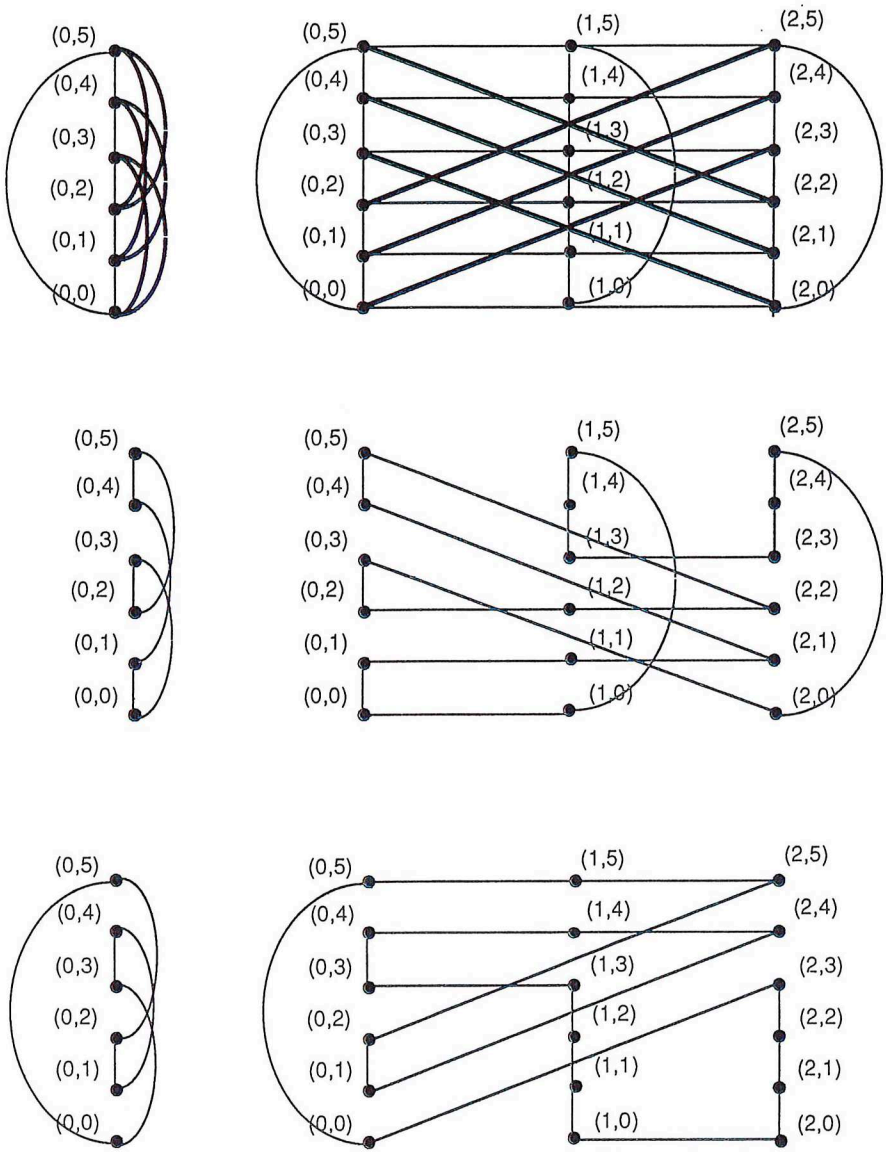

FIGURE 5

Note that the decompositions of these two last graphs are in fact obtained from Lemma 1 by using a hamiltonian decomposition of the multigraph shown in the left part of the Figs. 5 and 6.

Remarks. (1) There are many different possible decompositions of a graph of $\Gamma(\alpha, \beta)$ according to the order in which we apply the reduction, the manner we apply Lemma 1, and from what graph we start. Other constructions can also be obtained directly according to divisibility cases (such constructions can be sent to any interested reader).

(2) In the case of *multigraphs*, we can still apply Lemma 1, but Lemma 2 is not necessarily valid. Suppose $2a = 0$, then $\alpha = n/2$. The order

