



## Interconnection networks with each node on two buses

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### ► To cite this version:

Jean-Claude Bermond, Johny Bond, Claudine Peyrat. Interconnection networks with each node on two buses. International workshop on Parallel algorithms and architectures, North Holland, 1986, Luminy, France. hal-02500473

**HAL Id: hal-02500473**

**<https://inria.hal.science/hal-02500473>**

Submitted on 6 Mar 2020

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#### ABSTRACT

Interconnection Networks are the heart of parallel architecture. In this paper, we consider bus interconnection networks in which the processors are connected by buses. These networks are modelled by hypergraphs. We restrict here our attention to the important case where each processor is connected to only two buses. Two more constraints have to be satisfied. The traffic load of each bus is limited. If the network is symmetric, this implies that a bus is adjacent to a limited number of processors, that is, the edge size of the associated hypergraph is bounded. Furthermore, the transmission delay is also limited; that is the diameter (maximum distance between any two nodes) of the associated hypergraph must be small.

The main problem considered here is the construction of bus interconnection networks of maximal degree two, given edge size  $r$  and diameter  $D$ , having the maximum number of vertices. Using the idea of duality in hypergraphs, we give families of good such bus interconnection networks, which have more vertices than those already known. Furthermore, these networks are easy to describe and have simple routings. We solve completely the case  $D=2$  and give a table of the best known hypergraphs for  $D \leq 10$  and  $r \leq 16$ .

#### 1. Introduction

Recent advances in technology, especially the advent of very large scale integrated (VLSI) circuit technology have enabled very complex interconnection networks to be constructed. Thus it is of great interest to study the topologies of interconnection networks, and, in particular, their associated graphical properties. Usually point to point connections are considered and the computer network is modeled by a graph in which the nodes or vertices correspond to the computer centers in the network and the edges correspond to the communication links. Surveys on classical interconnection networks can be found in [1,5,8,13].

Here we consider the case in which the computers share a communication medium such as a bus; the network is modeled by a hypergraph, where the nodes correspond to the computer centers and the (hyper)edges to the buses.

This work has been supported by the French CNRS through the CRECO C3



Although these networks are now less used than the classical ones with point-to-point connections, their importance is growing and they might be very important in the near future. Our interest in this subject came from discussions with persons working at the French National Telecommunication Center (CNET), on a project called REBUS.

Examples of such networks are the well-known  $d$ -dimensional hypercubes of width  $r$ . The vertices can be seen as  $d$ -tuples  $(a_1, \dots, a_d)$ , whose coordinates can take the values  $0, 1, 2, \dots, r-1$ ; therefore the total number of vertices is  $r^d$ . An edge consists of all the vertices having  $d-1$  fixed coordinates. So a vertex belongs to exactly  $d$  edges and an edge contains exactly  $r$  vertices. Particular cases are the binary hypercube ( $r=2$ ) which is a graph, and the "grid" or "mesh" ( $d=2$ ). The grid structure is essentially that used in the computer interconnection systems Illiac IV or Micronet (see for example the article of Wittie and Van Tilborg [17]). Other such networks have been proposed by Bhuyan and Agrawal[6], Doty [11], Finkel and Solomon [14, 15] and Wittie [18]. In the design of general networks, several parameters are very important, for example message delay, message traffic density, reliability or fault tolerance, existence of efficient algorithms for routing messages, cost of the networks, ...

It is advantageous to make the length of the paths that the message must travel from one node to another as small as possible, since this will reduce the message delay and also the message traffic density in the links. It corresponds to minimize the diameter of the associated graph or hypergraph. Similarly other network characteristics correspond to parameters of graphs or hypergraphs, e.g. mean distance, symmetry, connectivity ... Furthermore physical limitations usually restrict the number of connections for each processor; for bus interconnection networks it means that a node must not belong to too many buses or equivalently that each vertex has a bounded degree.

For bus interconnection networks one more constraint is to be considered: the traffic load on each bus is limited. Since for general networks without symmetry properties the traffic per bus is not easily calculated, we will restrict our study to networks with some uniformity and, as a first approximation, replace the constraint of traffic load by the limitation of the number of processors connected to each bus. This means that the size  $r$  of the edges of the associated hypergraph is bounded.

It is easy to see that it is impossible to optimize all the criteria at the same time: a comparison of some different criteria on a few structures has been done by Wittie [18]. Here we restrict our attention to one kind of optimization problem.

