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Covert Cycle Stealing in a Single FIFO Server (extended version)

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Abstract

Consider a setting where Willie generates a Poisson stream of jobs and routes them to a single server that follows the first-in first-out discipline. Suppose there is an adversary Alice, who desires to receive service without being detected. We ask the question: what is the amount of service that she can receive covertly, i.e. without being detected by Willie? In the case where both Willie and Alice jobs have exponential service times with respective rates μ_1 and μ_2 , we demonstrate a phase-transition when Alice adopts the strategy of inserting a single job probabilistically when the server idles : over n busy periods, she can achieve a covert throughput of $\mathcal{O}(\sqrt{n})$ when $\mu_1 < 2\mu_2$, $\mathcal{O}(\sqrt{n/\log n})$ when $\mu_1 = 2\mu_2$, and $\mathcal{O}(n^{\mu_2/\mu_1})$ when $\mu_1 > 2\mu_2$. When both Willie and Alice jobs have general service times we establish an upper bound for the amount of service Alice can get covertly. This bound is related to the Fisher information. Additional upper bounds are obtained for more general insertion policies.

keywords: Cycle stealing; Covert communication; Queue.

1 Introduction

This paper considers the following problem. Willie has a sequence of jobs that arrive to a first-in first-out (FIFO) queue with a single server whose processing rate is known to Willie. There exists another actor, Alice, who may want to sneak jobs into the queue for the purpose of stealing processing cycles from Willie. This paper asks the following question: can Alice process her jobs without Willie being able to determine this occurrence beyond making a random guess and, if she can, what is her achievable job processing rate? Answer to this question may apply to several scenarios. Alice could administer a data center, contract to provide Willie with a server with a guaranteed performance, and then resell some of the processing cycles [8]. Similar considerations apply to network contracts. Willie could own a home computer and Alice could install malware for the purpose of stealing computational resources.

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In order to address this question of *covert cycle stealing*, we adopt the following model. Willie’s jobs arrive according to a Poisson process to a FIFO queue served by a single server with a specified processing rate. Service times of Willie’s jobs are assumed to be identically and independently distributed (iid) according to a general distribution. Alice can insert jobs as she wishes. Her service times are also iid coming from a general distribution that may differ from that of Willie. Once an Alice job starts service, it must remain in service until completion; this can interfere with the processing of Willie’s jobs. Last, both Willie and Alice know their own and the other party’s service time distributions and can observe the arrival and departure times of Willie’s jobs.

We formulate the problem as a statistical hypothesis testing problem where Willie’s task is to determine whether or not Alice is stealing cycles, based on observed arrivals and departures. We study two classes of policies that Alice can use for inserting her jobs into the queue. The first one, *Insert-at-End-of-Busy-Period* (IEBP) allows Alice to probabilistically insert a single job each time a Willie busy period (to be defined) ends. A variant of it, *Insert-at-Idle* (II) allows Alice to insert a job each time the server idles. The second one, *Insert-at-Idle-and-at-Arrival* (II-A) allows Alice to probabilistically insert both when the server idles and when a Willie job arrives. We obtain several results, of which the most interesting is for the IEBP and II policies when service times are exponentially distributed with rates μ_1 for Willie and μ_2 for Alice. In this case, we establish that, over n busy periods, Alice can achieve a covert throughput of $\mathcal{O}(\sqrt{n})$ when $\mu_1 < 2\mu_2$, $\mathcal{O}(\sqrt{n/\log n})$ when $\mu_1 = 2\mu_2$, and $\mathcal{O}(n^{\mu_1/\mu_2})$ when $\mu_1 > 2\mu_2$. This is interesting because of the phase transition at $\mu_1 = 2\mu_2$; earlier studies of covert communications and in steganography have never encountered such behavior. For example, [3] established the covert capacity of a Gaussian channel to be $\mathcal{O}(\sqrt{n})$ independent of channel parameters.

In addition to the above results for the IEBP policy when service times are exponentially distributed, we show that IEBP can also achieve a capacity of $\mathcal{O}(\sqrt{n})$ when Willie jobs have general service times and Alice jobs have (hyper-)exponential service times, under some constraints on the service rates. We also study the II-A policy and establish an upper bound on the number of jobs Alice can covertly introduce, namely that she cannot introduce $\omega(\sqrt{n})$ jobs without being detected. We also examine the effect of allowing Alice to insert batches of jobs and determine in some cases that this can seriously reduce her covert capacity.

The rest of the paper is organized as follows. Section 2 discusses related work. Section 3 introduces the model and needed background on hypothesis testing. This is followed by Section 4 where the IEBP policy is introduced and analyzed. Section 5 focuses on the II policy, a variant of the IEBP policy. Section 6 introduces the II-A policy and derives an asymptotic upper bound on Alice’s capacity. The section also derives tighter asymptotic upper bounds for the II-A policy when Alice introduces batches whose sizes are geometrically distributed. The paper concludes with Section 7.

2 Related work

Cycle stealing has been analyzed in the queueing literature in the context of task assignment in multi-server systems. The goal is to allow servers to borrow cycles from other servers while they are idle so as to reduce backlogs and latencies and prevent servers from being under-utilized [10, 14, 15]. These papers focus on the performance analysis of such systems, in particular, mean response times with or without the presence of switching costs. There is no attempt to hide or cover up the theft of cycles.

This paper focuses on the ability of an unknown user to steal cycles without the owner of the server detecting this. Thus it is an instance of a much broader set of techniques used in digital steganography and covert communications. Steganography is the discipline of hiding data in objects such as digital images. A steganographic system modifies fixed-size finite-alphabet covertext objects into stegotext containing hidden information. A fundamental result of steganography is the *square root law (SRL)*, $\mathcal{O}(\sqrt{n})$ symbols of an n symbol covertext may safely be altered to hide an $\mathcal{O}(\sqrt{n} \log n)$ -bit message [9]. Covert communications is

concerned with the transfer of information in a way that cannot be detected, even by an optimal detector. Here, there exists a similar SRL: suppose Alice may want to communicate to Bob in the face of a third party, Willie, but without being detected by Willie. When communication takes place over an additive white Gaussian noise (AWGN) channel, it has been established that Alice can transmit $O(\sqrt{n})$ bits of information in n channel uses [3]. This result has been extended to optical (Poisson) channels [2], binary channels [6], and many others [5, 18]. It has also been extended to include the presence of jammers [17], and to network settings [16]. Like our work, both steganography and covert communications rely on the use of statistical hypothesis testing. One difference from covert communications is that our underlying channel model is an *exponential* channel while it is an AWGN channel in the communications context. In addition, in the communications context, Alice has control over the power that she transmits at whereas in our context, Alice does not control the size of her jobs, only the rate at which they are introduced. This latter difference results in our inability to use arguments standard KL divergence arguments for establishing achievability as in [3].

This work also has ties to the detection of service level agreement (SLA) violation problem. Detecting SLA violation in today's complex computing infrastructures, such as clouds infrastructures, presents challenging research issues [8]. However no careful analysis of this problem has been conducted. Our work may provide an avenue to doing such.

3 Model and Background

This section gives details about the model we use in the present work and the needed background on hypothesis testing. As mentioned in the introduction, there is a legitimate user, Willie, who sends a sequence of jobs to a single server with known service rate. There is also an illegitimate user, Alice, who wants to introduce a sequence of jobs to be serviced. The questions that we address are the following: can Alice introduce her stream of jobs covertly, i.e. without Willie being able to tell with confidence whether she has introduced the stream or not, and if so, at what rate can she introduce her jobs? We answer these questions under the following assumptions:

1. Willie jobs arrive at the server according to a Poisson process with rate $\lambda \in (0, \infty)$;
2. the service times of all jobs are independent;
3. the service time distributions are known to both parties;
4. the server serves all jobs in a FIFO manner;
5. once in service, Alice jobs cannot be preempted;
6. Willie observes only his arrivals and departures;
7. Alice observes Willie's and her own arrivals and departures.

The first four assumptions are made mainly for tractability. When the fifth assumption is violated, Alice can hide her jobs during Willie idle periods and never affect his jobs. Note that allowing Alice to observe also Willie arrivals and departures gives her more power.

Assume that the system is empty at time 0. Denote by A_i and D_i the arrival and departure times of Willie's i -th job, respectively. Note that $0 < A_i < A_{i+1}$ and $0 < D_i < D_{i+1}$. Let $A_{1:m} = \{A_1, \dots, A_m\}$, $D_{1:m} = \{D_1, \dots, D_m\}$. Let $S_{1:m} = (S_1, \dots, S_m)$ denote the *reconstructed service times* of the first m jobs,

which satisfy the following recurrence relation,

$$S_i = \begin{cases} D_1 - A_1, & i = 1, \\ D_i - \max\{A_i, D_{i-1}\}, & i \geq 2. \end{cases} \quad (1)$$

Note that these are the service times perceived by Willie. Note that $(A_{1:m}, S_{1:m})$ and $(A_{1:m}, D_{1:m})$ contain the same information, as they uniquely determine each other. It is also all the information available to Willie in our model.

Willie tries to detect whether Alice has inserted jobs based on his observations of $(A_{1:m}, S_{1:m})$. The null hypothesis H_0 is that Alice does not insert jobs and the alternative hypothesis H_1 is that Alice inserts jobs. Willie's test may incorrectly accuse Alice when she does not insert jobs, i.e. he rejects H_0 when it is true. This is known as type I error or false alarm, and, its probability is denoted by P_{FA} [13]. On the other hand, Willie's test may fail to detect insertions of Alice's jobs, i.e. he accepts H_0 when it is false. This is known as type II error or missed detection, and its probability is denoted by P_{MD} . We assume that Willie uses classical hypothesis testing with equal prior probabilities of each hypothesis being true. Non equal priors are straightforward to handle; see [3, Sec. V.B] for an example in the context of covert communications. Then, the lower bound on the sum $P_E = P_{FA} + P_{MD}$ characterizes the necessary trade-off between false alarms and missed detections in the design of a hypothesis test.

We assume all service time distributions have densities throughout this paper. Denote by $p_i^{(m)}$ the probability density function (pdf) of $(A_{1:m}, S_{1:m})$ under hypothesis H_i for $i = 0, 1$. The optimal test using $(A_{1:m}, S_{1:m})$ that minimizes $P_E = P_{FA} + P_{MD}$ is given by the following,

Theorem 3.1. (Neyman-Pearson Lemma [13, Theorem 13.1.1]).

Any test that accepts H_0 if $p_0^{(m)}(A_{1:m}, S_{1:m}) > p_1^{(m)}(A_{1:m}, S_{1:m})$ and rejects H_0 if $p_0^{(m)}(A_{1:m}, S_{1:m}) < p_1^{(m)}(A_{1:m}, S_{1:m})$ minimizes P_E . Furthermore, the minimum P_E is given by

$$P_E^* = 1 - T_V(p_0^{(m)}, p_1^{(m)}),$$

where

$$T_V(f_0, f_1) = \frac{1}{2} \int_{\mathbb{R}^d} |f_0(x) - f_1(x)| dx \quad (2)$$

is the total variation distance between two distributions with densities u_0 and u_1 , respectively.

Given that Willie uses this optimal detector, Alice's insertions are covert provided that, for any $\epsilon > 0$,

$$\liminf_{m \rightarrow \infty} P_E^* \geq 1 - \epsilon, \quad (3)$$

or equivalently from Theorem 3.1, if for any $\epsilon > 0$,

$$\limsup_{m \rightarrow \infty} T_V(p_0^{(m)}, p_1^{(m)}) \leq \epsilon. \quad (4)$$

Here the limit is taken over the number of observations Willie makes. This covertness criterion was proposed in the context of low probability of detection (LPD) communications in [3].

Theorem 3.1 suggests using the total variation distance to analyze Willie's detectors. However, the total variation distance is often unwieldy even if $p_0^{(m)}$ and $p_1^{(m)}$ are products of pdfs. To overcome this drawback, it is common (e.g. see [3]) to use the following Pinsker's inequality (Lemma 11.6.1 in [7]):

$$T_V(u_0, u_1) \leq KL(u_0 || u_1), \quad (5)$$

where $KL(u_0||f_1) := \int_{\mathbb{R}^d} u_0(x) \ln \frac{u_0(x)}{u_1(x)} dx$ is the *Kullback-Leibler (KL) divergence* between the probability distributions with pdf f_0 and f_1 , respectively.

However, we will work with the Hellinger distance, which has the advantage of offering both lower and upper bounds on the total variation distance. The Hellinger distance between two probability distributions with pdf u_0 and u_1 respectively, denoted $H(f_0, f_1)$, is defined by

$$H(u_0, u_1) = \frac{1}{2} \int_{\mathbb{R}^d} \left(\sqrt{u_0(x)} - \sqrt{u_1(x)} \right)^2 dx. \quad (6)$$

Note that $H(u_0, f_1) = 1 - \int_{\mathbb{R}^d} \sqrt{u_0(x)u_1(x)} dx$ and $0 \leq H(u_0, u_1) \leq 1$. It is known [11, Lemma 4.1] that

$$H(u_0, u_1) \leq T_V(u_0, u_1) \leq \sqrt{2H(u_0, u_1)}. \quad (7)$$

The upper bound (resp. lower bound) in (7) will be used to establish covert (resp. non-covert) results.

The following property of the Hellinger distance will also be used. Given an iid sequence of rvs with common pdf f , denote by $f^{\otimes n}$ their joint pdf given by the n -fold tensor product of f . For every $n \geq 1$

$$\begin{aligned} H(u_0^{\otimes n}, u_1^{\otimes n}) &= 1 - \int_{\mathbb{R}^d} \prod_{j=1}^n \sqrt{u_0(x_j)u_1(x_j)} \prod_{j=1}^n dx_j \\ &= 1 - \left(\int_{\mathbb{R}^d} \sqrt{u_0(x)u_1(x)} dx \right)^n. \end{aligned} \quad (8)$$

For any $a \in [0, 1]$, let $\bar{a} := 1 - a$. The convolution of f and g is denoted by $f * g$. We use the shorthand notation $t_{i:j}$ for t_i, \dots, t_j .

4 Insert-at-End-of-Busy-Period Policy

In this section, we consider the strategy that Alice inserts a job probabilistically at the end of each *Willie Busy Period* (W-BP), which we call the *Insert-at-End-of-Busy-Period (IEBP) Policy*. By a W-BP we mean the time interval between the arrival of a Willie job that finds no other Willie jobs in the system and the first subsequent departure of a Willie job that leaves no other Willie jobs in the system. Note that there may be Alice jobs in the system at the start and the end of a W-BP. We call a *Willie Idle Period* (W-IP) the time interval between two successive W-BPs.

Throughout the section, we assume that Willie jobs have service time distribution G_1 with continuous pdf g_1 and finite mean $1/\mu_1 > 0$, and that Alice jobs have service time distribution G_2 with continuous pdf g_2 and finite mean $1/\mu_2 > 0$. Denote by $G_2^*(s) = \int_0^\infty e^{-sx} g_2(x) dx$ the Laplace transform (LT) of g_2 . Define $\rho_i = \lambda/\mu_i$, $i = 1, 2$. We also assume $\rho_1 < 1$ so that the system is stable under H_0 .

4.1 Introducing the IEBP Policy

To motivate the IEBP policy, we first find the minimum probability that an Alice job interferes with Willie's jobs. Suppose an Alice job is inserted at time t , with service time $\sigma_2 \sim G_2$. Let $U_t \geq 0$ be the unfinished work (of both Alice and Willie jobs) in the system just before time t . The newly inserted Alice job will affect Willie if he sends a job in the interval $(t, t + U_t + \sigma_2)$, the probability of which is

$$\begin{aligned} &\mathbb{P}(\text{at least one Willie job arrives in } (t, t + U_t + \sigma_2)) \\ &\geq \mathbb{P}(\text{at least one Willie job arrives in } (t, t + \sigma_2)) \end{aligned}$$

General	
λ	arrival rate Willie jobs
G_1, g_1	cdf, pdf Willie job service time
G_2, g_2	cdf, pdf Alice job service time
$1/\mu_1$	Willie job expected service time
$1/\mu_2$	Alice job expected service time
ρ_k	$\lambda/\mu_k, k = 1, 2$
H_0	null hypothesis (Alice does not insert jobs)
H_1	alternate hypothesis (Alice inserts jobs)
$T_V(u_0, u_1)$	variation distance between pdfs u_0 and u_1
$H(u_0, u_1)$	Hellinger distance between pdfs u_0 and u_1
X	random variable (rv) with pdf g_1
IEBP policy	
W-BP	Willie Busy Period
Y	Willie first job reconstructed service time
V	Willie idle period duration (exp. rate λ)
f_i	pdf of (Y, V) under $H_i, i = 0, 1$ ($f_0(x, v) = g_1(x)\lambda e^{-\lambda v}$)
q	probability Alice inserts a job ($\bar{q} = 1 - q$); depends on n , the number of W-BPs
$Z(q, x, v)$	$f_1(x, v)/f_0(x, v)$
C_0	Fisher information at origin ($=\mathbb{E}[\rho(X, V)^2]$, with $\rho(x, v)$ defined in (11))
\tilde{f}_i	pdf of Y under $H_i, i = 0, 1$ ($\tilde{f}_0(x) = g_1(x)$)
$\tilde{Z}(q, x)$	$\tilde{f}_1(x)/\tilde{f}_0(x)$
μ_r	μ_1
μ	μ_2
X_r	exponential rv, rate μ_r
$T(n)$	expected nb. Alice jobs inserted in n W-BPs ($T(n) = nq$)
$T_W(n)$	expected nb. of Willie jobs served in n W-BPs

Figure 1: Glossary of main notation

$$\begin{aligned}
&= \int_0^\infty \mathbb{P}(\text{at least one Willie job arrives in } (t, t+x)) g_2(x) dx \\
&= \int_0^\infty (1 - e^{-\lambda x}) g_2(x) dx = 1 - G_2^*(\lambda) := p.
\end{aligned} \tag{9}$$

Thus if Alice is to insert a single job then she should insert it when the system is idle so that the probability of being detected by Willie is minimized. Motivated by this observation, we introduce the IEBP policy below.

Alice's strategy. Alice inserts a job with probability q at the end of each W-BP. We refer to this as the *Insert-at-End-of-Busy-Period* (IEBP) policy. Given that Alice does insert a job, the probability that it interferes with a Willie job is given by p in (9). Thus pq is the probability that an interference occurs in a given W-BP.

4.2 Willie's Detector

We assume that Willie knows Alice's strategy and monitors the service times of all of his jobs. His detector uses the optimal test in Theorem 3.1 for the statistics $(A_{1:m}, S_{1:m})$. Let Y_j be the reconstructed service time of the first Willie job in the j th W-BP and V_j is the length of the j th W-IP, which precedes the j -th W-BP. Note (Y_j, V_j) are iid for different j . Let (Y, V) denote a generic (Y_j, V_j) . Denote by $f_i(x, v)$ the joint pdf of (Y, V) at $(Y = x, V = v)$ under H_i for $i = 0, 1$.

The following lemma shows the form of the likelihood ratio.

Lemma 4.1. *Suppose the observed values of $A_{1:m}, S_{1:m}$ are $a_{1:m}, s_{1:m}$, and those of (Y_j, V_j) are (y_j, v_j) . Let p_i be the likelihood under H_i . The likelihood ratio for the observation is*

$$\frac{p_1(a_{1:m}, s_{1:m})}{p_0(a_{1:m}, s_{1:m})} = \prod_{j=1}^{N_m} \frac{f_1(y_j, v_j)}{f_0(y_j, v_j)},$$

where N_m is the number W-BP for the first m Willie jobs, which is a function of $(a_{1:m}, s_{1:m})$.

Proof. With a slight abuse of notation, we use $p_i(v)$ to denote the pdf of a rv V at v under H_i . For notational simplicity, we do not explicitly indicate the dependency on V as V changes. For instance, $p_i(a_{1:j}, s_{1:l})$ denotes the pdf of $A_{1:j}, S_{1:l}$ at $a_{1:j}, s_{1:l}$ under H_i , $i = 0, 1$. Note that

$$\begin{aligned}
p_1(s_j \mid a_{1:j}, s_{1:(j-1)}) &= p_1(s_j \mid d_{j-1}, a_j) \\
&= \begin{cases} p_1(s_j \mid a_j - d_{j-1}), & \text{if } a_j > d_{j-1}; \\ g_1(s_j), & \text{if } a_j \leq d_{j-1}. \end{cases}
\end{aligned}$$

Thus

$$p_1(a_j, s_j \mid a_{1:(j-1)}, s_{1:(j-1)}) = \lambda e^{-\lambda(a_j - a_{j-1})} [g_1(s_j)]^{1-I_j} [p_1(s_j \mid a_j - d_{j-1})]^{I_j},$$

where $I_j = \mathbb{1}\{a_j > d_{j-1}\}$ is the indicator of whether the j -th Willie job initiates a W-BP. On the other hand,

$$p_0(a_j, s_j \mid a_{1:(j-1)}, s_{1:(j-1)}) = \lambda e^{-\lambda(a_j - a_{j-1})} g_1(s_j).$$

Using

$$p_i(a_{1:m}, s_{1:m}) = \prod_{j=1}^m p_i(a_j, s_j \mid a_{1:(j-1)}, s_{1:(j-1)}),$$

we obtain

$$\frac{p_1(a_{1:m}, s_{1:m})}{p_0(a_{1:m}, s_{1:m})} = \prod_{j=1}^m \left[\frac{p_1(s_j | a_j - d_{j-1})}{g_1(s_j)} \right]^{I_j}.$$

By the definition of Y and V , the above becomes,

$$\frac{p_1(a_{1:m}, s_{1:m})}{p_0(a_{1:m}, s_{1:m})} = \prod_{j=1}^{N_m} \frac{p_1(y_j | v_j)}{g_1(y_j)} = \prod_{j=1}^{N_m} \frac{f_1(y_j, v_j)}{f_0(y_j, v_j)},$$

where the last equality follows from the fact Y_j and V_j are independent under H_0 , and V_i is exponential with rate λ under both H_0 and H_1 . \blacksquare

This lemma motivates us to use the test statistics $(Y_{1:n}, V_{1:n})$ for n W-BP instead of $(A_{1:m}, S_{1:m})$ for m Willie jobs. For the non-covert results we will establish, we are allowed to do so since it suffices to find a good detector for Willie. For the covert results, there is no loss of optimality either. In fact, since the W-BPs form a renewal process,

$$\lim_{n \rightarrow \infty} \frac{N_m}{m} = \lim_{n \rightarrow \infty} \frac{n}{T_W(n)} = 1 - \rho_1$$

by Proposition 4.4 and (53), where $T_W(n)$ is the number of Willie jobs in n W-BPs. In other words, $N_m \sim m(1 - \rho_1)$ a.s. as m is large. Therefore, we will henceforth assume Willie uses the likelihood ratio test for the (asymptotically sufficient) statistics $(Y_{1:n}, V_{1:n})$ for n W-BPs.

