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A GRAPH THEORETICAL APPROACH TO EQUIVALENCE OF MULTISTAGE INTERCONNECTION NETWORKS*

Jean Claude BERMOND

L.R.I., UA410, C.N.R.S., Informatique, bât 490, Université Paris Sud, Orsay, France

Jean Michel FOURNEAU

Laboratoire d'Informatique des Systèmes Expérimentaux et de leur Modélisation, bât 490, Université Paris Sud, Orsay, France; and E.N.S.T. Département Informatique, Paris, France

Alain JEAN-MARIE

Laboratoire d'Informatique des Systèmes Expérimentaux et de leur Modélisation, bât 490, Université Paris Sud, Orsay, France; and Ecole Normale Supérieure Saint Cloud, France

We consider multistage interconnection networks for designed communication in multiprocessor architecture. We give a graph characterization of these networks and a set of properties which imply that a network is topologically equivalent to the Baseline network. In particular our characterization proves that the six classical multistage interconnection networks are topologically equivalent.

Keywords. Multistage interconnection networks, topological equivalence, multistage interconnection directed graphs, connected components, Baseline network.

1. Introduction

We consider here multistage interconnection networks. Such networks have been designed for communication in multiprocessors architectures and in telephone switching systems and therefore extensively studied (see for example the tutorial of Wu and Feng [8]). They consist of N inputs, N outputs and n stages of switching cells. The most common type of networks uses, as switching cells, 2×2 Crossbar networks. These cells are able to produce either a straight or a cross connection (Fig. 1).



Fig. 1. Straight connection (left) and cross connection (right).

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In this paper we consider only this type of networks, but the results we obtain can be generalized to networks using other sizes of cells. A minimal requirement is that any input can be connected to any output and this implies a lower bound on n . We restrict ourselves to the study of the case $N=2^n$. Such networks include the Omega [11], Flip [3], Indirect Binary Cube [12], Modified Data Manipulator [5], Baseline (see Fig. 2 for a drawing of this network) and Reverse Baseline [7] networks.

In [7], Wu and Feng have used algebraic description rules to prove that the six networks mentioned above are topologically equivalent. But their method is numerical and quite complex. We propose instead a graph theoretic approach which yields an easy characterization of Baseline equivalent networks.

Note that such an analysis has already been made by Agrawal in [1] (see also [2]) but the set of properties proposed is not sufficient to characterize these graphs as we will see in Remark 2.11.

2. Multistage interconnection directed graphs

Interconnection networks can easily be modeled by digraphs (directed graphs) in which nodes represent the switching cells and arcs the communication links. We do not represent inputs and outputs because we are only interested in the topological equivalence of networks and these inputs and outputs do not play any role in this equivalence. Note that this model is thus slightly different from Agrawal's one or from our previous model in [4]. In Fig. 2, we represent the Baseline-16 network and its associated digraph. In all the figures the arcs are directed from left to right.

Let us give now the precise definition of a multistage interconnection (MI) digraph according to the constraint of the class of networks we consider. In what follows we shall denote for a digraph (V, E) and for a set of vertices A by

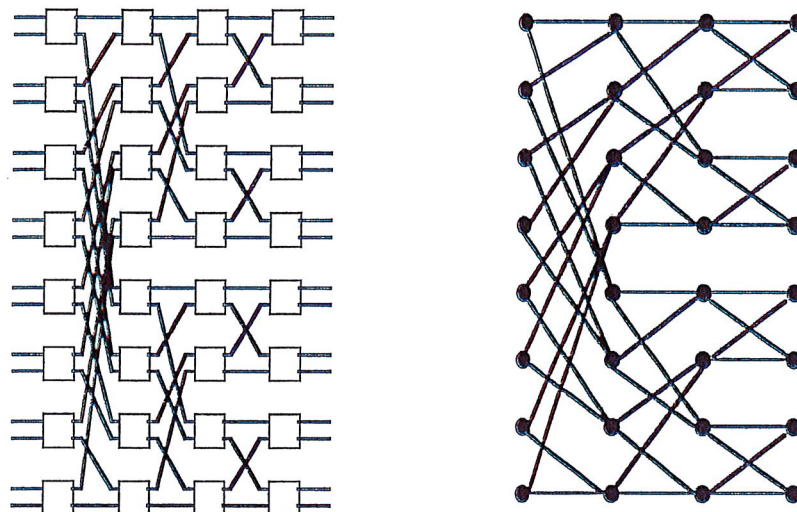


Fig.2. Baseline-16 network (left), Baseline-16 associated digraph (right).

