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# Guaranteed contraction of adaptive inexact $hp$ -refinement strategies with realistic stopping criteria\*

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February 20, 2020

## Abstract

The purpose of this contribution is to theoretically analyze the adaptive refinement strategies for conforming  $hp$ -finite element approximations of elliptic problems proposed for exact algebraic solvers in [Daniel, Ern, Smears, Vohralík, Comput. Math. Appl. **76** (2018), 967–983] and for inexact algebraic solvers in [Daniel, Ern, Vohralík, Comput. Methods Appl. Mech. Engrg. **359** (2020), 112607]. Both of these strategies are driven by guaranteed equilibrated flux energy error estimators. The employed  $hp$ -refinement criterion stems from solving two separate local residual problems posed only on the patches of elements around marked vertices, selected by a bulk-chasing criterion. In the above references, we have derived a fully computable guaranteed bound on the ratio of the error on two successive steps of the  $hp$ -adaptive loop. Here, our focus is to prove that this ratio is uniformly smaller than one, which implies a guaranteed contraction of the adaptive and adaptive inexact  $hp$ -refinement strategies. To be able to achieve this goal, we have to introduce some additional assumptions on the  $h$ - and  $p$ -refinements, namely the so-called interior node property, an extension of the marked region, and a sufficient polynomial degree increase. In the inexact case, a sufficiently precise stopping criterion for the algebraic solver is requested, but this criterion remains fully computable and also realistic in the sense that in numerical experiments, it does not request the algebraic error to be excessively small in comparison with the total error.

**Key words:** elliptic problem, finite element method, a posteriori error estimate, equilibrated flux,  $hp$ -adaptivity, inexact solver, error reduction, convergence analysis

## 1 Introduction

The adaptive finite element method has been used in practice and theoretically studied for several decades. Its recent development has been catalysed by the extensive research dedicated to efficient and reliable a posteriori error estimates, see, e.g., the survey books by Ainsworth and Oden [1] and Verfürth [42], and started with the pioneering work of Dörfler [21]. In the  $h$ -adaptive strategy for elliptic problems, Morin *et al.* [31] provided a plain convergence result, whereas Binev *et al.* [7] modified the method from [31] to prove not only optimal convergence rate but also optimal computational complexity. Other important results are to be found in Morin *et al.* [32, 33], Cascón *et al.* [14], and Carstensen *et al.* [13]. Most of the convergence results were stated for methods driven by residual-type a posteriori error estimates; the works addressing other types of estimators include Kreuzer and Siebert [29] and Cascón and Nochetto [15]. In contrast, convergence of  $hp$ -adaptive strategies has been addressed only recently in Dörfler and Heuveline [22], Bürg and Dörfler [10], and Bank, Parsania, and Sauter [3]. To our knowledge, the most recent state of the art optimality result is by Canuto *et al.* [11], hinging on an coarsening module due to Binev [6], and by Canuto *et al.* [12], hinging on a saturation condition assumption.

In a common adaptive finite element method, a (larger and larger) system of algebraic equations needs to be solved at each step. This can be practically quite costly and is actually not necessary upon employing

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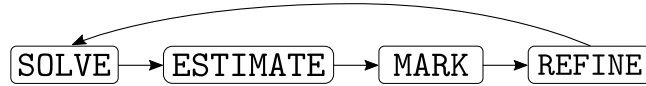
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the idea to balance the algebraic error to the level of the discretization one at each step of the adaptive procedure, as suggested in, e.g., [4, 7, 40, 41, 5, 2, 24, 37] and the references therein for linear elliptic problems and Holst *et al.* [27], Carstensen *et al.* [13], and Gantner *et al.* [26] for nonlinear elliptic problems. In the seminal works, Binev *et al.* [7] and Stevenson [40, 41] present their  $h$ -adaptive finite element algorithms in an abstract setting and the inexact approximations are assumed to be sufficiently/arbitrarily close to the exact ones, without, however, a practical estimate on the algebraic error. The algebraic error can be estimated for a specific solver under appropriate assumptions as in Arioli *et al.* [2], where a convergence of the inexact  $h$ -adaptive finite element algorithm employing the conjugate gradient method as the algebraic solver has been shown, provided a good estimate on the smallest eigenvalue of the finite element system matrix is available. Another approach is based on an a priori argument which requires that the employed iterative solver contracts the algebraic error at least by a factor  $\rho_{\text{it}} < 1$  on each iteration. In particular, in such a situation, Becker and Mao [5] design a convergent and quasi-optimal conforming  $h$ -adaptive finite element algorithm. We believe, though, that an optimal way to proceed is via a dedicated *a posteriori* error estimate on the *algebraic error*. This does not rely on a specific solver or assumptions and may be much more precise, leading to quite sharp identification of all total, algebraic, and discretization error components, see [28, 35, 34] and the references therein.

The goal of this work, stemming from Chapter 3 of the Ph.D. dissertation [16], is to complete our recently proposed  $hp$ -adaptive refinement strategy with *computable guaranteed bound* on the *error reduction factor* [17] and its counterpart with an *inexact algebraic solver setting* ([18]) by a rigorous convergence proof. As in [17, 18], we examine the Poisson equation with homogeneous Dirichlet boundary conditions. Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , be a polygonal/polyhedral domain (open, bounded, and connected set) with a Lipschitz boundary  $\partial\Omega$ , and let  $H_0^1(\Omega)$  denote the Sobolev space of all functions in  $L^2(\Omega)$  which have all their first-order weak derivatives in  $L^2(\Omega)$  and a vanishing trace on  $\partial\Omega$ . Assuming that  $f \in L^2(\Omega)$ , the model problem in its weak form reads: find  $u \in H_0^1(\Omega)$  such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega), \quad (1.1)$$

where  $(\cdot, \cdot)$  stands for the  $L^2(\Omega)$  or  $[L^2(\Omega)]^d$  inner product. The conforming  $hp$ -finite element method is used to discretize the model problem (1.1). Therein, we consider only matching simplicial meshes without hanging nodes.



**Scheme 1:** Paradigm of an  $hp$ -adaptive algorithm employing an exact algebraic solver.

The first part of this manuscript (Sections 2–5) is dedicated to the study of the  $hp$ -adaptive algorithm in the *exact setting*, i.e. we assume exact (up to machine precision) solution of all the resulting linear algebraic problems. Typically, such algorithms follow the well-established paradigm presented in Scheme 1. Particularly, in [17], we showed that between two consecutive steps of the adaptive loop from Scheme 1, it is possible to compute explicitly a real number  $C_{\ell, \text{red}} \in [0, 1]$  such that

$$\|\nabla(u - u_{\ell+1}^{\text{ex}})\| \leq C_{\ell, \text{red}} \|\nabla(u - u_{\ell}^{\text{ex}})\|, \quad (1.2)$$

where  $u_{\ell}^{\text{ex}}$  and  $u_{\ell+1}^{\text{ex}}$  denote the exact finite element solutions from the respective iterations  $\ell$  and  $\ell + 1$  of the adaptive loop. Even though we provided numerical evidence in [17, Section 6], we did not prove that the reduction factor  $C_{\ell, \text{red}}$  of [17, Section 5] is bounded by a generic constant strictly smaller than one; this is the subject of our study here. In order to achieve it, in contrast to [17], in Section 3 we introduce slightly modified versions of the modules MARK and REFINE of Scheme 1 with some additional assumptions on the employed  $h$ - and  $p$ -refinement methods. The key ingredients of the proof are the extension of the marked region in module MARK by one layer of elements, see Section 3.2, some interior node properties requested for  $h$ -refinement, a stronger  $p$ -refinement used within module REFINE, see Section 3.3, and the discrete stability of the local equilibrated fluxes computed within the module ESTIMATE, proved in Section 4. The only assumption is that we need to limit the maximal polynomial degree since the present analysis is not  $p$ -robust. This is restrictive from the theoretical viewpoint but completely reasonable in practice.