4.3 Main Results

Let $T(n)$ be the expected number of jobs that Alice inserts in n W-BPs. Under the IEBP policy,

$$T(n) = nq.$$

This section presents the main results that characterize $T(n)$ under various conditions as n is large. Implicit in all asymptotic results as $n \rightarrow \infty$ is that q is a function of n .

Recall that $f_i(x, v)$ is the joint pdf of (Y, V) at $(Y = x, V = v)$ under H_i for $i = 0, 1$. Note that f_1 depends on the parameter q . The likelihood ratio

$$Z(q, x, v) := \frac{f_1(x, v)}{f_0(x, v)}, \quad (10)$$

of $(Y = x, V = v)$ plays an important role in determining how many jobs Alice can insert covertly. Note that f_1 also depends on q , but we suppress that dependency in the notation when no confusion arises. It will be shown in Lemma 4.7 that Z has the following form,

$$Z(q, x, v) = 1 + q\rho(x, v),$$

where

$$\rho(x, v) := \frac{1}{g_1(x)} \int_0^x g_1(u)g_2(v + x - u)du - \bar{G}_2(v), \quad (11)$$

which is independent of q and only depends on the service time distributions.

Let X be a rv with pdf g_1 that is independent of V , which is exponential with rate λ . Define¹

$$C_0 := \mathbb{E}[\rho(X, V)^2]. \quad (12)$$

Proposition 4.2 gives our covert result for general service distributions. The proof is in Section 4.4.

¹Note that $C_0 = J(0)$, where $J(q) := \int_{[0, \infty)^2} \frac{1}{f_1(x, v)} \left(\frac{d}{dq} f_1(x, v) \right)^2 dx dv$ is the Fisher information of (Y, V) about the parameter q . The Fisher information evaluates the amount of information that a rv carries about an unknown parameter [12].

Proposition 4.2 (Covert result). Assume $C_0 < \infty$. Under the IEBP policy, the number of jobs Alice can insert covertly is $T(n) = \mathcal{O}(\sqrt{n})$ if $\mathbb{E}[\rho(X, V)] = 0$, and $T(n) = \mathcal{O}(1)$ if $\mathbb{E}[\rho(X, V)] \neq 0$.

Remark 1. $\mathbb{E}[\rho(X, V)] = 0$ for any g_1 if g_2 is the pdf of an exponential or an hyper-exponential rv. Indeed, when $g_2(x) = \mu_2 e^{-\mu_2 x}$, $\rho(x, v)$ in (11) writes

$$\rho(x, v) = e^{-\mu_2 v} \left(\frac{(g_1 * g_2)(x)}{g_1(x)} - 1 \right). \quad (13)$$

By the independence of X and V and the fact that $g_1 * g_2$ is a pdf,

$$\mathbb{E}[\rho(X, V)] = \mathbb{E}[e^{-\mu_2 V}] \cdot \mathbb{E} \left[\left(\frac{(g_1 * g_2)(X)}{g_1(X)} - 1 \right) \right] = 0. \quad (14)$$

The proof when g_2 is the pdf of an hyper-exponential rv is a simple generalization. It is worth noting that $\mathbb{E}[\rho(X, V)]$ does not always vanish. In particular, $\mathbb{E}[\rho(X, V)] \neq 0$ when G_2 is an Erlang distribution. Indeed, when $g_2(x) = \frac{(k\mu_2)^k}{(k-1)!} x^{k-1} e^{-k\mu_2 x}$, $k \geq 1$ (Alice service times follow a k -Erlang distribution with mean $1/\mu_2$), it is easy to show that for any pdf g_1 , $\mathbb{E}[\rho(X, V)] = (1 - G_2^*(\lambda)) \left(\frac{(k\mu_2)^k}{\mu_2} - 1 \right) \neq 0$ for all $k > 1$.

The next lemma gives conditions for $C_0 < \infty$ under various distributional assumptions. Its proof is found in Appendix B.

Lemma 4.3 (Finiteness of C_0).

1. Suppose both Alice and Willie have exponential service times, i.e. $g_i(x) = \mu_i e^{-\mu_i x}$ for $i = 1, 2$. Then $C_0 < \infty$ if and only if $\mu_1 < 2\mu_2$.
2. Suppose both Alice and Willie have hyperexponential service times, i.e.

$$g_i(x) = \sum_{l=1}^{K_i} p_{i,l} \mu_{i,l} e^{-\mu_{i,l} x},$$

where $\sum_{l=1}^{K_i} p_{i,l} = 1$, for $i = 1, 2$. Then $C_0 < \infty$ if and only if

$$\max_{1 \leq l \leq K_1} \mu_{1,l} < 2 \min_{1 \leq m \leq K_2} \mu_{2,m}.$$

3. Suppose Willie has Erlang service times and Alice has hyperexponential service times, i.e.

$$g_1(x) = \frac{\nu_1^{K_1}}{(K_1 - 1)!} x^{K_1 - 1} e^{-\nu_1 x}$$

and

$$g_2(x) = \sum_{l=1}^{K_2} p_{2,l} \mu_{2,l} e^{-\mu_{2,l} x},$$

where $\sum_{l=1}^{K_2} p_{2,l} = 1$. Then $C_0 < \infty$ if and only if

$$\nu_1 < 2 \min_{1 \leq l \leq K_2} \mu_{2,l}.$$

Proposition 4.2 gives sufficient conditions for Alice to be covert. This raises the following questions:

(Q1) When $C_0 < \infty$, can Alice insert covertly more than $\mathcal{O}(\sqrt{n})$ jobs on average during n W-BPs?

(Q2) When $C_0 = \infty$, what is the maximum number of jobs that Alice can insert covertly on average during n W-BPs?

We do not have a full answer to the above questions. Proposition 4.4 first gives a necessary condition for Alice to be covert under IEBP. Proposition 4.5 then provides a partial answer for the IEBP policy when both Alice and Willie have exponential service times. The proofs are found in Section 4.5 and Appendix C.

Proposition 4.4 (Necessary condition for covertness). *Under IEBP Alice cannot be covert if $\limsup_{n \rightarrow \infty} q > 0$.*

Proposition 4.5. *Assume that $g_i(x) = \mu_i e^{-\mu_i x}$ for $i = 1, 2$, and Alice uses the IEBP policy. She can be covert if*

$$T(n) = \begin{cases} \mathcal{O}(\sqrt{n}), & \text{if } \mu_1 < 2\mu_2; \\ \mathcal{O}(\sqrt{n/\log n}), & \text{if } \mu_1 = 2\mu_2; \\ \mathcal{O}(n^{\mu_2/\mu_1}), & \text{if } \mu_1 > 2\mu_2. \end{cases} \quad (15)$$

She cannot be covert if

$$T(n) = \begin{cases} \omega(\sqrt{n}), & \text{if } \mu_1 < 2\mu_2; \\ \omega(\sqrt{n/\log n}), & \text{if } \mu_1 = 2\mu_2; \\ \omega(n^{\mu_2/\mu_1}), & \text{if } \mu_1 > 2\mu_2. \end{cases} \quad (16)$$

The above results are in terms of $T(n)$, the expected number of jobs inserted by Alice over n successive W-BPs. It is interesting to determine also the expected number of Willie jobs served during these n W-BPs under the IEBP policy. Recall that $T_W(n)$ is this number. Proposition 4.6 below shows that $T_W(n)$ is of order n , which, roughly speaking, means that the IEBP policy bloats each W-BP by at most a constant factor. The proof is in Section 4.7.

Proposition 4.6. *Under IEBP, $T_W(n) = \Theta(n)$. More precisely, we have the following*

$$n \leq T_W(n) \leq n \left(\frac{1 + q\rho_2}{1 - \rho_1} \right). \quad (17)$$

If $g_2(x) = \mu_2 e^{-\mu_2 x}$, i.e. Alice job service times are exponentially distributed, then

$$T_W(n) = n \left(\frac{1 + pq\rho_2}{1 - \rho_1} \right). \quad (18)$$

4.4 Proof of Proposition 4.2

Clearly,

$$f_0(x, v) = g_1(x) \lambda e^{-\lambda v}. \quad (19)$$

The likelihood Z defined in (10) is given by the following

Lemma 4.7.

$$Z(q, x, v) = 1 + q\rho(x, v), \quad (20)$$

where $\rho(x, v)$ is defined in (11).

Proof. Consider a generic W-BP. Let σ_1 (resp. σ_2) denote a generic service time of a Willie (resp. Alice) job. Let A be the event that Alice inserts a job at the end of the W-BP. Then

$$Y = \sigma_1 + \mathbb{1}\{A\} \cdot (\sigma_2 - V)^+$$

where $(z)^+ = \max\{z, 0\}$. We first compute the conditional density $f_1(x | v)$ of Y given V . Given A^c , $Y = \sigma_1$, so

$$f_1(x | v, A^c) = g_1(x).$$

Given A and $V = v$, $Y = \sigma_1 + (\sigma_2 - v)^+$, so that

$$f_1(x | v, A) = g_1(x)G_2(v) + \int_0^x g_1(u)g_2(x + v - u)du.$$

Recall the probability of A under H_1 is q , so

$$\begin{aligned} f_1(x | v) &= qf_1(x | v, A) + \bar{q}f_1(x | v, A^c) \\ &= g_1(x) + q \left[\int_0^x g_1(u)g_2(x + v - u)du - g_1(x)\bar{G}_2(v) \right] \\ &= g_1(x)[1 + q\rho(x, v)], \end{aligned}$$

by using the definition of $\rho(x, v)$ in (11). Therefore,

$$Z(q, x, v) = \frac{f_1(x | v)\lambda e^{-\lambda v}}{f_0(x, v)} = 1 + q\rho(x, v),$$

by using (19), which concludes the proof. ■

Lemma 4.8. *Assume that $\mathbb{E}[\rho(X, V)] = 0$. Then, for every $n \geq 1$,*

$$T_V(f_0^{\otimes n}, f_1^{\otimes n}) \leq \frac{1}{2} \sqrt{(1 + q^2 C_0)^n - 1}, \quad (21)$$

where C_0 is defined in (12).

Proof. Let (X_j, V_j) be iid $\sim (X, V) \sim f_0$. By (2),

$$\begin{aligned} &2T_V(f_0^{\otimes n}, f_1^{\otimes n}) \\ &= \int_{[0, \infty)^{2n}} \left| \prod_{j=1}^n f_0(x_j, v_j) - \prod_{j=1}^n f_1(x_j, v_j) \right| \prod_{j=1}^n dx_j dv_j \\ &= \int_{[0, \infty)^{2n}} \prod_{j=1}^n f_0(x_j, v_j) \left| 1 - \prod_{j=1}^n Z(q, x_j, v_j) \right| \prod_{j=1}^n dx_j dv_j \\ &= \mathbb{E} \left[\left| 1 - \prod_{j=1}^n Z(q, X_j, V_j) \right| \right], \end{aligned} \quad (22)$$

where the second equality follows from (10). Using the inequality $\mathbb{E}|U| \leq \sqrt{\mathbb{E}[U^2]}$ in (22) yields

$$(2T_V(f_0^{\otimes n}, f_1^{\otimes n}))^2 \leq 1 - 2\mathbb{E} \left[\prod_{j=1}^n Z(q, X_j, V_j) \right] + \mathbb{E} \left[\left(\prod_{j=1}^n Z(q, X_j, V_j) \right)^2 \right]$$

$$\begin{aligned}
&= 1 - 2 [\mathbb{E}Z(q, X, V)]^n + [\mathbb{E}[Z(q, X, V)^2]]^n \\
&= -1 + (1 + 2q\mathbb{E}[\rho(X, V)] + q^2\mathbb{E}[\rho(X, V)^2])^n \\
&= -1 + (1 + q^2\mathbb{E}[\rho(X, V)^2])^n,
\end{aligned}$$

where we have used (20) and the assumption that $\mathbb{E}[\rho(X, V)] = 0$ to establish the last two identities. \blacksquare

Lemma 4.9. *The Hellinger distance between $f_0^{\otimes n}$ and $f_1^{\otimes n}$ satisfies*

$$H(f_0^{\otimes n}, f_1^{\otimes n}) = 1 - \left(\mathbb{E} \left[\sqrt{Z(q, X, V)} \right] \right)^n. \quad (23)$$

Proof. By (8) and (10),

$$\begin{aligned}
H(f_0^{\otimes n}, f_1^{\otimes n}) &= 1 - \left(\int_{[0, \infty)^2} \sqrt{f_0(x, v)f_1(x, v)} dx dv \right)^n \\
&= 1 - \left(\int f_0(x, v) \sqrt{Z(q, x, v)} dx dv \right)^n \\
&= 1 - \left(\mathbb{E} \left[\sqrt{Z(q, X, V)} \right] \right)^n.
\end{aligned}$$

\blacksquare

Lemma 4.10 (Lower and upper bounds). *For every $n \geq 1$,*

$$1 - \left(\mathbb{E} \left[\sqrt{Z(q, X, V)} \right] \right)^n \leq T_V(f_0^{\otimes n}, f_1^{\otimes n}) \leq \sqrt{2 \left(1 - \left(\mathbb{E} \left[\sqrt{Z(q, X, V)} \right] \right)^n \right)}. \quad (24)$$

Proof. Use (7), (6), and (33). \blacksquare

Proof of Proposition 4.2. In order to show achievability, assume that Willie uses the optimal detector, which is based on the sufficient statistic $\{(Y_j, V_j), j = 1, \dots, n\}$. Let

$$q = \frac{\delta}{\phi(n)}$$

with $\delta \in (0, 1]$ and $\phi : \{1, 2, \dots\} \rightarrow [1, \infty)$, so that

$$T(n) = \frac{\delta n}{\phi(n)}. \quad (25)$$

First consider the case $\mathbb{E}[\rho(X, V)] = 0$. Note that $T(n) = \mathcal{O}(\sqrt{n})$ implies

$$\limsup_{n \rightarrow \infty} \frac{\sqrt{n}}{\phi(n)} < \infty. \quad (26)$$

By Lemma 4.8

$$\sup_{k \geq n} T_V(f_0^{\otimes k}, f_1^{\otimes k}) \leq \sup_{k \geq n} \frac{1}{2} \sqrt{\left(1 + \frac{\delta^2 C_0}{\phi(k)^2} \right)^k - 1} \sim \frac{\delta \sqrt{C_0}}{2} \sup_{k \geq n} \frac{k}{\phi(k)^2}, \quad \text{as } n \rightarrow \infty$$

as $\lim_n \phi(n) = \infty$ by Lemma A.1 in Appendix A. Therefore,

$$\limsup_{n \rightarrow \infty} T_V(f_0^{\otimes n}, f_1^{\otimes n}) \leq \frac{\delta \sqrt{C_0}}{2} \left(\limsup_{n \rightarrow \infty} \frac{\sqrt{n}}{\phi(n)} \right)^2. \quad (27)$$

By making δ small enough, $\limsup_{n \rightarrow \infty} T_V(f_0^{\otimes n}, f_1^{\otimes n})$ can be made arbitrarily small. We then conclude from (3) that Alice is covert when $T(n) = \mathcal{O}(\sqrt{n})$, which completes the proof for the case $\mathbb{E}[\rho(X, V)]$.

Now consider the case $\mathbb{E}[\rho(X, V)] \neq 0$. Note that $T(n) = \frac{\delta n}{\phi(n)} = \mathcal{O}(1)$ implies there exist $k > 0$ and n_0 such that for all $n \geq n_0$, $0 \leq \frac{n}{\phi(n)} \leq k$. Using inequality (23) gives

$$\begin{aligned} & (2T_V(f_0^{\otimes n}, f_1^{\otimes n}))^2 \\ & \leq 1 - 2e^{n \log\left(1 + \frac{\delta}{\phi(n)} \mathbb{E}[\rho(X, V)]\right)} + e^{n \log\left(1 + \frac{2\delta}{\phi(n)} \mathbb{E}[\rho(X, V)] + \frac{\delta^2}{\phi(n)^2} C_0^2\right)} \\ & \sim 1 - 2e^{\mathbb{E}[\rho(X, V)] \frac{\delta n}{\phi(n)}} + e^{2\mathbb{E}[\rho(X, V)] \frac{\delta n}{\phi(n)} + \delta^2 C_0^2 \frac{n}{\phi(n)^2}} \end{aligned} \quad (28)$$

as $n \rightarrow \infty$. Since $\frac{n}{\phi(n)}$ and $\frac{n}{\phi(n)^2}$ are bounded away from infinity as $n \rightarrow \infty$, we see that the r.h.s. of (28) can be made arbitrarily close to 0 by letting $\delta \rightarrow 0$. We then conclude from (3) that Alice is covert when $T(n) = \mathcal{O}(1)$, which completes the proof. \blacksquare

4.5 Proof of Proposition 4.4

We first derive some properties for the rv Y . Denote by $\tilde{f}_i(x)$, $i = 0, 1$, the pdf of Y under H_i . Note $\tilde{f}_0 = g_1$. Define the likelihood ratio

$$\tilde{Z}(q, x) := \frac{f_1(x)}{f_0(x)} = \frac{f_1(x)}{g_1(x)}. \quad (29)$$

Since $\tilde{f}_i(x) = \int_0^\infty f_i(x, v) dv$, repeating the proof of Lemma 4.7, we obtain

$$\tilde{Z}(q, x) = 1 + q\tilde{\rho}(x), \quad (30)$$

where, by using (11) and (9),

$$\tilde{\rho}(x) = \int_0^\infty \lambda e^{-\lambda v} \rho(x, v) dv = \frac{(g_1 * \hat{g}_2)(x)}{g_1(x)} - p \quad (31)$$

and

$$\hat{g}_2(t) = \int_0^\infty \lambda e^{-\lambda v} g_2(v + t) dv, \quad (32)$$

The rvs Y_1, \dots, Y_n being iid, their joint pdf under H_i is $\tilde{f}_i^{\otimes n}(x) = \prod_{j=1}^n \tilde{f}_i(x)$. Similarly to the derivation of (23) we get

$$H(f_0^{\otimes n}, f_1^{\otimes n}) = 1 - \left(\mathbb{E} \left[\sqrt{\tilde{Z}(q, X)} \right] \right)^n. \quad (33)$$

Similar to Lemma 4.10, we have

$$\begin{aligned} 1 - \left(\mathbb{E} \left[\sqrt{\tilde{Z}(q, X)} \right] \right)^n & \leq T_V(f_0^{\otimes n}, f_1^{\otimes n}) \\ & \leq \sqrt{2 \left(1 - \left(\mathbb{E} \left[\sqrt{\tilde{Z}(q, X)} \right] \right)^n \right)}. \end{aligned} \quad (34)$$

Proof of Proposition 4.4. To prove that Alice is not covert is to enough to exhibit a detector such that (4) does not hold. Consider Willie's detector that only uses the samples (Y_1, \dots, Y_n) , with Y_j the reconstructed service time of his first job in the j th W-BP.

$$\mathbb{E} \left[\sqrt{\tilde{Z}(q, X)} \right] = \int \sqrt{\prod_{i=0}^1 \tilde{f}_i(x)} dx \leq \sqrt{\prod_{i=0}^1 \int \tilde{f}_i(x) dx} = 1, \quad (35)$$

where (35) follows from Cauchy-Schwarz inequality. Equality holds in (35) if and only if (see e.g. [1, p. 14]) $\tilde{f}_1(x) = c\tilde{f}_0(x)$ for some constant $c > 0$. Since both \tilde{f}_0 and \tilde{f}_1 are densities, integrating over $[0, \infty)$ yields $c = 1$, which is equivalent to $q = 0$ from (29) and (30). This shows that $\mathbb{E} \left[\sqrt{\tilde{Z}(q, X)} \right] < 1$ if $0 < q \leq 1$.

Assume that $\liminf_n \phi(n) = d$ with $d < \infty$. Then, there exists a subsequence of $\{\phi(n)\}_n$, say $\{\phi(k_n)\}_n$, such that $\phi(k_n) \geq d$ with $\lim_{n \rightarrow \infty} \phi(k_n) = d$.

Let $M := \sup_{1/d \leq q \leq 1} \mathbb{E} \left[\sqrt{\tilde{Z}(q, X)} \right]$. As $1/d > 0$ note that $M < 1$, as shown above. Therefore,

$$\mathbb{E} \left[\sqrt{\tilde{Z}(1/\phi(k_n), X)} \right] \leq M$$

for all n . As a result

$$\lim_{n \rightarrow \infty} \left(\mathbb{E} \left[\sqrt{\tilde{Z}(1/\phi(k_n), X)} \right] \right)^n = 0,$$

which implies from the lower bound in (34) that the covert criterion (4) is violated. \blacksquare

4.6 Paving the way to the proof of Proposition 4.5

Throughout this section, we assume that Alice and Willie job service times are exponentially distributed with rate μ_2 and μ_1 , respectively, namely, $g_i(x) = \mu_i e^{-\mu_i x}$ for $i = 1, 2$. Let $\mu_1 = r\mu$ and $\mu_2 = \mu$. For $r \neq 1$, define $\beta = \frac{r}{r-1}$ and note that $r = \frac{\beta}{\beta-1}$ and $1 - \beta = \frac{1}{r-1}$.

Let X_r denote an exponential rv with rate μr .

For $\theta \in [0, 1]$, $x \geq 0$, define

$$\Xi(\theta, x) = \begin{cases} 1 + \theta(\mu x - 1) & \text{if } r = 1 \\ 1 + \theta \left(\frac{e^{(r-1)\mu x}}{r-1} - \beta \right) & \text{if } r \neq 1. \end{cases} \quad (36)$$

With the definition of $Z(q, x, v)$ in (10) with $\rho(x, v)$ given in (82) in Appendix B, we get

$$Z(q, x, v) = \Xi(qe^{-\mu v}, x), \quad \forall q \in [0, 1], x \geq 0, v \geq 0. \quad (37)$$

Once the other hand, the identity $\tilde{Z}(q, x) = \int_0^\infty \lambda e^{-\lambda v} Z(q, x, v) dv$ and (37) yield

$$\tilde{Z}(q, x) = \Xi(pq, x), \quad \forall q \in [0, 1], x \geq 0. \quad (38)$$

Recall that $p = 1 - G_2^*(\lambda) = \lambda/(\mu_2 + \lambda)$ when Alice job service times are exponentially distributed.

For $\theta \in [0, 1]$, define

$$\xi_r(\theta) := \frac{(\beta - 1)\theta}{1 - \beta\theta} \quad (39)$$

for $r \geq 2$ (i.e. $1 < \beta \leq 2$), and

$$I_\beta := \beta \int_0^\infty \frac{1 + \frac{1}{2}t - \sqrt{t+1}}{t^{\beta+1}} dt \quad (40)$$

for $r > 2$. Since $\beta \in (1, 2)$ when $r > 2$, the generalized integral I_β is finite and positive.

