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# Cooperative colorings of trees and of bipartite graphs

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## Abstract

Given a system  $(G_1, \dots, G_m)$  of graphs on the same vertex set  $V$ , a cooperative coloring is a choice of vertex sets  $I_1, \dots, I_m$ , such that  $I_j$  is independent in  $G_j$  and  $\bigcup_{j=1}^m I_j = V$ . For a class  $\mathcal{G}$  of graphs, let  $m_{\mathcal{G}}(d)$  be the minimal  $m$  such that every  $m$  graphs from  $\mathcal{G}$  with maximum degree  $d$  have a cooperative coloring. We prove that  $\Omega(\log \log d) \leq m_{\mathcal{T}}(d) \leq O(\log d)$  and  $\Omega(\log d) \leq m_{\mathcal{B}}(d) \leq O(d/\log d)$ , where  $\mathcal{T}$  is the class of trees and  $\mathcal{B}$  is the class of bipartite graphs.

**Mathematics Subject Classifications:** 05C15, 05C69

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# 1 Introduction

A set of vertices in a graph is called *independent* if no two vertices in it form an edge. A *coloring* of a graph  $G$  is a covering of  $V(G)$  by independent sets. Given a system  $(G_1, \dots, G_m)$  of graphs on the same vertex set  $V$ , a *cooperative coloring* is a choice of vertex sets  $\{I_j \subseteq V : j \in [m]\}$  such that  $I_j$  is independent in  $G_j$  and  $\bigcup_{j=1}^m I_j = V$ . If all  $G_j$ 's are the same graph  $G$ , then a cooperative coloring is just a proper vertex coloring of  $G$  by  $m$  independent sets.

A basic fact about vertex coloring is that every graph  $G$  of maximum degree  $d$  is  $(d+1)$ -colorable. It is therefore natural to ask whether  $d+1$  graphs, each of maximum degree  $d$ , always have a cooperative coloring. This was shown to be false:

**Theorem 1** (Theorem 5.1 of Aharoni, Holzman, Howard and Sprüssel [AHHS15]). *For every  $d \geq 2$ , there exist  $d+1$  graphs of maximum degree  $d$  that do not have a cooperative coloring.*

Using the fundamental result on independent transversals of Haxell [Hax01, Theorem 2], it can be shown that  $2d$  graphs of maximum degree  $d$  always have a cooperative coloring. Let  $m(d)$  be the minimal  $m$  such that every  $m$  graphs of maximum degree  $d$  have a cooperative coloring. By the above,  $m(1) = 2$  and

$$d + 2 \leq m(d) \leq 2d, \text{ for every } d \geq 2. \quad (1)$$

The theorem of Loh and Sudakov [LS07, Theorem 4.1] on independent transversals in locally sparse graphs implies that  $m(d) = d + o(d)$ . Neither the lower bound nor the upper bound in (1) has been improved for general  $d$ ; even  $m(3)$  is not known. However, restricting the graphs to specific classes, better upper bounds can be obtained.

**Definition 2.** For a class  $\mathcal{G}$  of graphs, denote by  $m_{\mathcal{G}}(d)$  the minimal  $m$  such that every  $m$  graphs belonging to  $\mathcal{G}$ , each of maximum degree at most  $d$ , have a cooperative coloring.

For example, the following was proved:

**Theorem 3** (Corollary 3.3 of Aharoni et al. [ABZ07] and Theorem 6.6 of Aharoni et al. [AHHS15]). *Let  $\mathcal{C}$  be the class of chordal graphs and let  $\mathcal{P}$  be the class of paths. Then  $m_{\mathcal{C}}(d) = d + 1$  for all  $d$ , and  $m_{\mathcal{P}}(2) = 3$ .*

In this paper, we prove some bounds on  $m_{\mathcal{G}}(d)$  for two more classes:

**Theorem 4.** *Let  $\mathcal{T}$  be the class of trees, and let  $\mathcal{B}$  be the class of bipartite graphs. Then for  $d \geq 2$ ,*

$$\begin{aligned} \log_2 \log_2 d \leq m_{\mathcal{T}}(d) &\leq (1 + o(1)) \log_{4/3} d, \\ \log_2 d \leq m_{\mathcal{B}}(d) &\leq (1 + o(1)) \frac{2d}{\ln d}. \end{aligned}$$

*Remark 5.* Let  $\mathcal{F}$  be the class of forests. It is evident that  $m_{\mathcal{F}}(d) \geq m_{\mathcal{T}}(d)$  as  $\mathcal{F} \supset \mathcal{T}$ . Conversely, when  $d \geq 2$ , given  $m = m_{\mathcal{T}}(d)$  forests  $F_1, \dots, F_m$  of maximum degree  $d$ , we can add edges to  $F_i$  to obtain a tree  $F'_i$  of maximum degree  $d$ , and the cooperative coloring for  $F'_1, \dots, F'_m$  is also a cooperative coloring for  $F_1, \dots, F_m$ . Therefore  $m_{\mathcal{F}}(d) = m_{\mathcal{T}}(d)$  for  $d \geq 2$ .