**Scheme 2:** Paradigm of an adaptive loop employing an inexact algebraic solver.

In the second part of this manuscript (starting with Section 6), we extend our results concerning the convergence (guaranteed contraction) of the *hp*-adaptive algorithm from the exact setting to the *inexact setting*. Following [18], we incorporate the use of an arbitrary inexact algebraic solver within the framework of adaptive algorithm of Scheme 1 by replacing the module SOLVE by an adaptive sub-loop consisting of modules ONE\_SOLVER\_STEP and ESTIMATE, see Scheme 2. We treat carefully our choice of a practical *adaptive stopping criterion* for the algebraic solver in order to ensure the convergence of the adaptive algorithm. In contrast to [7, 40, 41, 5, 2], the particular advantage of the present work is that no request on the algebraic solver is made and our bounds on algebraic, discretization, and total error do not contain any generic constant and remain fully computable, see Section 6.1. Consequently, the values of the stopping parameter  $\gamma_\ell$ , which expresses the percentage of the algebraic error with respect to the total error, do not need to theoretically take excessively small (and unknown) values. The proper choice of stopping criterion together with the modifications of modules MARK and REFINE, introduced in the exact setting, then allow us to show, in extension of (1.2), the error reduction property between two consecutive inexact approximations  $u_\ell$  and  $u_{\ell+1}$  produced by the adaptive loop of Scheme 2 in the form

$$\|\nabla(u - u_{\ell+1})\| \leq C_{\ell,\text{red}} \|\nabla(u - u_\ell)\|, \quad (1.3)$$

with the fully computable factor  $0 \leq C_{\ell,\text{red}} \leq C < 1$ . This theoretical result improves in particular the developments of [18, Theorem 5.4], where the reduction factor, derived therein in a slightly different form in comparison to the present  $C_{\ell,\text{red}}$ , was not showed to be bounded by a generic constant strictly smaller than one. In particular, the reduction property (1.3), showed in Section 8 here, implies the convergence of the adaptive algorithm prescribed by Scheme 2. The quick numerical assessment of Section 9, that complements the extensive numerical studies from [17] and [18], illustrates that the generated sequences of meshes and polynomial degree distributions still lead to asymptotic exponential convergence, and this for the stopping parameter  $\gamma_\ell$  with the very-reasonable-in-practice value around 0.05 in the inexact solver case.

## 2 Framework and notation

Within the adaptive loops of Schemes 1 and 2, a sequence  $\{(\mathcal{T}_\ell, \mathbf{p}_\ell)\}_{\ell \geq 0}$ , with  $\ell \geq 0$  the iteration counter, is generated. Each pair  $(\mathcal{T}_\ell, \mathbf{p}_\ell)$  consists of a matching simplicial mesh  $\mathcal{T}_\ell$  of the computational domain  $\Omega$ , i.e. a finite collection of (closed) simplices  $K \in \mathcal{T}_\ell$  covering  $\bar{\Omega}$  and such that the intersection of two different simplices is either empty or their  $d'$ -dimensional common face,  $0 \leq d' \leq d-1$ , and of a polynomial-degree distribution vector  $\mathbf{p}_\ell := \{p_{\ell,K}\}_{K \in \mathcal{T}_\ell}$  which assigns a degree  $p_{\ell,K} \in \mathbb{N}_{\geq 1}$  to each simplex  $K \in \mathcal{T}_\ell$ . Moreover, each pair  $(\mathcal{T}_\ell, \mathbf{p}_\ell)$  prescribes a discrete finite-dimensional space  $V_\ell$ , defined as

$$V_\ell := \mathbb{P}_{\mathbf{p}_\ell}(\mathcal{T}_\ell) \cap H_0^1(\Omega), \quad \forall \ell \geq 0,$$

where  $\mathbb{P}_{\mathbf{p}_\ell}(\mathcal{T}_\ell)$  denotes the space of piecewise polynomials of total degree at most  $p_{\ell,K}$  on each simplex  $K \in \mathcal{T}_\ell$ . Let us denote by  $N_\ell$  the dimension of the  $\ell$ -th level space  $V_\ell$ . Note that we enforce the  $H_0^1(\Omega)$ -conformity of the spaces  $V_\ell$  for all  $\ell \geq 0$ . In addition, we make the following nestedness assumption:

$$V_\ell \subset V_{\ell+1}, \quad \forall \ell \geq 0. \quad (2.1)$$

The initial pair  $(\mathcal{T}_0, \mathbf{p}_0)$  is assumed to be given. Then, the purpose of each step  $\ell \geq 0$  of the adaptive loops of Schemes 1 and 2 is to determine the next pair  $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$ . The nestedness property (2.1) gives us two crucial restrictions on the meshes and polynomial-degree distributions defining the spaces  $V_\ell$ : (i) the sequence of meshes  $\{\mathcal{T}_\ell\}_{\ell \geq 0}$  needs to be *hierarchically nested*, i.e., for all  $\ell \geq 1$  the mesh  $\mathcal{T}_\ell$  is a refinement of  $\mathcal{T}_{\ell-1}$  such that for all  $K \in \mathcal{T}_\ell$ , there is a unique simplex  $\tilde{K} \in \mathcal{T}_{\ell-1}$ , called the parent of  $K$ , satisfying  $K \subseteq \tilde{K}$ ;

(ii) The local polynomial degree is *locally increasing*, i.e., for all  $\ell \geq 1$  and all  $K \in \mathcal{T}_\ell$ ,  $p_{\ell,K} \geq p_{\ell-1,\tilde{K}}$ , where  $\tilde{K} \in \mathcal{T}_{\ell-1}$  is the parent of  $K$ . Moreover, we assume the following standard shape-regularity property: There exists a constant  $\kappa_{\mathcal{T}} > 0$  such that  $\max_{K \in \mathcal{T}_\ell} h_K/\rho_K \leq \kappa_{\mathcal{T}}$  for all  $\ell \geq 0$ , where  $h_K$  is the diameter of  $K$  and  $\rho_K$  is the diameter of the largest ball inscribed in  $K$ . This can in practice be ensured by using e.g. the newest-vertex bisection mesh refinement algorithm [39].

We denote by  $\mathcal{V}_\ell$  (respectively  $\mathcal{F}_\ell$ ) the set of vertices (respectively  $(d-1)$ -dimensional faces) of the mesh  $\mathcal{T}_\ell$ . These are decomposed into interior vertices  $\mathcal{V}_\ell^{\text{int}}$  ( $(d-1)$ -dimensional faces  $\mathcal{F}_\ell^{\text{int}}$ ) and boundary vertices  $\mathcal{V}_\ell^{\text{ext}}$  ( $(d-1)$ -dimensional faces  $\mathcal{F}_\ell^{\text{ext}}$ ). For each vertex  $\mathbf{a} \in \mathcal{V}_\ell$ ,  $\ell \geq 0$ , the so-called hat function  $\psi_\ell^{\mathbf{a}}$  is the continuous, piecewise affine function that takes the value 1 at the vertex  $\mathbf{a}$  and the value 0 at all the other vertices of  $\mathcal{V}_\ell$ ; the function  $\psi_\ell^{\mathbf{a}}$  is contained in the space  $V_\ell$  for all vertices  $\mathbf{a} \in \mathcal{V}_\ell^{\text{int}}$  and all polynomial degrees  $\mathbf{p}_\ell$ . Furthermore, we consider the simplex patch  $\mathcal{T}_\ell^{\mathbf{a}} \subset \mathcal{T}_\ell$  which is the collection of the simplices sharing the vertex  $\mathbf{a} \in \mathcal{V}_\ell$ , with  $\omega_\ell^{\mathbf{a}}$  the corresponding open subdomain of  $\Omega$ , coinciding with the support of  $\psi_\ell^{\mathbf{a}}$ . Let  $\mathcal{F}_\ell^{\mathbf{a}} \subset \mathcal{F}_\ell$  denote the set of all the  $(d-1)$ -dimensional faces in the patch  $\mathcal{T}_\ell^{\mathbf{a}}$ . This set can be further decomposed into  $\mathcal{F}_\ell^{\mathbf{a},\text{int}}$ , the subset of faces from  $\mathcal{F}_\ell^{\mathbf{a}}$  that share the vertex  $\mathbf{a}$  and are shared by two distinct simplices in  $\mathcal{T}_\ell^{\mathbf{a}}$ , and  $\mathcal{F}_\ell^{\mathbf{a},\text{ext}}$ , the faces from  $\mathcal{F}_\ell^{\mathbf{a}}$  lying in  $\partial\omega_\ell^{\mathbf{a}}$ , so that  $\mathcal{F}_\ell^{\mathbf{a}} = \mathcal{F}_\ell^{\mathbf{a},\text{int}} \cup \mathcal{F}_\ell^{\mathbf{a},\text{ext}}$ . For any face  $F \in \mathcal{F}_\ell$ ,  $\mathbf{n}_F$  stands for a unit vector normal to  $F$  with an arbitrary but fixed orientation. The operator  $[\![\cdot]\!]$  yields the jump, in the direction of  $\mathbf{n}_F$ , of the traces of the argument from the two simplices that share  $F \in \mathcal{F}_\ell^{\text{int}}$ , and the actual trace for  $F \in \mathcal{F}_\ell^{\text{ext}}$ . Finally, for each simplex  $K \in \mathcal{T}_\ell$ ,  $\mathcal{V}_{\ell,K}$  denotes the set of vertices of  $K$  and  $\mathcal{F}_{\ell,K}$  denotes the set of  $(d-1)$ -dimensional faces of element  $K \in \mathcal{T}_\ell$ .