The lemma below is a key ingredient to prove Proposition 4.5. The proof of Proposition 4.5 can be found in Appendix C.

Lemma 4.11. For $\theta \in [0, 1]$, define

$$F_r(\theta) := \begin{cases} \frac{1-r}{4(r-2)} & \text{if } 0 < r < 1 \\ \frac{1}{4}\xi_2^2(\theta) \log \xi_2(\theta) + \Delta_2(\xi_2(\theta)) & \text{if } r = 2 \\ -I_\beta \xi_r^\beta(\theta) + \Delta_r(\xi_r(\theta)) & \text{if } r > 2, \end{cases} \quad (41)$$

where, for $t > 0$,

$$\Delta_r(t) := \begin{cases} o(t^2) & \text{if } 0 < r < 1 \\ o(t^2 \log t) & \text{if } r = 2 \\ o(t^\beta) & \text{if } r > 2. \end{cases} \quad (42)$$

Then, for $r \in (0, 1) \cup [2, \infty)$,

$$\mathbb{E} \left[\sqrt{\Xi(\theta, X_r)} \right] = 1 + F_r(\theta), \quad 0 \leq \theta \leq 1. \quad (43)$$

Proof. Assume that $0 < r < 1$ (i.e. $\beta < 0$). Easy algebra from (29) gives

$$\begin{aligned} & \mathbb{P} \left(\sqrt{\Xi(\theta, X_r)} > z \right) \\ &= \begin{cases} 0 & \text{if } z > \sqrt{1 - \theta\beta} \\ \left(\frac{1 - \theta\beta - z^2}{\theta(1 - \beta)} \right)^{-\beta} & \text{if } \sqrt{1 - \theta} \leq z \leq \sqrt{1 - \theta\beta} \\ 1 & \text{if } z < \sqrt{1 - \theta}, \end{cases} \end{aligned}$$

which yields

$$\begin{aligned} \mathbb{E} \left[\sqrt{\Xi(\theta, X_r)} \right] &= \int_0^\infty \mathbb{P} \left(\sqrt{Y(\theta, X_r)} > z \right) dz \\ &= \sqrt{1 - \theta} + (\theta(1 - \beta))^\beta \int_{\sqrt{1 - \theta}}^{\sqrt{1 - \theta\beta}} (-y^2 + 1 - \theta\beta)^{-\beta} dy \\ &= \sqrt{1 - \theta} + \left(\frac{\theta(1 - \beta)}{1 - \theta\beta} \right)^\beta (1 - \theta\beta)^{1/2} \int_{\sqrt{\frac{1 - \theta}{1 - \theta\beta}}}^1 (1 - y^2)^{-\beta} dy. \end{aligned} \quad (44)$$

Recall that $\xi_r(\theta) = (1 - \beta)\theta/(1 - \theta\beta)$, so that $\theta = \xi_r/(1 - \beta + \beta\xi_r(\theta))$. Substitution into (44) yields (with $\xi_r \equiv \xi_r(\theta)$ with a slight abuse of notation)

$$\begin{aligned} \mathbb{E} \left[\sqrt{\Xi(\theta, X_r)} \right] &= \sqrt{1 - \frac{\xi_r}{1 - \beta + \beta\xi_r}} + \xi_r^\beta \left(1 + \frac{\beta\xi_r}{1 - \beta} \right)^{-1/2} \int_{\sqrt{1 - \xi_r}}^1 (1 - y^2)^{-\beta} dy \\ &= \sqrt{1 - \frac{\xi_r}{1 - \beta + \beta\xi_r}} + \frac{1}{2} \xi_r^\beta \left(1 + \frac{\beta\xi_r}{1 - \beta} \right)^{-1/2} \int_0^{\xi_r} \frac{x^{-\beta}}{\sqrt{1 - x}} dx. \end{aligned}$$

When x is small, $x^{-\beta}/\sqrt{1 - x} \sim x^{-\beta} + x^{1-\beta}/2$, so that $\int_0^{\xi_r} \frac{x^{-\beta}}{\sqrt{1 - x}} dx \sim \xi_r^{1-\beta}/(1 - \beta) + \xi_r^{2-\beta}/(2(2 - \beta))$ as ξ_r is small. With this, we obtain

$$\mathbb{E} \left[\sqrt{\Xi(\theta, X_r)} \right] = 1 + \frac{1 - 4\beta + 2\beta^2}{4(2 - \beta)(1 - \beta)^2} \xi_r^2 + o(\xi_r^2)$$

$$= 1 + \frac{1 - 4\beta + 2\beta^2}{4(2 - \beta)}\theta^2 + o(\theta^2). \quad (45)$$

Since $\frac{1-4\beta+2\beta^2}{4(2-\beta)} = \frac{1-r}{4(r-2)}$, this proves the lemma when $r < 1$.

Consider now the case where $r \geq 2$. Notice that $1 < \beta \leq 2$ when $r \geq 2$. It is easily seen from (29) that, for $r > 1$,

$$\frac{d}{dz} \mathbb{P}(\Xi(\theta, X_r) < z) = \begin{cases} \frac{\beta(\theta(\beta-1))^\beta}{(z-1+\theta\beta)^{\beta+1}} & \text{if } z \geq 1 - \theta \\ 0 & \text{if } z < 1 - \theta, \end{cases}$$

which yields

$$\begin{aligned} \mathbb{E} \left[\sqrt{\Xi(\theta, X_r)} \right] &= \beta\theta^\beta(\beta-1)^\beta \int_{1-\theta}^{\infty} \frac{\sqrt{z}}{(z-1+\theta\beta)^{\beta+1}} dz \\ &= \frac{2\beta\theta^\beta(\beta-1)^\beta}{(1-\theta\beta)^{\beta-1/2}} \int_{\sqrt{\frac{1-\theta}{1-\theta\beta}}}^{\infty} \frac{t^2}{(t^2-1)^{\beta+1}} dt. \end{aligned} \quad (46)$$

We are now ready to address the case when $r \geq 2$.

Assume first that $r = 2$, so that $\beta = 2$, $\xi_2 \equiv \xi_2(\theta) = \frac{\theta}{1-2\theta}$, and $\theta = \frac{\xi}{2\xi+1}$. By (46) we have

$$\begin{aligned} \mathbb{E} \left[\sqrt{\Xi(\theta, X_2)} \right] &= 4(1+2\xi_2)^{-1/2} \xi_2^2 \int_{\sqrt{\xi_2+1}}^{\infty} \frac{t^2}{(t^2-1)^3} dt \\ &= 2(1+2\xi_2)^{-1/2} \xi_2^2 \int_{\xi_2}^{\infty} \frac{\sqrt{y+1}}{y^3} dy. \end{aligned}$$

Let $h(y) = \frac{1+\frac{1}{2}y-\sqrt{y+1}}{y^3}$. We have

$$\lim_{\xi_2 \rightarrow 0} \frac{2\xi_2^2 \int_{\xi_2}^{\infty} \frac{\sqrt{y+1}}{y^3} dy - 1 - \xi_2}{2\xi_2^2 \log \xi_2} = \lim_{\xi_2 \rightarrow 0} \frac{-\int_{\xi_2}^{\infty} h(y) dy}{\log \xi_2} = \lim_{\xi_2 \rightarrow 0} \frac{h(\xi_2)}{\xi_2^{-1}} = \lim_{\xi_2 \rightarrow 0} \frac{1 + \frac{1}{2}\xi_2 - \sqrt{\xi_2+1}}{\xi_2^2} = \frac{1}{8},$$

and

$$2\xi_2^2 \int_{\xi_2}^{\infty} \frac{\sqrt{y+1}}{y^3} dy = 1 + \xi_2 + \frac{1}{4}\xi_2^2 \log \xi_2 + o(\xi_2^2 \log \xi_2).$$

It follows that

$$\begin{aligned} \mathbb{E} \left[\sqrt{\Xi(\theta, X_2)} \right] &= (1+2\xi_2)^{-1/2} 2\xi_2^2 \int_{\xi_2}^{\infty} \frac{\sqrt{y+1}}{y^3} dy \\ &= (1-\xi_2 + O(\xi_2^2)) \left(1 + \xi_2 + \frac{1}{4}\xi_2^2 \log \xi_2 + o(\xi_2^2 \log \xi_2) \right) \\ &= 1 + \frac{1}{4}\xi_2^2 \log \xi_2 + o(\xi_2^2 \log \xi_2). \end{aligned} \quad (47)$$

This proves the lemma when $r = 1$.

Finally, assume that $r > 2$. Recall that $\xi_r \equiv \xi_r(\theta) = \frac{(\beta-1)\theta}{1-\beta\theta}$, so that $\theta = \frac{\xi_r}{\beta\xi_r+\beta-1}$ and, by (46),

$$\mathbb{E} \left[\sqrt{\Xi(\theta, X_r)} \right] = 2\beta \left(1 + \frac{\beta}{\beta-1}\xi_r \right)^{-1/2} \xi_r^\beta \int_{\sqrt{\xi_r+1}}^{\infty} \frac{x^2}{(x^2-1)^{\beta+1}} dx$$

$$= \beta \left(1 + \frac{\beta}{\beta-1} \xi_r\right)^{-1/2} \xi_r^\beta \int_{\xi_r}^{\infty} \frac{\sqrt{y+1}}{y^{\beta+1}} dy.$$

Let

$$h(y) = \frac{1 + \frac{1}{2}y - \sqrt{y+1}}{y^{\beta+1}}.$$

Note that $h(y) > 0$ for $y > 0$. As $y \rightarrow \infty$, $h(y) \sim \frac{1}{2}y^{-\beta}$. As $y \rightarrow 0$, $h(y) \sim \frac{1}{8}y^{1-\beta}$. Since $\beta \in (1, 2)$ for $r > 2$, the generalized integral $I_\beta := \beta \int_0^\infty h(y) dy$ is finite and positive.

Therefore,

$$\begin{aligned} \lim_{\xi_r \rightarrow 0} \frac{\beta \xi_r^\beta \int_{\xi_r}^{\infty} \frac{\sqrt{y+1}}{y^{\beta+1}} dy - 1 - \frac{\beta}{2(\beta-1)} \xi_r}{\beta \xi_r^\beta} &= \lim_{\xi_r \rightarrow 0} \int_{\xi_r}^{\infty} \frac{\sqrt{y+1}}{y^{\beta+1}} dy - \frac{1}{\beta} \xi_r^{-\beta} - \frac{1}{2(\beta-1)} \xi_r^{1-\beta} \\ &= \lim_{\xi_r \rightarrow 0} - \int_{\xi_r}^{\infty} h(y) dy = -\frac{I_\beta}{\beta}, \end{aligned}$$

and

$$\beta \xi_r^\beta \int_{\xi_r}^{\infty} \frac{\sqrt{y+1}}{y^{\beta+1}} dy = 1 + \frac{\beta}{2(\beta-1)} \xi_r - I_\beta \xi_r^\beta + o(\xi_r^\beta).$$

It follows that

$$\begin{aligned} \mathbb{E} \left[\sqrt{\Xi(\theta, X_r)} \right] &= \left(1 + \frac{\beta}{\beta-1} \xi_r\right)^{-1/2} \beta \xi_r^\beta \int_{\xi_r}^{\infty} \frac{\sqrt{y+1}}{y^{\beta+1}} dy \\ &= \left(1 - \frac{\beta}{2(\beta-1)} \xi_r + O(\xi_r^2)\right) \left(1 + \frac{\beta}{2(\beta-1)} \xi_r - I_\beta \xi_r^\beta + o(\xi_r^\beta)\right) \\ &= 1 - I_\beta \xi_r^\beta + o(\xi_r^\beta). \end{aligned} \tag{48}$$

This proves the lemma when $r > 2$. ■

4.7 Proof of Proposition 4.6

Recall that N_j is the number of Willie jobs served during the j th W-BP. Since $\{N_j\}_{j \geq 1}$ is an iid sequence, the expected number of Willie jobs served during n W-BPs is $T_W(n) = n\mathbb{E}[N]$, with N a rv with the same distribution as N_j . Let $\mathcal{G}_N(z) = \mathbb{E}[z^N]$ be its generating function. Below, we obtain $\mathcal{G}_N(z)$ by observing that under the IEBP policy the system behaves as an M/G/1 queue with an exceptional first job in each busy period.

Let $\tau^*(s)$ be the LT of the reconstructed service time Y of the first Willie job in a W-BP under the IEBP policy. Since the LT of all the other Willie jobs in a W-BP is $G_1^*(s)$, we get from [4],

$$\mathcal{G}_N(z) = z\tau^*(\lambda(1-d(z))), \quad |z| \leq 1, \tag{49}$$

where $d(z)$ is the root with the smallest modulus of the equation $t = zG_1^*(\lambda(1-t))$.

Noting $d(1) = 1$ and $d'(1) = \frac{1}{1-\rho_1}$, we obtain from (49)

$$\mathbb{E}[N] = \frac{1 - \rho_1 + \lambda\mathbb{E}[Y]}{1 - \rho_1}, \tag{50}$$

provided that the stability condition $\rho_1 < 1$ holds. It remains to find $\mathbb{E}[Y]$. For that, we will use the identity $\mathbb{E}[Y] = -\frac{d\tau^*(s)}{ds}|_{s=0}$. Let us calculate $\tau^*(s)$.

Under IEBP, Y has pdf \tilde{f}_1 . Using Eqs. (29)-(32),

$$\begin{aligned}\tau^*(s) &= \int_{x=0}^{\infty} e^{-sx} \tilde{f}_1(x) dx \\ &= \int_{x=0}^{\infty} e^{-sx} [(1-pq)g_1(x) + q(g_1 * \hat{g}_2)(x)] dx \\ &= G_1^*(s) \left(1 - pq + q \int_0^{\infty} e^{-st} \hat{g}_2(t) dt \right),\end{aligned}\tag{51}$$

Differentiating (51) with respect to s at $s = 0$ and using the identity² $\int_0^{\infty} \hat{g}_2(t) dt = p$ yields

$$\begin{aligned}\mathbb{E}[Y] &= \frac{1-pq}{\mu_1} + \frac{q}{\mu_1} \int_0^{\infty} \hat{g}_2(t) dt + q \int_0^{\infty} t \hat{g}_2(t) dt \\ &= \frac{1}{\mu_1} + q \int_0^{\infty} t \hat{g}_2(t) dt,\end{aligned}$$

By 50,

$$\mathbb{E}[N] = \frac{1}{1-\rho_1} + \frac{\lambda q}{1-\rho_1} \int_0^{\infty} t \hat{g}_2(t) dt,\tag{52}$$

and

$$T_W(n) = n\mathbb{E}[N] = \frac{n}{1-\rho_1} + \frac{\lambda q n}{1-\rho_1} \int_0^{\infty} t \hat{g}_2(t) dt.\tag{53}$$

Now we upper bound the integral in (52) and (53). By (32),

$$\begin{aligned}\int_0^{\infty} t \hat{g}_2(t) dt &= \int_0^{\infty} \lambda e^{-\lambda v} \int_0^{\infty} t g_2(v+t) dv dt \\ &\leq \int_0^{\infty} \lambda e^{-\lambda v} \int_0^{\infty} (t+v) g_2(v+t) dv dt \\ &= \int_0^{\infty} \lambda e^{-\lambda v} \int_v^{\infty} u g_2(u) du dv \\ &\leq \int_0^{\infty} \lambda e^{-\lambda v} \int_0^{\infty} u g_2(u) du dv = \frac{1}{\mu_2}.\end{aligned}$$

Hence, $\mathbb{E}[N] \leq \frac{1+q\rho_2}{1-\rho_1}$ and $T_W(n) \leq n \left(\frac{1+q\rho_2}{1-\rho_1} \right)$. This shows the upper bound in (17). The lower bound is trivial.

If $g_2(x) = \mu_2 e^{-\mu_2 x}$, from (32) and (9) we find $p = \frac{\lambda}{\mu_2 + \lambda}$ and $\hat{g}_2(x) = \frac{\lambda \mu_2}{(\lambda + \mu_2)} e^{-\mu_2 x} = p \mu_2 e^{-\mu_2 x}$, which yields (18).

5 Insert-at-Idle policy

In this section, we consider the variant of the IEBP policy where each time the server idles, Alice inserts a job with probability q and stops with probability \bar{q} (before she tries again at the end of a new W-BP). We call this policy the *Insert-at-Idle (II)* policy. The difference between the IEBP and II policies is that under the former Alice may only insert one job between the end of a W-BP and the start of the next W-BP, whereas under the II policy she may insert more than one job during this time period.

We will show that when Alice job service times are exponentially distributed all covert/non-covert results obtained under the IEBP policy hold under the II policy. The intuition behind this is that when Alice job

² $\int_0^{\infty} \hat{g}_2(t) dt = \int_0^{\infty} \lambda e^{-\lambda v} \int_v^{\infty} g_2(t) dt dv = \int_0^{\infty} \lambda e^{-\lambda v} (1 - G_2(v)) dv = 1 - G_2^*(\lambda) = p$.

service times are exponentially distributed, Willie sees "the same system behavior" under either policy; indeed, under either policy a job of his can interfere with at most one Alice job in a W-BP, whose remaining service time is exponentially distributed.

Throughout this section quantities with the subscript "+" refer to the II policy. Let $Y_{+,j}$ be the reconstructed service time of the first Willie job in the j -th W-BP, and $V_{+,j}$ the duration of the idle period between the $(j-1)$ -th and the j -th W-BPs. The rvs $(Y_{+,j}, V_{+,j})$, $j = 1, \dots, n$, are iid, and we denote by (Y_+, V_+) a generic element with the same distribution.

The argument in Section 4 to prove that $\{Y_j, V_j\}_j$ is a sufficient statistics under the IEBP policy can be reproduced to argue that $\{Y_{+,j}, V_{+,j}\}_j$ is a sufficient statistics; this is the case, as, similar to the IEBP policy, only the first Willie job in a W-BP may interfere with an Alice job under the II policy.

Introduce $f_{+,i}$ the pdf of (Y_+, V_+) under H_i for $i = 0, 1$, so that the joint pdf of $\{(Y_{+,j}, V_{+,j})\}_{j=1}^n$ under H_i is $f_{+,i}^{\otimes n}$. Similarly, let $\tilde{f}_{+,i}$ be the pdf of Y_+ under H_i , $i = 0, 1$, with $\tilde{f}_{+,i}^{\otimes n}$ the pdf under H_i of the iid rvs $\{Y_{+,j}\}_{j=1}^n$.

Clearly, $f_{+,0}(x, v) = f_0(x, v) = \lambda e^{-\lambda v} g_1(x)$ (cf. (19)) and $\tilde{f}_{+,0}(x) = \tilde{f}_0(x) = g_1(x)$.

The following lemma is proved in Appendix E:

Lemma 5.1 (pdfs $f_{+,1}$ and $\tilde{f}_{+,1}$ under the II_+ policy).

For any pdf g_1 and $g_2(x) = \mu_2 e^{-\mu_2 x}$,

$$f_{+,1}(x, v) = \lambda e^{-\lambda v} g_1(x) \left[1 + q e^{-\mu_2 \bar{q} v} \left(\frac{(g_1 * g_2)(x)}{g_1(x)} - 1 \right) \right] \quad (54)$$

$$\tilde{f}_{+,1}(x) = g_1(x) \left[1 + \frac{pq}{1 - \bar{p}q} \left(\frac{(g_1 * g_2)(x)}{g_1(x)} - 1 \right) \right], \quad (55)$$

for all $x \geq 0$, $v \geq 0$.

From now on $g_2(x) = \mu_2 e^{-\mu_2 x}$. Recall that X is a rv with pdf g_1 and V is an exponential rv with rate λ , independent of X .

By replacing f_i by $f_{+,i}$, $i = 0, 1$, in the derivation of (23) in the proof of Lemma 4.8, we obtain

$$\begin{aligned} (2T_V)^2 (f_{+,0}^{\otimes n}, f_{+,1}^{\otimes n}) &\leq 1 - 2 \left(\mathbb{E} \left[\frac{f_{+,1}(X, V)}{f_{+,0}(X, V)} \right] \right)^n + \left(\mathbb{E} \left[\frac{f_{+,1}(X, V)^2}{f_{+,0}(X, V)^2} \right] \right)^n \\ &= 1 - 2 \left(1 + \mathbb{E} [e^{-\mu_2 \bar{q} V}] \mathbb{E} \left[\frac{g_1 * g_2(X)}{g_1(X)} - 1 \right] \right)^n + \left(\mathbb{E} \left[\left(1 + q e^{-\mu_2 \bar{q} V} \left(\frac{(g_1 * g_2)(X)}{g_1(X)} - 1 \right) \right)^2 \right] \right)^n \\ &= -1 + \left(1 + q^2 \mathbb{E} \left[e^{-2\mu_2 \bar{q} V} \left(\frac{(g_1 * g_2)(X)}{g_1(X)} - 1 \right)^2 \right] \right)^n, \end{aligned} \quad (56)$$

where the latter equality follows from the identity $\mathbb{E} \left[\frac{g_1 * g_2(X)}{g_1(X)} \right] = \int_0^\infty g_1 * g_2(x) dx = 1$ and the independence of the rvs X and V . Define $C_1 = \mathbb{E} \left[e^{-2\mu_2 \bar{q} V} \left(\frac{(g_1 * g_2)(X)}{g_1(X)} - 1 \right)^2 \right]$. Inequality (56) then rewrites

$$T_V (f_{+,0}^{\otimes n}, f_{+,1}^{\otimes n}) \leq \frac{1}{2} \sqrt{(1 + q^2 C_1)^n - 1}, \quad n \geq 1. \quad (57)$$

Let $T_{+,n}$ be the expected number of jobs inserted by Alice over n W-BPs. Observe that

$$T_+(n) = n \times q(p + \bar{p}q) \sum_{i \geq 0} (i+1) (\bar{p}q)^i = \frac{nq}{1 - \bar{p}q}. \quad (58)$$

Mimicking now the proof of Proposition 4.2 with (57) replacing (21), and $T_+(n)$ in (58) replacing $T(n) = nq$, we obtain the following covert result:

Proposition 5.2. *Assume that $g_2(x) = \mu_2 e^{-\mu_2 x}$ and $C_1 < \infty$. Under the II policy, Alice can achieve a covert throughput of $T_+(n) = \mathcal{O}(\sqrt{n})$.*

The lemma below gives the Hellinger distances between $f_{+,0}^{\otimes n}$ and $f_{+,1}^{\otimes n}$, and between $\tilde{f}_{+,0}^{\otimes n}$ and $\tilde{f}_{+,1}^{\otimes n}$ when Alice and Willie job service times are exponentially distributed.