$$\Gamma^+(A) = \{y \in V : \exists x \in A \text{ and } (x, y) \in E\},$$

$$\Gamma^-(A) = \{y \in V : \exists x \in A \text{ and } (y, x) \in E\},$$

and by induction $\Gamma^{+k}(A) = \Gamma^+(\Gamma^{+(k-1)}(A))$.

Definition 2.1 (MI digraph). By a multistage interconnection directed graph (MI digraph) with n rows, we denote a digraph $G = (V, E)$ such that:

(H₁) There exists a partition of V into $(V_1, V_2, \dots, V_{n-1}, V_n)$ with

$$\Gamma^+(V_i) = V_{i+1} \quad \forall i = 1, \dots, n-1,$$

$$\Gamma^-(V_i) = V_{i-1} \quad \forall i = 2, \dots, n.$$

(H₂) If x is a node of V_1 , then its indegree is 0; otherwise the indegree is 2.

(H₃) If x is a node of V_n , then its outdegree is 0; otherwise the outdegree is 2.

(H₄) Every V_i has the same cardinality: 2^{n-1} .

Let V_i be the i th row of G , and let M_n be the set of n -stage MI digraphs.

From this definition, we shall say that two networks are topologically equivalent if their MI digraphs are isomorphic.

Definition 2.2 (Automorphisms on MI digraphs). S is an automorphism operating on an MI digraph G if and only if:

- the restriction of S to V_i is a one-to-one mapping denoted s_i .
- the adjacency relationship is invariant by S .

We denote the image of G under S as $S(G)$ and denote the components of S by (s_1, \dots, s_n) .

Definition 2.3 (Banyan property). A MI digraph satisfies the Banyan property if and only if there exists a single path connecting any input node to any output node.

Let F_n be the set of Banyan MI digraphs with n rows. The two following lemmas can be easily proved.

Lemma 2.4. *If G is a Banyan MI digraph, and u is a node of V_i , then:*

$$|\Gamma^{+k}(u)| = 2^k \quad \forall k = 1, \dots, n-i,$$

$$|\Gamma^{-k}(u)| = 2^k \quad \forall k = 1, \dots, i-1.$$

Lemma 2.5. *If G is an MI digraph which does not satisfy the Banyan property, then the following two conditions hold:*

- (i) *There exists a node x of V_1 and a node y of V_n such that there is no path from x to y .*
- (ii) *There exists a node z of V_1 and a node w of V_n such that there are at least two paths from z to w .*

The usual definition of the Banyan property insures that one of these two conditions must hold. One can easily prove that the two conditions are equivalent according to the definitions of MI digraphs.

Definition 2.6. We will denote by $(G)_{i,j}$ the subdigraph induced by the vertices from row i to row j .

Definition 2.7 (Connected components). The connected components of a MI digraph are the components of the underlying undirected graph (obtained by omitting the orientation of the arcs).

Definition 2.8 ($P(i,j)$ property). We shall say that an MI digraph with n rows satisfies the $P(i,j)$ property for $1 \leq i \leq j \leq n$ if and only if the subdigraph $(G)_{i,j}$ has $2^{n-1-(j-i)}$ connected components.

Definition 2.9 ($P(*,*)$ property). We shall say that an MI digraph satisfies the

- $P(*,j)$ property if and only if it satisfies the $P(i,j)$ property for every i such that $i \leq j$;
- $P(i,*)$ property if and only if it satisfies the $P(i,j)$ property for every j such that $i \leq j$;
- $P(*,*)$ property if and only if it satisfies the $P(i,j)$ property for every (i,j) such that $i \leq j$.