### 3 The $hp$ -adaptive algorithm with exact solver

In this section we first recall the modules **SOLVE** and **ESTIMATE** as defined in [17]. Afterwards, we introduce the slightly modified versions of the remaining modules **MARK** and **REFINE** from the adaptive loop of Scheme 1, which will allow us to prove the convergence of such an adaptive algorithm. In order to avoid technicalities with data oscillation, we suppose:

**Assumption 3.1** (Source term  $f$ ). *In the following analysis, the datum  $f$  is assumed to be piecewise polynomial of variable degree at most  $\mathbf{p}_f$  with respect to the coarsest partition  $\mathcal{T}_0$ ,  $f \in \mathbb{P}_{\mathbf{p}_f}(\mathcal{T}_0)$ , such that  $\mathbf{p}_f \leq \mathbf{p}_0 - 1$ .*

#### 3.1 The modules **SOLVE** and **ESTIMATE**

Let  $\ell \geq 0$  denote the current iteration counter. The module **SOLVE** takes as input the current finite element space  $V_\ell \subset H_0^1(\Omega)$  and outputs the Galerkin approximation  $u_\ell^{\text{ex}} \in V_\ell$  of the weak solution  $u$  of (1.1) defined as the unique solution of

$$(\nabla u_\ell^{\text{ex}}, \nabla v_\ell) = (f, v_\ell) \quad \forall v_\ell \in V_\ell. \quad (3.1)$$

Note that obtaining  $u_\ell^{\text{ex}}$  is equivalent to computing the exact solution of the system of linear algebraic equations

$$\mathbb{A}_\ell \mathbf{U}_\ell^{\text{ex}} = \mathbf{F}_\ell, \quad (3.2)$$

where  $\{\psi_\ell^n\}_{1 \leq n \leq N_\ell}$  is the basis of the  $\ell$ -th level space  $V_\ell$  such that  $u_\ell^{\text{ex}} := \sum_{n=1}^{N_\ell} (\mathbf{U}_\ell^{\text{ex}})_n \psi_\ell^n$ ,  $(\mathbb{A}_\ell)_{k,m} = (\nabla \psi_\ell^m, \nabla \psi_\ell^k)$ , and  $(\mathbf{F}_\ell)_k = (f, \psi_\ell^k)$ ,  $1 \leq k, m \leq N_\ell$ .

Following [19, 8, 25, 20], see also the references therein, the module **ESTIMATE** relies on an equilibrated flux a posteriori error estimate on the energy error  $\|\nabla(u - u_\ell^{\text{ex}})\|$ . The module **ESTIMATE** takes as input the finite element solution  $u_\ell^{\text{ex}}$  and outputs a collection of local error indicators  $\{\eta_K\}_{K \in \mathcal{T}_\ell}$ .

The equilibrated flux is constructed locally on the simplex patches  $\mathcal{T}_\ell^{\mathbf{a}}$  attached to each vertex  $\mathbf{a} \in \mathcal{V}_\ell$ . For this construction, we consider as in [20, 17] the local polynomial degree  $p_{\mathbf{a}}^{\text{est}} := \max_{K \in \mathcal{T}_\ell^{\mathbf{a}}} p_{\ell,K}$  (any other choice so that  $p_{\mathbf{a}}^{\text{est}} \geq \max_{K \in \mathcal{T}_\ell^{\mathbf{a}}} p_{\ell,K}$  can also be employed). For a fixed vertex  $\mathbf{a} \in \mathcal{V}_\ell$ , let the broken space

$$\mathbf{RTN}_p(\mathcal{T}_\ell^{\mathbf{a}}) := \{\mathbf{v}_\ell \in [L^2(\omega_\ell^{\mathbf{a}})]^d; \mathbf{v}_\ell|_K \in \mathbf{RTN}_p(K), \quad \forall K \in \mathcal{T}_\ell^{\mathbf{a}}\},$$

where  $\mathbf{RTN}_p(K) := [\mathbb{P}_p(K)]^d + \mathbb{P}_p(K)\mathbf{x}$  is the usual  $p$ -th order Raviart–Thomas–Nédélec space (cf. [9, 38]) on a simplex  $K \in \mathcal{T}_\ell$ . Then, the patchwise normal-trace-continuous spaces  $\mathbf{V}_\ell^{\mathbf{a}}$  with homogeneous Neumann

boundary conditions in which the local equilibration will be performed are defined by

$$\mathbf{V}_\ell^{\mathbf{a}} := \begin{cases} \{\mathbf{v}_\ell \in \mathbf{RTN}_{p_{\mathbf{a}}^{\text{est}}}(\mathcal{T}_\ell^{\mathbf{a}}) \cap \mathbf{H}(\text{div}, \omega_\ell^{\mathbf{a}}); \mathbf{v}_\ell \cdot \mathbf{n}_{\omega_\ell^{\mathbf{a}}} = 0 \text{ on } \partial\omega_\ell^{\mathbf{a}}\} & \text{if } \mathbf{a} \in \mathcal{V}_\ell^{\text{int}} \\ \{\mathbf{v}_\ell \in \mathbf{RTN}_{p_{\mathbf{a}}^{\text{est}}}(\mathcal{T}_\ell^{\mathbf{a}}) \cap \mathbf{H}(\text{div}, \omega_\ell^{\mathbf{a}}); \mathbf{v}_\ell \cdot \mathbf{n}_{\omega_\ell^{\mathbf{a}}} = 0 \text{ on } \partial\omega_\ell^{\mathbf{a}} \setminus \partial\Omega\} & \text{if } \mathbf{a} \in \mathcal{V}_\ell^{\text{ext}}. \end{cases} \quad (3.3)$$

**Definition 3.2** (Equilibrated flux  $\sigma_\ell$  by local minimizations). *Let  $u_\ell^{\text{ex}}$  be the solution of (3.1). For each vertex  $\mathbf{a} \in \mathcal{V}_\ell$ , let the local equilibrated flux  $\sigma_\ell^{\mathbf{a}} \in \mathbf{V}_\ell^{\mathbf{a}}$  be defined by the local minimization problem*

$$\sigma_\ell^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_\ell \in \mathbf{V}_\ell^{\mathbf{a}}, \\ \nabla \cdot \mathbf{v}_\ell = \psi_\ell^{\mathbf{a}} f - \nabla \psi_\ell^{\mathbf{a}} \cdot \nabla u_\ell^{\text{ex}}}} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \mathbf{v}_\ell\|_{\omega_\ell^{\mathbf{a}}}. \quad (3.4)$$

Then, extending each local contribution  $\sigma_\ell^{\mathbf{a}}$  by zero outside of the patch domain  $\omega_\ell^{\mathbf{a}}$ , the global  $\mathbf{H}(\text{div}, \Omega)$ -conforming equilibrated flux  $\sigma_\ell$  is constructed as

$$\sigma_\ell := \sum_{\mathbf{a} \in \mathcal{V}_\ell} \sigma_\ell^{\mathbf{a}}.$$

Note that the Neumann compatibility condition  $\int_{\omega_\ell^{\mathbf{a}}} \psi_\ell^{\mathbf{a}} f - \nabla \psi_\ell^{\mathbf{a}} \cdot \nabla u_\ell = 0$  for the problem (3.4) is satisfied for all  $\mathbf{a} \in \mathcal{V}_\ell^{\text{int}}$  as a direct consequence of (3.1) (consider  $\psi_\ell^{\mathbf{a}}$  as a test function in (3.1)). The global  $\mathbf{H}(\text{div}, \Omega)$ -conformity of  $\sigma_\ell$  follows from imposing the zero normal trace of functions in the local spaces  $\mathbf{V}_\ell^{\mathbf{a}}$ , cf. the definition (3.3).

Then, as stated in [19, 8], see also [20, Theorem 3.3], the following guaranteed upper bound on the energy error holds true:

$$\|\nabla(u - u_\ell^{\text{ex}})\| \leq \eta(u_\ell^{\text{ex}}, \mathcal{T}_\ell) := \left\{ \sum_{K \in \mathcal{T}_\ell} \eta_K^2(u_\ell^{\text{ex}}) \right\}^{\frac{1}{2}}, \quad \eta_K := \|\nabla u_\ell^{\text{ex}} + \sigma_\ell\|_K. \quad (3.5)$$

Note that due to the simplifying Assumption 3.1, the local estimator  $\eta_K$  does not include the so-called data oscillation term, as opposed to [20, Theorem 3.3].

### 3.2 The module MARK

The module MARK takes as input the local error estimators from (3.5) and proceeds in two phases.