Lemma 5.3. *Assume that $g_i(x) = \mu_i e^{-\mu_i x}$, $i = 0, 1$. Then, for every $n \geq 1$,*

$$H\left(f_{+,0}^{\otimes n}, f_{+,1}^{\otimes n}\right) = 1 - \left(\mathbb{E}\left[\sqrt{\Xi(qe^{-\mu_2 \bar{q}V}, X)}\right]\right)^n \quad (59)$$

$$H\left(\tilde{f}_{+,0}^{\otimes n}, \tilde{f}_{+,1}^{\otimes n}\right) = 1 - \left(\mathbb{E}\left[\sqrt{\Xi(pq/(1-\bar{p}q), X)}\right]\right)^n, \quad (60)$$

with X an exponential rv with rate μ_1 , where the mapping Ξ is defined in (36).

For the sake of comparison, recall that under the IEBP policy the Hellinger distances corresponding to (59) and (60) are given by (Hint: introduce (37) and (38) in (23), respectively)

$$H\left(f_0^{\otimes n}, f_1^{\otimes n}\right) = 1 - \left(\mathbb{E}\left[\sqrt{\Xi(qe^{-\mu_2 V}, X)}\right]\right)^n \quad (61)$$

$$H\left(\tilde{f}_0^{\otimes n}, \tilde{f}_1^{\otimes n}\right) = 1 - \left(\mathbb{E}\left[\sqrt{\Xi(pq, X)}\right]\right)^n, \quad (62)$$

respectively, when $g_i(x) = \mu_i e^{-\mu_i x}$ for $i = 0, 1$.

We then see that, for q small, $H\left(f_{+,0}^{\otimes n}, f_{+,1}^{\otimes n}\right) \sim H\left(f_0^{\otimes n}, f_1^{\otimes n}\right)$ and $H\left(\tilde{f}_{+,0}^{\otimes n}, \tilde{f}_{+,1}^{\otimes n}\right) \sim H\left(\tilde{f}_0^{\otimes n}, \tilde{f}_1^{\otimes n}\right)$, thereby explaining why Proposition 4.5 holds under the II policy, as announced earlier (a rigorous proof mimicks the (very lengthy) proof of Proposition 4.5).

In conclusion, as $n \rightarrow \infty$, policies IEBP and II behave the same as far as covert/non-covert results are concerned when Alice job service times are exponentially distributed. This means that Alice should rather use the II policy since the expected number of jobs that she inserts over a *finite* number n of W-BPs, given by $nq/(1-\bar{p}q)$, is larger under the II policy than it is under the IEBP policy (given by nq).

6 Insert-at-Idle-and-at-Arrivals Policy

Throughout this section we assume that the service times of Willie and Alice are exponentially distributed with rate μ_1 and μ_2 , respectively.

We have observed at the beginning of Section 4.1 that Alice should preferably insert jobs at idle times; this was the motivation for introducing and investigating the IEBP policy in Section 4 and its variant, the II policy investigated in Section 5.

But can Alice submit more jobs covertly if she also inserts jobs at other times than at idle times, typically, just after an arrival /departure of a Willie job? This is the question we try to answer in this section. Note that, because of the FIFO assumption, Alice cannot benefit from inserting a job at a time $t+$ if time t is neither an arrival time nor a departure time of a Willie job.

In this section, we assume that Alice inserts jobs at idle times *and* at arrival times (see Remark 3). More precisely,

- each time the server idles Alice inserts one job with probability q and does not insert a job with probability \bar{q} ;
- after the arrival of each Willie job, Alice inserts a batch of $s \geq 0$ jobs with probability $qQ(s)$ and with probability $1 - q$ she does not insert any job.

These policies are called *Insert-at-Idle-and-at-Arrivals* (II-A) policies. Let \mathcal{A} be the set of all such policies. A policy in \mathcal{A} is fully characterized by the pair (q, Q) , with Q a pdf with support in $\{0, 1, \dots\}$. Notice that the II-A policy reduces to the II policy when $Q_B(0) = 1$ (no job inserted at arrival times).

For the time being we do not make any assumption on Q (later on we will assume that it has a finite support). We assume that successive job batch sizes inserted by Alice just after arrival epochs are mutually independent rvs.

Let

$$\mathcal{G}_Q(z) = \sum_{s \geq 0} z^s Q(s)$$

be the generating function of Q and denote by B the expected batch size. For the time being, the pair (q, Q) is fixed, that is, we focus on a particular policy in \mathcal{A} . Under this policy, all of Willie jobs are susceptible to interference from Alice. This is in contrast with the IEBP policy, where only the first of Willie jobs in a W-BP can be affected by Alice.

Unlike for the IEBP policy (cf. Proposition 4.5), we have not been able to find a sufficient statistics composed of iid rvs. As a result, we will only focus on obtaining an upper bound for $T(n)$. To obtain such a result, recall that Willie does not need to work with a sufficient statistics as it is enough for Willie to use a detector that prevents Alice from being covert (this argument was used to prove the non-covert part of Proposition 4.5).

The non-covert result is stated in Proposition 6.4. We will see that it gives a loose upper bound since, in particular, it does not reduce to the non-covert result obtained under the II policy (see the remark after the proof of Proposition 6.5). This is due to the fact that Willie does not use the full information he has about Alice jobs or, equivalently, he does not use a sufficient statistics. This said, we conjecture that the results in Proposition 4.5 should hold for all policies in \mathcal{A} and also for a much broader class of policies (e.g. stationary policies) provided service times are exponentially distributed.

Recall the definition of a *Willie Busy Period* (W-BP) and *Willie Idle Period* (W-IP) introduced at the beginning of Section 4. We call a *cycle* the period consisting of a W-IP followed by a W-BP. Denote by N_A and N_W the expected number of Alice jobs and Willie jobs served during a cycle, respectively.

Lemma 6.1. *Under the II-A policy the queue is stable iff $\rho_1 + q\rho_2 B < 1$. In this case,*

$$\mathbb{E}[N_W] = \left[1 + q \frac{\rho_2 p}{1 - q\bar{p}} \left(\frac{1}{1 - q\bar{p}} - \frac{(1+p)Q(1)}{\bar{p}} \right) \right. \quad (63)$$

$$\left. + q^2 \frac{\rho_2}{1 - q\bar{p}} \left(pB - \frac{p}{1 - qp} (1 - \mathcal{G}_Q(\bar{p})) \right) \right] \times \frac{1}{1 - \rho_1 - q\rho_2 B}$$

$$= \frac{1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_1} \left(p - (2-p)Q(1) - \frac{B}{1 - \rho_1} \right) q + o(q), \quad (64)$$

and

$$\mathbb{E}[N_A] = qB\mathbb{E}[N_W] + q \frac{\bar{p}\bar{q} + p}{(1 - q\bar{p})^2} (\bar{q}\bar{Q}(0) + \mathcal{G}_Q(\bar{p})) \quad (65)$$

$$= q \left(\frac{B}{1 - \rho_1} + \bar{Q}(0) + \mathcal{G}_Q(\bar{p}) \right) + o(q).$$

The proof is given in Appendix F. From now on we assume that $q \in [0, q_0)$ with $q_0 := (1 - \rho_1)/(\rho_2 B)$ so that the stability condition $\rho_1 + q\rho_2 B < 1$ holds (recall that $\rho_1 < 1$ – see Section 4). In particular, W-BPs have finite expected lengths when $\rho_1 + q\rho_2 B < 1$.

To apply the results of Section 4, Willie needs to come up with a detector built in such a way that the reconstructed service times form an iid sequence. To this end, he will use the following detector, hereafter refers to as D_W : from each of the first n W-BP, he picks a job uniformly at random and reconstructs its service time. Under the enforced assumptions (Poisson arrivals and exponential service times), this detector produces an iid sequence of reconstructed service times.

We consider a generic cycle and denote by $J \in \{1, \dots, N_W\}$ the identity of the Willie job picked at random in the W-BP. Let $\pi_J := \mathbb{P}(J = 1)$ be the probability that the first job is picked. Let Y be the reconstructed service time of the randomly picked job J (recall that in Section 4 Y denotes the reconstructed service time of the first job, corresponding to $\pi_J = 1$. We use the same notation here for the sake of simplicity, as no confusion should arise).

We have

$$Y = \sigma_1 + \sum_{r=1}^{S_J} \tau_r, \quad (66)$$

where σ_1 is a generic service time for a Willie job, τ_1, \dots, τ_r are the service times of r different Alice' s jobs, and S_J is the number of Alice jobs that interferes with J . Define

$$w_i(x) := \frac{d}{dx} \mathbb{P}_{H_i}(Y < x), \quad i = 0, 1, \quad (67)$$

the pdf of the Willie job reconstructed service time in a W-BP under H_i . Clearly, $w_0(x) = \frac{d}{dx} \mathbb{P}_{H_0}(Y < x) = g_1(x)$. The Hellinger distance between the pdfs w_0 and w_1 is (cf. (6))

$$H(w_0, w_1) = 1 - \mathbb{E} \left[\sqrt{W(q, X)} \right], \quad (68)$$

where X is an exponential rv with parameter μ_1 , and

$$W(q, x) = \frac{w_1(x)}{g_1(x)}. \quad (69)$$

The mapping W corresponds to the mapping Z for the IEBP policy (see (10)). The lemma below determines $W(q, x)$. The proof is provided in Appendix G.

Lemma 6.2. For $q \in [0, q_0)$,

$$W(q, x) = \Delta_1(q) + \Delta_2(q)\Phi_1(x) + q\Phi_2(x), \quad x \geq 0, \quad (70)$$

where

$$\Delta_1(q) := \frac{\bar{q}}{1 - q\bar{p}} (\bar{q} + q\mathcal{G}_Q(\bar{p})) \pi_J + (1 - q\bar{Q}(0))\bar{\pi}_J \geq 0 \quad (71)$$

$$\Delta_2(q) := q \frac{\bar{p}(1 - q\bar{Q}(0)) + \mathcal{G}_Q(\bar{p}) - Q(0)}{\bar{p}(1 - q\bar{p})} \geq 0 \quad (72)$$

$$\Phi_1(x) := p\pi_J \frac{(g_1 * h_1)(x)}{g_1(x)} \quad (73)$$

$$\begin{aligned}\Phi_2(x) &:= p\pi_J \sum_{s \geq 2} \frac{(g_1 * h_s)(x)}{g_1(x)} \sum_{l \geq s} Q(l) \bar{p}^{l-s} \\ &\quad + \bar{\pi}_J \sum_{s \geq 1} \frac{(g_1 * h_s)(x)}{g_1(x)} Q(s),\end{aligned}\tag{74}$$

where $h_s(x) = \mu_2^s x^{s-1} e^{-\mu_2 x} / (s-1)!$ is the pdf of a s -stage Erlang rv with mean k/μ_2 .

Lemma 6.3. *Assume that the support of Q is finite. If $\mu_1 < 2\mu_2$ there exists a finite constant c_0 such that*

$$\mathbb{E} \left[\sqrt{W(q, X)} \right] = 1 + c_0 q^2 + o(q^2),\tag{75}$$

where X is an exponential rv with rate μ_1 .

The proof is provided in Appendix H. Below is the main result of this section.

Proposition 6.4. *Assume that the support of Q is finite. For all μ_1 and μ_2 , Alice is not covert if $T(n) = \omega(\sqrt{n})$.*

Proof. Without loss of generality assume that the support of Q is contained in $\{0, 1, \dots, S\}$ with $S < \infty$.

Let Y_1, \dots, Y_n be the Willie reconstructed job service times over n W-BPs. Under H_1 (resp. H_0), the joint pdf of Y_1, \dots, Y_n is $w_1^{\otimes n}$ (resp. $w_0^{\otimes n} = g_1^{\otimes n}$) since these rvs are iid. Hence, from (8),

$$H(w_0^{\otimes n}, w_1^{\otimes n}) = 1 - \left(\mathbb{E} \left[\sqrt{W(q, X)} \right] \right)^n, \quad \forall n \geq 1,$$

which in turn gives, by using (7),

$$1 - \left(\mathbb{E} \left[\sqrt{W(q, X)} \right] \right)^n \leq T_V(w_0^{\otimes n}, w_1^{\otimes n}), \quad \forall n \geq 1.\tag{76}$$

Let $T_n = \frac{n}{\phi(n)} = \omega(\sqrt{n})$ or, equivalently, $\lim_{n \rightarrow \infty} \frac{\phi(n)^2}{n} = 0$. Upon replacing $\tilde{f}_1(x)$ by $w_1(x)$, the same argument in the proof of Proposition 4.4 shows that Alice is not covert if $\liminf_n \phi(n) < \infty$. Therefore, we assume from now on that $\lim_n \phi(n) = \infty$.

Assume first that $\mu_1 < 2\mu_2$. By Lemma 6.3, we conclude that

$$\lim_{n \rightarrow \infty} \left(\mathbb{E} \left[\sqrt{W(1/\phi(n), X)} \right] \right)^n = 0\tag{77}$$

which proves, thanks to (76) and (4), that Alice is not covert if $T(n) = \omega(\sqrt{n})$ and $\mu_1 < 2\mu_2$.

Let us show that (77) holds when $\mu_1 \geq 2\mu_2$ which will complete the proof. From (140), we see that for each $s = 1, \dots, S$, the mapping $x \rightarrow (g_1 * h_s)(x)/g_1(x)$ is non-decreasing in $[0, \infty)$. On the other hand, notice that both sums in (74) are finite under the assumption that Q has a finite support, and observe that each term $(g_1 * h_s)(x)/g_1(x)$, $s = 1, \dots, S$, is multiplied by a non-negative constant. Therefore, the mappings $x \rightarrow \Phi_i(x)$, $i = 1, 2$, are non-decreasing in $[0, \infty)$, which in turn shows that the mapping $x \rightarrow W(q, x)$ (given in (70)) is non-decreasing in $[0, \infty)$ for all $q \in [0, 1]$, since $\Delta_i(q) \geq 0$ for $i = 1, 2$ (Hint: in (72) $\mathcal{G}_Q(\bar{p}) \geq Q(0)$ by definition of the generating function \mathcal{G}_Q).

Let \tilde{X}_ν be an exponential rv with rate ν . Take ν_1 such that $\mu_1 \geq 2\mu_2 > \nu_1$. The stochastic inequality $X = \tilde{X}_{\mu_1} \leq_{st} \tilde{X}_{\nu_1}$ together with increasingness (shown above) of the mapping $x \rightarrow W(q, x)$ for $q \in [0, 1]$, yield

$$\mathbb{E} \left[\sqrt{W(q, X)} \right] \leq \mathbb{E} \left[\sqrt{W(q, \tilde{X}_{\nu_1})} \right], \quad \forall q \in [0, 1].\tag{78}$$

Then, (77) and (78) imply

$$\lim_{n \rightarrow \infty} \left(\mathbb{E} \left[\sqrt{W(1/\phi(n), X)} \right] \right)^n = 0$$

when $T(n) = \omega(\sqrt{n})$. This completes the proof. \blacksquare

In general, the asymptotic upper bound in Proposition 6.4 is loose as Willie's detector lacks of information (cf. discussion at the beginning of this section). This is the case when $Q_B(0) = 1$ (i.e. the II-A policy reduces to the II policy) as the bound is larger than the bound for $\mu_1 = 2\mu_2$ ($\omega(\sqrt{n/\log n})$) and for $\mu_1 > 2\mu_2$ ($\omega(n^{\mu_2/\mu_1})$) under the II policy (see Section 5).

Last, we consider a variant of the II-A policies where Alice inserts a batch of jobs that is geometrically distributed with mean $1/a$ at times the server becomes idle and immediately after the arrival of Willie job, both with probability q . We have the following result:

Proposition 6.5. *Assume that $g_i(x) = \mu_i e^{-\mu_i x}$ for $i = 1, 2$. When Willie uses detector D_W , Alice is not covert if she inserts*

- (a) $\omega(\sqrt{n})$ jobs when $\mu_1 < 2a\mu_2$
- (b) $\omega(\sqrt{n/\log n})$ jobs when $\mu_1 = 2a\mu_2$
- (c) $\omega(n^{a\mu_2/\mu_1})$ jobs when $\mu_1 > 2a\mu_2$

on average over n W-BPs.

Proof. Note that each batch of Alice jobs incurs a total amount of service time that is exponentially distributed with rate $a\mu_2$. This coupled with Willie detector produces a pdf for the hypothesis H_1 of the form (10) for some $p > \lambda/(\lambda + a\mu)$ in (11). The arguments leading to the converse in Proposition 4.5 apply to this case to yield the desired result. \blacksquare

Remark 2 (Geometric batch size). *Note that geometric batching provably reduces covert throughput in the range $2a\mu_2 \leq \mu_1 < 2\mu_2$ under a variant of the II-A policy using batches with finite support. This appears to be due to the exponential tail. We conjecture that batches of size greater than one can only reduce covert throughput. A similar result holds for a variant of the IEBP policy where Alice introduces a batch of jobs with probability q each time the server becomes idle where the batch is geometrically distributed with mean $1/a$ leading to a considerably smaller covert throughput than is possible when Alice introduces only one job at a time. This is evidence that batching again may be harmful and that Alice should introduce only one job at a time.*

Remark 3 (Insert-at-Idle-and-at-Departure). *The analysis of the policy, called II-D, where Alice may insert a job each time the server idles and may also insert a batch of jobs after each Willie job departure (provided the system is not empty) is more involved than that of the II-A policy. This is so because the reconstructed service time of a Willie job in a W-BP depends on what happened in this busy period prior to the arrival of this job. To illustrate this, assume first that the j th Willie job ($j > 1$) in a W-BP arrives during the service time of the 1st Willie job in this W-BP. Then, job j will not be affected by any Alice's insertions in this W-BP. But if job j arrives during the service time of the $(j - 1)$ st Willie job then it may be affected by 0, 1 or up to $j - 2$ Alice batches, depending on how many batches Alice insert at departures of Willie jobs $1, 2, \dots, j - 2$. This is in contrast with the II-A policy, where job $j > 1$ in a W-BP will be affected by at most one Alice's batch (the batch inserted after the arrival of customer $j - 1$, if any).*

7 Concluding remarks

In this paper we have studied covert cycle stealing in an M/G/1 queue. We have obtained a phase transition result on the expected number of jobs that Alice can covertly insert in n busy periods when both Alice and Willie's jobs have exponential service times and established a partial achievability result for arbitrary service times. Supported by results obtained for batch insertions and, in particular, geometric batches, we conjecture that batching reduces covert throughput. One future research direction consists in weakening the assumption that Willie's detectors generate iid rvs, which would allow us to enrich our achievability results

References

- [1] G. Bachmann and L. Narici. *Fourier and Wavelet Analysis*. Springer Science & Business Media, 2012.
- [2] B.A. Bash, A.H. Gheorghe, M. Patel, J.L. Habif, D. Goeckel, D. Towsley, and S. Guha. Quantum-secure covert communication on bosonic channels. In *Nature Communications*, volume 6, 2015.
- [3] B.A. Bash, D. Goeckel, and D. Towsley. Square root law for communication with low probability of detection on awgn channels. In *Proc. IEEE Int. Symp. Inform. Theory (ISIT)*, pages 448–452, Cambridge, MA, USA, July 2012.
- [4] U.N. Bhat. On the busy period of a single server bulk queue with modified service mechanism. *Calcutta Statist. Assoc. Bull.*, 13:163–171, 1964.
- [5] M.R. Bloch. Covert communication over noisy channels: A resolvability perspective. *IEEE Trans. Inf. Theory*, 62(5):2334–2354, 2016.
- [6] P.H. Che, M. Bakshi, C. Chan, and S. Jaggi. Reliable deniable communication with channel uncertainty. In *Proc. IEEE Int. Symp. Inform. Theory Workshop*, pages 448–452, July 2013.
- [7] T.M. Cover and J.A. Thomas. *Elements of Information Theory*. John Wiley & Sons, 2nd edition, 2002.
- [8] V. Emeakaroha, M. Netto T. Ferreto, I. Brandic, and C. De Rose. Casvid: Application level monitoring for SLA violation detection in clouds. In *IEEE Annual Computer Software and Applications Conference (COMPSAC)*, pages 499–508, Cambridge, MA, USA, July 2012.
- [9] J. Fridrich. *Steganography in Digital Media: Principles, Algorithms, and Applications*. Cambridge University Press, 2009.
- [10] M. Harchol-Balter, C. Li, T. Osogami, A. Scheller-Wolf, and M. Squillante. Analysis of task assignment with cycle stealing under central queue. In *Proc. Fifteenth ACM Annual Symposium on Parallel Algorithms and Architectures*, pages 274–285, San Diego, June 2003.
- [11] C.H. Kraft. *Some Conditions for Consistency and Uniform Consistency of Statistical Procedures*, volume 2, pages 125–142. Univ. California Publications in Statist., 1955.
- [12] S. Kullback. *Information Theory and Statistics*. Dover, 1968.
- [13] E. Lehmann and J. Romano. *Testing Statistical Hypotheses*. New York: Springer, 3rd edition, 2005.
- [14] T. Osogami, M. Harchol-Balter, and A. Scheller-Wolf. Analysis of cycle stealing with switching times and thresholds. In *ACM Sigmetrics*, pages 184–195, San Diego, CA, Junw 2003.

- [15] T. Osogami, M. Harchol-Balter, and A. Scheller-Wolf. Analysis of cycle stealing with switching times and thresholds. *Performance Evaluation*, 61(4):347–369, 2005.
- [16] A. Sheikholeslami, M. Ghaderi, D. Towsley, B. Bash, S. Guha, and D. Goeckel. Multi-hop routing in covert wireless networks. *IEEE Trans. Wireless Communications*, 17(6):3656–3669, 2018.
- [17] T.V. Sobers, B.A. Bash, S.K. Guha, D. Towsley, and D. Goeckel. Covert communication in the presence of an uninformed jammer. *IEEE Trans. Wireless Communications*, 16(9):6193–6206, 2017.
- [18] L. Wang, G.W. Wornell, and L. Zheng. Fundamental limits of communication with low probability of detection. *IEEE Trans. on Information Theory*, 62(6):3493–3503, 2016.

A Appendix

Lemma A.1. *Let $f, g : \mathbb{N} \rightarrow [0, \infty)$. If $\limsup_{n \rightarrow \infty} \frac{g(n)}{f(n)} < \infty$ with g a non-decreasing function such that $\lim_{n \rightarrow \infty} g(n) = \infty$ then $\lim_{n \rightarrow \infty} f(n) = \infty$. The result remains true if g is asymptotically increasing.*

Proof. By assumption, there exist $L < \infty$ and n_0 such that for all $n > n_0$

$$\sup_{k \geq n} \frac{g(k)}{f(k)} < L.$$

Since g is non-decreasing

$$L > \sup_{k \geq n} \frac{g(k)}{f(k)} \geq \frac{g(n)}{\inf_{k \geq n} f(k)} \geq \frac{g(n)}{f(n)}.$$

Hence, for $n > n_0$, $f(n) > L^{-1}g(n)$, which proves the lemma since $\lim_{n \rightarrow \infty} g(n) = \infty$. The proof is the same if g is asymptotically increasing. ■

B Appendix: Proof of Lemma 4.3

Let $g_2(x) = \sum_{l=1}^{K_2} p_{2,l} g_{2,l}(x)$ with $g_{2,l}(x) := \mu_{2,l} e^{-\mu_{2,l}x}$, $p_{2,l} \geq 0$ for all l and $\sum_{l=1}^{K_2} p_{2,l} = 1$, namely, Alice job service times follow an hyper-exponential distribution with mean $1/\mu_2 = \sum_{l=1}^{K_2} 1/\mu_{2,l}$. Denote by $G_1^*(s) = \int_0^\infty e^{-sx} g_1(x) dx$ the Laplace transform of Willie job service times.