## **2. An optimization problem: $(\Delta, D, r)$ hypergraphs**

When direct node to node connections are considered, a problem which has been extensively studied is to interconnect the maximum number of nodes, subject to restrictions on the maximum node interdistance and on the maximum number of connections per node. This is known as the  $(\Delta, D)$  graph problem, which can be stated in graph theoretical terms as follows: find the maximum number of vertices of a graph with given maximum degree  $\Delta$  and given diameter  $D$  (see [1] or [5] for recent results on this problem and references).



For bus interconnection networks the problem can be stated in a similar manner: find bus interconnection networks having the maximum number of nodes possible subject to restrictions on the maximum node interdistance (diameter), on the maximum number of connections per node (maximum degree) and on the maximum traffic load on each bus (maximum edge size).

To state exactly the problem let us give some definitions and notations.

$H = (X, E)$  will denote a hypergraph with vertex-set  $X$  and (hyper)edge-set  $E$ .

The *degree* of a vertex is the number of edges containing it.

$\Delta$  will denote the maximum degree over all vertices of  $H$ .

$r$  will denote the maximal size of a (hyper)edge (or a bus).

A *path* connecting  $x$  and  $y$  in a hypergraph will be a sequence  $x = x_1, E_1, x_2, \dots, x_i, E_i, x_{i+1}, \dots, E_{p-1}, x_p = y$  with  $\{x_i, x_{i+1}\} \subseteq E_i$ . Its *length* will be  $p-1$ , the number of edges of the path. The *distance*  $d_H(x, y)$  between two different vertices  $x$  and  $y$  is the length of a shortest  $x-y$  path in  $H$ . The *diameter* is the maximum distance over all the pairs of vertices. It will be denoted by  $D$ .

A  $(\Delta, D, r)$  hypergraph (or  $(\Delta, D, r)$  bus network) will be a hypergraph of diameter  $D$ , maximum degree  $\Delta$  and maximum edge size  $r$ . The problem mentioned above can be stated as follows: construct  $(\Delta, D, r)$  hypergraphs having the maximum number of vertices. Let us denote by  $n(\Delta, D, r)$  this maximum number of vertices. (Remark : Note that our notations are different from Doty[12] and other's ones. They called the maximum degree  $d$ , the diameter  $k$  and the maximum size of edges -called bus degree-  $b$ . We use here the classical notations of hypergraph and design theory.)

Moore bound and Moore networks :

A bound analogous to the classic one in graph theory can easily be obtained for  $n(\Delta, D, r)$ . Each node belonging to at most  $\Delta$  buses and each bus containing at most  $r$  nodes, there are at most  $\Delta(r-1)$  nodes at distance one from a given node. Similarly, by using all these nodes we have at most  $\Delta(r-1)(\Delta-1)(r-1)$  nodes at distance 2 and more generally  $\Delta(\Delta-1)^{i-1}(r-1)^i$  nodes at distance  $i$ . Therefore we have :

$$\text{Proposition : } n(\Delta, D, r) \leq 1 + \Delta(r-1) \sum_{i=0}^{D-1} (\Delta-1)^i (r-1)^i$$

This bound is in general not attained. In fact, it can be attained only if  $D \leq 2$ . If  $D=2$ , there are only few possibilities, for  $r=2, r=4, r=5$ . If  $D=1$ , the problem is related to the existence of  $(n, r, 1)$ - designs ( see [1] § 6, for more details and references for the general case, and the article [3] for the case  $D=1$ ). In the case  $D=1$ , we want that each pair of processors communicate directly. In the classical node to node connection network, it corresponds to a complete graph or network and we have at most  $\Delta+1$  processors. For bus communication networks the situation is non trivial, and is extensively studied in [3] and in [4].

Here, we focus our attention on a case of practical interest, which was asked by the CNET and which is considered in most of the papers [6, 14, 15, 18], the case  $\Delta=2$ .

Therefore, in what follows, we will suppose that **each processor is connected to at most 2 buses** (other ports being used to communicate with the outside).

### 3. Case $\Delta = 2$

#### 3.1. Results

In table 1 below, we summarize the characteristics of the known bus networks with maximum degree  $\Delta = 2$  and maximum edge size  $r$ . We refer the reader to the corresponding articles for a complete definition of these networks. The bipartite graph associated to a bus network is defined as follows : one set of vertices correspond to the nodes of the network, the other to the buses. Two vertices are joined if the corresponding node and bus are incident.