The first phase corresponds to the module MARK used in [17]. We select the smallest subset of marked vertices  $\mathcal{V}_\ell^\theta \subset \mathcal{V}_\ell$  using a bulk-chasing criterion inspired by the well-known Dörfler's criterion (cf. [21])

$$\eta\left(u_\ell^{\text{ex}}, \bigcup_{\mathbf{a} \in \mathcal{V}_\ell^\theta} \mathcal{T}_\ell^{\mathbf{a}}\right) \geq \theta \eta(u_\ell^{\text{ex}}, \mathcal{T}_\ell), \quad (3.6)$$

where  $\theta \in (0, 1]$  is a fixed threshold parameter, and where, for a subset  $\mathcal{S} \subset \mathcal{T}_\ell$ , we adopt the notation  $\eta(u_\ell^{\text{ex}}, \mathcal{S}) := \{\sum_{K \in \mathcal{S}} \eta_K(u_\ell^{\text{ex}})^2\}^{1/2}$ . Then, letting

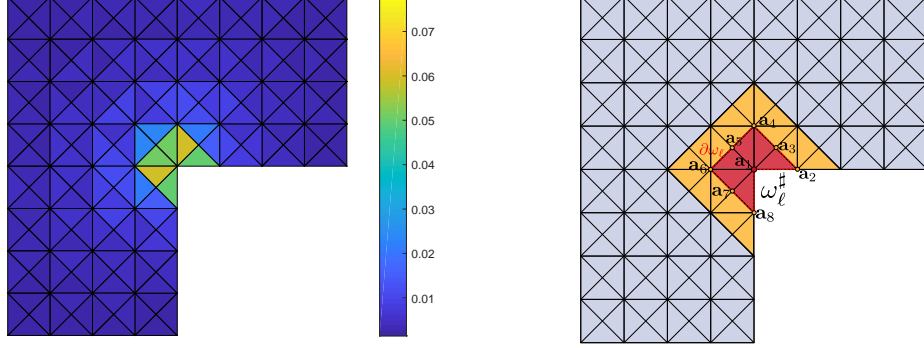
$$\mathcal{M}_\ell^\theta := \bigcup_{\mathbf{a} \in \mathcal{V}_\ell^\theta} \mathcal{T}_\ell^{\mathbf{a}} \subset \mathcal{T}_\ell$$

be the collection of all the simplices that belong to a patch associated with a marked vertex, we observe that (3.6) means that  $\eta(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta) \geq \theta \eta(u_\ell^{\text{ex}}, \mathcal{T}_\ell)$ . Let us denote by  $\omega_\ell := \bigcup_{\mathbf{a} \in \mathcal{V}_\ell^\theta} \omega_\ell^{\mathbf{a}}$  the open subdomain corresponding to the set of marked elements  $\mathcal{M}_\ell^\theta$ . To select a set  $\mathcal{V}_\ell^\theta$  of minimal cardinality, the mesh vertices in  $\mathcal{V}_\ell$  are sorted by comparing the vertex-based error estimators  $\eta(u_\ell, \mathcal{T}_\ell^{\mathbf{a}})$  for all  $\mathbf{a} \in \mathcal{V}_\ell$ , and a greedy algorithm is employed to build the set  $\mathcal{V}_\ell^\theta$ . Possible reductions of the computational cost of such an algorithm are proposed in [21, Section 5.2] and in Stevenson [41, Section 5].

In a second phase, we define an extended set of marked vertices  $\tilde{\mathcal{V}}_\ell^\#$  with the corresponding set of marked elements  $\mathcal{M}_\ell^\#$  in the following way

$$\tilde{\mathcal{V}}_\ell^\# := \bigcup_{K \in \mathcal{M}_\ell^\#} \mathcal{V}_{\ell, K} \quad \text{and} \quad \mathcal{M}_\ell^\# := \bigcup_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} \mathcal{T}_\ell^{\mathbf{a}}.$$

In other words, we extend the set  $\mathcal{M}_\ell^\theta$  by one more layer of neighbouring elements in contact with the boundary  $\partial\omega_\ell$ , see Figure 1 for an illustration. In addition, we define by  $\omega_\ell^\sharp := \bigcup_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp} \omega_\ell^\mathbf{a}$  the open subdomain



**Figure 1:** An example of local error estimators  $\eta_K(u_\ell^{\text{ex}})$  from (3.5) (left) and illustration of the corresponding set of marked vertices  $\mathcal{V}_\ell^\theta = \{\mathbf{a}_1\}$  and its extension  $\tilde{\mathcal{V}}_\ell^\sharp = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_8\}$ ,  $\theta = 0.5$  (right). The region highlighted in red color corresponds to the subdomain  $\omega_\ell$ , while its union with the yellow region amounts to the extended subdomain  $\omega_\ell^\sharp$ .

corresponding to the set of marked elements  $\mathcal{M}_\ell^\sharp$ . This extension is motivated by the structure of the error estimate (3.5), stemming from equilibrated flux  $\sigma_\ell$  composed of local patchwise contributions  $\sigma_\ell^\mathbf{a}$ . It plays a particular role later in our convergence proof, when we decompose the error estimate  $\eta(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta)$  into a sum of local patchwise contributions in (5.7) with their corresponding domains of definitions possibly exceeding the original marked subdomain  $\omega_\ell$ , but always included in the extended marked subdomain  $\omega_\ell^\sharp$ . The present theoretical analysis requires such extension of the marked region; this is the price we pay for the precision of our estimates on the error reduction factor  $C_{\ell, \text{red}}$ :

**Remark 3.3** (Marking and analysis without the additional layer). *Instead of the elementwise estimate (3.5), one could employ the following patchwise form of the error estimate, cf. [25, Lemma 3.22] and [12, Proposition 3.1]:*

$$\|\nabla(u - u_\ell^{\text{ex}})\| \leq \sqrt{d+1} \left\{ \sum_{\mathbf{a} \in \mathcal{V}_\ell} \eta_\mathbf{a}(u_\ell^{\text{ex}})^2 \right\}^{\frac{1}{2}}, \quad \eta_\mathbf{a}(u_\ell^{\text{ex}}) := \|\psi_\ell^\mathbf{a} \nabla u_\ell^{\text{ex}} + \sigma_\ell^\mathbf{a}\|_{\omega_\ell^\mathbf{a}}. \quad (3.7)$$

*This would result in no need to extend the obtained set of marked vertices  $\mathcal{V}_\ell^\theta$  by the additional layer, as well as in an overall a simpler hp-adaptive algorithm. However, the presence of the constant  $\sqrt{d+1}$  already in the error estimate (3.7) would lead to a deterioration of the bound on the error reduction factor  $C_{\ell, \text{red}}$ , which was designed as sharp as possible in [17, 18].*

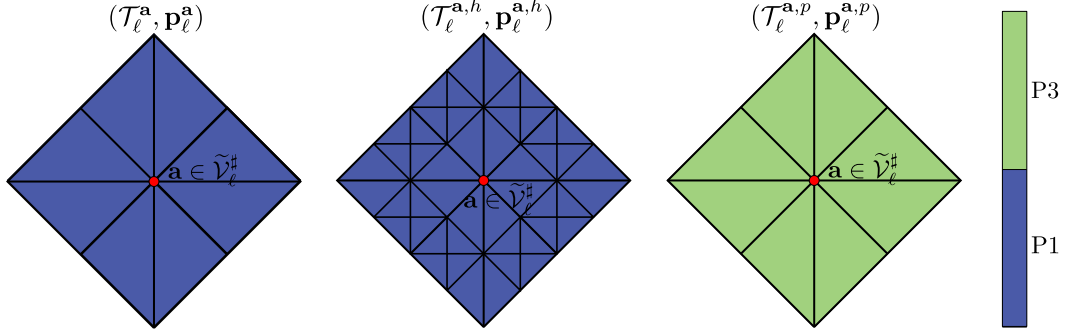
### 3.3 The module REFINE

The module **REFINE** takes as input the extended set of marked vertices  $\tilde{\mathcal{V}}_\ell^\sharp$  and outputs the mesh  $\mathcal{T}_{\ell+1}$  and the polynomial-degree distribution  $\mathbf{p}_{\ell+1}$  to be used at the next iteration of the adaptive loop from Scheme 1. In the following, we only highlight the changes with respect to the module **REFINE** proposed in [17, 18] and provide a summary of the present slightly modified version in Algorithm 1.