The notions of cooperative coloring and of list coloring have a common generalization: given a system  $(G_1, \dots, G_m)$  of graphs with vertex sets  $V_1, \dots, V_m$  (which are not necessarily the same vertex set), a *cooperative list coloring* is then a choice of independent sets in  $G_i$  whose union equals  $V := V_1 \cup \dots \cup V_m$ . The notion of cooperative coloring is obtained by taking  $V_i = V$ , and list colorings are formed when  $G_i$  is an induced subgraph of the same graph  $G$  for all  $i$ . The upper bounds in Theorem 4 generalize to cooperative list colorings. For example, our proof of Theorem 4 for bipartite graphs readily gives the following result.

**Theorem 6.** *For every system  $(G_1, \dots, G_m)$  of bipartite graphs with maximum degree  $d$  with vertex sets  $V_1, \dots, V_m$ , there is a cooperative list coloring if for every  $v \in V_1 \cup \dots \cup V_m$ , the number of its occurrences in  $V_1, \dots, V_m$ , that is  $|\{i \in [m] : v \in V_i\}|$ , is at least  $(1 + o(1)) \frac{2d}{\ln d}$ .*

A conjecture of Alon and Krivelevich [AK98, Conjecture 5.1] states that the choice number of any bipartite graph with maximum degree  $d$  is at most  $O(\log d)$  (see [AR08] for a result in this direction). This conjecture would follow if the term  $(1 + o(1)) \frac{2d}{\ln d}$  in Theorem 6 was strengthened to  $\Omega(\log d)$ .

The rest of the paper is organized as follows. In Section 2 and Section 3, we prove Theorem 4 for trees and bipartite graphs respectively. In Section 4 we discuss a further generalization of cooperative colorings.

## 2 Trees

*Proof of the lower bound on  $m_{\mathcal{T}}(d)$ .* Note that the system  $\mathcal{T}_2$ , consisting of two paths in Figure 1 (one in thin red, the other in bold blue), does not have a cooperative coloring.

Suppose now that  $\mathcal{S} = (F_1, F_2, \dots, F_m)$  is a system of forests on a vertex set  $V$ , not having a cooperative coloring. We shall construct a system  $\mathcal{Q}(\mathcal{S})$  of  $m + 1$  new forests  $F'_1, F'_2, \dots, F'_m, F'_{m+1}$ , again not having a cooperative coloring.

The vertex set common to the new forests is  $V' = (V \cup \{z\}) \times V$ , namely the vertex set consists of  $|V| + 1$  copies of  $V$ . For every  $u \in V \cup \{z\}$  and every  $i \in [m]$ , take a copy



Figure 1: Construction of two paths without a cooperative coloring.

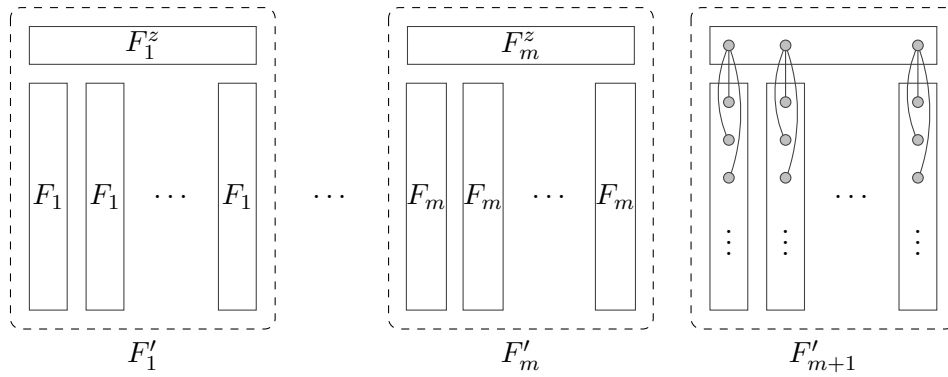


Figure 2: Construction of  $Q(\mathcal{S}) = (F'_1, \dots, F'_m, F'_{m+1})$  from  $\mathcal{S} = (F_1, \dots, F_m)$ .