By using (11), we find

$$\rho(x, v) = \frac{1}{g_1(x)} \sum_{l=1}^{K_2} p_{2,l} e^{-\mu_{2,l}v} (g_1 * g_{2,l})(x) - \sum_{l=1}^{K_2} p_{2,l} e^{-\mu_{2,l}v},$$

so that

$$\mathbb{E} [\rho(X, V)^2] = \alpha_1 - 2\alpha_2 + \alpha_3$$

with

$$\begin{aligned} \alpha_1 &:= \int_{[0, \infty)^2} \frac{\lambda e^{-\lambda v}}{g_1(x)} \left[\sum_{l=1}^{K_2} p_{2,l} e^{-\mu_{2,l}v} (g_1 * g_{2,l})(x) \right]^2 dv dx \\ &= \sum_{l=1}^{K_2} \sum_{m=1}^{K_2} \frac{\lambda p_{2,l} p_{2,m}}{\mu_{2,l} + \mu_{2,m}} \int_0^\infty \frac{(g_1 * g_{2,l})(x) \times (g_1 * g_{2,m})(x)}{g_1(x)} dx; \end{aligned}$$

$$\begin{aligned}
\alpha_2 &:= \sum_{\substack{l=1 \\ m=1}}^{K_2} p_{2,l} p_{2,m} \int_{[0,\infty)^2} \lambda e^{-(\lambda+\mu_{2,l})v} (g_1 * g_{2,l})(x) e^{\mu_{2,m}x} dv dx \\
&= \sum_{\substack{l=1 \\ m=1}}^{K_2} p_{2,l} p_{2,m} \frac{\lambda \mu_{2,l} G_1^*(\mu_{2,m})}{(\lambda + \mu_{2,l})(\mu_{2,l} + \mu_{2,m})} \leq 1; \\
\alpha_3 &:= \sum_{\substack{l=1 \\ m=1}}^{K_2} p_{2,l} p_{2,m} \int_{[0,\infty)^2} \lambda e^{-\lambda v} g_1(x) e^{-(\mu_{2,l}+\mu_{2,m})x} dv dx \\
&= \sum_{\substack{l=1 \\ m=1}}^{K_2} p_{2,l} p_{2,m} G_1^*(\mu_{2,l} + \mu_{2,m}) \leq 1.
\end{aligned}$$

We conclude from the above that $\mathbb{E}[\rho(X, V)^2] < \infty$ if and only if

$$\beta_{l,m} := \int_0^\infty \frac{(g_1 * g_{2,l})(x) \times (g_1 * g_{2,m})(x)}{g_1(x)} dx < \infty \quad (79)$$

for all $l, m = 1, \dots, K_2$.

Case 1: $g_1(x) = \sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} e^{-\mu_{1,i}x}$, $p_{1,i} \geq 0$ for all i and $\sum_{i=1}^{K_1} p_{1,i} = 1$, namely, Willie job service times follow an hyper-exponential distribution with mean $1/\mu_1 = \sum_{i=1}^{K_1} 1/\mu_{1,i}$.

We have

$$(g_1 * g_{2,l})(x) = \sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} \mu_{2,l} \left[x e^{-\mu_{2,l}x} \mathbf{1}(\mu_{1,i} = \mu_{2,l}) + \frac{e^{-\mu_{2,l}x} - e^{-\mu_{1,i}x}}{\mu_{1,i} - \mu_{2,l}} \mathbf{1}(\mu_{1,i} \neq \mu_{2,l}) \right]$$

for $l = 1, \dots, K_2$, so that

$$\begin{aligned}
(g_1 * g_{2,l})(x) \times (g_1 * g_{2,m})(x) &= \sum_{\substack{i=1 \\ j=1}}^{K_1} p_{1,i} p_{1,j} \mu_{1,i} \mu_{1,j} \left[P_{i,l}(x) P_{j,m}(x) e^{-(\mu_{2,l}+\mu_{2,m})x} \right. \\
&\quad \left. - a_{i,l} b_{j,m} x e^{-(\mu_{1,i}+\mu_{2,m})x} - a_{j,m} b_{i,l} x e^{-(\mu_{1,j}+\mu_{2,l})x} + b_{i,l} b_{j,m} e^{-(\mu_{1,i}+\mu_{1,j})x} \right],
\end{aligned}$$

with

$$\begin{aligned}
a_{i,l} &:= \mu_{2,l} \mathbf{1}(\mu_{1,i} = \mu_{2,l}) \\
b_{i,l} &:= \frac{\mu_{2,l}}{\mu_{1,i} - \mu_{2,l}} \mathbf{1}(\mu_{1,i} \neq \mu_{2,l}) \\
P_{i,l}(x) &:= a_{i,l} x + b_{i,l},
\end{aligned}$$

for $i = 1, \dots, K_1$, $l = 1, \dots, K_2$. Define $\mu_1^* = \max_{1 \leq i \leq K_1} \mu_{1,i}$. Then,

$$\begin{aligned}
\beta_{l,m} &= \sum_{\substack{i=1 \\ j=1}}^{K_1} p_{1,i} p_{1,j} \mu_{1,i} \mu_{1,j} \int_0^\infty \frac{P_{i,l}(x) P_{j,m}(x) e^{-(\mu_{2,l}+\mu_{2,m}-\mu_1^*)x}}{\sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} e^{(\mu_1^*-\mu_{1,i})x}} dx \\
&\quad - \sum_{j=1}^{K_1} p_{1,j} \mu_{1,j} b_{j,m} \int_0^\infty \frac{\sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} a_{i,l} e^{-\mu_{1,i}x}}{\sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} e^{-\mu_{1,i}x}} x e^{-\mu_{2,m}x} dx
\end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^{K_1} p_{1,j} \mu_{1,j} b_{j,l} \int_0^\infty \frac{\sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} a_{i,m} e^{-\mu_{1,i} x}}{\sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} e^{-\mu_{1,i} x}} x e^{-\mu_{2,l} x} dx \\
& + \sum_{j=1}^{K_1} p_{1,j} \mu_{1,j} b_{j,m} \int_0^\infty \frac{\sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} b_{i,l} e^{-\mu_{1,i} x}}{\sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} e^{-\mu_{1,i} x}} e^{-\mu_{1,j} x} dx.
\end{aligned} \tag{80}$$

The second, third, and fourth integrals in the r.h.s. of (80) are finite since $\lim_{x \rightarrow \infty} \frac{\sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} b_{i,l} e^{-\mu_{1,i} x}}{\sum_{i=1}^{K_1} p_{1,i} \mu_{1,i} e^{-\mu_{1,i} x}}$ is finite for any $l = 1, \dots, K_2$. The first integral is finite if and only if

$$\mu_1^* = \max_{1 \leq i \leq K_1} \mu_{1,i} \leq 2 \min_{1 \leq l \leq K_2} \mu_{2,l}. \tag{81}$$

This shows that $C_0 < \infty$ when (81) holds.

In particular, when $K_1 = K_2 = 1$ (exponential service times for both Alice and Willie jobs) then $C_0 < \infty$ if and only if $\mu_1 < 2\mu_2$. For further reference, note that

$$\rho(x, v) = \begin{cases} e^{-\mu_1 v} (\mu_1 x - 1) & \text{if } \mu_1 = \mu_2 \\ e^{-\mu_2 v} \left(\frac{\mu_2 e^{-(\mu_2 - \mu_1)x} - \mu_1}{\mu_1 - \mu_2} \right) & \text{if } \mu_1 \neq \mu_2 \end{cases} \tag{82}$$

when $g_i(x) = \mu_i e^{-\mu_i x}$, $i = 1, 2$.

Case 2: $g_1(x) = \nu_1^{K_1} x^{K_1-1} e^{-\nu_1 x} / (K_1 - 1)!$ with $1/\mu_1 = K_1/\nu_1$ (Willie job service times follow a K_1 -stage Erlang pdf with mean $1/\mu_1$).

We have, with $\eta_l := \frac{\nu_1^{K_1} \mu_{2,l}}{(K_1 - 1)!}$,

$$(g_1 * g_{2,l})(x) = \begin{cases} \eta_l \frac{x^{K_1}}{K_1} e^{-\mu_{2,l} x} & \text{if } \nu_1 = \mu_{2,l} \\ \eta_l e^{-\mu_{2,l} x} \int_0^x u^{K_1-1} e^{-(\nu_1 - \mu_{2,l})u} du & \text{if } \nu_1 \neq \mu_{2,l}, \end{cases}$$

for $l = 1, \dots, K_2$.

Define $\xi_l(k) = \int_0^x u^{k-1} e^{-(\nu_1 - \mu_{2,l})u} du$ for $k \geq 1$. Integrating by part gives

$$\xi_l(k) = \frac{-x^{k-1} e^{-(\nu_1 - \mu_{2,l})x}}{\nu_1 - \mu_{2,l}} + \frac{k-1}{\nu_1 - \mu_{2,l}} \xi_l(k-1), \quad k \geq 2,$$

which yields (use that $\xi_l(1) = (1 - e^{-(\nu_1 - \mu_{2,l})x}) / (\nu_1 - \mu_{2,l})$)

$$\xi_l(k) = -e^{-(\nu_1 - \mu_{2,l})x} \sum_{i=1}^k \frac{(k-1)!}{(k-i)!} \frac{x^{k-i}}{(\nu_1 - \mu_{2,l})^i} + \frac{(k-1)!}{(\nu_1 - \mu_{2,l})^k}.$$

Therefore,

$$(g_1 * g_{2,l})(x) = Q_{1,l}(x) e^{-\mu_{2,l} x} - Q_{2,l}(x) e^{-\nu_1 x}, \tag{83}$$

with

$$Q_{1,l}(x) := \eta_l \frac{x^{K_1}}{K_1} \mathbf{1}(\nu_1 = \mu_{2,l}) + \eta_l \frac{(K_1 - 1)!}{(\nu_1 - \mu_{2,l})^{K_1}} \mathbf{1}(\nu_1 \neq \mu_{2,l}) \tag{84}$$

$$Q_{2,l}(x) := \eta \sum_{i=1}^{K_1} \frac{(K_1 - 1)!}{(K_1 - i)!} \frac{x^{K_1 - i}}{(\nu_1 - \mu_{2,l})^i} \mathbf{1}(\nu_1 \neq \mu_{2,l}). \quad (85)$$

When $\nu_1 = \mu_{2,l}$ or $\nu_1 = \mu_{2,m}$ it is easily seen from (83)-(85) that $\beta(l, m) < \infty$ if and only if $\nu_1 < \mu_{2,l} + \mu_{2,m}$. Let us investigate the (less trivial) remaining case when $\nu_1 \neq \mu_{2,l}$ and $\nu_1 \neq \mu_{2,m}$. In this case we have, from (83)-(85),

$$\begin{aligned} & \frac{(g_1 * g_{2,l})(x)(g_1 * g_{2,m})(x)}{g_1(x)} \\ &= \frac{\eta \eta_m}{\nu_1^{K_1} x^{K_1} e^{-\nu_1 x}} \left[\frac{(K_1 - 1)!}{(\nu_1 - \mu_{2,l})^{K_1}} e^{-\mu_{2,l} x} - \sum_{i=1}^{K_1} \frac{(K_1 - 1)!}{(K_1 - i)!} \frac{x^{K_1 - i}}{(\nu_1 - \mu_{2,l})^i} e^{-\nu_1 x} \right] \\ & \quad \times \left[\frac{(K_1 - 1)!}{(\nu_1 - \mu_{2,m})^{K_1}} e^{-\mu_{2,m} x} - \sum_{i=1}^{K_1} \frac{(K_1 - 1)!}{(K_1 - i)!} \frac{x^{K_1 - i}}{(\nu_1 - \mu_{2,m})^i} e^{-\nu_1 x} \right] \\ &= \frac{\eta \eta_m ((K_1 - 1)!)^2}{\nu_1^{K_1} (\nu_1 - \mu_{2,l})^{K_1} (\nu_1 - \mu_{2,m})^{K_1}} \\ & \quad \times \left[e^{(\nu_1 - \mu_{2,l})x} - \sum_{j=0}^{K_1 - 1} \frac{(x(\nu_1 - \mu_{2,l}))^j}{j!} \right] \times \frac{1}{x^{K_1}} \left[e^{-\mu_{2,m} x} - e^{-\nu_1 x} \sum_{j=0}^{K_1 - 1} \frac{(x(\nu_1 - \mu_{2,m}))^j}{j!} \right], \end{aligned}$$

which shows that $(g_1 * g_{2,l})(x)(g_1 * g_{2,m})(x)/g_1(x)$ is well-defined when $x \rightarrow 0$ and is $[0, \infty)$ -integrable if and only if $\nu_1 < \mu_{2,l} + \mu_{2,m}$.

In summary, $C_0 < \infty$ if and only if $\nu_1 < 2 \min_{1 \leq l \leq K_2} \mu_{2,l}$ or, equivalently, if and only if $\mu_1 < \frac{2}{K_1} \min_{1 \leq l \leq K_2} \mu_{2,l}$.

C Appendix: Proof of Proposition 4.5

To prove (15) one needs to prove it for any detector that Willie may use. It will be true if one proves (15) for the detector $\{(Y_i, V_i)\}_i$ since this is a sufficient statistic. This is in contrast with the proof of (16), where it is enough to exhibit a detector such that (16) holds; we will prove (16) for the detector $\{Y_i\}_i$.

When $\liminf_{n \rightarrow \infty} \phi(n) < \infty$ we know by Proposition 4.4 that Alice is not covert. Hence, in particular, Alice is not covert when $T(n) = \omega(\sqrt{n})$ (resp. $T(n) = \omega(\sqrt{n/\log n})$, $T(n) = \omega(n^{\mu_2/\mu_1})$) and $\liminf_{n \rightarrow \infty} \phi(n) < \infty$. On the other hand, Lemma A.1 shows that if $T(n) = \mathcal{O}(\sqrt{n})$ (resp. $T(n) = \mathcal{O}(\sqrt{n/\log n})$, $T(n) = \mathcal{O}(n^{\mu_2/\mu_1})$) then $\lim_{n \rightarrow \infty} \phi(n) = \infty$. As a result, we only have to consider that

$$\lim_{n \rightarrow \infty} \phi(n) = \infty. \quad (86)$$

Throughout the proof we will assume that (86) holds. From (86) we see that there exists n_1 such that $\phi(n) > 3$ for all $n \geq n_1$. Since we will only be interested by asymptotic results as $n \rightarrow \infty$ we will assume, without loss of generality, that $n_1 = 1$, that is,

$$\phi(n) > 3 \quad \text{for all } n \geq 1. \quad (87)$$

Last, since the function $\xi_r(\theta)$ defined in (39) will only be evaluated at $\theta = qe^{-\mu\nu}$ with $q = \delta/\phi(n)$ and $\delta \in [0, 1]$, giving $\xi_r(\theta) = \frac{(\beta-1)\delta}{e^{\mu\nu}\phi(n) - \beta\delta}$, we will skip the argument θ in $\xi_r(\theta)$ to keep notation simple.

Conditioning the expectation in the r.h.s. of (23) with respect to $V = v$ gives, by using the independence of X_r and V ,

$$H(f_0^{\otimes n}, f_1^{\otimes n}) = 1 - \left(\int_0^\infty \lambda e^{-\lambda v} \mathbb{E} \left[\sqrt{Z(q, X_r, v)} \right] dv \right)^n. \quad (88)$$

Proof of (15) for $\mu_1 < 2\mu_2$: The proof follows from Proposition 4.2 since $\mathbb{E}[\rho(X, V)] = 0$ when Alice and Wille job service times are exponentially distributed (cf. Remark 1) and since $C_0 < \infty$ when $\mu_1 < 2\mu_2$, as shown in Lemma 4.3-(1).

Proof of (15) for $\mu_1 = 2\mu_2$: Assume that $r = 2$. Note that $\beta = 2$ when $r = 2$. Recall that $T(n) = \frac{\delta n}{\phi(n)}$ with $\delta \in (0, 1]$. By Lemma 4.11,

$$\begin{aligned} \left(\mathbb{E} \left[\sqrt{Z(\delta/\phi(n), X_2, V)} \right] \right)^n &= \left(1 + \int_0^\infty \lambda e^{-\lambda v} \left(\frac{1}{4} \xi_2^2 \log \xi_2 + \Delta_2(\xi_2) \right) dv \right)^n \\ &= e^{n \log \left(1 + \int_0^\infty \lambda e^{-\lambda v} \xi_2^2 \log \xi_2 \times \left(\frac{1}{4} + \frac{\Delta_2(\xi_2)}{\xi_2^2 \log \xi_2} \right) dv \right)}, \end{aligned} \quad (89)$$

with $\Delta_2(z) = o(z^2 \log z)$ and $\xi_2 = \frac{\delta e^{-\mu v}}{\phi(n) - 2\delta e^{-\mu v}} > 0$ for all $n \geq 1$ and $v \geq 0$ thanks to (87). For $v \geq 0$ notice that $\xi_2 > 0$ for all n and $\xi_2 \rightarrow 0$ as $n \rightarrow \infty$. Define

$$D_n := \int_0^\infty \lambda e^{-(\lambda+2\mu)v} \frac{\log \xi_2}{(\phi(n) - 2\delta e^{-\mu v})^2} \left(\frac{1}{4} + \frac{\Delta_2(\xi_2)}{\xi_2^2 \log \xi_2} \right) dv, \quad (90)$$

(89) rewrites

$$\left(\mathbb{E} \left[\sqrt{Z(\delta/\phi(n), X_2, V)} \right] \right)^n = e^{n \log(1 + \delta^2 D_n)}. \quad (91)$$

Let us show that $D_n \rightarrow 0$ as $n \rightarrow \infty$.

Fix $\epsilon > 0$. Since $\Delta_2(z) = o(z^2 \log z)$ there exists $z_\epsilon > 0$ such that for all $0 < z < z_\epsilon$, $\left| \frac{\Delta_2(z)}{z^2 \log z} \right| < \epsilon$. Since for all n such that $\frac{\delta}{\phi(n) - 2\delta} < z_\epsilon$ we have $\xi_2 = \frac{\delta e^{-\mu v}}{\phi(n) - 2\delta e^{-\mu v}} < z_\epsilon$ for all $v \geq 0$ (Hint: the mapping $v \rightarrow \xi_2$ is nonincreasing in $[0, \infty)$ and $\xi_2 = \frac{\delta}{\phi(n) - 2\delta}$ when $v = 0$), we conclude that for n large enough,

$$\sup_{v \geq 0} \left| \frac{\Delta_2(\xi_2)}{\xi_2^2 \log \xi_2} \right| < \epsilon. \quad (92)$$

Hence, for n large enough,

$$\begin{aligned} |D_n| &\leq \left(\frac{1}{4} + \epsilon \right) \int_0^\infty \lambda e^{-(\lambda+2\mu)v} \left| \frac{\log \xi_2}{(\phi(n) - 2\delta e^{-\mu v})^2} \right| dv \\ &= \left(\frac{1}{4} + \epsilon \right) \int_0^\infty \lambda e^{-(\lambda+2\mu)v} \left| \frac{\log(\phi(n) - 2\delta e^{-\mu v}) + \mu v - \log \delta}{(\phi(n) - 2\delta e^{-\mu v})^2} \right| dv. \end{aligned} \quad (93)$$

For n large enough

$$\begin{aligned} a_n(v) &:= \left| \frac{\log(\phi(n) - 2\delta e^{-\mu v}) + \mu v - \log \delta}{(\phi(n) - 2\delta e^{-\mu v})^2} \right| \\ &\leq \mu v - \log \delta + \frac{|\log(\phi(n) - 2\delta e^{-\mu v})|}{(\phi(n) - 2\delta e^{-\mu v})^2} \end{aligned} \quad (94)$$

for all $v \geq 0$. It is easy to check that for all n such that $\phi(n) > 2\delta + \sqrt{e}$, the mapping $v \rightarrow \frac{\log(\phi(n) - 2\delta e^{-\lambda v})}{(\phi(n) - 2\delta e^{-\lambda v})^2}$ is non-decreasing in $[0, \infty)$. Therefore, for all n such that $\phi(n) > 2\delta + \sqrt{e}$,

$$\frac{\log(\phi(n) - 2\delta e^{-\mu v})}{(\phi(n) - 2\delta e^{-\mu v})^2} \leq \frac{\log(\phi(n) - 2\delta)}{(\phi(n) - 2\delta)^2} \quad \text{for all } v \geq 0.$$

This shows that for all n such that $\phi(n) > 2\delta + \sqrt{e}$ [Hint: $\phi(n) - 2\delta e^{-\lambda v} > 1$ for all $v \geq 0$ when $\phi(n) > 2\delta + \sqrt{e}$]

$$\begin{aligned} \frac{|\log(\phi(n) - 2\delta e^{-\mu v})|}{(\phi(n) - 2\delta e^{-\mu v})^2} &= \frac{\log(\phi(n) - 2\delta e^{-\mu v})}{(\phi(n) - 2\delta e^{-\mu v})^2} \\ &\leq \frac{\log(\phi(n) - 2\delta)}{(\phi(n) - 2\delta)^2} \quad \forall v \geq 0. \end{aligned} \quad (95)$$

We conclude from (94) and (95) that for n large enough [Hint: for n large enough, $\log(\phi(n) - 2\delta)/(\phi(n) - 2\delta)^2 < 1$ since $\log t/t^2 \rightarrow 0$ as $t \rightarrow \infty$ and $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$]

$$0 \leq a_n(v) \leq \mu v + 1 - \log \delta \quad \text{for all } v \geq 0.$$

Since for every $v \geq 0$, $a_n(v) \rightarrow 0$ as $n \rightarrow \infty$ (cf. (94)), and

$$\int_0^\infty \lambda e^{-(\lambda+2\mu)v} (\mu v + 1 - \log \delta) dv = \frac{\lambda\mu}{(\lambda+2\mu)^2} + \frac{\lambda(1 - \log \delta)}{\lambda+2\mu} < \infty,$$

we may apply the Bounded Convergence Theorem to the sequence $\{a_n(v)\}_n$, to get from (93) that

$$\lim_{n \rightarrow \infty} |D_n| \leq \left(\frac{1}{4} + \epsilon\right) \lim_{n \rightarrow \infty} \int_0^\infty \lambda e^{-(\lambda+2\mu)v} a_n(v) dv = \left(\frac{1}{4} + \epsilon\right) \int_0^\infty \lambda e^{-(\lambda+2\mu)v} \lim_{n \rightarrow \infty} a_n(v) dv = 0.$$

This shows that

$$D_n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (96)$$

We now assume that $T(n) = \frac{\delta n}{\phi(n)} \in \mathcal{O}(\sqrt{n/\log n})$, which is equivalent to $\liminf_{n \rightarrow \infty} \frac{\phi(n)}{\sqrt{n \log n}} = a$ for some $a > 0$. Two cases will be considered:

$$(i) \quad \limsup_{n \rightarrow \infty} \frac{\phi(n)}{\sqrt{n \log n}} < \infty$$

$$(ii) \quad \liminf_{n \rightarrow \infty} \frac{\phi(n)}{\sqrt{n \log n}} = \infty.$$

We have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \left(\mathbb{E} \left[\sqrt{Z(\delta/\phi(n), X_2, V)} \right] \right)^n &= \liminf_{n \rightarrow \infty} e^{n \log(1 + \delta^2 D_n)} \quad \text{from (91)} \\ &= e^{\liminf_{n \rightarrow \infty} n \log(1 + \delta^2 D_n)} \\ &= e^{\delta^2 \liminf_{n \rightarrow \infty} n D_n \left(1 + \frac{o(D_n)}{D_n}\right)} \\ &= e^{\delta^2 \liminf_{n \rightarrow \infty} n D_n}, \end{aligned} \quad (97)$$

since $\lim_{n \rightarrow \infty} o(D_n)/D_n = 0$ from (96).