Note that in the case  $\Delta = 2$ , the Moore bound is :

$$n(2, D, r) \leq 1 + 2(r-1) \sum_{i=0}^{D-1} (r-1)^i = 1 + \frac{2(r-1)((r-1)^D - 1)}{r-2}$$

For a given  $D$ , this bound is of the order of  $r^D$ . The networks constructed in [6, 14, 15, 18] are far from this bound. Those of [11] have  $\left(\frac{r}{2}\right)^D$  vertices. Here, we will construct networks having a greater number of vertices, easier to describe and having simple algorithms. The remaining part of the paper is organized as follows : we give the main idea which uses duality. Then a first application give us networks with  $\left(\frac{r}{2}\right)^D + \left(\frac{r}{2}\right)^{D-1}$  vertices. Using bipartite graphs, we obtain a network with  $2\left(\frac{r}{2}\right)^D$  vertices. Then, we solve completely the case  $D=2$ . Finally, we give a construction of Rote of networks with  $3\left(\frac{r}{2}\right)^D$  vertices. Other better results are obtained for small values of  $D$  and  $r$  (see Table 2) and particular congruences of  $D$ .

#### 3.2 Main Idea

The main idea is simple and uses the duality in hypergraphs. Let us recall that if  $H = (X, E)$  is a hypergraph, its *dual hypergraph*  $H^* = (E^*, X^*)$  is the hypergraph whose vertices (resp. edges) are in one-to-one correspondence with the edges (resp. vertices) of  $H$ , and in which the vertex  $e_j^*$  belongs to the edges  $X_i^*$  if and only if  $x_i$  belongs to  $E_j$ . Clearly, the maximum degree of  $H$  is equal to the maximum edge size of  $H^*$  and vice-versa. Therefore, if  $H$  is a  $(\Delta, D, r)$ -hypergraph with  $n$  vertices and  $m$  edges, then  $H^*$  is a  $(r, D^*, \Delta)$ -hypergraph with  $m$  vertices and  $n$  edges.

Note that in the particular case  $\Delta=2$ , the dual of a  $(2, D, r)$  bus network is a graph. Therefore, we will use the known results on graphs to obtain results for  $(2, D, r)$  bus networks by applying the following proposition :

**Proposition :** If  $G$  is a graph with maximum degree  $r$ , diameter  $D$  and  $m$  edges, then its dual hypergraph  $H$  is a  $(2, D^*, r)$  bus network on  $m$  vertices with maximum edge size  $r$ , diameter  $D^*$  where  $D-1 \leq D^* \leq D+1$ .



Proof. Indeed, let  $E_1$  and  $E_2$  be two vertices of the hypergraph  $H$ . They correspond to edges  $e_1$  and  $e_2$  of  $G$ . Let  $x_1$  be a vertex of  $e_1$  and  $x_2$  be a vertex of  $e_2$ . As  $d_G(x_1, x_2) \leq D$ , then  $d_H(E_1, E_2) \leq D+1$ . Therefore  $D^* \leq D+1$ . By duality,  $D \leq D^*+1$  that is  $D-1 \leq D^*$ .  $\square$

### 3.3. First construction

The preceding proposition suffices to give us good constructions of  $(2, D, r)$  bus networks. Indeed, if we know a "good" graph with maximum degree  $r$  and diameter  $D-1$ , we have by duality a  $(2, D, r)$  bus network. For example, in the particular case where we want  $D=2$ , we obtain by taking the best  $(r, 1)$  graph known, that is the complete graph  $K_{r+1}$ , a hypergraph with  $\frac{r(r+1)}{2}$  vertices, that is  $n(2, D, r) \geq \frac{r(r+1)}{2}$  (see fig 1 for an example).