$F_i^u$  of  $F_i$  on the vertex set  $\{(u, v) : v \in V\}$ . Let  $F'_i$  consist of  $|V| + 1$  disjoint copies of  $F_i$ :

$$F'_i := \bigcup_{u \in V \cup \{z\}} F_i^u, \quad \text{for all } i \in [m].$$

To these we add the  $(m + 1)$ st forest  $F'_{m+1}$  obtained by joining  $(z, u)$  to  $(u, v)$  for all  $u, v \in V$ . So  $F'_{m+1}$  is a disjoint union of stars, each with  $|V|$  leaves.

Assume that there is a cooperative coloring  $(I_1, I_2, \dots, I_m, I_{m+1})$  for the system  $Q(\mathcal{S})$ . Since the forests  $F_1^u, F_2^u, \dots, F_m^u$  do not have a cooperative coloring,  $I_{m+1}$  must contain a vertex from  $\{u\} \times V$  for all  $u \in V \cup \{z\}$ . In particular,  $I_{m+1}$  contains a vertex  $(z, u) \in I_{m+1}$  for some  $u \in V$  and a vertex  $(u, v)$  for some  $v \in V$ . Since  $(z, u)$  is connected in  $F'_{m+1}$  to  $(u, v)$ , this is contrary to our assumption that  $I_{m+1}$  is independent.

Note that  $|V'| = |V|^2 + |V| \leq 2|V|^2$ . Note also that the maximum degree of  $Q(\mathcal{S})$  is attained in  $F'_{m+1}$ , and it is equal to  $|V|$ . Recursively define the system  $\mathcal{T}_m := Q(\mathcal{T}_{m-1})$  consisting of  $m$  forests for  $m \geq 3$ . Because the base  $\mathcal{T}_2$  has 4 vertices, one can check inductively that  $|V(\mathcal{T}_m)|$  is at most  $2^{3 \cdot 2^{m-2} - 1}$  using  $|V(\mathcal{T}_m)| \leq 2|V(\mathcal{T}_{m-1})|^2$ . Thus the maximum degree of  $\mathcal{T}_m$  is at most  $2^{3 \cdot 2^{m-3} - 1} \leq 2^{2^{m-1}}$ .

Given the maximum degree  $d \geq 2$ , choose  $m := \lceil \log_2 \log_2 d \rceil$ . By the choice of  $m$ , the maximum degree of  $\mathcal{T}_m$  is at most  $2^{2^{m-1}} \leq d$ . By adding a few edges between the leaves in each forest of  $\mathcal{T}_m$ , we can obtain a system of  $m$  trees of maximum degree  $d$  that does not have a cooperative coloring. This means  $m_{\mathcal{T}}(d) > m > \log_2 \log_2 d$ .  $\square$

*Proof of the upper bound on  $m_{\mathcal{T}}(d)$ .* Let  $(T_1, T_2, \dots, T_m)$  be a system of trees of maximum degree  $d$ . We shall find a cooperative coloring by a random construction if  $m \geq (1 + o(1)) \log_{4/3} d$ .

Choose arbitrarily for each tree  $T_i$  a root so that we can specify the parent or a sibling of a vertex that is not the root of  $T_i$ . For each  $T_i$ , choose independently a random vertex set  $S_i$ , in which each vertex is included in  $S_i$  independently with probability  $1/2$ . Set

$$R_i := \{v \in S_i : \text{the parent of } v \text{ is not in } S_i, \text{ or } v \text{ is a root}\}.$$

Since among any two adjacent vertices in  $T_i$  one is the parent of the other,  $R_i$  is independent in  $T_i$ .