We will show that $\liminf_{n \rightarrow \infty} n D_n$ is finite and nonnegative in case (i) and $\liminf_{n \rightarrow \infty} n D_n = 0$ in case (ii).

Define

$$f_n(v) := -\frac{n \log \xi_2}{(\phi(n) - 2\delta e^{-\mu v})^2} \left(\frac{1}{4} + \frac{\Delta_2(\xi_2)}{\xi_2^2 \log \xi_2} \right) \quad (98)$$

$$= \frac{n (\log(\phi(n) - 2\delta e^{-\mu v}) - \log \delta + \mu v)}{(\phi(n) - 2\delta e^{-\mu v})^2} \left(\frac{1}{4} + \frac{\Delta_2(\xi_2)}{\xi_2^2 \log \xi_2} \right), \quad (99)$$

by using the definition of ξ_2 . Notice that

$$nD_n = - \int_0^\infty \lambda e^{-(\lambda+2\mu)v} f_n(v) dv$$

so that

$$\liminf_{n \rightarrow \infty} nD_n = - \limsup_{n \rightarrow \infty} \int_0^\infty \lambda e^{-(\lambda+2\mu)v} f_n(v) dv. \quad (100)$$

Observe that $\inf_{v \geq 0} f_n(v) \geq 0$ for n large enough since, from (92), $\frac{\Delta_2(\xi_2)}{\xi_2^2 \log \xi_2}$ can be made arbitrarily small by letting $n \rightarrow \infty$ and since, by assumption, $\phi(n) > 3$ for all $n \geq 1$, which implies that $\inf_{v \geq 0} \log(\phi(n) - 2\delta e^{-\mu v}) \geq 0$.

Case (i): $\liminf_n \frac{\phi(n)}{\sqrt{n \log n}} = a_{\text{inf}} > 0$, $\limsup_n \frac{\phi(n)}{\sqrt{n \log n}} < \infty$.

Define

$$L_2 := \frac{\lambda}{\lambda + 2\mu}, \quad L_3 := \frac{\lambda\mu}{(\lambda + 2\mu)^2}.$$

Fix $\epsilon \in (0, 1/4)$. Since $1/(\phi(n) - 2\delta e^{-\mu v})^2 < 1/(\phi(n) - 2\delta)^2$ and $0 < \phi(n) - 2\delta e^{-\mu v} < \phi(n)$ for all n and $v \geq 0$, we get from (99) and (92) that for n large enough (recall that $\inf_{v \geq 0} f_n(v) \geq 0$ when n is large enough)

$$0 \leq f_n(v) \leq \frac{n}{(\phi(n) - 2\delta)} (\log \phi(n) - \log \delta + \mu v) \left(\frac{1}{4} + \epsilon \right)$$

for all $v \geq 0$. Therefore for n large enough

$$\begin{aligned} 0 &\leq \int_0^\infty \lambda e^{-(\lambda+2\mu)v} f_n(v) dv \\ &\leq \frac{nL_2 (\log \phi(n) - \log \delta + L_3)}{(\phi(n) - 2\delta)^2} \left(\frac{1}{4} + \epsilon \right) \\ &= \frac{L_2(1/4 + \epsilon)}{\left(\frac{\phi(n)}{\sqrt{n \log n}} - \frac{2\delta}{\sqrt{n \log n}} \right)^2} \left(\frac{\log(\phi(n)/\sqrt{n \log n})}{\log n} + \frac{1}{2} + \frac{\log \log n}{2 \log n} + \frac{L_3 - \log \delta}{\log n} \right) \\ &\leq \sup_{k \geq n} \frac{L_2(1/4 + \epsilon)}{\left(\frac{\phi(k)}{\sqrt{k \log k}} - \frac{2\delta}{\sqrt{k \log k}} \right)^2} \left(\frac{\log(\phi(k)/\sqrt{k \log k})}{\log k} + \frac{1}{2} + \frac{\log \log k}{2 \log k} + \frac{L_3 - \log \delta}{\log k} \right) \\ &\leq \frac{L_2(1/4 + \epsilon)}{\inf_{k \geq n} \left(\frac{\phi(k)}{\sqrt{k \log k}} - \frac{2\delta}{\sqrt{k \log k}} \right)^2} \sup_{k \geq n} \left(\frac{\log(\phi(k)/\sqrt{k \log k})}{\log k} + \frac{1}{2} + \frac{\log \log k}{2 \log k} + \frac{L_3 - \log \delta}{\log k} \right) \\ &\rightarrow \frac{L_2(1/4 + \epsilon)}{a_{\text{inf}}^2} \times \frac{1}{2} < \infty \quad \text{as } n \rightarrow \infty, \end{aligned}$$

by using both properties of $\phi(n)$ in case (i). We have thus shown that

$$0 \leq \limsup_{n \rightarrow \infty} \int_0^\infty \lambda e^{-(\lambda+2\mu)v} f_n(v) dv \leq \frac{L_2(1/4 + \epsilon)}{2a_{\text{inf}}^2} < \infty.$$

The above and (100) show that $\liminf_{n \rightarrow \infty} nD_n < \infty$ and, by (97), $\liminf_n \left(\mathbb{E} \left[\sqrt{Z(\delta/\phi(n), X_2, V)} \right] \right)^n$ can be made arbitrarily small by letting $\delta \rightarrow 0$. We then obtain from (88) and the upper bound in Lemma 4.10 that $\lim_{n \rightarrow \infty} T_V(f_0^{\otimes n}, f_1^{\otimes n})$ can be made arbitrary small by letting $\delta \rightarrow 0$. By invoking now the upper bound in Lemma 4.10, we conclude that Alice is covert in case (i) when $T(n) = \mathcal{O}(\sqrt{n/\log n})$ and $r = 2$.

Case (ii): $\liminf_n \frac{\phi(n)}{\sqrt{n \log n}} = a_{\text{inf}} > 0$, $\limsup_n \frac{\phi(n)}{\sqrt{n \log n}} = \infty$.

We have

$$\sup_{k \geq n} \frac{\phi(k)}{\sqrt{k \log k}} \leq \frac{\sup_{k \geq n} \phi(k)}{\inf_{k \geq n} \sqrt{k \log k}} = \frac{\sup_{k \geq n} \phi(k)}{\sqrt{n \log n}}.$$

Taking the limit in both sides gives

$$\infty = \limsup_n \frac{\phi(n)}{\sqrt{n \log n}} \leq \frac{\limsup_n \phi(n)}{\lim_n \sqrt{n \log n}} = \frac{\lim_n \phi(n)}{\lim_n \sqrt{n \log n}} = \lim_{n \rightarrow \infty} \frac{\phi(n)}{\sqrt{n \log n}}, \quad (101)$$

where the 2nd equality holds since $\lim_n \phi(n)$ exists (and is equal to ∞). Let $L_4 := L_2(1/4 + \epsilon)$. The definition of $f_n(v)$ in (99) gives

$$\begin{aligned} 0 &\leq \int_0^\infty \lambda e^{-(\lambda+2\mu)v} f_n(v) dv \\ &\leq \frac{L_4}{\frac{\phi(n)}{\sqrt{n \log n}} - \frac{2\delta}{\sqrt{n \log n}}} \left(\frac{\log(\phi(n)/\sqrt{n \log n})}{\phi(n)/\sqrt{n \log n}} \cdot \frac{\phi(n)/\sqrt{n \log n}}{\frac{\phi(n)}{\sqrt{n \log n}} - \frac{2\delta}{\sqrt{n \log n}}} \cdot \frac{1}{\log n} \right) \\ &\quad + \frac{L_4}{\left(\frac{\phi(n)}{\sqrt{n \log n}} - \frac{2\delta}{\sqrt{n \log n}} \right)^2} \left(\frac{1}{2} + \frac{\log \log n}{2 \log n} + \frac{L_3 - \log \delta}{\log n} \right) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

by using $\lim_n \phi(n)/\sqrt{n \log n} = \infty$ (cf. (101)).

This shows from (97) and (100) that $\liminf_n \left(\mathbb{E} \left[\sqrt{Z(\delta/\phi(n), X_2, V)} \right] \right)^n = 0$ (note that, unlike in case (i), we do not need to let $\delta \rightarrow 0$ to establish this limit). We may then apply the argument used at the end case (i), which yields that Alice is covert in case (ii) when $T(n) = \mathcal{O}(\sqrt{n/\log n})$ and $r = 2$.

Proof of (15) for $\mu_1 > 2\mu_2$: Fix $r > 2$ so that $1 < \beta < 2$. Recall that $T(n) = \frac{\delta n}{\phi(n)}$, $\delta \in (0, 1]$, with $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Lemma 4.11

$$\begin{aligned} \left(\mathbb{E} \left[\sqrt{Z(\delta/\phi(n), X_r, V)} \right] \right)^n &= e^{n \log \left(1 + \int_0^\infty \lambda e^{-\lambda v} \left(-I_\beta \xi_r^\beta + \Delta_r(\xi_r) \right) dv \right)} \\ &= e^{n \log(1 + \delta^\beta (\beta-1)^\beta E_n)} \end{aligned} \quad (102)$$

with $\Delta_r(z) = o(z^\beta)$, $\xi_r = \frac{\delta(\beta-1)}{e^{\mu v} \phi(n) - \beta \delta}$, and

$$E_n := - \int_0^\infty \lambda e^{-\lambda v} \frac{I_\beta - \Delta_r(\xi_r)/\xi_r^\beta}{(e^{\mu \beta} \phi(n) - \delta \beta)^\beta} dv. \quad (103)$$

Let us show that $E_n \rightarrow 0$ as $n \rightarrow \infty$.

Fix $\epsilon > 0$. Since $\Delta_r(z) = o(z^\beta)$ when $r > 2$, there exists $z_\epsilon > 0$ such that for all $0 < z < z_\epsilon$, $\left| \frac{\Delta_r(z)}{z^\beta} \right| < \epsilon$. Since for all n such that $\frac{\delta(\beta-1)}{\phi(n)-\delta\beta} < z_\epsilon$ we have $\xi_r = \frac{\delta(\beta-1)}{e^{\mu v} \phi(n) - \delta\beta} < z_\epsilon$ for all $v \geq 0$ (Hint: the mapping $v \rightarrow \xi_r$ is nonincreasing in $[0, \infty)$ and $\xi_r = \frac{\delta(\beta-1)}{\phi(n)-\delta\beta}$ when $v = 0$), we conclude that for n large enough,

$$\sup_{v \geq 0} \left| \frac{\Delta_r(\xi_r)}{\xi_r^\beta} \right| < \epsilon. \quad (104)$$

Hence, for n large enough,

$$\begin{aligned} |E_n| &\leq (I_\beta + \epsilon) \int_0^\infty \frac{\lambda e^{-\lambda v}}{(\phi(n)e^{\mu v} - \delta\beta)^\beta} dv \\ &\leq \frac{I_\beta + \epsilon}{(\phi(n) - \delta\beta)^\beta} \int_0^\infty \lambda e^{-\lambda v} dv \\ &= \frac{I_\beta + \epsilon}{(\phi(n) - \delta\beta)^\beta} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Consequently, cf. (102),

$$\liminf_{n \rightarrow \infty} \left(\mathbb{E} \left[\sqrt{\delta/\phi(n), X_r, V} \right] \right)^n = e^{\delta^\beta (\beta-1)^\beta} \liminf_{n \rightarrow \infty} n E_n. \quad (105)$$

We have, cf. (103),

$$\liminf_n E_n = - \limsup_n \int_0^\infty \lambda e^{-\lambda v} k_n(v) dv. \quad (106)$$

with

$$k_n(v) := n \frac{I_\beta - \Delta_r(\xi_r)/\xi_r^\beta}{(e^{\mu\beta v} \phi(n) - \delta\beta)^\beta} \quad (107)$$

$$\begin{aligned} &= \frac{I_\beta - \Delta_r(\xi_r)/\xi_r^\beta}{(e^{\mu\beta v} \phi(n)/n^{1/\beta} - \delta\beta/n^{1/\beta})^\beta} \\ &\leq \frac{I_\beta - \Delta_r(\xi_r)/\xi_r^\beta}{(\phi(n)/n^{1/\beta} - \delta\beta/n^{1/\beta})^\beta}, \end{aligned} \quad (108)$$

for all $n \geq 1$ and $v \geq 0$. Recall that $I_\beta > 0$. Let $\epsilon < I_\beta$ in (104). From (108) we see that for n large enough

$$0 \leq k_n(v) \leq \frac{I_\beta + \epsilon}{(\phi(n)/n^{1/\beta} - \delta\beta/n^{1/\beta})^\beta} \quad \text{for all } v \geq 0. \quad (109)$$

$T(n) = \mathcal{O}(n^{1/r})$ or, equivalently since $T(n) = \delta n / \phi(n)$

$$\liminf_{n \rightarrow \infty} \phi(n)/n^{1/\beta} = b_{\text{inf}},$$

for some $b_{\text{inf}} > 0$. From (109) we obtain

$$\begin{aligned} 0 &\leq \limsup_n \int_0^\infty \lambda e^{-\lambda v} k_n(v) dv \\ &\leq \limsup_n \frac{I_\beta + \epsilon}{(\phi(n)/n^{1/\beta} - \delta\beta/n^{1/\beta})^\beta} \end{aligned}$$

$$= \frac{I_\beta + \epsilon}{(\liminf_{n \rightarrow \infty} \phi(n)/n^{1/\beta})^\beta} = \frac{I_\beta + \epsilon}{b_{\text{inf}}^\beta}. \quad (110)$$

This shows from (106) that

$$-\frac{I_\beta + \epsilon}{b_{\text{inf}}^\beta} \leq \liminf_{n \rightarrow \infty} E_n \leq 0.$$

Hence, by (105), $\liminf_{n \rightarrow \infty} \left(\mathbb{E} \left[\sqrt{\delta/\phi(n), X_r, V} \right] \right)^n$ can be made arbitrarily close to 1 by taking δ small enough. This in turn shows by (88) and the u.b. in Lemma 4.10 that $\lim_{n \rightarrow \infty} T_V(f_0^{\otimes n}, f_1^{\otimes n})$ can be made arbitrary small by letting $\delta \rightarrow 0$. By invoking now the upper bound in Lemma 4.10, we conclude that Alice is covert when $T(n) = \mathcal{O}(\sqrt{n^{1/r}})$ and $r > 2$.

Proof of (16) for $\mu_1 < 2\mu_2$: Assume that $T(n) = n/\phi(n) = \omega(\sqrt{n})$ or, equivalently, $\lim_{n \rightarrow \infty} n/\phi(n)^2 = \infty$.

Assume first that $0 < r < 1$. From (43) we obtain

$$\begin{aligned} \left(\mathbb{E} \left[\sqrt{\tilde{Z}(p/\phi(n), X_r)} \right] \right)^n &= e^{n \log \left(1 + \frac{p^2}{4} \left(\frac{1-r}{r-2} \right) / \phi(n)^2 + o(1/\phi(n)^2) \right)} \\ &\sim e^{\frac{n}{\phi(n)^2} \frac{p^2}{4} \left(\frac{1-r}{r-2} \right)} \quad \text{from (86)} \\ &\sim 0 \end{aligned} \quad (111)$$

when n is large, since $\frac{1-r}{r-2} < 0$ when $0 < r < 1$. This allows us to conclude from the lower bound in (34) in Lemma 4.10 that

$$\lim_n T_V \left(\tilde{f}_0^{\otimes n}, \tilde{f}_1^{\otimes n} \right) = 1, \quad (112)$$

thereby proving that Alice is not covert when $T(n) = \omega(\sqrt{n})$ and $0 < r < 1$, since (112) violates the covert criterion in (4).

It remains to show that Alice is not covert for $1 \leq r < 2$ when $T(n) = \omega(\sqrt{n})$ with $\lim_{n \rightarrow \infty} n/\phi(n)^2 = \infty$. Without any additional effort, we will prove the stronger result (to be used in the proof of the case $\mu_1 = 2\mu_2$ of (16)) that Alice is not covert when $T(n) = \omega(\sqrt{n})$ and $r \geq 1$. By applying Lemma D.1 in Appendix D to (38), we obtain

$$\mathbb{E} \left[\sqrt{\tilde{Z}(p/\phi(n), X_{r'})} \right] \leq \mathbb{E} \left[\sqrt{\tilde{Z}(p/\phi(n), X_r)} \right] \quad (113)$$

for any $r' \geq r$. Combining now (113) and (111) readily yields

$$\lim_{n \rightarrow \infty} \left(\mathbb{E} \left[\sqrt{\tilde{Z}(p/\phi(n), X_{r'})} \right] \right)^n = 0$$

for any $r' \geq 1$. Similarly to the case $0 < r < 1$ we then conclude that Alice is not covert $T(n) = \omega(\sqrt{n})$ and $r \geq 1$. In summary, we have shown that Alice is not covert for all $r > 0$ when $T(n) = \omega(\sqrt{n})$.

Proof of (16) for $\mu_1 = 2\mu_2$: Assume that $T(n) = n/\phi(n) = \omega(\sqrt{n/\log n})$ or, equivalently,

$$\lim_n \frac{\phi(n)}{\sqrt{n \log n}} = 0. \quad (114)$$

By (43),

$$\left(\mathbb{E} \left[\sqrt{\tilde{Z}(p/\phi(n), X_2)} \right] \right)^n = e^{n \log \left(1 + \frac{1}{4} \xi_2^2 \log \xi_2 + o(\xi_2^2 \log \xi_2) \right)}, \quad (115)$$

with $\xi_2 = \frac{p}{\phi(n)-2p}$. Since $\xi_2 \sim 0$ as $n \rightarrow \infty$ under (86), we have

$$\xi_2 \log \xi_2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, from (115),

$$\left(\mathbb{E} \left[\sqrt{\tilde{Z}(p/\phi(n), X_2)} \right] \right)^n \sim e^{\frac{n}{4}\xi_2^2 \log \xi_2} \quad \text{as } n \rightarrow \infty. \quad (116)$$

We have proved in the proof of the case $\mu_1 < 2\mu_2$ of (16) that Alice is not covert for all $r > 0$ when $T(n) = \omega(\sqrt{n})$. As a result, it is enough to focus on $T(n)$ satisfying (114) and such that $T(n) \neq \omega(\sqrt{n})$. The latter is equivalent to $\phi(n) = \Omega(\sqrt{n})$, that is,

$$\liminf_{n \rightarrow \infty} \frac{\phi(n)}{\sqrt{n}} > 0. \quad (117)$$

We have

$$\begin{aligned} n\xi_2^2 \log \xi_2 &= \frac{p^2}{\left(\frac{\phi(n)}{\sqrt{n} \log n} - \frac{2p}{\sqrt{n} \log n} \right)^2} \left(\frac{\log p}{\log n} - \frac{\log(\phi(n) - 2p)}{\log n} \right) \\ &\sim \frac{-p^2}{\left(\frac{\phi(n)}{\sqrt{n} \log n} - \frac{2p}{\sqrt{n} \log n} \right)^2} \times \frac{\log(\phi(n) - 2p)}{\log n}, \end{aligned} \quad (118)$$

as $n \rightarrow \infty$. By (114) the first factor in the r.h.s. of (118) converges to $-\infty$ as $n \rightarrow \infty$. Let us focus on the second factor. We have

$$\frac{\log(\phi(n) - 2p)}{\log n} = \frac{1}{2} + \frac{\log\left(\frac{\phi(n)-2p}{\sqrt{n}}\right)}{\log n} \sim \frac{1}{2} + \frac{\log\left(\frac{\phi(n)}{\sqrt{n}}\right)}{\log n}$$

as $n \rightarrow \infty$. Assumption (117) ensures that $\frac{\log\left(\frac{\phi(n)}{\sqrt{n}}\right)}{\log n} \rightarrow 0$ as $n \rightarrow \infty$ and

$$\frac{\log(\phi(n) - 2p)}{\log n} \rightarrow \frac{1}{2} \quad \text{as } n \rightarrow \infty.$$

In summary, we have shown that $n\xi_2^2 \log \xi_2 \rightarrow -\infty$ as $n \rightarrow \infty$ which, in turn, implies from (116) that $\lim_n \left(\mathbb{E} \left[\sqrt{\tilde{Z}(p/\phi(n), X_2)} \right] \right)^n = 0$. Invoking now the lower bound in (34) in Lemma 4.10 we may conclude that Alice is not covert if $r = 2$ and $T(n) = \omega(\sqrt{n/\log n})$.

Proof of (16) for $\mu_1 > 2\mu_2$: Assume that $T(n) = n/\phi(n) = \omega(n^{\mu_2/\mu_1})$ or, equivalently,

$$\lim_{n \rightarrow \infty} \frac{\phi(n)}{n^\beta} = 0. \quad (119)$$

Let $r > 2$ so that $\beta \in (1, 2)$. From (43),

$$\begin{aligned} \left(\mathbb{E} \left[\sqrt{\tilde{Z}(p/\phi(n), X_r)} \right] \right)^n &= e^{n \log(1 - I_\beta \xi_r^\beta + o(\xi_r^\beta))} \\ &\sim e^{-I_\beta n \xi_r^\beta} \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (120)$$

since $\xi_r = \frac{(\beta-1)p}{\phi(n)-\beta p} \rightarrow 0$ as $n \rightarrow \infty$ thanks to (86). We have

$$n\xi_r^\beta = \frac{(\beta-1)p^\beta}{\left(\frac{\phi(n)}{n^\beta} - \frac{\beta p}{n^\beta} \right)^\beta} \rightarrow +\infty \quad \text{as } n \rightarrow \infty.$$

Introducing the above limit in (120) and using the finiteness and positiveness of I_β for $\beta \in (1, 2)$, gives $\lim_n \left(\mathbb{E} \left[\sqrt{\tilde{Z}(p/\phi(n), X_r)} \right] \right)^n = 0$, which shows by using again the lower bound in (34) that Alice is not covert if $r > 2$ and $T(n) = \omega(n^{\mu_2/\mu_1})$.