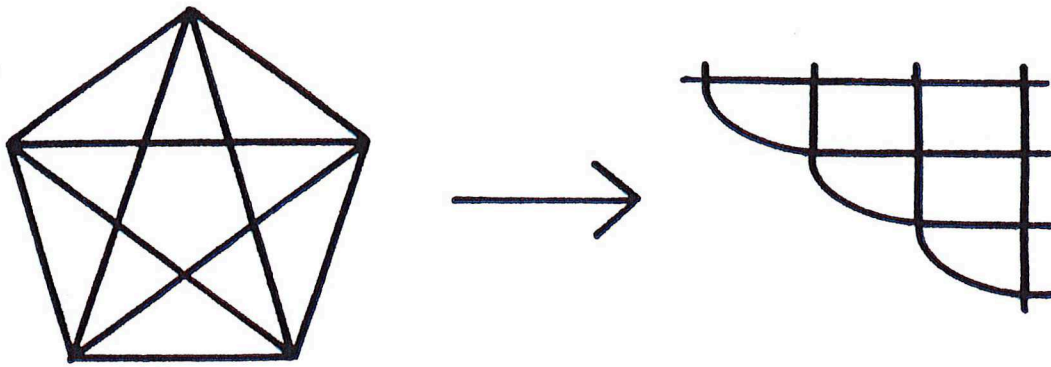


Figure 1

In the case  $D=3$ , using the best  $(r, 2)$  graphs, which are associated to projective planes and have about  $r^2$  vertices, we obtain  $(2, 3, r)$  bus networks with about  $\frac{r^3}{2}$  vertices and so on. In table 2, the value of the best known bus networks with  $r=4$  and  $D=10$  ( $n=7416$ ) is obtained from the best known  $(4, 9)$  graphs on 3708 vertices due to Doty [12].

A general good family of graphs is given by the Kautz-graphs. Recall that a Kautz digraph of degree  $\delta$  and diameter  $D$  has as vertices the words of  $D$  letters where two consecutive letters are different and where each letter can take  $\delta+1$  values. There is an arc from  $x$  to  $y$  if the  $D-1$  first letters of  $x$  are the  $D-1$  last letters of  $y$ . By deleting the orientation, one obtains a multigraph. Kautz graphs are simple graphs obtained by deleting double edges (for example between  $(a, b, a, b, \dots)$  and  $(b, a, b, a, \dots)$ ).

Here we will consider the multigraph; indeed if the double edges were useless for the  $(\Delta, D)$ -graph problem, here they will give rise to different vertices in the dual bus network. A Kautz multigraph has  $\delta^D + \delta^{D-1}$  vertices, is regular of degree  $2\delta$ , has diameter  $D$  and  $\delta(\delta^D + \delta^{D-1})$  edges. By taking the dual of a Kautz multigraph, with diameter  $D-1$  and degree  $r$ , we have a  $(2, D, r)$  bus network with  $(\frac{r}{2})^D + (\frac{r}{2})^{D-1}$  edges. Therefore we obtain the following result :

**Theorem :** There exist  $(2, D, r)$  bus network with  $(\frac{r}{2})^D + (\frac{r}{2})^{D-1}$  vertices.

*Remark:* One can define in fact, this network directly. The vertices are words of length  $D$  constructed on an alphabet of  $\delta + 1$  letters where two consecutive letters are different. The buses are the words of length  $D-1$  with no two identical consecutive letters, the vertex  $(a_1, \dots, a_D)$  belongs to the two buses  $(a_1, \dots, a_{D-1})$  and  $(a_2, \dots, a_D)$ . The bus  $(b_1, \dots, b_{D-1})$  contains the  $r=2\delta$  vertices  $(*, b_1, \dots, b_{D-1})$  and  $(b_1, \dots, b_{D-1}, *)$ .

Note that the routing is very simple in such a bus network : let  $(a_1, \dots, a_D)$  and  $(a'_1, \dots, a'_D)$  be two vertices, and let  $i$  be the greatest value such that  $(a_1, \dots, a_i) = (a'_{D-i+1}, \dots, a'_D)$  or  $(a'_1, \dots, a'_i) = (a_{D-i+1}, \dots, a_D)$ .

Suppose for example that  $(a_1, \dots, a_i) = (a'_{D-i+1}, \dots, a'_D)$ . Then the routing uses the vertices  $(a_1, \dots, a_D)$  ;  $(a'_{D-i}, a_1, \dots, a_{D-1})$  ; ... ;  $(a'_{D-i-j}, \dots, a'_{D-i}, a_1, \dots, a_{D-1-j})$  ; ... ;  $(a'_1, \dots, a'_D)$ .

### 3.4. Constructions using bipartite graphs

To find better networks, we must construct graphs such that the diameter of the dual network  $D^*$  is less or equal than  $D$ . Note that the characterization of such graphs seems to be a difficult problem.

However if the graph is bipartite, we have the following proposition.

**Proposition :** If  $G$  is a bipartite graph with maximum degree  $r$ , diameter  $D$  and  $m$  edges, then its dual hypergraph  $H$  is a  $(2, D^*, r)$  bus network with diameter  $D^* \leq D$  and  $m$  vertices.