Remark 2.10. The $P(i,i)$ property, for every i , is equivalent to the assumption (H_4) . The buddy property introduced by Agrawal and Kim in [2] is equivalent to the $P(i, i+1)$ property for every i such that $1 \leq i < n$. As shown in [2, 4] this property is not sufficient to characterize the isomorphic MI digraphs. Such an MI digraph, non-isomorphic to the Baseline MI digraph, is shown in Fig. 3. Indeed it contains a cycle of length 10 drawn in bold while the Baseline MI digraph does not contain such a cycle.

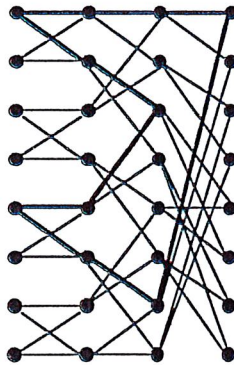


Fig.3. A non-isomorphic MI digraph satisfying the buddy property.

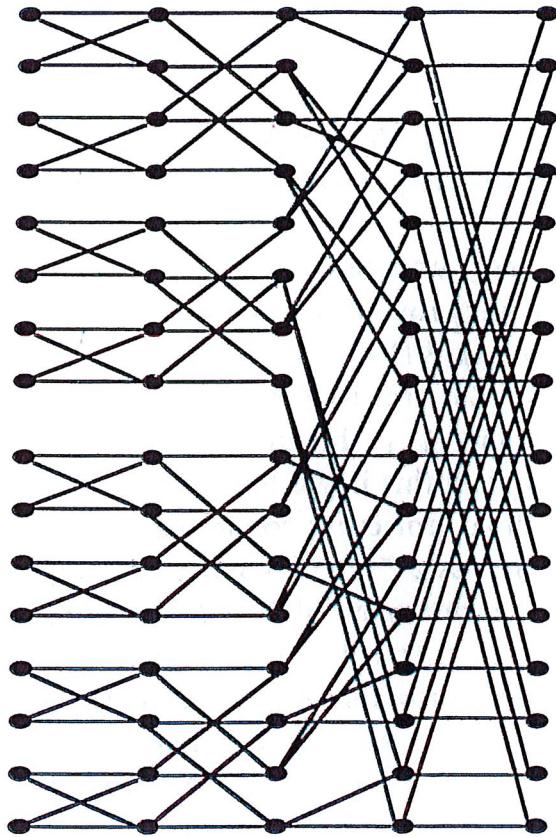


Fig.4. A non-isomorphic MI digraph satisfying the strict buddy property.

Remark 2.11 (*Agrawal's characterization*). In [1] Agrawal introduces the strict buddy property, which turns out to be equivalent to the conjunction of $P(i, i+1)$ and $P(i, i+2)$ properties for every i . Agrawal claimed that all the MI digraphs with n stages satisfying the Banyan and the strict buddy property are isomorphic. Unfortunately that is true only if $n \leq 4$. For $n = 5$, it can be checked that the MI digraph of Fig.4 satisfies the Banyan and the strict buddy properties. However it is not equivalent to the Baseline network; indeed it does not satisfy the $P(1,4)$ property. Similarly one can show that a local characterization is not possible; more exactly that there exist for $n \geq k+3$ non-isomorphic Banyan digraphs satisfying the properties $P(i, i+j)$ for every $j \leq k$ and $i \leq n-j$ (see [4] for counterexamples).

Remark 2.12. The $P(i, j)$ property is invariant by the application of an automorphism on MI digraph; indeed the image of a connected component is a connected component of the image of the MI digraph. So the $P(*, *)$ property is a necessary condition for topological equivalence with the Baseline MI digraph.

In the remainder of this section, we shall prove that if K is a connected component of a Banyan MI digraph G satisfying the $P(*, *)$ property then K satisfies both properties (i.e. Banyan and $P(*, *)$). In Section 3, we shall study further properties of the Baseline MI digraph. We shall present two constructive operations on MI

digraphs and show their relationships to the Baseline network. The fundamental lemma and the two characterization theorems will be presented in Section 4.

Lemma 2.13. *If G is a Banyan MI digraph satisfying the $P(i, j)$ property and K is a connected component of $(G)_{i, j}$, then K is in F_{j-i+1} .*

Proof. (H_1) , (H_2) and (H_3) are easily verified; and K has $j-i+1$ rows.