We denote the polynomial-degree distribution in the patch  $\mathcal{T}_\ell^\mathbf{a}$  by the vector  $\mathbf{p}_\ell^\mathbf{a} := \{p_{\ell, K}\}_{K \in \mathcal{T}_\ell^\mathbf{a}}$ . The idea employed in [17, 18] is to emulate separately the effects of  $h$ - and  $p$ -refinement on a patch assigned to a marked vertex using two distinct local patch-based spaces. For this purpose, let us introduce a matching simplicial refinement  $\mathcal{T}_\ell^{\mathbf{a}, h}$  (see the center panel of Figure 2), obtained from  $\mathcal{T}_\ell^\mathbf{a}$  by dividing each simplex  $K \in \mathcal{T}_\ell^\mathbf{a}$  into at least six children simplices such that a node interior to all the elements  $K \in \mathcal{T}_\ell^\mathbf{a}$  as well as all the faces  $F \in \mathcal{F}_\ell^{\mathbf{a}, \text{int}}$  is created, cf. the requirement of having an interior node and its implementation in two and three dimensions in [31]. This requirement can actually be relaxed, as done previously in, e.g., [23, Lemma 11]. In particular, when the space dimension  $d = 2$  and either  $f = 0$  or  $p_{f, K} \leq p_{\ell, K} - 2$  for all  $K \in \mathcal{T}_\ell^\mathbf{a}$ , then only the faces from  $\mathcal{F}_\ell^{\mathbf{a}, \text{int}}$  need to have an interior node but not the elements from



$\mathcal{T}_\ell^{\mathbf{a}}$ . The polynomial-degree distribution  $\mathbf{p}_\ell^{\mathbf{a},h}$  corresponding to  $\mathcal{T}_\ell^{\mathbf{a},h}$  is obtained from  $\mathbf{p}_\ell^{\mathbf{a}}$  by assigning to each newly-created simplex the same polynomial degree as that of its parent, with a possible increase to match the polynomial degree of its marked neighbors. More precisely, for each  $K \in \mathcal{T}_\ell^{\mathbf{a}}$ , we set  $\bar{p}_{\ell,K}$  as the maximum of the polynomial degrees  $p_{\ell,K}$  and all  $p_{\ell,K'}$  where  $K' \in \mathcal{T}_\ell^{\mathbf{a}}$  shares a face with  $K$ . Next, let the polynomial-degree distribution vector  $\mathbf{p}_\ell^{\mathbf{a},p}$  be set by assigning to each simplex  $K \in \mathcal{T}_\ell^{\mathbf{a},p} := \mathcal{T}_\ell^{\mathbf{a}}$  the polynomial degree  $\max\{\bar{p}_{\ell,K} + d - 1, p_{f,K} + d + 1\}$ , with  $p_{f,K}$  the local polynomial degree of  $f$  on  $K$  (see the right panel of Figure 2). We note that the employed refinement methods slightly differ from those suggested in [17, 18]; in particular, the stricter requirements on  $h$ - and  $p$ -refinements are motivated by the theory developed in Section 4.



**Figure 2:** An example of patch  $\mathcal{T}_\ell^{\mathbf{a}}$  together with its polynomial-degree distribution  $\mathbf{p}_\ell^{\mathbf{a}}$  (left), its  $h$ -refined version (center), and its  $p$ -refined version (right).

Following [17, 18], the  $hp$ -decision is made on the basis of the two local primal solves on the patches  $\mathcal{T}_\ell^{\mathbf{a}}$ , but this time, for each vertex  $\mathbf{a}$  from the extended set of marked vertices  $\tilde{\mathcal{V}}_\ell^\#$ , not only for the vertices from the set  $\mathcal{V}_\ell^\theta$  as in [17, 18]. Hence, for each vertex  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#$ , we consider the local patch-based spaces  $V_\ell^{\mathbf{a},h}$  and  $V_\ell^{\mathbf{a},p}$  given by

$$V_\ell^{\mathbf{a},h} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},h}}(\mathcal{T}_\ell^{\mathbf{a},h}) \cap H_0^1(\omega_\ell^{\mathbf{a}}), \quad V_\ell^{\mathbf{a},p} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},p}}(\mathcal{T}_\ell^{\mathbf{a},p}) \cap H_0^1(\omega_\ell^{\mathbf{a}}). \quad (3.8)$$

We then let  $r^{\mathbf{a},h} \in V_\ell^{\mathbf{a},h}$  and  $r^{\mathbf{a},p} \in V_\ell^{\mathbf{a},p}$  be the  $h$ - and  $p$ -refinement residual liftings respectively given by

$$(\nabla r^{\mathbf{a},h}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\text{ex}}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},h} \in V_\ell^{\mathbf{a},h}, \quad (3.9a)$$

$$(\nabla r^{\mathbf{a},p}, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\text{ex}}, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},p} \in V_\ell^{\mathbf{a},p}. \quad (3.9b)$$

The two above local primal problems use Dirichlet boundary conditions; we refer to [18] for the possible use of Neumann boundary conditions. The rest of the module **REFINE** including the  $hp$ -decision criterion and an actual refinement leading to the new pair  $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$  is outlined in Algorithm 1. We note that in order to prove convergence, we need to limit the maximal polynomial degree by a user-defined value  $p_{\max}$ . In order to not exceed this threshold,  $h$ -refinement may be employed also for elements for which  $p$ -refinement is suggested by our refinement criterion, cf. lines 7–8 in Algorithm 1. Though this is a theoretical restriction, such limitations of maximal polynomial degree are typically present in practical implementations of  $hp$ -methods.

### 3.4 Discrete lower bound on the incremental error on marked simplices

Once the new pair  $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$  is determined within the **REFINE** module of Section 3.3 (recall that  $V_{\ell+1} := \mathbb{P}_{\mathbf{p}_{\ell+1}}(\mathcal{T}_{\ell+1}) \cap H_0^1(\Omega)$ ), the finite element space to be used on iteration  $(\ell+1)$  of the adaptive loop of Scheme 1 is at our disposal. We now proceed by extending the discrete lower bound of [17, Lemma 5.1] to the present setting. For all the vertices  $\mathbf{a}$  from the extended set of marked vertices  $\tilde{\mathcal{V}}_\ell^\#$ , let us set, in extension of (3.8),

$$V_\ell^{\mathbf{a},hp} := V_{\ell+1}|_{\omega_\ell^{\mathbf{a}}} \cap H_0^1(\omega_\ell^{\mathbf{a}}) \quad (3.10)$$



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**Algorithm 1 (module REFINE)**


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1: module REFINE ( $\tilde{\mathcal{V}}_\ell^\sharp$ )
2:    $\triangleright$  Input: extended set of marked vertices  $\tilde{\mathcal{V}}_\ell^\sharp$ 
3:    $\triangleright$  Output: new pair  $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$ 
4:   for all  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp$  do
5:     Compute the residual liftings  $r^{\mathbf{a},h}, r^{\mathbf{a},p}$  given by (3.9a) and (3.9b)
6:   end for

7:   Set  $\tilde{\mathcal{V}}_\ell^p := \{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp \mid \|\nabla r^{\mathbf{a},h}\|_{\omega_\ell^\mathbf{a}} < \|\nabla r^{\mathbf{a},p}\|_{\omega_\ell^\mathbf{a}} \text{ and } p_{\mathbf{a}}^{\text{est}} \leq (p_{\max} - d)\}$ 
8:   Set  $\tilde{\mathcal{V}}_\ell^h := \tilde{\mathcal{V}}_\ell^\sharp \setminus \tilde{\mathcal{V}}_\ell^p$ 

9:   Select  $\mathcal{M}_\ell^h := \{K \in \mathcal{T}_\ell \mid \mathcal{V}_{\ell,K} \cap \tilde{\mathcal{V}}_\ell^h \neq \emptyset\} \subset \mathcal{M}_\ell^\sharp$ 
10:  Select  $\mathcal{M}_\ell^p := \{K \in \mathcal{T}_\ell \mid \mathcal{V}_{\ell,K} \cap \tilde{\mathcal{V}}_\ell^p \neq \emptyset\} \subset \mathcal{M}_\ell^\sharp$ 

11:  Build  $\mathcal{T}_{\ell+1}$  from  $\mathcal{T}_\ell$  and  $\mathcal{M}_\ell^h$ , such that  $\mathcal{T}_\ell^{\mathbf{a},h} \subset \mathcal{T}_{\ell+1} \ \forall \mathbf{a} \in \tilde{\mathcal{V}}_\ell^h$ 
12:  For each face  $\tilde{F} \in \mathcal{F}_\ell^{\text{int}}$ , set  $p_{\ell,\tilde{F}} := \max\{p_{\ell,\tilde{K}_1}, p_{\ell,\tilde{K}_2}\}$  the maximal polynomial degree of the elements
13:     $\tilde{K}_1, \tilde{K}_2$  that share it
14:  For each element  $\tilde{K} \in \mathcal{T}_\ell$ , set  $\bar{p}_{\ell,\tilde{K}} := \max_{\tilde{F} \in \mathcal{V}_{\ell,\tilde{K}}, \tilde{F} \in \cup_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp} \mathcal{F}_\ell^{\mathbf{a},\text{int}}} p_{\ell,\tilde{F}}$ 

15:  for all  $K \in \mathcal{T}_{\ell+1}$  with its parent element  $\tilde{K} \in \mathcal{T}_\ell$  do
16:     $p_{\ell+1,K} := \begin{cases} \bar{p}_{\ell,\tilde{K}} & \text{if } \tilde{K} \notin \mathcal{M}_\ell^p \\ \max\{\bar{p}_{\ell,\tilde{K}} + d - 1, p_{f,\tilde{K}} + d + 1\} & \text{if } \tilde{K} \in \mathcal{M}_\ell^p \end{cases}$ 
17:  end for
18: end module