*Proof:*

Let  $E_1$  and  $E_2$  be two vertices of  $H$ . They correspond to edges  $e_1=(x_1, y_1)$  and  $e_2=(x_2, y_2)$  in  $G$ . If  $d_G(x_1, x_2) \leq D-1$ , then  $d_H(E_1, E_2) \leq D$ . Suppose  $d_G(x_1, x_2) = D$ . As  $G$  is a bipartite graph,  $G$  has no cycle of odd length, therefore  $d_G(x_1, y_2) \neq D$ . As  $G$  has diameter  $D$ ,  $d_G(x_1, y_2) \leq D-1$  and  $d_H(E_1, E_2) \leq D$  and therefore  $D^* \leq D$ . □

Now we can use large bipartite  $(r, D)$  graphs to construct large  $(2, D, r)$  bus network. For  $D=2$  the best bipartite graph is the complete bipartite graph  $K_{r,r}$  which has  $r^2$  edges. Its dual is nothing but the grid which has  $r^2$  vertices,  $r$  horizontal and  $r$  vertical buses and in which it is also easy to find a simple routing (see fig. 2 for an example).



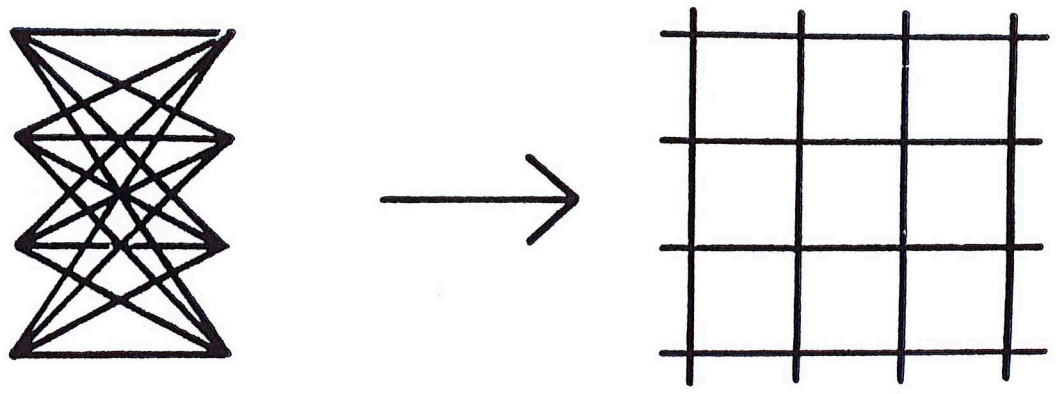


Figure 2

Similarly for  $D=3$ , we obtain  $(2,3,r)$  bus network with about  $r^3$  vertices by using the bipartite graph associated to projective planes.

Generally, we obtain an asymptotically good family of  $(2,D,r)$  bus network by using Delorme's work. Let  $G$  be a particular case of the  $C_d$ -graphs defined in [10]. A vertex of  $G$  is defined by its coordinates  $(i, a_1, \dots, a_k)$ , where  $i$  belongs to  $\mathbb{Z}/2\mathbb{Z}$  and  $a_j$  belongs to an alphabet  $A$  of  $\delta$  letters. The vertex  $(i, a_1, \dots, a_k)$  is adjacent with all the vertices  $(1-i, \lambda, a_1, \dots, a_{k-1})$  and  $(1-i, a_2, \dots, a_k, \lambda)$  where  $\lambda$  is any letter from  $A$ . If we keep the double edges, then  $G$  is regular of degree  $2\delta$ , has  $2\delta^k$  vertices and is of diameter  $D=k+1$ . Therefore its dual hyperbus network has maximum edge size  $r=2\delta$ , diameter  $D$  and order  $2 \left(\frac{r}{2}\right)^D$ .

**Theorem :** There exist  $(2,D,r)$  bus networks with  $2 \left(\frac{r}{2}\right)^D$  vertices.

For some values of  $D$ , one can also use Delorme's work to get better bounds. For example, if  $D$  is even, there exists  $(2,D,r)$  bus networks with  $4 \left(\frac{r}{2}\right)^D$  vertices. See

Delorme's article for other values.

The table 2 is mainly based on such bipartite graphs (see [7] for an updated table of the largest known bipartite graphs). However, we use a small improvement which enables us to add  $\left\lfloor \frac{r}{2} \right\rfloor$  more vertices.

**Proposition :** Let  $G$  be a bipartite graph of maximum degree  $r$ , diameter  $D$  and  $m$  edges. Let  $\bar{G}$  be a multigraph of maximum degree  $r$  obtained from  $G$  by splitting a vertex  $v$  into two vertices  $v_1$  and  $v_2$  in such a way that  $\left\lfloor \frac{r}{2} \right\rfloor$  edges incident to  $v$  are in-



cident to  $v_1$  and the remaining ones to  $v_2$  and by adding  $\left\lfloor \frac{r}{2} \right\rfloor$  parallel edges between  $v_1$  and  $v_2$ . Then  $H$ , the dual bus network of  $\bar{G}$  has maximum edge size  $r$ , diameter  $D^* \leq D$  and  $m + \left\lfloor \frac{r}{2} \right\rfloor$  vertices.