First let us show that every row of K has the same number of elements. Suppose that for some k ($i \leq k \leq j-1$) $|K \cap V_k| = \alpha$ and $|K \cap V_{k+1}| = \beta$. As every vertex of V_k has outdegree 2, there are 2α arcs in K between V_k and V_{k+1} . Similarly there are 2β arcs in K between V_k and V_{k+1} , therefore $\alpha = \beta$. Lemma 2.4 implies that a node of V_i has 2^{j-i} sons in V_j and this is the lower bound for the number of nodes in $K \cap V_j$. But this bound holds for every connected component of $(G)_{i, j}$. The $P(i, j)$ property means that $(G)_{i, j}$ has $2^{n-1-(j-i)}$ connected components like K . According to the definition of G , the cardinality of V_j is 2^{n-1} . Therefore the number of nodes in a row of K is exactly 2^{j-i} , and this proves (H_4) .

The two arguments above show that K is an MI digraph. We must now verify that K satisfies the Banyan property. According to Lemma 2.5, if K is not a Banyan MI digraph, we can find in K two vertices connected by two paths. However this implies that G itself does not satisfy the Banyan property. \square

Lemma 2.14. *If G is a Banyan MI digraph satisfying the $P(*, *)$ property, then for every ordered pair (i, j) , the connected components K of $(G)_{i, j}$ satisfy the $P(*, *)$ property.*

Proof. Every connected component of $(K)_{k, l}$ is a connected component of $(G)_{i+k-1, i+l-1}$. According to Lemma 2.13, all the connected components of $(G)_{i+k-1, i+l-1}$ have the same size (i.e. 2^{l-k} nodes per row). As it holds for every K , and as K has 2^{j-i} nodes per row, the number of connected components in $(K)_{k, l}$ is $2^{(j-i)-(l-k)}$. As K has $j-i+1$ rows, it satisfies the $P(*, *)$ property. \square

3. The Baseline MI digraph

We shall give here the definition of the Baseline MI digraph and the fundamental properties of this graph.

Definition 3.1 (*Baseline MI digraph*). According to the definition of the Baseline network in [6], we define the Baseline MI digraph as follows: the node x of the i th row whose binary representation is $(x_{n-1}, x_{n-2}, \dots, x_2, x_1)$ has two sons in the $(i+1)$ th row whose labelings are respectively $(y_{n-1}, y_{n-2}, \dots, y_2, y_1)$ and $(z_{n-1}, z_{n-2}, \dots, z_2, z_1)$ with

$$\begin{aligned}
y_{n-i} &= 0, \\
z_{n-i} &= 1, \\
z_{n-j-1} &= y_{n-j-1} = x_{n-j} \quad \forall n-1 > j \geq i, \\
z_{n-j} &= y_{n-j} = x_{n-j} \quad \forall 1 < j < i.
\end{aligned}$$

In what follows we shall denote the Baseline MI digraph with n rows by B_n .

Definition 3.2 (*Automorphism Σ*). We define Σ by its components

$$\Sigma = (\Sigma_1, \dots, \Sigma_n),$$

where the one-to-one mapping Σ_i applied to V_i makes the negation of the $(n-i)$ bits of lower weight (i.e. $\Sigma_i(x_{n-1}, \dots, x_{n-i}, \dots, x_1) = (x_{n-1}, \dots, \overline{x_{n-i}}, \dots, x_1)$).

One can easily verify the following lemma.

Lemma 3.3. Σ is an automorphism of the Baseline MI digraph and $\Sigma(B_n) = B_n$. It means that the application of Σ to a Baseline MI digraph leaves it invariant.

The two operations we shall now introduce will make the subsequent proofs more readable.

Definition 3.4 (*Superposition of MI digraphs \oplus*). Let G_0 and G_1 be two elements of M_{n-1} and G an element of M_n . We will say that subgraph $(G)_{2,n}$ is the superposition of G_0 and G_1 , and we will note it:

$$(G)_{2,n} = G_0 \oplus G_1$$

if and only if:

- (i) The node x of V_i ($i \geq 2$) of G whose binary representation is $(x_{n-1}, x_{n-2}, \dots, x_1)$ is the node numbered (x_{n-2}, \dots, x_1) in the $(i-1)$ th row of G_0 if $x_{n-1} = 0$ or G_1 if $x_{n-1} = 1$.
- (ii) Two nodes x and y of $V_i \times V_{i+1}$ ($1 < i < n$) are connected in G if and only if they are in the same G_j and they are connected in this G_j .