This concludes the proof of the proposition.

D Appendix

Lemma D.1. *For any $\theta \in [0, 1]$, $r' \geq r$,*

$$\mathbb{E} \left[\sqrt{\Xi(\theta, X_{r'})} \right] \leq \mathbb{E} \left[\sqrt{\Xi(\theta, X_r)} \right].$$

Proof. Fix $\theta \in [0, 1]$. When $r' \geq r$, $X_{r'} \leq_{st} X_r$, which in turn implies that $\Xi(\theta, X_{r'}) \leq_{st} \Xi(\theta, X_r)$ as the mapping $x \rightarrow \Xi(\theta, x)$ in (36) is nondecreasing in $[0, \infty)$,

Therefore,

$$\mathbb{E} \left[\sqrt{\Xi(\theta, X_{r'})} \right] \leq \mathbb{E} \left[\sqrt{\Xi(\theta, X_r)} \right]$$

as the mapping $x \rightarrow \sqrt{x}$ is nondecreasing in $[0, \infty)$, and finally

$$\mathbb{E} \left[\sqrt{\Xi(\theta, X_{r'})} \right] \leq \mathbb{E} \left[\sqrt{\Xi(\theta, X_r)} \right].$$

■

E Proof of Lemma 5.1

Proof. We denote by $f^{(k)}$ the k th convolution of f with itself. For the time being we do not make any assumption on $g_2(x)$.

Let A be the event that Alice inserts a job at the end of a W-BP, with $\mathbb{P}(A) = q$. Given that Alice inserts a job at the end of a W-BP, let B_i ($i \geq 1$) be the event that Alice i th job inserted after the end of a W-BP affects Willie first job. Notice that

$$\mathbb{P}(B_i | A, V_* = t) = q^{i-1} \mathbb{P} \left(\sum_{l=1}^{i-1} \sigma_{2,l} < t < \sum_{l=1}^i \sigma_{2,l} \right), \quad i \geq 1.$$

Let us calculate $\mathbb{P}_{H_1}(Y_* < x, V_* < v)$. For the sake of simplicity we will drop the subscript H_1 . We have

$$\begin{aligned} \mathbb{P}(Y_* < x, V_* < v) &= \\ & q\mathbb{P}(Y_* < x, V_* < v | A) + \bar{q}\mathbb{P}(Y_* < x, V_* < v | A^c) \\ &= q \int_0^v P(Y_* < x | A, V_* = t) \lambda e^{-\lambda t} dt \\ & \quad + \bar{q}G_1(x)(1 - e^{-\lambda v}). \end{aligned} \tag{121}$$

Let us focus on $P(Y_* < x | A, V_* = t)$. We have

$$\mathbb{P}(Y_* < x | A, V_* = t) = \sum_{i \geq 1} P(\{Y_* < x\} \cap B_i | A, V_* = t)$$

$$\begin{aligned}
& +\mathbb{P}(\{Y_* < x\} \cap (\cup_{l \geq 1} B_l)^c) \\
= & P(t < \sigma_{2,1} < x + t - \sigma_1) \\
& + \sum_{i \geq 2} \int_{u=0}^t q^{i-1} P(t-u < \sigma_{2,i} < x+t-u+\sigma_1) \\
& \times g_2^{(i-1)}(u) du + G_1(x) \mathbb{P}((\cup_{l \geq 1} B_l)^c)
\end{aligned}$$

Let us find $\mathbb{P}((\cup_{l \geq 1} B_l)^c | A, V = t)$. This is the probability that no Alice job intersects with a Willie job given that Alice inserts a job at the end of a W-BP and that $V_* = t$. Given A and $V_* = t$, there is no interference if Alice inserts $i \geq 1$ jobs successfully and that she does not insert an $(i+1)$ -st job. The probability of this event is $\bar{q} q^{i-1} \mathbb{P}(\sigma_{2,1} + \dots + \sigma_{2,i} < t)$. Therefore,

$$\begin{aligned}
\mathbb{P}((\cup_{l \geq 1} B_l)^c | A, V = t) & = \\
& \bar{q} G_1(x) \sum_{l \geq 1} q^{l-1} \mathbb{P}(\sigma_{2,1} + \dots + \sigma_{2,l} < t) \\
& = \bar{q} G_1(x) \sum_{i \geq 1} q^{i-1} G_2^{(i)}(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}(Y < x | A, V = t) & = P(t < \sigma_{2,1} < x + t - \sigma_1) \\
& + \sum_{i \geq 2} \int_{u=0}^t q^{i-1} P(t-u < \sigma_{2,i} < x+t-u+\sigma_1) g_2^{*(i-1)}(u) du \\
& + \bar{q} G_1(x) \sum_{l \geq 1} q^{l-1} G_2^{*(l)}(t).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{P}(Y < x, V < v) & = q \int_0^v \lambda e^{-\lambda t} P(t < \sigma_{2,1} < x + t - \sigma_1) dt \\
& + \int_{t=0}^v \lambda e^{-\lambda t} \left[\int_{u=0}^t \mathbb{P}(t-u < \sigma_{2,i} < x+t-u-\sigma_1) \right. \\
& \times \sum_{i \geq 2} q^i g_2^{*(i-1)}(u) du + \bar{q} G_1(x) \sum_{l \geq 1} q^l G_2^{*(l)}(t) \left. \right] dt \\
& + \bar{q} G_1(x) (1 - e^{-\lambda v}),
\end{aligned}$$

which gives after conditioning on σ_1

$$\begin{aligned}
\mathbb{P}(Y < x, V < v) & = \\
& = q \int_{t=0}^v \lambda e^{-\lambda t} \int_{y=0}^x (G_2(x-y+t) - G_2(t)) g_1(y) dy dt \\
& + q \int_{t=0}^v \lambda e^{-\lambda t} \left[\int_{u=0}^t \sum_{i \geq 1} q^i g_2^{*(i)}(u) \right. \\
& \times \int_{y=0}^x (G_2(x-y+t-u) - G_2(t-u)) g_1(y) dy du \left. \right] dt
\end{aligned}$$

$$+\bar{q}G_1(x) \int_0^v \lambda e^{-\lambda t} \sum_{l \geq 1} q^l G_2^{*(l)}(t) dt + \bar{q}G_1(x)(1 - e^{-\lambda v}).$$

From now on we will assume that $G_2(t) = 1 - e^{-\mu_2 t}$ (Alice service times are exponential), which implies that

$$G_2^{*(l)}(t) = 1 - e^{-\mu_2 t} \sum_{m=0}^{l-1} \frac{(\mu_2 t)^m}{m!}, \quad l \geq 1.$$

Easy algebra gives (Hint: $g^{*(i)}(t) = \frac{d}{dt} G_2^{*(i)}(t)$)

$$\sum_{l \geq 1} q^l G_2^{*(l)}(t) = \frac{q}{\bar{q}} (1 - e^{-\mu_2 \bar{q} t}) \quad (122)$$

$$\sum_{i \geq 1} q^i g_2^{*(i)}(u) du = \mu_2 q e^{-\mu_2 \bar{q} u}. \quad (123)$$

Lengthy but easy algebra using (122)-(123) gives

$$\begin{aligned} \mathbb{P}(Y < x, V < v) &= (1 - e^{-\lambda v}) G_1(x) \\ &\quad - \frac{pq}{1 - \bar{p}q} \left(1 - e^{-(\lambda + \mu_2 \bar{q})v}\right) \frac{(g_1 * g_2)(x)}{\mu_2}. \end{aligned}$$

Again after easy algebra, we finally find

$$\begin{aligned} f_{+,1}(x, v) &= \frac{\partial^2}{\partial x \partial v} \mathbb{P}(Y < x, V < v) \\ &= \lambda e^{-\lambda v} g_1(x) \left[1 + q e^{-\mu_2 \bar{q} v} \left(\frac{(g_1 * g_2)(x)}{g_1(x)} - 1 \right) \right]. \end{aligned}$$

and

$$\begin{aligned} \tilde{f}_{+,1}(x) &= \int_0^\infty f_{+,1}(x, v) dv \\ &= g_1(x) \left[1 + \frac{pq}{1 - \bar{p}q} \left(\frac{(g_1 * g_2)(x)}{g_1(x)} - 1 \right) \right]. \end{aligned}$$

■

F Proof of Lemma 6.1

Under the II policy the queue behaves as an M/M/1 queue with an exceptional first customer. Let $\hat{\sigma}$ be the expected service time of this first customer and let $\hat{\tau}$ be the expected service time of the other customers. Then ([4] - see also Section 4),

$$\mathbb{E}[N_W] = \frac{1 - \lambda \hat{\sigma} + \lambda \hat{\tau}}{1 - \lambda \hat{\sigma}}. \quad (124)$$

$\hat{\tau}$ is the sum of the Willie's job expected service time (given by $1/\mu_1$) and of the expected time needed to serve all Alice's jobs inserted just after a Willie's job arrival. The latter quantity is given by qB/μ_2 . Hence, $\hat{\tau} = \frac{1}{\mu_1} + \frac{qB}{\mu_2}$. $\hat{\sigma}$ is the sum of Willie's job expected service time (given by $1/\mu_1$) and of the expected

time needed to serve all Alice's jobs present in the queue at the beginning of a W-BP. The probability that there are s such jobs is given by $\mathbb{P}(\mathcal{E}(s))$ in (134)-(135) in Appendix G. Therefore,

$$\hat{\sigma} = \frac{1}{\mu_1} + \frac{1}{\mu_2} \sum_{s \geq 1} \mathbb{P}(\mathcal{E}(s))s.$$

Elementary algebra then gives

$$\hat{\sigma} = \frac{1}{\mu_1} + \frac{qp}{\mu_2(1 - q\bar{p})} \left(\frac{1 - q(1 - \mathcal{G}_Q(\bar{p}))}{1 - q\bar{p}} + \frac{B - (1 + p)Q(1)}{p} \right).$$

Introducing $\hat{\tau}$ and $\hat{\sigma}$ into (124) gives (63).

During a W-BP, $qB\mathbb{E}[N_W]$ Alice's jobs are inserted on average. Therefore, $\mathbb{E}[N_A]$ is the sum of these jobs and of the expected number of jobs that Alice inserts during a W-IP. Let call $\mathbb{E}[N_{A,IP}]$ this number. Let κ be the number of Alice's jobs in the system at the beginning of a W-IP. Note that these jobs were inserted just after the arrival of the last Willie's job served in the previous W-BP, so that $\mathbb{P}(\kappa = k) = qQ(k)$ if $k \geq 1$ and $\mathbb{P}(\kappa = 0) = \bar{q} + qQ(0)$. If $\kappa = 0$ Alice inserts $i \geq 1$ jobs in a W-IP if either she inserts successfully i jobs and stops there (prob. $(q\bar{p})^i \bar{q}$) or if she inserts i jobs but the last one is not successful (prob. $(q\bar{p})^{i-1} qp$) giving the overall prob. $q(q\bar{p})^{i-1}(\bar{p}\bar{q} + p)$. Hence, the expected number of Alice's jobs inserted in a W-IP given that $\kappa = 0$ is $q(\bar{p}\bar{q} + p) \sum_{i \geq 1} (q\bar{p})^{i-1} i = q(\bar{p}\bar{q} + p)/(1 - q\bar{p})^2$. If $\kappa > 0$ Alice will not insert any job in a W-IP if a Willie's job arrives within the time to serve these κ jobs, the probability of this event being \bar{p}^κ and otherwise she will insert $i \geq 1$ jobs with the prob. $\bar{p}^\kappa q(\bar{p}\bar{q} + p)(q\bar{p})^{i-1}$. Hence, the expected number of Alice's jobs inserted in a W-IP given that $\kappa \geq 1$ is $q(\bar{p}\bar{q} + p)\bar{p}^\kappa \sum_{i \geq 1} (q\bar{p})^{i-1} i = q(\bar{p}\bar{q} + p)\bar{p}^\kappa/(1 - q\bar{p})^2$. Finally,

$$\begin{aligned} \mathbb{E}[N_{A,IP}] &= \frac{q(\bar{p}\bar{q} + p)}{(1 - q\bar{p})^2} \left(\bar{q} + qQ(0) + \sum_{k \geq 1} Q(k)\bar{p}^k \right) \\ &= \frac{q(\bar{p}\bar{q} + p)}{(1 - q\bar{p})^2} (\bar{q} + qQ(0)) + \mathcal{G}_Q(\bar{p}) - Q(0) \end{aligned}$$

and

$$\mathbb{E}[N_A] = qB\mathbb{E}[N_W] + q \frac{\bar{p}\bar{q} + p}{(1 - q\bar{p})^2} (\bar{q}\bar{Q}(0) + \mathcal{G}_Q(\bar{p})),$$

which concludes the proof.

G Proof of Lemma 6.2

Throughout the proof we will skip the subscript H_1 in $\mathbb{P}_{H_1}(Y < x)$ for the sake of conciseness. In this appendix U_λ denotes an exponential rv with rate λ .

Define the events

$$\begin{aligned} \mathcal{E}_s &= \{s \text{ Alice's jobs interfere with } J \text{ given } J = 1\} \\ \mathcal{F}_s &= \{s \text{ Alice's jobs interfere with } J \text{ given } J \neq 1\} \\ \mathcal{G}_l &= \{\text{Alice inserts } l \text{ jobs after the arrival of a Willie's job}\} \end{aligned}$$

for $s \geq 0, l \geq 0$. We have

$$\mathbb{P}(\mathcal{F}_0) = \bar{q} + qQ(0) = 1 - q\bar{Q}(0) \tag{125}$$

$$\mathbb{P}(\mathcal{F}_s) = qQ(s), \quad s \geq 1 \quad (126)$$

$$\mathbb{P}(\mathcal{G}_0) = \bar{q} + qQ(0) = 1 - q\bar{Q}(0) \quad (127)$$

$$\mathbb{P}(\mathcal{G}_l) = qQ(l), \quad l \geq 1. \quad (128)$$

Let T^- be the time at which a W-BP ends. Time T is the time at which Alice inserts one job with probability q and 0 job with probability \bar{q} is the system if empty at T^- . Let us determine $\mathbb{P}(\mathcal{E}_s)$ for $s \geq 0$. We have

$$\mathbb{P}(\mathcal{E}_0) = \mathbb{P}(\mathcal{E}_0 | \mathcal{G}_0)(1 - q\bar{Q}(0)) + q \sum_{l \geq 1} \mathbb{P}(\mathcal{E}_0 | \mathcal{G}_l)Q(l). \quad (129)$$

Recall that $\bar{p} = \frac{\mu_2}{\mu_2 + \lambda}$ is the probability that no Willie's job arrives during the service time of a Alice's job. Throughout, we will use that

$$P \left(U_\lambda > \sum_{r=1}^k \tau_r \right) = \bar{p}^k \quad (130)$$

when τ_1, τ_2, \dots are iid exponential rvs with rate μ_2 . Given \mathcal{G}_0 , there is no interference if Alice does not submit a job when an idle period starts (prob. \bar{q}) or if Alice submits one job (prob. q) and that during the service time of this Alice's job there is no arrival of a Willie job (prob. p) and Alice does not submit another job when the system becomes idle again (prob. \bar{q}), etc. This gives (same argument/result as in (58))

$$\mathbb{P}(\mathcal{E}_0 | \mathcal{G}_0) = \bar{q} + \bar{q}q\bar{p} + \bar{q}(q\bar{p})^2 + \bar{q}(q\bar{p})^3 + \dots = \bar{q} \sum_{i=0}^{\infty} (q\bar{p})^i = \frac{\bar{q}}{1 - q\bar{p}}. \quad (131)$$

For $l \geq 1$,

$$\begin{aligned} \mathbb{P}(\mathcal{E}_0 | \mathcal{G}_l) &= \bar{q}\mathbb{P} \left(U_\lambda > \sum_{r=1}^l \tau_r \right) + \bar{q}q\mathbb{P} \left(U_\lambda > \sum_{r=1}^{l+1} \tau_r \right) \\ &\quad + \bar{q}q^2\mathbb{P} \left(U_\lambda > \sum_{r=1}^{l+2} \tau_r \right) + \dots \\ &= \bar{q} \sum_{i=0}^{\infty} P \left(U_\lambda > \sum_{r=1}^{l+i} \tau_r \right) q^i = \bar{q}\bar{p}^l \sum_{i=0}^{\infty} (q\bar{p})^i \\ &= \frac{\bar{q}\bar{p}^l}{1 - q\bar{p}}. \end{aligned}$$

In summary,

$$\mathbb{P}(\mathcal{E}_0 | \mathcal{G}_l) = \frac{\bar{q}\bar{p}^l}{1 - q\bar{p}}, \quad \forall l \geq 0. \quad (132)$$

Therefore, from (129)-(132),

$$\begin{aligned} \mathbb{P}(\mathcal{E}_0) &= \frac{\bar{q}}{1 - q\bar{p}} \left(1 - q\bar{Q}(0) + q \sum_{l \geq 1} \bar{p}^l Q(l) \right) \\ &= \frac{\bar{q}}{1 - q\bar{p}} (1 - q\bar{Q}(0) + q\mathcal{G}_Q(\bar{p}) - qQ(0)) \\ &= \frac{\bar{q}}{1 - q\bar{p}} (\bar{q} + q\mathcal{G}_Q(\bar{p})). \end{aligned} \quad (133)$$

Consider now $\mathbb{P}(\mathcal{E}_s | \mathcal{G}_l)$ for $s \geq 1$. We will investigate separately the case $s = 1$ and $s \geq 2$. For $s = 1$, $l \geq 1$, we have

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_1 | \mathcal{G}_l) &= \mathbb{P}\left(\sum_{r=1}^{l-1} \tau_r < U_\lambda < \sum_{r=1}^l \tau_r\right) + q\mathbb{P}\left(\sum_{r=1}^l \tau_r < U_\lambda < \sum_{r=1}^{l+1} \tau_r\right) \\
&\quad + q^2\mathbb{P}\left(\sum_{r=1}^{l+1} \tau_r < U_\lambda < \sum_{r=1}^{l+2} \tau_r\right) + \dots \\
&= \sum_{i \geq 0} q^i \mathbb{P}\left(\sum_{r=1}^{l+i-1} \tau_r < U_\lambda < \sum_{r=1}^{l+i} \tau_r\right) \\
&= \sum_{i \geq 0} q^i \mathbb{P}\left(\sum_{r=1}^{l+i} \tau_r > U_\lambda\right) - \sum_{i \geq 0} q^i \mathbb{P}\left(\sum_{r=1}^{l+i-1} \tau_r > U_\lambda\right) \\
&= \sum_{i \geq 0} q^i (1 - \bar{p}^{l+i}) - \sum_{i \geq 0} q^i (1 - \bar{p}^{l+i-1}) \\
&= \bar{p}^{l-1} \sum_{i \geq 0} (q\bar{p})^i - \bar{p}^l \sum_{i \geq 0} (q\bar{p})^i \\
&= \frac{\bar{p}\bar{p}^{l-1}}{1 - q\bar{p}}.
\end{aligned}$$

For $s = 1$ and $l = 0$, then

$$\mathbb{P}(\mathcal{E}_1 | \mathcal{G}_0) = 1 - \mathbb{P}(\mathcal{E}_0 | \mathcal{G}_0) = \frac{qp}{1 - q\bar{p}}.$$

We then obtain

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_1) &= \mathbb{P}(\mathcal{E}_1 | \mathcal{G}_0)\mathbb{P}(\mathcal{G}_0) + \sum_{l \geq 1} \mathbb{P}(\mathcal{E}_1 | \mathcal{G}_l)\mathbb{P}(\mathcal{G}_l) \\
&= \frac{qp}{1 - q\bar{p}}(1 - q\bar{Q}(0)) + \sum_{l \geq 1} \frac{\bar{p}\bar{p}^{l-1}}{1 - q\bar{p}} qQ(l) \\
&= \frac{qp}{1 - q\bar{p}} \left(1 - q\bar{Q}(0) + \frac{\mathcal{G}(Q)(\bar{p})}{\bar{p}} - \frac{Q(0)}{\bar{p}}\right). \tag{134}
\end{aligned}$$

Assume now that $s \geq 2$. There can be two interferences or more only if Willie interferes with jobs in the system at time T^- . Therefore,

$$\mathbb{P}(\mathcal{E}_s | \mathcal{G}_l) = 0 \quad \text{if } 0 \leq l < s$$

and, for $l \geq s$,

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_s | \mathcal{G}_l) &= \mathbb{P}\left(\sum_{r=1}^{l-s} \tau_r < U_\lambda < \sum_{r=1}^{l-s+1} \tau_r\right) \\
&= \mathbb{P}\left(U_\lambda < \sum_{r=1}^{l-s+1} \tau_r\right) - \mathbb{P}\left(U_\lambda < \sum_{r=1}^{l-s} \tau_r\right) \\
&= 1 - \bar{p}^{l-s+1} - (1 - \bar{p}^{l-s}) = \bar{p}^{l-s} p.
\end{aligned}$$

We then obtain

$$\mathbb{P}(\mathcal{E}_s) = \sum_{l \geq s} \mathbb{P}(\mathcal{E}_s | \mathcal{G}_l)\mathbb{P}(\mathcal{G}_l) = \frac{qp}{\bar{p}^s} \sum_{l=s}^{\infty} Q(l)\bar{p}^l, \quad \forall s \geq 2. \tag{135}$$

Under H_1 , the cdf of Y , Willie job reconstructed service time, is given by

$$\begin{aligned}
& \mathbb{P}(Y < x) \\
&= \mathbb{P}(Y < x \mid J = 1)\pi_J + \mathbb{P}(Y < x \mid J \neq 1)\bar{\pi}_J \\
&= \mathbb{P}(Y < x \mid J = 1, \mathcal{E}_0)\mathbb{P}(\mathcal{E}_0)\pi_J + \sum_{s \geq 1} \mathbb{P}(Y < x \mid J = 1, \mathcal{E}_s)\mathbb{P}(\mathcal{E}_s)\bar{\pi}_J \\
&\quad + \mathbb{P}(Y < x \mid J \neq 1, \mathcal{F}_0)\mathbb{P}(\mathcal{F}_0)\bar{\pi}_J + \sum_{s \geq 1} \mathbb{P}(Y < x \mid J \neq 1, \mathcal{F}_s)\mathbb{P}(\mathcal{F}_s)\bar{\pi}_J \\
&= \mathbb{P}(\sigma < x) (\mathbb{P}(\mathcal{E}_0)\pi_J + \mathbb{P}(\mathcal{F}_0)\bar{\pi}_J) + \sum_{s \geq 1} \mathbb{P}\left(\sigma + \sum_{r=1}^s \tau_r < x\right) (\mathbb{P}(\mathcal{E}_s)\pi_J + \mathbb{P}(\mathcal{F}_s)\bar{\pi}_J) \\
&= G_1(x) (\mathbb{P}(\mathcal{E}_0)\pi_J + \mathbb{P}(\mathcal{F}_0)\bar{\pi}_J) + \sum_{s \geq 1} (G_1 * h_s)(x) (\mathbb{P}(\mathcal{E}_s)\pi_J + \mathbb{P}(\mathcal{F}_s)\bar{\pi}_J).
\end{aligned}$$

From (125), (126), (133), (134), and (135) we find

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_0)\pi_J + \mathbb{P}(\mathcal{F}_0)\bar{\pi}_J &= \frac{\bar{q}}{1 - q\bar{p}}(\bar{q} + q\mathcal{G}_Q(\bar{p}))\pi_J + (1 - q\bar{Q}(0))\bar{\pi}_J \\
\mathbb{P}(\mathcal{E}_1)\pi_J + \mathbb{P}(\mathcal{F}_1)\bar{\pi}_J &= \frac{qp}{1 - q\bar{p}}\left(1 - q\bar{Q}(0) + \frac{\mathcal{G}_Q(\bar{p}) - Q(0)}{\bar{p}}\right)\pi_J \\
&\quad + qQ(1)\bar{\pi}_J \\
\mathbb{P}(\mathcal{E}_s)\pi_J + \mathbb{P}(\mathcal{F}_s)\bar{\pi}_J &= qp\pi_J \sum_{l \geq s} Q(l)\bar{p}^{l-s} + qQ(s)\bar{\pi}_J, \quad \forall s \geq 2.
\end{aligned}$$

Hence, for $x \geq 0$,

$$\begin{aligned}
\mathbb{P}(y < x) &= G_1(x) \left(\frac{\bar{q}}{1 - q\bar{p}}(\bar{q} + q\mathcal{G}_Q(\bar{p}))\pi_J + (1 - q\bar{Q}(0))\bar{\pi}_J \right) + q(G_1 * h_1)(x) \\
&\quad \times \left(\frac{p}{1 - q\bar{p}}\left(1 - q\bar{Q}(0) + \frac{\mathcal{G}_Q(\bar{p}) - Q(0)}{\bar{p}}\right)\pi_J + Q(1)\bar{\pi}_J \right) \\
&\quad + q \sum_{s \geq 2} (G_1 * h_s)(x) \left(p\pi_J \sum_{l \geq s} Q(l)\bar{p}^{l-s} + Q(s)\bar{\pi}_J \right). \tag{136}
\end{aligned}$$

Dividing both sides of (136) by $g_1(x)$ gives (70).