**Lemma:** Let  $G$  be a bipartite graph of diameter  $D$ , let  $e_1$  and  $e_2$  be two edges of  $G$ . Then, there exist either a path of length at most  $D-2$  or two disjoint paths of length  $D-1$  between the two edges.

*Proof of the lemma*

Suppose there is no path of length less than  $D-1$  between  $e_1 = (x_1, y_1)$  and  $e_2 = (x_2, y_2)$ . Then as  $d(x_1, x_2)$  and  $d(x_1, y_2)$  are of different parities ( $G$  being bipartite). We can suppose  $d(x_1, x_2) = D-1$ ,  $d(x_1, y_2) = D$ ,  $d(y_1, y_2) = D-1$  and  $d(y_1, x_2) = D$ . Let  $P_1$  (resp.  $P_2$ ) be a shortest path between  $x_1$  and  $y_1$  (resp.  $x_2$  and  $y_2$ ). Then  $P_1$  and  $P_2$  are disjoint paths. Otherwise  $P_1 \cup P_2$  will contain a path  $Q_1$  between  $x_1$  and  $y_2$  and a path  $Q_2$  between  $y_1$  and  $x_2$ . The sum of the lengths of  $Q_1$  and  $Q_2$  is at most the sum of the lengths of  $P_1$  and  $P_2$ , that is  $2D-2$ , which contradicts  $d(x_1, y_2) = D$  and  $d(x_2, y_1) = D$ . □

*Proof of the proposition:*

Let  $E_1$  and  $E_2$  be two vertices of  $H$ . They correspond to edges  $e_1 = (x_1, y_1)$  and  $e_2 = (x_2, y_2)$  of  $\bar{G}$ .

$\alpha)$   $e_1$  and  $e_2$  are edges of  $G$ .

Then from the lemma  $d_H(E_1, E_2) \leq D$ .

$\beta)$   $e_1$  or  $e_2 = (v_1, v_2)$ .

Suppose  $e_1 = (v_1, v_2)$ . If  $e_2$  has at least one endvertex in  $\{v_1, v_2\}$ , then  $d_H(E_1, E_2) = 1$ . Else,  $e_2$  belongs to  $G$ .  $G$  being bipartite  $x_2$  or  $y_1$  is at distance at most  $D-1$  of  $v$  in  $G$  and of  $v_1$  or  $v_2$  in  $\bar{G}$ . Hence,  $d_H(E_1, E_2) \leq D$ .

$\gamma)$   $e_1$  has  $v_1$  or  $v_2$  as endvertex.

We can suppose that  $e_2$  is an edge of  $G$ , otherwise the distance between  $e_1$  and  $e_2$  is at most 2. Let  $z$  be the endvertex of  $e_1$ , which is not  $v_1$  or  $v_2$ . Then,  $z$  is at distance at most  $D-1$  in  $G$  from  $x_2$  or  $y_2$  (say  $x_2$ ). If a shortest path between  $z$  and  $x_2$  does not contain  $v$ , then  $d_H(E_1, E_2) \leq D$ . Else  $d_G(x_2, v) \leq D-2$  and therefore  $d_{\bar{G}}(v_1, x_2) = D-2$  or  $D-1$ . Hence,  $d_H(E_1, E_2) \leq D$ . □

### 3.5. Case $D=2$ .

At this stage, let us go further on the study of the  $(2,2,r)$  bus networks. The Moore bound shows that they have at most  $2r^2 - 2r + 1$  vertices. We got construction of  $(2,2,r)$  bus network with  $\frac{r^2}{2}$  vertices and then  $r^2$  vertices by taking the grid. However  $(2,2,r)$  bus networks with  $\frac{5}{4}r^2$  vertices can be constructed if  $r$  is even. Although we found them in a complete different way, they correspond to the dual of a non bipartite graphs of diameter  $D$  with diameter  $D^* \leq D$ . Indeed let  $G$  be a graph obtained from a cycle of length five by replacing each edge by a complete bipartite graph  $K_{\frac{r}{2}, \frac{r}{2}}$ .

Let  $H$  its dual bus network, then  $H$  has diameter  $D$ , maximum edge size  $r$  and  $\frac{5}{4}r^2$  vertices (see fig. 3 for an example).