In other words, $(G)_{2,n}$ is the superposition of two graphs of M_{n-1} if no vertex of a row i , $i \geq 2$, of G numbered j , $j < 2^{n-2}$ is connected to a vertex of row $i+1$ numbered k with $k \geq 2^{n-2}$. The way V_1 is connected to V_2 in G is irrelevant. To illustrate this property, we state the classical ‘‘left-recursive’’ property of the Baseline network as:

Lemma 3.5. $(B_n)_{2,n} = B_{n-1} \oplus B_{n-1}$.

$(B)_{2,n}$ possesses two connected components which are themselves Baseline MI digraphs of size $n-1$. These components are connected via the first row V_1 . This

property is equivalent to the left-recursive construction of the Baseline network [6].

By induction, one can now verify using Lemma 3.5 that the Baseline MI digraph satisfies the Banyan property.

Definition 3.6 (*Total shuffle of MI digraphs $\dot{\cup}^*$*). Let G_0 and G_1 be two elements of M_{n-1} and G be an element of M_n . We shall say that the subgraph $(G)_{1,n-1}$ is the total shuffle of G_0 and G_1 and we shall note:

$$(G)_{1,n-1} = G_0 \dot{\cup}^* G_1,$$

when the following two conditions are satisfied:

(i) The node x of V_i ($i < n$) of G whose binary representation is $(x_{n-1}, \dots, x_{n-i+1}, x_{n-i}, x_{n-i-1}, \dots, x_1)$ is the node numbered $(x_{n-1}, \dots, x_{n-i+1}, x_{n-i-1}, \dots, x_1)$ in the i th row of G_0 if $x_{n-i} = 0$, or G_1 if $x_{n-i} = 1$.

(ii) Two nodes x and y of $V_i \times V_{i+1}$ ($1 \leq i < n - 1$) are connected in G if and only if they are in the same G_j and they are connected in this G_j .

Lemma 3.7. $(B_n)_{1,n-1} = B_{n-1} \dot{\cup}^* B_{n-1}$ (Fig. 5).

Proof. Let us introduce the following partition of the nodes of B_n into two sets C^0 and C^1 .

Let x be a node of V_i whose binary representation is (x_{n-1}, \dots, x_1) . If x_{n-i} equals 0, then x is in C^0 ; x is in C^1 otherwise (see Fig. 5).

We shall prove now that the two graphs G^0 and G^1 whose nodes are in C^0 and C^1 are the connected components of $(B)_{1,n-1}$.

Let us consider the node x in V_i numbered (x_{n-1}, \dots, x_1) . The node x has two sons in V_{i+1} which we denote by y and z . According to Definition 3.1, we have the relations

$$y_{n-i-1} = z_{n-i-1} = x_{n-i}.$$

Therefore x , y , and z are in the same set C^0 or C^1 . The two fathers of x are in the same set too. Therefore the graphs G^0 and G^1 are disconnected. But the Baseline

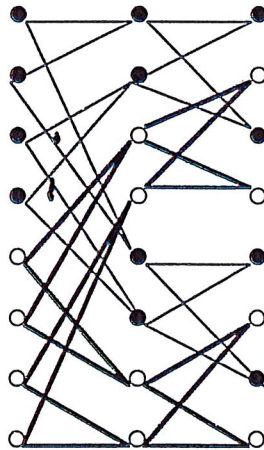


Fig. 5. The graph $(B_4)_{1,3}$

MI digraph satisfies the Banyan property; and this implies that the graph $(G)_{1,n-1}$ has at most two connected components. Therefore the graphs G^0 and G^1 are the connected components of $G_{1,n-1}$:

$$(B_n)_{1,n-1} = G^0 \overset{*}{\cup} G^1.$$

Now we have to prove that G^0 is a B_{n-1} . If x is a node of $C^0 \cap V_i$, then as a node of $(G)_{1,n-1}$ it is labeled $(x_{n-1}, \dots, x_{n-i+1}, 0, x_{n-i-1}, \dots, x_1)$, and as a node of the i th row of C^0 , it is labeled $(x_{n-1}, \dots, x_{n-i+1}, x_{n-i-1}, \dots, x_1)$. If we consider as previously the nodes x , y and z of C^0 , the relations between x_i , y_i and z_i allow us to verify that C^0 is a Baseline MI digraph of size $n-1$. \square