We shall show that with positive probability the sets  $R_i$  form a cooperative coloring. For each vertex  $v$ , let  $B_v$  be the event that  $v \notin \bigcup_{i=1}^m R_i$ . If  $v$  is the root of  $T_i$ , then  $\Pr(v \in R_i) = 1/2$ ; otherwise  $\Pr(v \in R_i) = 1/4$ . In any case,  $\Pr(v \notin R_i) \leq 3/4$ , and so  $\Pr(B_v) \leq (3/4)^m$ . Notice that  $B_v$  is only dependent on the events  $B_u$  for  $u$  that is the parent, a sibling or a child of  $v$  in some  $T_i$ . Since the degree of  $v$  is at most  $d$ , it follows that  $B_v$  is dependent on less than  $2md$  other events. By the symmetric version of the Lovász Local Lemma (see for example [AS16, Chapter 5]), if

$$e \times \left(\frac{3}{4}\right)^m \times 2md \leq 1, \tag{2}$$

then with positive probability no  $B_v$  occurs, meaning that the sets  $R_i$  form a cooperative coloring. The inequality (2) indeed holds under the assumption that  $m \geq (1+o(1)) \log_{4/3} d$ .  $\square$

### 3 Bipartite graphs

*Proof of the lower bound on  $m_{\mathcal{B}}(d)$ .* Given  $d$ , take  $m = \lceil \log_2 d \rceil$ . Let the vertex set be  $\{0, 1\}^m$ , and for  $j \in [m]$  let  $G_j$  be the complete bipartite graph between  $V_j^0$  and  $V_j^1$  where

$$V_j^k = \{v \in \{0, 1\}^m : v_j = k\}, \quad \text{for } k \in \{0, 1\}.$$

Note that the degree of  $G_j$  is  $2^{m-1} \leq d$ .

Suppose that  $I_1, \dots, I_m$  are independent sets in  $G_1, \dots, G_m$  respectively. As each  $G_j$  is a complete bipartite graph,  $I_j \subseteq V_j^{k_j}$  for some  $k_j \in \{0, 1\}$ . Thus  $(1 - k_1, \dots, 1 - k_m)$  is not in any  $I_j$ , and so  $I_1, \dots, I_m$  do not form a cooperative coloring. This means  $m_{\mathcal{B}}(d) > m \geq \log_2 d$ .  $\square$

*Proof of the upper bound on  $m_{\mathcal{B}}(d)$ .* Let  $\mathcal{G} = (G_1, \dots, G_m)$  be a system of bipartite graphs on the same vertex set  $V$  with maximum degree  $d$ . By a semi-random construction, we shall find a cooperative coloring if  $m \geq (1 + \varepsilon) \frac{2d}{\ln d}$  for fixed  $\varepsilon > 0$  and  $d$  sufficiently large. We may assume that  $m = O(d)$  because of (1).

For each  $j \in [m]$ , let  $(L_j, R_j)$  be a bipartition of  $G_j$ . Define  $J_L(v) := \{j \in [m] : v \in L_j\}$  and  $J_R(v) := \{j \in [m] : v \in R_j\}$  for each vertex  $v \in V$ , and let  $A := \{v \in V : |J_L(v)| \geq m/2\}$ . Set  $B := V \setminus A$ . Clearly, we have

$$|J_L(a)| \geq m/2, \quad \text{for all } a \in A; \tag{3a}$$

$$|J_R(b)| \geq m/2, \quad \text{for all } b \in B. \tag{3b}$$

Consider the following random process.

1. For each  $a \in A$ , choose  $j = j(a) \in J_L(a)$  uniformly at random, and put  $a$  in the set  $I_j$ .
2. For each  $b \in B$ , choose arbitrarily  $j \in J_R(b) \setminus \{j(a) : a \in A, (a, b) \in E(G_j)\} =: J'_R(b)$  as long as it is possible, and put  $b$  in the set  $I_j$ .

For any  $a, a' \in A \cap I_j$ ,  $a, a' \in L_j$  and so  $(a, a') \notin G_j$ . This means  $A \cap I_j$  is independent, and similarly  $B \cap I_j$  is independent. For any  $b \in B \cap I_j$  and  $(a, b) \in E(G_j)$ , by the definition of  $J'_R(b)$ ,  $j(a) \neq j$  and so  $a \notin I_j$ . Therefore  $I_j$  is independent for all  $j \in [m]$ .

To prove the existence of a cooperative coloring it suffices to show that  $J'_R(b)$  is nonempty for all  $b \in B$  with positive probability. For a vertex  $b \in B$ , let  $E_b$  be the contrary event, that is, the event that  $J'_R(b)$  is empty.