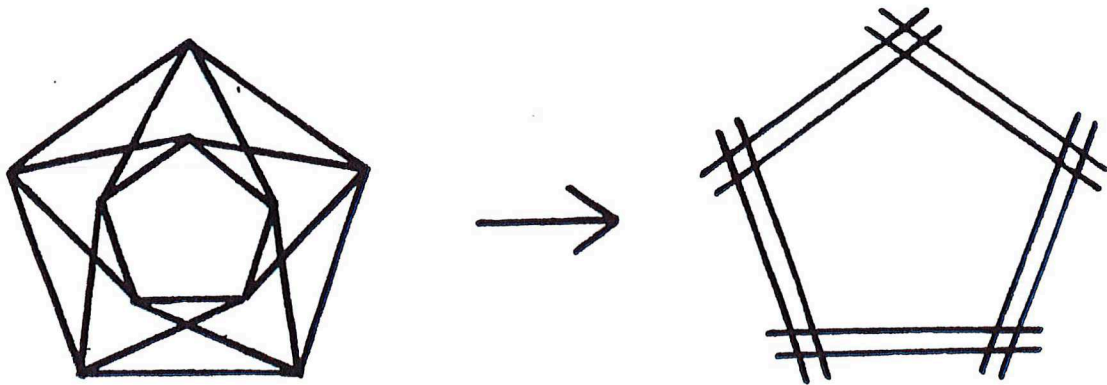


Figure 3

We conjectured in 1982, that this bound was best possible. This was proved by Kleitman (private communication 83) and gave rise to further interesting graph studies (see the article of Kleitman, Paoli and Trotter [16]). Indeed, one is interested in the maximum number of edges of graphs which do not contain an induced graph consisting of two independent edges. We summarize all that in

**Theorem:**  $n(2,2,r) = \frac{5}{4}r^2$ ,  $r$  even.

In the case  $r$  odd, we do a similar construction by splitting the vertices in  $\left\lfloor \frac{r}{2} \right\rfloor$  or  $\left\lceil \frac{r}{2} \right\rceil$  vertices, that is by replacing one edge by a  $K_{\left\lfloor \frac{r}{2} \right\rfloor, \left\lceil \frac{r}{2} \right\rceil}$  and the 4 others by  $K_{\left\lfloor \frac{r}{2} \right\rfloor, \left\lfloor \frac{r}{2} \right\rfloor}$ .

Therefore, if  $r=2t+1$ ,  $H$  has  $5t^2 + 4t = \frac{5}{4}(r-1)^2 + 5(r-1)$  vertices.

### 3.6 Other constructions

The pentagon idea described above can be refixed by replacing one edge by a kind of  $C_D$  structure instead of a complete bipartite graph. The graph obtained were first described by Delorme and Fahri [9] as  $C_5$  graphs. Let the vertices of  $G$  be labelled  $(i, a_1, \dots, a_k)$  where  $i$  belongs to  $\mathbb{Z}/5\mathbb{Z}$  and  $a_i$  belongs to an alphabet  $A$  of  $\delta$  letters. The vertex  $(i, a_1, \dots, a_k)$  is joined to the vertices  $(i+1, \lambda, a_1, \dots, a_{k-1})$  and therefore to  $(i-1, a_2, \dots, a_k, \lambda)$ , where  $\lambda$  is any letter from  $A$ . This graph has  $5\delta^k$  vertices, degree  $2\delta$  and diameter  $k+1$  if and only if  $k \equiv 1$  or  $4 \pmod{5}$ . Indeed two vertices whose first coordinate differ exactly by one are at distance at most  $k$ . Furthermore, as any two edges have a pair of such endvertices, the diameter of the dual network is at



most  $k+1$ . Therefore by taking  $D = k+1$ , we have the following theorem :

**Theorem** : If  $D \equiv 0$  or  $2 \pmod{5}$ , then  $n(2, D, r) \geq 5 \left( \frac{r}{2} \right)^D$ .

This theorem gives us the values indicated in table 2 for  $D = 10$  and  $r = 8, 12, 14$  and  $16$  (otherwise the values obtained by the construction using bipartite graphs are better).

For general diameter, one can use a similar construction with triangles. This nice idea is due to Gunter Rote during this colloquium (thanks to the napkins of the CIRM). The definition is the same as above with  $i$  in  $\mathbb{Z}/3\mathbb{Z}$ . Similarly we have :

**Theorem** (G. Rote) :  $n(2, D, r) \geq 3 \left( \frac{r}{2} \right)^D$ .

#### 4. Conclusion

We hope to have show how simple ideas of graph theory can give constructions of good bus network, with maximum degree 2. Note that as we show in 3.3, there exist very simple routing in all these networks. Therefore, these networks might be of great use in future parallel architectures. This paper covered the interesting case  $\Delta = 2$ ; for results concerning the general case see [2].