As a corollary we have

Lemma 3.8. *If G is an element of F_n that satisfies*

- $(G)_{1,n-1} = B_{n-1} \overset{*}{\cup} B_{n-1}$, and
- *the relationship of adjacency between the two last rows is the same as in the Baseline MI digraph,*

then G is the Baseline MI digraph of size n .

Lemmas 3.5 and 3.7 imply also:

Lemma 3.9. *The Baseline MI digraph satisfies the $P(*, *)$ property.*

4. Theorems of characterization

Lemma 4.1 (Fundamental Lemma). *If G is a Banyan MI digraph such that:*

- $(G)_{2,n} = B_{n-1} \oplus B_{n-1}$, and
- G *satisfies the $P(*, *)$ property,*

then G and B_n are isomorphic, and the n th component of the isomorphism is the identity.

Proof. By induction on n .

$n=2$. There is only one MI digraph in F_2 . It is the B_2 MI digraph, and it satisfies the conditions of the lemma.

Suppose the lemma holds for every p ($p < n$), and let G be a graph of F_n which satisfies the hypothesis. In particular $(G)_{2,n} = B_{n-1} \oplus B_{n-1}$ (see Fig. 6). Applying Lemma 3.7 to each of the B_{n-1} we get:

$$(G)_{2,n-1} = (B_{n-2} \overset{*}{\cup} B_{n-2}) \oplus (B_{n-2} \overset{*}{\cup} B_{n-2}).$$

To these four B_{n-2} correspond four subgraphs of G : R^0 , Y^0 , R^1 , Y^1 , which contain in particular the vertices of V_2 numbered j , $0 \leq j < 2^{n-3}$, $2^{n-3} \leq j < 2^{n-2}$, $2^{n-2} \leq j < 3 \cdot 2^{n-3}$, $3 \cdot 2^{n-3} \leq j < 2^{n-1}$ respectively (as shown in Fig. 7).

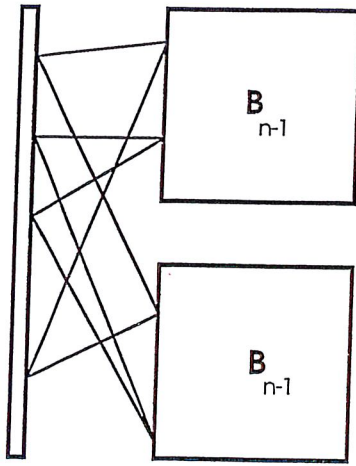


Fig. 6. Construction of G .

But $(G)_{1,n-1}$ possesses two connected components C^0 and C^1 since G satisfies the $P(1, n-1)$ property. $(C^0)_{2,n-1}$ is therefore the union of two connected components chosen among R^0, Y^0, R^1, Y^1 . If we suppose that C^0 contains R^0 , then it cannot also contain Y^0 because R^0 and Y^0 are already connected via the last row of G which satisfies the Banyan property. Therefore, only the two configurations shown in Fig. 7 remain.

Case 1: C^0 contains Y^1 . In this case we apply to G the isomorphism S defined so that

- the restriction of S to $R_0 \cup Y_0$ is the identity automorphism,
- the restriction of S to $R^1 \cup Y^1$ is the Σ automorphism,
- the restriction of S to the first row of G is the identity permutation.

This isomorphism exchanges R^1 and Y^1 and it does not modify the last row. Therefore, we have reduced the problem to the next case.

Case 2: C^0 contains R^1 (and C^1 contains Y^0 and Y^1). In this case, every node of the first row which is connected to R^0 is also connected to R^1 ; otherwise it is connected to Y^0 and Y^1 . We can construct a sorting isomorphism T (see Fig. 8) such that:

- the application of the restriction of T to the first row sorts the nodes in the following manner: every node of V_1 connected to R^0 has a lower number than every node connected to Y^0 ;
- T does not modify the rest of G .