For a fixed  $b \in B$ , let us estimate from above the probability of  $E_b$ . For every  $j \in J_R(b)$ , let  $E^j$  be the event that  $j \notin J'_R(b)$ , that is the event that  $j(a) = j$  for some  $a \in A$  that is a neighbor of  $b$  in  $G_j$ . For each  $a \in A$  that is a neighbor of  $b$  in  $G_j$ , we have

$$\Pr(j(a) = j) = \frac{1}{|J_L(a)|} \stackrel{(3a)}{\leq} \frac{2}{m} \leq \frac{\ln d}{(1 + \varepsilon)d}.$$

As there are at most  $d$  neighbors of  $b$  in  $G_j$ , we have for sufficiently large  $d$  that

$$1 - \Pr(E^j) \geq \left(1 - \frac{\ln d}{(1 + \varepsilon)d}\right)^d \geq \exp(-(1 - \varepsilon) \ln d) = d^{\varepsilon - 1} \geq \frac{8 \ln d}{m}. \quad (4)$$

We claim that the events  $E^j$ ,  $j \in J_R(b)$ , are negatively correlated. This is easier to see with the complementary events  $\bar{E}^j$ ,  $j \in J_R(b)$ . We have to show that for any choice of indices  $j_1, \dots, j_t \in J_R(b)$  there holds

$$\Pr(E^j \mid \bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \dots \cap \bar{E}^{j_t}) \geq \Pr(E^j).$$

The event  $\bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \dots \cap \bar{E}^{j_t}$  means that for all  $a \in A$  if  $a$  is a neighbor of  $b$  in  $G_{j_i}$  then  $j(a) \neq j_i$ . Then, for any  $j \notin \{j_1, \dots, j_t\}$ , for those vertices  $a \in A$  that are neighbors of  $b$  in  $G_j$ , knowing that  $j(a) \neq j_i$  for certain  $i \in [t]$  increases the probability that  $j(a) = j$ , and therefore increases the probability of  $E^j$ .

By the claim, the inequality (4) and the fact that  $E_b = \bigcap_{j \in J_R(b)} E^j$ , we have

$$\Pr(E_b) \leq \prod_{j \in J_R(b)} \Pr(E^j) \stackrel{(3b)}{\leq} \left(1 - \frac{8 \ln d}{m}\right)^{\frac{m}{2}} \leq \exp\left(-\frac{8 \ln d}{m} \cdot \frac{m}{2}\right) = \frac{1}{d^4}.$$

The event  $E_b$  is dependent on less than  $md^2$  other events  $E_{b'}$ , since for such dependence to exist it is necessary that  $b' \in B$  is at distance at most 2 from  $b$  in some graph  $G_j$ . Thus, by the Lovász Local Lemma, for the positive probability that none of  $E_b$  occurs it suffices that

$$e \times \frac{1}{d^4} \times md^2 \leq 1,$$

which indeed holds for  $d$  sufficiently large as  $m = O(d)$ . □

## 4 Cooperative covers

Cooperative coloring of graphs is a special case of a more general concept.

**Definition 7.** Given a system  $(C_1, \dots, C_n)$  of (abstract) simplicial complexes, all sharing the same vertex set  $V$ , a *cooperative cover* is a choice of faces  $f_i \in C_i$  such that  $\bigcup_{i=1}^n f_i = V$ .

A cooperative coloring for  $(G_1, \dots, G_n)$  is the special case in which  $C_i$  is the independence complex  $I(G_i)$  of  $G_i$ , that is, the collection of all independent sets in  $G_i$ .

**Definition 8.** Given a hypergraph  $C$  with vertex set  $V$ , the *edge covering number*  $\rho(C)$  is the minimal number of hyperedges from  $C$  whose union is  $V$ . For a class  $\mathcal{C}$  of simplicial complexes, let  $n_{\mathcal{C}}(b)$  denote the minimal number  $n$ , such that every system  $(C_1, \dots, C_n)$  of simplicial complexes belonging to  $\mathcal{C}$  on the same vertex set  $V$  satisfying  $\rho(C_i) \leq b$  for all  $i \leq n$ , has a cooperative cover. Let  $n_{\mathcal{C}}(b) = \infty$  if no such  $n$  exists.

For example, consider the class  $\mathcal{I}$  of all the independence complexes of graphs. If  $G$  is bipartite, then  $\rho(I(G)) \leq 2$ . Hence the fact that  $m_{\mathcal{B}}(d) \geq \log_2(d)$  for all  $d \geq 2$  (see Theorem 4) implies  $n_{\mathcal{I}}(2) = \infty$ .