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Name	Defined in	Characterization	Number of vertices	Number of edges	Diameter
p-cube	[14]	Dual of an r-cube on $[0,1]$	$r \cdot 2^{r-1}$ $\sim D 2^{D-1}$	$2^r$	$r$
Snowflake	[14]	The associated bipartite graph is a tree	$r^n$ $\sim r^{\log_2 D}$	$\frac{r^n - 1}{r - 1}$	$2^n - 1$
Dense Snowflake	[14]	The associated bipartite graph is defined recursively	$r^n$ $\sim r^{\frac{D}{2}}$	$2 \cdot r^{n-1} - 1$	$2^n - 1$
Star	[14]	The associated bipartite graph is a tree	$\frac{r(r-1)^n - r(r-1)}{r-2} + 2$ $\sim r^{\frac{D}{2}}$	$\frac{r(r-1)^{n-1} - r + 1}{r-2}$	$2n$
Lens	[15]	The associated bipartite graph is defined recursively	$\frac{(r-1)^{n+1} - 1}{r-2}$ $\sim r^{\frac{D}{2}}$	$\frac{(r-1)^{n+1} - 1}{r-2}$	$2n$
Dual bus hypercube	[18]	From the n-cube, only some edges are covered by the buses	$r^n$ $\sim r^{\frac{D+1}{2}}$	$2 \cdot r^{n-1}$	$2n-1$
General hyperbus structure	[6]	Dual of an r-cube on $\{0, \dots, \frac{r}{D}\}$	$\frac{r}{2} (\frac{r}{D})^D$	$(\frac{r}{D})^D$	$D+1$
de Bruijn networks	[11]	The edges of a de Bruijn graph are covered by buses	$(\frac{r}{2})^D$	$(\frac{r}{2})^{D-1}$	$D$
bus hinging graph	[11]	By its bipartite associated graph	$\frac{r(r-1)^n - 2r}{r-2}$ $\sim r^{\frac{D}{2}}$	$2 \frac{r(r-1)^{n-2}}{r-2}$	$2n$
Kautz networks	Here	Dual of a Kautz graph	$(\frac{r}{2})^D + (\frac{r}{2})^{D-1}$	$(\frac{r}{2})^{D-1} + (\frac{r}{2})^{D-1}$	$D-2$
Delorme networks	Here	Dual of Delorme's bipartite graphs	$2(\frac{r}{2})^D$	$2(\frac{r}{2})^{D-1}$	$D$
Triangular networks	Here	Dual of a $C_3$ -graph	$3(\frac{r}{2})^D$	$3(\frac{r}{2})^{D-1}$	$D$
Pentagon networks	Here		$\frac{5}{4} r^2$	$5 \frac{r}{2}$	$2$

Table 1

# LARGEST BUS NETWORKS

(with each vertex on two buses)

D r	3	4	5	6	7	8	9	10
3	22	46	65	190	235	337	541	757
4	54	162	230	1 458	1 670	2 018	5 602	*7 416
5	107	427	602	6 627	7 462	12 502	26 652	73 127
6	189	939	1 533	23 439	24 833	57 333	131 043	393 123
7	276	1 214	3 279	31 475	82 029	218 739	546 843	1 998 627
8	460	3204	6 244	156 868	162 348	390 628	1 238 404	**5 242 880
9	661	5 269	10 604	337 045	529 420	1 764 724	5 294 164	26 470 804
10	915	8 205	20 005	664 305	1 123 475	4 119 395	13 481 645	74 898 005
11	1 017	9 542	32 180	783 557	2 192 195	8 768 765	30 690 830	182 682 505
12	1 602	17 574	49 206	2 125 878	3 965 806	16 147 446	44 640 966	**302 330 880
13	1 748	19 922	74 626	2 433 944	6 909 090	32 242 398	128 969 574	661 332 477
14	2 569	33 327	103 327	5 631 283	12 400 927	60 032 791	178 573 255	**1 412 376 245
15	2767	37 102	153 727	6 315 367	18 601 387	96 536 167	434 412 727	3 354 631 567
16	3 180	41 080	218 408	7 072 504	32 178 728	181 298 888	597 757 336	**5 368 709 120

Table 2