H Proof of Lemma 6.3

For any mapping $h(q)$, we denote by $h'(q)$ its 1st derivative and by $h''(q)$ its 2nd derivative at q when they do exist.

Define

$$F(q) = \mathbb{E} \left[\sqrt{W(q, X)} \right] = \int_0^\infty \sqrt{g_1(x)} \sqrt{g_1(x)W(q, x)} dx. \tag{137}$$

Notice that, by Cauchy-Schwarz inequality,

$$0 \leq F(q) \leq \left(\int_0^\infty g_1(x) dx \right)^{1/2} \left(\int_0^\infty g_1(x)W(q, x) dx \right)^{1/2} = 1,$$

since $g_1(x)W(q, x)$ is the density of a nonnegative rv, which implies that $\int_0^\infty g_1(x)W(q, x) dx = 1$.

Define $f(q, x) = \sqrt{\Delta_1(q) + \Delta_2(q)\Phi_1(x) + q\Phi_2(x)}$, so that (cf. (137))

$$F(q) = \int_0^\infty \mu_1 e^{-\mu_1 x} f(q, x) dx.$$

For later use, note that

$$\frac{df(q, x)}{dq} = \frac{\Delta_1'(q) + \Delta_2'(q)\Phi_1(x) + \Phi_2(x)}{2f(q, x)} \quad (138)$$

$$\frac{d^2 f(q, x)}{dq^2} = \frac{\Delta_1''(q) + \Delta_2''(q)\Phi_1(x)}{2f(q, x)} - \frac{(\Delta_1'(q) + \Delta_2'(q)\Phi_1(x) + \Phi_2(x))^2}{4f(q, x)^3}. \quad (139)$$

Assume that there exist $0 < q < \min\left\{1, \frac{1-\rho_1}{\rho_2 B}\right\}$ (recall that the queue is stable when $\rho_1 + q\rho_2 B < 1$ – see Lemma 6.1) and three non-negative mappings h_i , $i = 0, 1, 2$, satisfying $\int_0^\infty e^{-\mu_1 x} h_i(x) dx < \infty$, $i = 0, 1, 2$, such that

1. for all $x \geq 0$, $q \rightarrow f(q, x)$ is continuous in $[0, q_1]$;
2. for all $q \in [0, q_1]$, $x \rightarrow f(q, x)$ is continuous in $[0, \infty)$;
3. for all $(q, x) \in [0, q_1] \times [0, \infty)$, $|f(q, x)| \leq h_0(x)$;
4. for all $x \geq 0$, $q \rightarrow \frac{df(q, x)}{dq}$ is continuous $q \in [0, q_1]$;
5. for all $q \in [0, q_1]$, $q \rightarrow \frac{df(q, x)}{dq}$ is continuous in $[0, \infty)$;
6. for all $(q, x) \in [0, q_1] \times [0, \infty)$, $\left|\frac{df(q, x)}{dq}\right| \leq h_1(x)$;
7. for all $x \geq 0$, $q \rightarrow \frac{d^2 f(q, x)}{dq^2}$ is continuous in $[0, q_1]$;
8. for all $q \in [0, q_1]$, $q \rightarrow \frac{d^2 f(q, x)}{dq^2}$ is continuous in $[0, \infty)$;
9. for all $(q, x) \in [0, q_1] \times [0, \infty)$, $\left|\frac{d^2 f(q, x)}{dq^2}\right| \leq h_2(x)$.

Then, by the Dominated Convergence Theorem, (1)-(9) will ensure that $F(q)$ is twice differentiable in $[0, q_1]$.

When $g_1(x) = \mu_1 e^{-\mu_1 x}$,

$$\begin{aligned} \frac{g_1 * h_s(x)}{g_1(x)} &= \frac{\mu_2^s}{(s-1)!} \int_0^x \mu_2^s (x-t)^{s-1} e^{-(\mu_1 - \mu_2)(x-t)} dt \\ &= \frac{\mu_2^s}{(s-1)!} \int_0^x \mu_2^s u^{s-1} e^{-(\mu_1 - \mu_2)u} du, \end{aligned} \quad (140)$$

which, for each $s \geq 1$, is continuous in $[0, \infty)$. We may then apply the Beppo Levi Monotone Convergence Theorem to $\Phi_2(x)$ (as both sums in $\Phi_2(x)$ have non-negative terms) to get that the mapping $x \rightarrow \Phi_2(x)$ is continuous on $[0, \infty)$, and so is the mapping $x \rightarrow \Phi_1(x)$.

On the other hand, it is easily from the definitions of $\Delta_1(q)$ and $\Delta_2(q)$ that

$$x \rightarrow \{\Delta_1(q), \Delta_1'(q), \Delta_1''(q), \Delta_2(q), \Delta_2'(q), \Delta_2''(q)\}$$

are all continuous mappings in $[0, 1]$. This shows (1) and (2); this will also show (4), (5), (7) and (8) if we can show that there exists $q_2 \in (0, q_1)$ such that $f(q, x) \neq 0$ for all $q \in [0, q_2]$, $x \geq 0$. From the definition of $f(q, x)$, we see that

$$f(q, x) \geq \Delta_1(q) \geq \bar{q}(\bar{q} + q\mathcal{G}_Q(\bar{p}))\pi_J, \quad (141)$$

for all $q \in [0, 1]$, $x \geq 0$. It is easily seen that the mapping $q \rightarrow \zeta(q) := \bar{q}(\bar{q} + q\mathcal{G}_Q(\bar{p}))\pi_J$ is strictly decreasing in $[0, 1]$ with $\zeta(0) = 1$ and $\zeta(1) = 0$. Pick $\delta \in (0, 1)$. The above implies that there exists $q_2 \in (0, q_1)$ such that $(\bar{q} + q\mathcal{G}_Q(\bar{p}))\pi_J \geq \delta$ for all $q \in [0, q_2]$, which in turn implies that

$$f(q, x) \geq \delta > 0, \quad (142)$$

for all $q \in [0, q_2]$, $x \geq 0$. This establishes the validity of (4), (5), (7) and (8).

We are left with proving (3), (6), and (9). For $q \in [0, q_1]$, $x \geq 0$, we have

$$f(q, x) \leq 1 + \eta_1 + \eta_2\Phi_1(x) + \Phi_2(x) := k_1(x) \quad (143)$$

$$\left| \frac{df(q, x)}{dq} \right| \leq \frac{1}{2\delta}(\eta_3 + \eta_4\Phi_1(x) + \Phi_2(x)) := k_2(x) \quad (144)$$

$$\left| \frac{d^2f(q, x)}{dq^2} \right| \leq \frac{1}{2\delta}(\eta_5 + \eta_6\Phi_1(x)) + \frac{1}{4\delta^3}(\eta_3 + \eta_4\Phi_1(x) + \Phi_2(x))^2 := k_3(x), \quad (145)$$

where

$$\begin{aligned} \eta_1 &:= \sup_{q \in [0, q_1]} |\Delta_1(q)|, & \eta_2 &:= \sup_{q \in [0, q_1]} |\Delta_1'(q)| \\ \eta_3 &:= \sup_{q \in [0, q_1]} |\Delta_1''(q)|, & \eta_4 &:= \sup_{q \in [0, q_1]} |\Delta_2(q)| \\ \eta_5 &:= \sup_{q \in [0, q_1]} |\Delta_2'(q)|, & \eta_6 &:= \sup_{q \in [0, q_1]} |\Delta_2''(q)|. \end{aligned}$$

The constants η_i , $i = 1, \dots, 6$ are all finite as the mappings

$$q \rightarrow \{\Delta_1(q), \Delta_1'(q), \Delta_1''(q), \Delta_2(q), \Delta_2'(q), \Delta_2''(q)\}$$

are all continuous in $[0, q_1]$ from the very definition of $\Delta_1(q)$ and $\Delta_2(q)$ in (71)-(72).

By Beppo Levi Monotone Convergence Theorem (which applies here as all terms in the sums in (74) are non-negative as already noticed) we get

$$\begin{aligned} \int_0^\infty g_1(x)\Phi_2(x)dx &= \sum_{s \geq 2} \sum_{l \geq s} Q(l)p^{l-s} \int_0^\infty g_1 * h_s(x)dx \\ &= \sum_{s \geq 2} \sum_{l \geq s} Q(l)p^{l-s} < \infty, \end{aligned}$$

where the finiteness of $\sum_{s \geq 2} \sum_{l \geq s} Q(l)p^{l-s}$ is shown in Lemma I.1 in Appendix I. This shows that $\int_0^\infty h_i(x)dx < \infty$, $i = 1, 2$, where $h_1(x)$ and $h_2(x)$ are defined in (143)-(144), and proves the validity of (3) and (6).

It is shown in Lemma I.3 in Appendix I that when $\mu_1 < 2\mu_2$

$$\int_0^\infty \mu_1^{-\mu_1 x} (\eta_4\Phi_1(x) + \Phi_2(x))^2 dx < \infty,$$

which implies together with the finiteness of $\int_0^\infty g_1(x)\Phi_i(x)dx$, $i = 1, 2$, that $\int_0^\infty \mu_1 e^{-\mu_1 x} h_3(x)dx$, where $h_3(x)$ is defined in (145). This proves (9) when $\mu_1 < 2\mu_2$.

We have therefore shown that there exists $q_1 \in (0, q_0)$ such $F(q)$ is twice differentiable in $[0, q_1)$ when $\mu_1 < 2\mu_2$. Application of Leibniz's differentiation rule gives

$$\begin{aligned} F(0) &= 1 \\ F'(0) &= \frac{1}{2} \int_0^\infty \mu_1 e^{-\mu_1 x} (\alpha + \beta \Phi_1(x) + \Phi_2(x)) dx \end{aligned} \quad (146)$$

$$\begin{aligned} F''(0) &= -\frac{1}{4} \int_0^\infty \mu_1 e^{-\mu_1 x} (\alpha + \beta \Phi_1(x) + \Phi_2(x))^2 dx \\ &\quad + (p - \mathcal{G}_Q(\bar{p})) \left(p\pi_J - \int_0^\infty \mu_1 e^{-\mu_1 x} \Phi_1(x) dx \right) \\ &= -\frac{1}{4} \int_0^\infty \mu_1 e^{-\mu_1 x} (\alpha + \beta \Phi_1(x) + \Phi_2(x))^2 dx \end{aligned} \quad (147)$$

where

$$\alpha := -(p + \overline{\mathcal{G}_Q}(\bar{p}))\pi_J - \overline{Q}(0)\bar{\pi}_J \quad (148)$$

$$\beta := 1 + \frac{\mathcal{G}_Q(\bar{p}) - Q(0)}{\bar{p}}. \quad (149)$$

Note that (147) holds since $\int_0^\infty \mu_1 e^{-\mu_1 x} \Phi_1(x) dx = p\pi_J \int_0^\infty g_1 * h_s(x) dx = p\pi_J$ from the definition of $\Phi_1(x)$ in (73), and since $g_1 * h_s$ is a pdf on $[0, \infty)$.

It is shown in Lemma I.2 in Appendix I that $F'(0) = 0$. Hence, by Taylor's Theorem,

$$F(q) = 1 + c_0 q^2 + o(q^2), \quad (150)$$

with $c_0 := \frac{1}{2}F''(0) < \infty$ when $\mu_1 < 2\mu_2$, which completes the proof.

I Appendix

Lemma I.1.

$$\sum_{s \geq 2} \sum_{l \geq s} Q(l) \bar{p}^{l-s} = \frac{1 - \bar{p} \mathcal{G}_Q(\bar{p})}{p} - \frac{\mathcal{G}_Q(\bar{p})(1 + \bar{p}) - Q(0)}{\bar{p}}.$$

Proof.

$$\begin{aligned} \sum_{s \geq 2} \sum_{l \geq s} Q(l) \bar{p}^{l-s} &= \sum_{s \geq 0} \sum_{l \geq s} Q(l) \bar{p}^{l-s} - \sum_{l \geq 1} Q(l) \bar{p}^{l-1} - \sum_{l \geq 0} Q(l) \bar{p}^l \\ &= \sum_{s \geq 0} \sum_{l \geq s} Q(l) \bar{p}^{l-s} - \frac{\mathcal{G}_Q(\bar{p}) - Q(0)}{\bar{p}} - \mathcal{G}_Q(\bar{p}) \\ &= \sum_{l \geq 0} Q(l) \bar{p}^l \sum_{s=0}^l \left(\frac{1}{\bar{p}} \right)^s - \frac{\mathcal{G}_Q(\bar{p})(1 + \bar{p}) - Q(0)}{\bar{p}} \\ &= \sum_{l \geq 0} Q(l) \bar{p}^l \left(\frac{1 - \bar{p}^{-(l+1)}}{1 - \bar{p}^{-1}} \right) - \frac{\mathcal{G}_Q(\bar{p})(1 + \bar{p}) - Q(0)}{\bar{p}} \\ &= -\frac{1}{\bar{p}} \sum_{l \geq 0} Q(l) \bar{p}^l (1 - \bar{p}^{-1}) - \frac{\mathcal{G}_Q(\bar{p})(1 + \bar{p}) - Q(0)}{\bar{p}} \end{aligned}$$

$$= \frac{1 - \bar{p}\mathcal{G}_Q(\bar{p})}{p} - \frac{\mathcal{G}_Q(\bar{p})(1 + \bar{p}) - Q(0)}{\bar{p}}.$$

■

Lemma I.2.

$$F'(0) = 0,$$

where $F'(0)$ is given in (146).

Proof. The definitions of $\Phi_1(x)$ and $\Phi_2(x)$ in (73)-(74) yield

$$\int_0^\infty g_1(x)\Phi_1(x)dx = p\pi_j \int_0^\infty g_1 * h_1(x)dx = p\pi_j, \quad (151)$$

with $g_1(x) = \mu_1 e^{-\mu_1 x}$, and

$$\begin{aligned} & \int_0^\infty \mu_1 e^{-\mu_1 x} \Phi_2(x) dx \\ &= p\pi_J \int_0^\infty \sum_{s \geq 2} g_1 * h_s(x) \sum_{l \geq s} Q(l) \bar{p}^{l-s} dx + \bar{\pi}_J \int_0^\infty \sum_{s \geq 1} g_1 * h_s(x) Q(x) dx \\ &= p\pi_J \sum_{s \geq 2} \left(\int_0^\infty g_1 * h_s(x) dx \right) \sum_{l \geq s} Q(l) \bar{p}^{l-s} + \bar{\pi}_J \sum_{s \geq 1} \int_0^\infty g_1 * h_s(x) Q(s) dx \end{aligned} \quad (152)$$

$$= p\pi_J \sum_{s \geq 2} \sum_{l \geq s} Q(l) \bar{p}^{l-s} + \bar{\pi}_J (1 - Q(0)), \quad (153)$$

since $\int_0^\infty g_1 * h_s(x) dx = 1$ and $\sum_{s \geq 0} Q(s) = 1$, where the interchange of the integrals and sums in (152) is justified by the Beppo Levi's Monotone Convergence Theorem. Thus, by using (146), the definition of α and β in (148)-(149), (151), (153), and Lemma I.1, we obtain

$$\begin{aligned} 2F'(0) &= -(p + \bar{\mathcal{G}}_Q(\bar{p}))\pi_J - \bar{Q}(0)\bar{\pi}_J + \left(1 + \frac{\mathcal{G}_Q(\bar{p}) - Q(0)}{\bar{p}}\right) p\pi_J \\ &\quad + \pi_J(1 - \bar{p}\mathcal{G}_Q(\bar{p})) - \frac{\mathcal{G}_Q(\bar{p})(1 + \bar{p}) - Q(0)}{\bar{p}} p\pi_J + \bar{\pi}_J(1 - Q(0)) \\ &= Q(0) \left(\bar{\pi}_J - \frac{p\bar{\pi}_J}{\bar{p}} + \frac{p\bar{\pi}_J}{\bar{p}} - \bar{\pi}_J \right) + \mathcal{G}_Q(\bar{p}) \left(\pi_J + \frac{p\pi_J}{\bar{p}} - \bar{p}\pi_J - \frac{1 + \bar{p}}{\bar{p}} p\pi_J \right) = 0. \end{aligned}$$

■

Lemma I.3. Assume that the support of Q_B is finite. Then,

$$\int_0^\infty \mu_1 e^{-\mu_1 x} (\eta_4 \Phi_1(x) + \Phi_2(x))^2 dx < \infty$$

if $\mu_1 < 2\mu_2$.

Proof. Without any loss of generality, assume that the support of Q_B is in $\{0, 1, \dots, S\}$ with $S \leq 1$. From (73)-(74) we obtain

$$\begin{aligned} & (\eta_1 \Phi_1(x) + \Phi_2(x))^2 \leq \max(\eta_4^2, 1) (\Phi_1(x) + \Phi_2(x))^2 \\ & \leq \max(\eta_4^2, 1) \left(\sum_{s=1}^S \frac{g_1 * h_s(x)}{g_1(x)} \sum_{l=s}^S Q(l) \bar{p}^{l-s} + \bar{\pi}_J \sum_{s=1}^S \frac{g_1 * h_s(x)}{g_1(x)} Q(s) + \pi_J \frac{g_1 * h_1(x)}{g_1(x)} \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq \max(\eta_4^2, 1) \left(\sum_{s=1}^S \frac{g_1 * h_s(x)}{g_1(x)} \left[Q(s) + \sum_{l=s}^S Q(l) \bar{p}^{l-s} \right] \right)^2 \\
&\leq \max(\eta_4^2, 1) (S+1)^2 \left(\sum_{s=1}^S \frac{g_1 * h_s(x)}{g_1(x)} \right)^2 \\
&= \max(\eta_4^2, 1) (S+1)^2 \sum_{i=1}^S \sum_{j=1}^S \frac{g_1 * h_i(x)}{g_1(x)} \frac{g_1 * h_j(x)}{g_1(x)}, \tag{154}
\end{aligned}$$

by using the identity $(\sum_i a_i)^2 = \sum_i \sum_j a_i a_j$. With $g_i(x) = \mu_i e^{-\mu_i x}$ for $i = 1, 2$ it is easily seen that

$$\begin{aligned}
\frac{g_1 * h_s(x)}{g_1(x)} &= \frac{\mu_2^s}{(s-1)!} \int_0^x t^{s-1} e^{-(\mu_2 - \mu_1)t} dt \\
&= \begin{cases} \frac{(\mu_1 x)^s}{s!} & \text{if } \mu_1 = \mu_2 \\ \frac{1}{(s-1)!} \left(\frac{\mu_2}{(\mu_2 - \mu_1)} \right)^s - \mu_2^s e^{-(\mu_2 - \mu_1)x} \\ \quad \times \sum_{i=0}^{s-1} \frac{1}{(\mu_2 - \mu_1)^{i+1}} \frac{x^{s-1-i}}{(s-1-i)!} & \text{if } \mu_1 \neq \mu_2. \end{cases} \tag{155}
\end{aligned}$$

A glance at (154) and (155) shows that the r.h.s. of (154) is a finite sum of terms of the form x^k , $e^{-(\mu_2 - \mu_1)x} x^k$, and $e^{-2(\mu_2 - \mu_1)x} x^k$. Now, since integrals

$$\int_0^\infty \mu_1 e^{-\mu_1 x} x^k dx = \frac{k!}{\mu_1^k}$$

and

$$\int_0^\infty \mu_1 e^{-\mu_1 x} e^{-(\mu_2 - \mu_1)x} x^k dx = \frac{\mu_1 k!}{\mu_2^{k+1}}$$

are finite, and the integral

$$\int_0^\infty \mu_1 e^{-\mu_1 x} e^{-2(\mu_2 - \mu_1)x} x^k dx = \frac{\mu_1 k!}{(2\mu_2 - \mu_1)^{k+1}}$$

is finite for $\mu_1 < 2\mu_2$, the lemma is proved. ■