There are natural classes  $\mathcal{C}$  of hypergraphs for which  $n_{\mathcal{C}}$  is finite. One of these is the class of simplicial complexes associated to polymatroids, as introduced in [Edm70]. A *polymatroid*  $(V, r)$  is defined via a rank function  $r: 2^V \rightarrow \mathbb{N}$ , that is submodular, monotone increasing and is 0 on the empty set. A *k-polymatroid* is a polymatroid in which every singleton set has rank at most  $k$ . For example, a  $k$ -uniform hypergraph  $H$  endowed with the function  $r(E) = |\cup E|$ , for every subset of hyperedges  $E$  in  $H$ , is a  $k$ -polymatroid.

Following the notation in [LP86, Section 11], given a  $k$ -polymatroid  $(V, r)$ , a set  $M \subseteq V$  is called a *matching* if  $r(M) = k|M|$ . By the submodularity of the rank function  $r$ , the matchings in a  $k$ -polymatroid form a simplicial complex on  $V$ , which we call the *matching complex* of a  $k$ -polymatroid.

**Theorem 9.** Let  $\mathcal{M}_k$  be the class of all the matching complexes of  $k$ -polymatroids. Then  $n_{\mathcal{M}_k}(b) \leq kb$  for every  $b$ .

The proof uses the (homotopic) connectivity  $\eta(C)$  of a complex  $C$ . We refer to [AB06, Section 2] for background. We shall use the following two topological tools. Given a complex  $C$  on  $V$  and  $U \subseteq V$ , we denote by  $C[U]$  the simplicial subcomplex induced on  $U$ .

**Theorem 10** (Topological Hall's theorem). Let  $C$  be a simplicial complex on the vertex set  $V$  and let  $\bigcup_{i=1}^m W_i$  be a partition of  $V$ . If for all  $I \subseteq [m]$

$$\eta \left( C \left[ \bigcup_{i \in I} W_i \right] \right) \geq |I|,$$

then  $C$  contains a face  $\sigma$  such that  $|\sigma \cap W_i| = 1$  for all  $i \in [m]$ .

**Theorem 11.** If  $C$  is a matching complex on  $V$  of a  $k$ -polymatroid, then the connectivity  $\eta(C)$  of  $C$  is at least  $\nu(C)/k$ , where  $\nu(C)$  is the maximal size of faces in  $C$ .

The above formulation of Theorem 10 first appeared in [Mes01], attributed to the first author of the present paper (see the remark after Theorem 1.3 in [Mes01]). Theorem 11 is an unpublished result of the first two authors. The special case, where the  $k$ -polymatroid is the sum of  $k$  matroids on the same vertex set, is proved in [AB06, Theorem 6.5].



*Proof of Theorem 9.* Let  $n = kb$ , and let  $C_1, \dots, C_n$  be simplicial complexes associated to  $k$ -polymatroids  $(V, r_1), \dots, (V, r_n)$  on the same vertex set  $V$  such that the edge covering number of each  $C_i$  is at most  $b$ . Let  $C$  be the join of  $C_1, \dots, C_n$  on  $V \times [n]$ , that is,

$$C := \left\{ \bigcup_{i=1}^n \sigma_i \times \{i\} : \sigma_i \in C_i \text{ for all } i \in [n] \right\}.$$

A cooperative cover can be viewed as a face  $\sigma \in C$  such that  $|\sigma \cap (\{v\} \times [n])| = 1$  for all  $v \in V$ . By the topological Hall's theorem, it suffices to prove that

$$\eta(C[U \times [n]]) \geq |U| \text{ for all } U \subseteq V.$$

Let  $U$  be a subset of  $V$ . Note that  $C_i[U]$  is the matching complex of the  $k$ -polymatroid  $(U, r_i|_U)$ . By Theorem 11,  $\eta(C_i[U]) \geq \nu(C_i[U])/k$ . Since  $\nu(C_i[U])$  is the maximal size of faces in  $C_i[U]$  and the edge covering number of  $C_i[U]$  is at most  $b$ , we obtain  $\nu(C_i[U])b \geq |U|$ , and so  $\eta(C_i[U]) \geq |U|/(kb)$ . Notice that  $C[U \times [n]]$  is the join of  $C_1[U], \dots, C_n[U]$ . Using the superadditivity of  $\eta$  with respect to the join operator and Theorem 11, we obtain the required condition for the topological Hall's theorem

$$\eta(C[U \times [n]]) \geq \sum_{i=1}^n \eta(C_i[U]) \geq \sum_{i=1}^n |U|/(kb) = |U|. \quad \square$$

*Remark 12.* It is of interest to explore the sharpness of this result.

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