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and construct the residual lifting  $r^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}$  by solving

$$(\nabla r^{\mathbf{a},hp}, \nabla v^{\mathbf{a},hp})_{\omega_\ell^\mathbf{a}} = (f, v^{\mathbf{a},hp})_{\omega_\ell^\mathbf{a}} - (\nabla u_\ell^{\text{ex}}, \nabla v^{\mathbf{a},hp})_{\omega_\ell^\mathbf{a}} \quad \forall v^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}. \quad (3.11)$$

Then, after extending  $r^{\mathbf{a},hp}$  by zero outside  $\omega_\ell^\mathbf{a}$ , we have for the current approximation  $u_\ell^{\text{ex}} \in V_\ell$  and the next level's approximation  $u_{\ell+1}^{\text{ex}} \in V_{\ell+1}$ , by [17, Lemma 5.1] used with the extended set  $\tilde{\mathcal{V}}_\ell^\sharp$  in place of  $\mathcal{V}_\ell^\theta$ :

$$\|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell^{\text{ex}})\|_{\omega_\ell^\sharp} \geq \eta_{\mathcal{M}_\ell^\sharp}, \quad \eta_{\mathcal{M}_\ell^\sharp} := \begin{cases} \frac{\sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp} \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^\mathbf{a}}^2}{\left\| \nabla \left( \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp} r^{\mathbf{a},hp} \right) \right\|_{\omega_\ell^\sharp}^2} & \text{if } \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp} r^{\mathbf{a},hp} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.12)$$

Moreover, the above lower bound can be further localized using the fact that each simplex has  $(d+1)$  vertices as

$$\eta_{\mathcal{M}_\ell^\sharp} \geq \frac{\{\sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp} \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^\mathbf{a}}^2\}^{1/2}}{\sqrt{d+1}}; \quad (3.13)$$

This can be seen from

$$\begin{aligned} \left\| \nabla \left( \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp} r^{\mathbf{a},hp} \right) \right\|_{\omega_\ell^\sharp}^2 &= \sum_{K \in \mathcal{M}_\ell^\sharp} \left\| \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp \cap \mathcal{V}_K} \nabla r^{\mathbf{a},hp} \right\|_K^2 \\ &\leq \sum_{K \in \mathcal{M}_\ell^\sharp} (d+1) \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp \cap \mathcal{V}_K} \|\nabla r^{\mathbf{a},hp}\|_K^2 \\ &= (d+1) \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp} \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^\mathbf{a}}^2. \end{aligned}$$

## 4 Discrete stability of equilibrated fluxes in an exact setting

Our main tool will be the following:

**Proposition 4.1** (Discrete stability of the local flux equilibration). *Let  $u_\ell^{\text{ex}} \in V_\ell$  satisfy the hat function orthogonality*

$$(f, \psi_\ell^{\mathbf{a}})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\text{ex}}, \nabla \psi_\ell^{\mathbf{a}})_{\omega_\ell^{\mathbf{a}}} = 0 \quad \forall \mathbf{a} \in \mathcal{V}_\ell^{\text{int}},$$

*let the local equilibrated flux  $\sigma_\ell^{\mathbf{a}}$  be constructed by (3.4), and let  $r^{\mathbf{a},hp}$  be given by (3.11). Then there exists a constant  $C_{\text{st}} \geq 1$  only depending on the space dimension  $d$ , the mesh shape-regularity  $\kappa_{\mathcal{T}}$ , and the maximal polynomial degree  $p_{\text{max}}$  such that*

$$\|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \sigma_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}} \leq C_{\text{st}} \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}} \quad \forall \mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#.$$
 (4.1)

To prove Proposition 4.1 we will rely on two auxiliary results employing the bubble function technique, cf. [42]. From now on, we use the shorthand notation  $x_1 \lesssim x_2$  when there exists a generic positive constant  $C$  that only depends on the space dimension  $d$ , the shape-regularity  $\kappa_{\mathcal{T}}$  of the underlying hierarchy of meshes, and the polynomial degree of approximation employed locally such that  $x_1 \leq Cx_2$ .

**Lemma 4.2** (Discrete stability of the element residuals). *Let a vertex  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#$  and the corresponding  $r^{\mathbf{a},hp}$  of (3.11) be given. Then*

$$h_K \|f + \Delta u_\ell^{\text{ex}}\|_K \lesssim \|\nabla r^{\mathbf{a},hp}\|_K \quad \forall K \in \mathcal{T}_\ell^{\mathbf{a}}.$$
 (4.2)

*Proof.* Fix an element  $K \in \mathcal{T}_\ell^{\mathbf{a}}$  and set

$$v_K := (f + \Delta u_\ell^{\text{ex}})|_K.$$
 (4.3)

We note that  $v_K \in \mathbb{P}_{\max\{p_{\ell,K}-2, p_{f,K}\}}(K)$ , using Assumption 3.1. Let us define  $\psi_K$ , the bubble function on element  $K$ , depending on which refinement has been applied within the **REFINE** module of Section 3.3 for the vertex  $\mathbf{a}$ , to be a function

$$\psi_K \in \begin{cases} \mathbb{P}_1(\mathcal{T}_{\ell+1}|_K) \cap H_0^1(K) & \text{if } \mathbf{a} \in \tilde{\mathcal{V}}_\ell^h \\ \mathbb{P}_{d+1}(K) \cap H_0^1(K) & \text{if } \mathbf{a} \in \tilde{\mathcal{V}}_\ell^p, \end{cases} \quad \text{with } \|\psi_K\|_{\infty,K} = 1;$$
 (4.4)

more precisely, in the first case,  $\psi_K$  is a piecewise affine function taking nonzero values in the(/all) interior nodes in  $K$ , whereas in the second case,  $\psi_K$  is the usual element bubble function. The equivalence of norms on finite-dimensional spaces gives

$$(v_K, v_K)_K \lesssim (v_K, \psi_K v_K)_K.$$
 (4.5)

This in combination with (4.3) yields

$$\|v_K\|_K^2 \lesssim (v_K, \psi_K v_K)_K = (f, \psi_K v_K)_K + (\Delta u_\ell^{\text{ex}}, \psi_K v_K)_K.$$

Then, noting that  $(\nabla u_\ell^{\text{ex}})|_K \in \mathbf{H}(\text{div}, K)$  and  $\psi_K v_K \in H_0^1(K)$ , employing the Green theorem and using (3.11) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \|v_K\|_K^2 &\lesssim (f, \psi_K v_K)_K - (\nabla u_\ell^{\text{ex}}, \nabla(\psi_K v_K))_K = (\nabla r^{\mathbf{a},hp}, \nabla(\psi_K v_K))_K, \\ &\leq \|\nabla r^{\mathbf{a},hp}\|_K \|\nabla(\psi_K v_K)\|_K, \end{aligned}$$
 (4.6)

where we have crucially employed that  $\psi_K v_K$  extended by zero is contained in the space  $V_\ell^{\mathbf{a},hp}$  due to (3.10) and (4.4). In particular, we use the fact that either  $h$ -refinement has been employed and then  $\mathcal{T}_{\ell+1}|_K$  contains an interior node, or  $p$ -refinement has been applied and then the polynomial degree has been increased to at least  $\max\{p_{\ell,K} + d - 1, p_{f,K} + d + 1\}$ . Then, using the inverse inequality (cf., e.g. [36, Proposition 6.3.2]), we have

$$\|\nabla(\psi_K v_K)\|_K \lesssim h_K^{-1} \|\psi_K v_K\|_K.$$
 (4.7)

Moreover, from the definition of the bubble function  $\psi_K$ , there holds

$$\|\psi_K v_K\|_K \leq \|\psi_K\|_{\infty,K} \|v_K\|_K = \|v_K\|_K.$$
 (4.8)

Finally, (4.6) with (4.7) and (4.8) lead to the assertion of the lemma when there is an interior node for all the elements  $K \in \mathcal{T}_\ell^{\mathbf{a}}$  as well as all the faces  $F \in \mathcal{F}_\ell^{\mathbf{a}, \text{int}}$ .