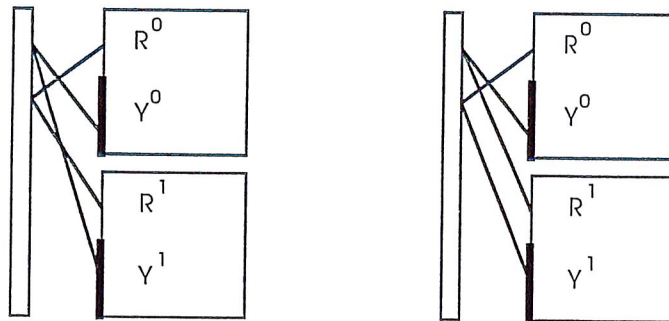


Fig. 7. The two constructions of C^0 and C^1 .

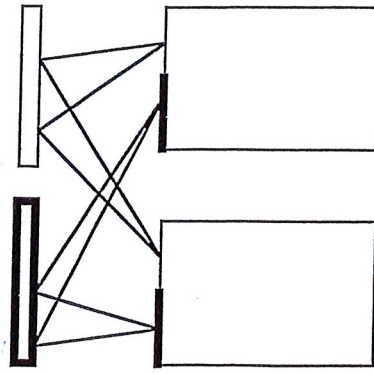


Fig.8. Construction of $T(G)$.

The isomorphisms S and T have changed G into a structure of total shuffle of C^0 and C^1 and both of them are the superposition of two B_{n-2} MI digraphs:

$$(G)_{1,n-1} = C^0 \dot{\cup} C^1, \quad (C^0)_{2,n-1} = R^0 \oplus R^1, \quad (C^1)_{2,n-1} = Y^0 \oplus Y^1$$

and according to Lemma 2.14, C^0 and C^1 satisfy the $P(*, *)$ property.

It is therefore possible to apply the induction hypothesis: there exist two isomorphisms S^0 and S^1 such that $S^0(C^0)$ and $S^1(C^1)$ are Baseline MI digraphs of size $n-1$. We build then the last isomorphism U such that

- the restriction of U to C^0 is S^0 ,
- the restriction of U to C^1 is S^1 ,
- the restriction of U to the last row of G is the identity permutation.

Then the digraph $U(G)$ satisfies:

$$(U(G))_{1,n-1} = B_{n-1} \dot{\cup} B_{n-1}.$$

Note that the three isomorphisms S , T , U do not modify the last row. Furthermore the adjacency relationship between the two last rows of G is also invariant. Indeed, S leaves B_{n-1} invariant and T does not modify the $(n-1)$ th row as well as U by induction assumption. Therefore Lemma 3.8 proves that $U(G)$ is the B_n MI digraph. \square

Theorem 4.2. *Let G be a Banyan MI digraph. G is topologically equivalent to the Baseline MI digraph if and only if G satisfies the $P(*, *)$ property.*

Proof. By induction on n .

As the properties $P(*, *)$ and Banyan are preserved by graph isomorphisms, these properties are necessary conditions.

When $n=2$, the theorem holds since B_2 is the unique graph of F_2 .

When $n>2$, suppose the theorem holds for every $p<n$ and let G be an element of F_n that satisfies the $P(*, *)$ property. As G satisfies the $P(2, n)$ property, $(G)_{2,n}$

has two connected components C^0 and C^1 , and using Lemma 2.14 each of them satisfies the $P(*, *)$ property.

Lemma 2.13 shows that C^0 and C^1 are members of F_{n-1} . Therefore, we can construct an isomorphism S that sorts C^0 and C^1 so that every node of C^0 has a lower number than every node of C^1 , and S leaves the first row of G unchanged. Then $(S(G))_{2,n-1} \cong C^0 \oplus C^1$.

By induction, there exist two isomorphisms that transform C^0 and C^1 into Baseline MI digraph of size $n-1$. Then $(G)_{2,n-1} = B_{n-1} \oplus B_{n-1}$.

According to Lemma 4.1, we know that G is isomorphic to the Baseline MI digraph for size n . \square

We now have a simple characterization of Baseline equivalent digraphs. However, while the Banyan property is simple to verify, the $P(i, j)$ property is a strong topological property which might be expensive to check. Fortunately enough, Theorem 4.5 will give a simpler characterization. Let us prove two preliminary technical lemmas.