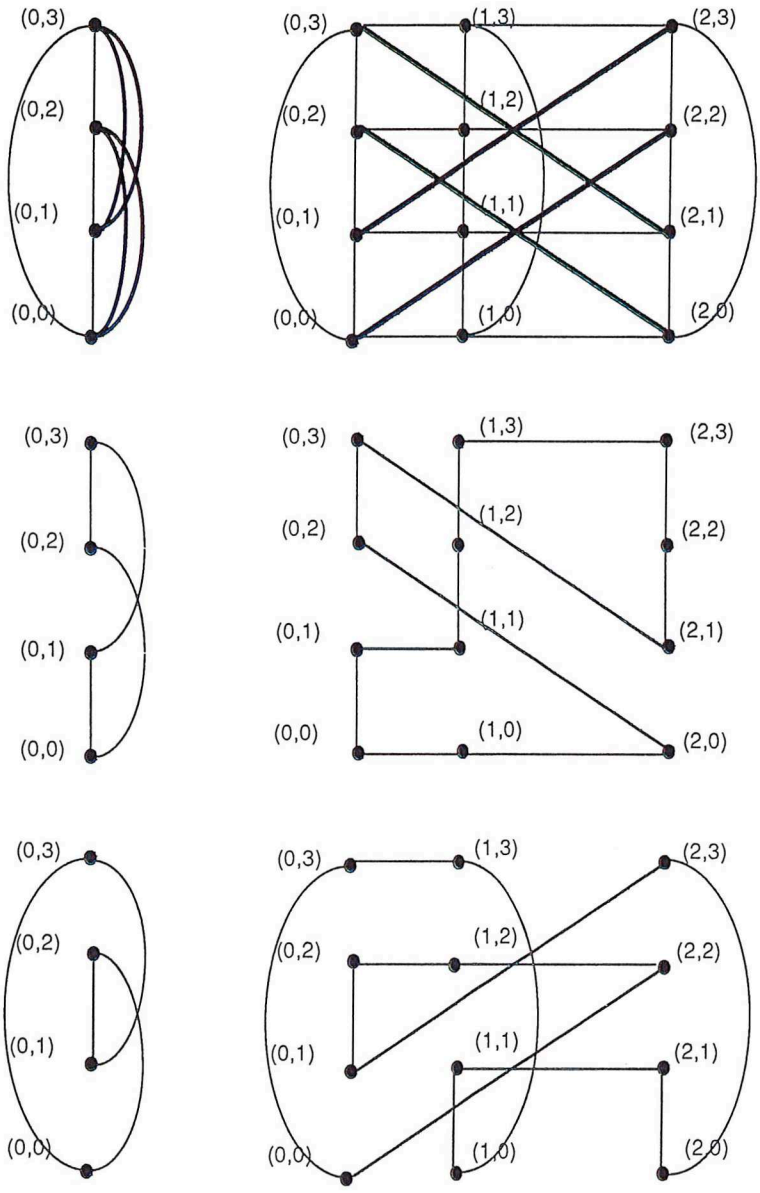

FIGURE 6

of b being greater or equal to α can only be $n/2$ or n and then $\beta = 2$ or $\beta = 1$. The multigraph admits a hamiltonian decomposition if and only if:

(i) $\alpha = n/2$ *is odd and* $\beta = 1$,

(ii) $\alpha = n/2$ *is even and* $\beta = 2$,

(iii) $\alpha = 2, \beta = 1$.

Conclusion. This paper supports Alspach's conjecture; maybe one can solve the case of 6-regular graphs (decomposition in three cycles) by using the work of Aubert and Schneider [2]. However the general case seems difficult. It implies the following conjecture: if the graphs G_i, $1 \leqslant i \leqslant p$, can be decomposed into n_i hamiltonian cycles, then the cartesian sum of these graphs can also be decomposed into hamiltonian cycles (see the survey [3] or the recent survey by B. Alspach, J.-C. Bermond, and D. Sotteau, Decompositions into cycles I: Hamilton decompositions in cycles and rays, *in* "Proc. Workshop Montreal April 1987," Reidel ASI series, in press).

Finally let us note an interesting corollary to the main theorem which answers a problem asked by M. Laurent. Let the pth power of a cycle be the graph whose vertices are the integers modulo n and where vertex i is joined to vertex j if $|i-j| \leqslant p$. Then this graph can be decomposed into p hamiltonian cycles. Indeed if p is even this graph is the union of $p/2$ double loop graphs with $a = 2i+1$ and $b = 2i+2$, $0 \leqslant i \leqslant p/2 - 1$. If p is odd it is the union of the cycle plus $(p-1)/2$ double loop graphs with $a = 2i$ and $b = 2i+1$, $1 \leqslant i \leqslant (p-1)/2$.

Acknowledgments

We thank C. Delorme, C. Peyrat, and the referees for their helpful comments and Olivier Bermond for his artwork.

References

1. B. Alspach, Unsolved problem 4.5, *Ann. Discrete Math.* **27** (1985), 464.
2. J. Aubert and B. Schneider, Decomposition de la somme cartésienne d'un cycle et de l'union de deux cycles hamiltoniens en cycles hamiltoniens, *Discrete Math.* **38** (1982), 7–16.
3. J.-C. Bermond, Hamiltonian décomposition of graphs and hypergraphs, *in* "Advances in Graph Theory" (B. Bollobas, Ed.), pp. 21–28, Annals of Discrete Mathematics, Vol. 3, 1978.
4. J.-C. Bermond, G. Illiades, and C. Peyrat, An optimization problem in distributed loop computer networks, *in* "Proc. Coll. 3rd International Conference on Combinatorial Mathematics, New York, June 1985," Annals New York Acad. Sci., in press.

5. F. Boesch and R. Tindell, Circulants and their connectivities, *J. Graph Theory* **8** (1984), 487–499.
6. C. Delorme, O. Favaron, and M. Maheo, Isomorphisms of Cayley multigraphs of degree 4 on finite abelian groups, submitted for publication.
7. M. F. Foregger, Hamiltonian decompositions of product of cycles, *Discrete Math.* **24** (1978), 251–260.
8. Yixiu Huang, On hamiltonian decompositions of Cayley graphs on cyclic groups, *in* "Proc. First China–USA International Conference on Graph Theory and its Applications, Jinan, China, June 1986."
9. Loo Soaping, Some results about circulant graphs, *in* "Proc. First China–USA International Conference on Graph Theory and its Applications, Jinan, China, June 1986."
10. Liang Sun, Some results on circulant graphs, *in* "Proc. First China–USA International Conference on Graph Theory and its Applications, Jinan, China, June 1986."
11. D. Witte and J. A. Gallian, A survey: Hamiltonian cycles in Cayley graphs, *Discrete Math.* **51** (1984), 293–304.