For the relaxation mentioned in Section 3.3, where only the faces from  $\mathcal{F}_\ell^{\mathbf{a}, \text{int}}$  have an interior node for the case  $d = 2$  and either  $f = 0$  or  $p_{f,K} \leq p_{\ell,K} - 2$  for all  $K \in \mathcal{T}_\ell^{\mathbf{a}}$ , there is no change for the  $p$ -refinement. For the  $h$ -refinement, one rather uses  $\psi_K \in \mathbb{P}_2(\mathcal{T}_{\ell+1}|_K) \cap H_0^1(K)$  with  $\|\psi_K\|_{\infty,K} = 1$ , the edge bubble function of the newly-added edge, in place of the first choice in (4.4).  $\square$

**Lemma 4.3** (Discrete stability of the face residuals). *Let a vertex  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#$  and the corresponding  $r^{\mathbf{a}, hp}$  of (3.11) be given. Let  $\mathcal{T}_F$  denote the set of elements containing the two elements  $K \in \mathcal{T}_\ell^{\mathbf{a}}$  that share a face  $F \in \mathcal{F}_\ell^{\mathbf{a}, \text{int}}$ , and let  $\omega_F$  be the corresponding open subdomain. Then*

$$h_F^{\frac{1}{2}} \|\llbracket \nabla u_\ell^{\text{ex}} \cdot \mathbf{n}_F \rrbracket\|_F \lesssim \|\nabla r^{\mathbf{a}, hp}\|_{\omega_F} \quad \forall F \in \mathcal{F}_\ell^{\mathbf{a}, \text{int}}.$$

*Proof.* Fix an edge  $F \in \mathcal{F}_\ell^{\mathbf{a}, \text{int}}$ . This time, we set

$$v_F := \llbracket \nabla u_\ell^{\text{ex}} \cdot \mathbf{n}_F \rrbracket_F \quad (4.9)$$

and note that  $v_F \in \mathbb{P}_{\max\{p_{\ell,K}-1, p_{\ell,K'}-1\}}$ , where  $K, K' \in \mathcal{T}_\ell^{\mathbf{a}}$  share the face  $F$ ; this is the motivation for the local polynomial degree increase in the REFIN module from Section 3.3. Let  $\psi_F$  be the bubble function on  $\mathcal{T}_F$ , depending on the refinement method applied within the REFIN module of Section 3.3; namely, for the vertex  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#$ ,

$$\psi_F \in \begin{cases} \mathbb{P}_1(\mathcal{T}_{\ell+1}|\mathcal{T}_F) \cap H_0^1(\omega_F) & \text{if } \mathbf{a} \in \tilde{\mathcal{V}}_\ell^h \\ \mathbb{P}_d(\mathcal{T}_F) \cap H_0^1(\omega_F) & \text{if } \mathbf{a} \in \tilde{\mathcal{V}}_\ell^p. \end{cases} \quad \text{with } \|\psi_F\|_{\infty, \omega_F} = 1; \quad (4.10)$$

similarly to (4.4),  $\psi_K$  is respectively a piecewise affine function taking nonzero values in the(/all) interior nodes in  $F$ , or the usual face bubble function. Then, by equivalence of norms on finite-dimensional spaces, similarly to (4.5), there holds

$$(v_F, v_F)_F \lesssim (v_F, \psi_F v_F)_F. \quad (4.11)$$

Let us keep the same notation for the extension of the function  $v_F$ , with its original domain of definition being only the face  $F$ , to a function defined on the two simplices of  $\mathcal{T}_F$ . The extension is done by constant values in the direction of the barycenter of the face towards the vertex opposite to  $F$ . Then, the following estimate holds true

$$\|v_F\|_{\omega_F} \lesssim h_F^{\frac{1}{2}} \|v_F\|_F. \quad (4.12)$$

Employing (4.11), (4.9), and expanding the jump term with  $\mathcal{T}_F := \{K, K'\}$  and the convention that  $\mathbf{n}_F$  points from  $K$  to  $K'$ , we have

$$\begin{aligned} \|v_F\|_F^2 &\lesssim (v_F, \psi_F v_F)_F = ((\nabla u_\ell^{\text{ex}}|_K)|_F \cdot \mathbf{n}_F, \psi_F v_F)_F - ((\nabla u_\ell^{\text{ex}}|_{K'})|_F \cdot \mathbf{n}_F, \psi_F v_F)_F \\ &= ((\nabla u_\ell^{\text{ex}}|_K)|_{\partial K} \cdot \mathbf{n}_{\partial K}, \psi_F v_F)_{\partial K} + ((\nabla u_\ell^{\text{ex}}|_{K'})|_{\partial K'} \cdot \mathbf{n}_{\partial K'}, \psi_F v_F)_{\partial K'}. \end{aligned} \quad (4.13)$$

We have also used the fact that  $(\psi_F v_F)|_{\partial \omega_F} = 0$  due to the definition of the bubble function  $\psi_F$ . Moreover, for each simplex  $K \in \mathcal{T}_F$ ,  $(\psi_F v_F)|_K \in H^1(K)$  and  $(\nabla u_\ell^{\text{ex}}|_K) \in \mathbf{H}(\text{div}, K)$ , so we are able to apply the Green theorem for both terms on the right-hand side of (4.13). Adding and subtracting  $(f, \psi_F v_F)_{\omega_F}$ , using that  $\psi_F v_F$  extended by zero is in  $V_\ell^{\mathbf{a}, hp}$ , and recalling the discrete problem (3.11), we finally obtain

$$\begin{aligned} \|v_F\|_F^2 &\lesssim (\nabla u_\ell^{\text{ex}}, \nabla(\psi_F v_F))_{\omega_F} - (f, \psi_F v_F)_{\omega_F} + (f + \Delta u_\ell^{\text{ex}}, \psi_F v_F)_{\omega_F} \\ &= -(\nabla r^{\mathbf{a}, hp}, \nabla(\psi_F v_F))_{\omega_F} + (f + \Delta u_\ell^{\text{ex}}, \psi_F v_F)_{\omega_F}. \end{aligned} \quad (4.14)$$

Again, the face interior node property/appropriate polynomial degree increase were important to ensure that  $\psi_F v_F$  extended by zero is in  $V_\ell^{\mathbf{a}, hp}$ . From the definition of the bubble function  $\psi_F$ , there holds

$$\|\psi_F v_F\|_{\omega_F} \leq \|\psi_F\|_{\infty, \omega_F} \|v_F\|_{\omega_F} = \|v_F\|_{\omega_F}, \quad (4.15)$$

while the inverse inequality and the shape-regularity of the mesh  $\mathcal{T}_\ell$  give

$$\|\nabla(\psi_F v_F)\|_{\omega_F} \lesssim h_F^{-1} \|\psi_F v_F\|_{\omega_F}. \quad (4.16)$$

Then, the following chain of inequalities holds true due to the Cauchy–Schwarz inequality, (4.15), (4.16), and (4.12):

$$\begin{aligned}\|v_F\|_F^2 &\lesssim \|\nabla r^{\mathbf{a},hp}\|_{\omega_F} \|\nabla(\psi_F v_F)\|_{\omega_F} + \|f + \Delta u_\ell^{\text{ex}}\|_{\omega_F} \|\psi_F v_F\|_{\omega_F} \\ &\lesssim \|\nabla r^{\mathbf{a},hp}\|_{\omega_F} h_F^{-1} \|v_F\|_{\omega_F} + \|f + \Delta u_\ell^{\text{ex}}\|_{\omega_F} \|v_F\|_{\omega_F} \\ &\lesssim (\|\nabla r^{\mathbf{a},hp}\|_{\omega_F} + h_F \|f + \Delta u_\ell^{\text{ex}}\|_{\omega_F}) h_F^{-\frac{1}{2}} \|v_F\|_F.\end{aligned}$$

The assertion of the lemma then follows from the above result combined with (4.2) and the shape-regularity of the mesh  $\mathcal{T}_\ell$ .  $\square$

*Proof of Proposition 4.1.* Let  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\sharp$  be fixed. We will crucially employ two reformulations of the local minimization problem (3.4). The Euler–Lagrange conditions for (3.4), imposing the divergence constraint via a Lagrange multiplier, amount to solving the local Neumann mixed finite element problem: seek the pair  $(\boldsymbol{\sigma}_\ell^{\mathbf{a}}, \delta_\ell^{\mathbf{a}}) \in \mathbf{V}_\ell^{\mathbf{a}} \times Q_\ell^{\mathbf{a}}$  such that

$$(\boldsymbol{\sigma}_\ell^{\mathbf{a}}, \mathbf{v}_\ell)_{\omega_\ell^{\mathbf{a}}} - (\delta_\ell^{\mathbf{a}}, \nabla \cdot \mathbf{v}_\ell)_{\omega_\ell^{\mathbf{a}}} = -(\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}}, \mathbf{v}_\ell)_{\omega_\ell^{\mathbf{a}}} \quad \forall \mathbf{v}_\ell \in \mathbf{V}_\ell^{\mathbf{a}}, \quad (4.17a)$$

$$(\nabla \cdot \boldsymbol{\sigma}_\ell^{\mathbf{a}}, q_\ell)_{\omega_\ell^{\mathbf{a}}} = (\psi_\ell^{\mathbf{a}} f - \nabla \psi_\ell^{\mathbf{a}} \cdot \nabla u_\ell^{\text{ex}}, q_\ell)_{\omega_\ell^{\mathbf{a}}} \quad \forall q_\ell \in Q_\ell^{\mathbf{a}}, \quad (4.17b)$$

with the patchwise discontinuous piecewise polynomial spaces  $Q_\ell^{\mathbf{a}}$  defined by

$$Q_\ell^{\mathbf{a}} := \begin{cases} \{q_\ell \in \mathbb{P}_{p_{\mathbf{a}}^{\text{est}}}(\mathcal{T}_\ell^{\mathbf{a}}); (q_\ell, 1)_{\omega_\ell^{\mathbf{a}}} = 0\} & \text{if } \mathbf{a} \in \mathcal{V}_\ell^{\text{int}}, \\ \mathbb{P}_{p_{\mathbf{a}}^{\text{est}}}(\mathcal{T}_\ell^{\mathbf{a}}) & \text{if } \mathbf{a} \in \mathcal{V}_\ell^{\text{ext}}. \end{cases} \quad (4.18)$$