Lemma 4.3. *If G is an element of F_n satisfying the $P(1, *)$ property, then for every couple (i, k) , $2 \leq i < i+k \leq n$, every connected component of $(G)_{1,i+k}$ is in F_{i+k} and is the union of 2^k connected components of $(G)_{1,i}$ connected by the k rows of G from $i+1$ to $i+k$.*

Proof. We use Lemma 2.13 which proves that:

- the connected components of $(G)_{1,i+k}$ are in F_{i+k} ,
- the connected components of $(G)_{1,i}$ are in F_i .

We also use the argument that the components of $(G)_{1,i}$ are included in the components of $(G)_{1,i+1}$. A size argument gives the numerical result. \square

Lemma 4.4. *If G is an element of F_n satisfying the $P(*, n)$ property, then for every pair (i, k) , $1 \leq i-k < i \leq n$, every connected component of $(G)_{i-k,n}$ is in $F_{n+k-i+1}$ and contains 2^k components of $(G)_{i,n}$.*

Theorem 4.5. *If G is a Banyan MI digraph, then G satisfies the $P(*, *)$ property if and only if G satisfies both the $P(*, n)$ and $P(1, *)$ properties.*

Proof. If G satisfies the $P(*, *)$ property, then it satisfies the $P(*, n)$ and $P(1, \cdot)$ properties.

Conversely, let us consider a pair (i, j) such that $1 \leq i \leq j \leq n$ and let α be the number 2^{j-i} . We want to prove that $(G)_{i,j}$ possesses $2^{n-1-(j-i)}$ connected components. Obviously if i equals j this property holds. Let H be a connected component of $(G)_{i,j}$ and let K be the last row of H . G is a Banyan MI digraph, so K has at least α nodes.

However all the nodes of K are in distinct components of $(G)_{j,n}$. Indeed K is included in some components of $(G)_{1,j}$ and this component is Banyan because $(G)_{1,j}$ satisfies $P(1, *)$ property. Therefore for any pair (x, y) of elements of $K \cap V_j$ there is a node w in the first row of G connected to x and y . If x and y are in the same Banyan component of $(G)_{j,n}$, there exists also a node z in the last row of G connected to x and y , and there exists two distinct paths in G from w to z , which leads to a contradiction.

Then if K has more than α elements, the row from i to j connect more than α connected components of $(G)_{j,n}$. This statement is in contradiction with Lemma 4.4. \square

Due to the symmetry of the conditions of Theorem 4.5, if G is a Baseline isomorphic digraph, then the digraph G^{-1} obtained from G by changing the orientation of all the arcs is also a Baseline isomorphic digraph. The digraph G^{-1} is associated to what is called the "reverse network" in the literature [7]. So we have the following corollary

Corollary 4.6. *If a network is equivalent to the Baseline network then its reverse network is also equivalent to the Baseline network.*

The reader may check that the other networks like Omega, Indirect Binary Cube (and therefore their reverse Flip and Modified Data Manipulator) satisfy the conditions of Theorem 4.5 (see [9] for proof concerning the Omega network).

5. Conclusion

We have defined a graph model of multistage interconnection networks which yields an easy characterization of Baseline equivalent networks. Checking the conditions can be formal using the networks definitions, or algorithmic in the more general case of directed graphs. As the algorithm consists mainly in covering the graph, its complexity is linear in the number of nodes.

The results obtained can be generalized to the case where the switching cells are $m \times m$ Crossbar networks (see [9, 13]). Furthermore relations between this model and the algebraic formalization of the multistage interconnection networks are made clear in [9]. Note that another approach have been used by Kruskal and Snir [10]. They use graph theory plus labeling schemes of the routings in the network. They defined a network isomorphism as a graph isomorphism which furthermore preserves the labels of vertices. They give a sufficient condition, called the bidelta property, to insure that a graph is isomorphic in their sense to the classical ones. Our result is different in the sens that our characterization uses only graph (or topological) properties. Finally both models are useful to study other general properties of these networks like rearrangeability [14], reliability and performance.

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