Next, imposing the normal trace continuity constraint on faces from  $\mathcal{F}_\ell^{\mathbf{a},\text{int}}$  and the no-flux condition on all faces from  $\mathcal{F}_\ell^{\mathbf{a},\text{ext}}$  for  $\mathbf{a} \in \mathcal{V}_\ell^{\text{int}}$  and only faces from  $\mathcal{F}_\ell^{\mathbf{a},\text{ext}}$  not lying on  $\partial\Omega$  for  $\mathbf{a} \in \mathcal{V}_\ell^{\text{ext}}$  via a Lagrange multiplier leads to the hybridized formulation: Seek  $\boldsymbol{\sigma}_\ell^{\mathbf{a}}$  in the broken space  $\mathbf{RTN}_{p_{\mathbf{a}}^{\text{est}}}(\mathcal{T}_\ell^{\mathbf{a}})$ ,  $\delta_\ell^{\mathbf{a}} \in Q_\ell^{\mathbf{a}}$ , and  $\lambda_\ell^F \in \mathbb{P}_{p_{\mathbf{a}}^{\text{est}}}(F)$  for all  $F \in \mathcal{F}_\ell^{\mathbf{a}} \setminus \mathcal{F}_\ell^{\mathbf{a},\partial\Omega}$  such that

$$\begin{aligned}\sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} (\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}, \mathbf{v}_\ell)_K - \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} (\nabla \cdot \mathbf{v}_\ell, \delta_\ell^{\mathbf{a}})_K + \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} \sum_{F \in \mathcal{F}_{\ell,K} \setminus \mathcal{F}_\ell^{\mathbf{a},\partial\Omega}} (\mathbf{v}_\ell \cdot \mathbf{n}_K, \lambda_\ell^F)_F = 0 \\ \forall \mathbf{v}_\ell \in \mathbf{RTN}_{p_{\mathbf{a}}^{\text{est}}}(\mathcal{T}_\ell^{\mathbf{a}}),\end{aligned} \quad (4.19a)$$

$$\sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} (\nabla \cdot \boldsymbol{\sigma}_\ell^{\mathbf{a}}, q_\ell)_K = \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} (\psi_\ell^{\mathbf{a}} f - \nabla \psi_\ell^{\mathbf{a}} \cdot \nabla u_\ell^{\text{ex}}, q_\ell)_K \quad \forall q_\ell \in Q_\ell^{\mathbf{a}}, \quad (4.19b)$$

$$-\sum_{K \in \mathcal{T}_F} (\boldsymbol{\sigma}_\ell^{\mathbf{a}}, \mathbf{n}_K, \xi_\ell)_F = 0 \quad \forall \xi_\ell \in \mathbb{P}_{p_{\mathbf{a}}^{\text{est}}}(F), \forall F \in \mathcal{F}_\ell^{\mathbf{a}} \setminus \mathcal{F}_\ell^{\mathbf{a},\partial\Omega}, \quad (4.19c)$$

where we used the notation

$$\mathcal{F}_\ell^{\mathbf{a},\partial\Omega} := \begin{cases} \emptyset & \text{if } \mathbf{a} \in \mathcal{V}_\ell^{\text{int}}, \\ \mathcal{F}_\ell^{\mathbf{a},\text{ext}} \cap \mathcal{F}_\ell^{\text{ext}} & \text{if } \mathbf{a} \in \mathcal{V}_\ell^{\text{ext}}. \end{cases} \quad (4.20)$$

We note that  $\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}} \in \mathbf{RTN}_{p_{\mathbf{a}}^{\text{est}}}(\mathcal{T}_\ell^{\mathbf{a}})$ , as (3.3) uses the maximal local polynomial degree  $p_{\mathbf{a}}^{\text{est}}$ . Thus we can use it as a test function  $\mathbf{v}_\ell$  in (4.19a). Afterwards, employing  $\delta_\ell^{\mathbf{a}}$  as a test function  $q_\ell$  in (4.19b) and  $\lambda_\ell^F$  as  $\xi_\ell$  in (4.19c), summing (4.19c) over all  $F \in \mathcal{F}_\ell^{\mathbf{a}} \setminus \mathcal{F}_\ell^{\mathbf{a},\partial\Omega}$ , and finally summing (4.19a)–(4.19c), we obtain

$$\begin{aligned}\|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}^2 &= \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} (\psi_\ell^{\mathbf{a}} f - \nabla \psi_\ell^{\mathbf{a}} \cdot \nabla u_\ell^{\text{ex}} + \nabla \cdot (\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}}), \delta_\ell^{\mathbf{a}})_K \\ &\quad - \sum_{F \in \mathcal{F}_\ell^{\mathbf{a}} \setminus \mathcal{F}_\ell^{\mathbf{a},\partial\Omega}} ([\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} \cdot \mathbf{n}_F], \lambda_\ell^F)_F \\ &= \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} (\psi_\ell^{\mathbf{a}} (f + \Delta u_\ell^{\text{ex}}), \delta_\ell^{\mathbf{a}})_K - \sum_{F \in \mathcal{F}_\ell^{\mathbf{a}} \setminus \mathcal{F}_\ell^{\mathbf{a},\partial\Omega}} (\psi_\ell^{\mathbf{a}} [\nabla u_\ell^{\text{ex}} \cdot \mathbf{n}_F], \lambda_\ell^F)_F \\ &\leq \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} \|\psi_\ell^{\mathbf{a}} (f + \Delta u_\ell^{\text{ex}})\|_K \|\delta_\ell^{\mathbf{a}}\|_K + \sum_{F \in \mathcal{F}_\ell^{\mathbf{a},\text{int}}} \|\psi_\ell^{\mathbf{a}} [\nabla u_\ell^{\text{ex}} \cdot \mathbf{n}_F]\|_F \|\lambda_\ell^F\|_F,\end{aligned} \quad (4.21)$$

where for the estimate on the right-hand side of (4.21) we have also used the fact that the hat function  $\psi_\ell^{\mathbf{a}}|_F = 0 \ \forall F \in \mathcal{F}_\ell^{\mathbf{a},\text{ext}}$  for vertices  $\mathbf{a} \in \mathcal{V}_\ell^{\text{int}}$ , and the boundary faces are excluded from the sum and  $\psi_\ell^{\mathbf{a}}|_F = 0 \ \forall F \in \mathcal{F}_\ell^{\mathbf{a},\text{ext}} \setminus \mathcal{F}_\ell^{\mathbf{a},\partial\Omega}$  for  $\mathbf{a} \in \mathcal{V}_\ell^{\text{ext}}$ .

We now proceed by bounding the terms in the estimate of (4.21). Firstly, by the inf-sup stability of the mixed discretization (4.17), see e.g. [9] or [43, Theorem 5.9], we have

$$\|\delta_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}} \lesssim h_{\omega_\ell^{\mathbf{a}}} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}. \quad (4.22)$$

Secondly,  $\|\psi_\ell^{\mathbf{a}}\|_{\infty,K} \leq 1$  together with Lemma 4.2 yield the estimate

$$\|\psi_\ell^{\mathbf{a}}(f + \Delta u_\ell^{\text{ex}})\|_K \leq \|f + \Delta u_\ell^{\text{ex}}\|_K \lesssim h_K^{-1} \|\nabla r^{\mathbf{a},hp}\|_K \quad \forall K \in \mathcal{T}_\ell^{\mathbf{a}}. \quad (4.23)$$

Combining (4.21) with (4.23), the Cauchy–Schwarz inequality, the shape-regularity yielding  $h_K \approx h_{\omega_\ell^{\mathbf{a}}}$ , and (4.22), we obtain the estimate on the first term

$$\begin{aligned} \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} \|\psi_\ell^{\mathbf{a}}(f + \Delta u_\ell^{\text{ex}})\|_K \|\delta_\ell^{\mathbf{a}}\|_K &\lesssim \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} h_K^{-1} \|\nabla r^{\mathbf{a},hp}\|_K \|\delta_\ell^{\mathbf{a}}\|_K \\ &\lesssim h_{\omega_\ell^{\mathbf{a}}}^{-1} \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}} \|\delta_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}} \\ &\lesssim \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}. \end{aligned} \quad (4.24)$$

We continue by bounding the second term in the estimate of (4.21). By the characterization of the degrees of freedom in the Raviart–Thomas–Nédélec spaces, if  $F \in \mathcal{F}_\ell^{\mathbf{a},\text{int}}$  is a face of an element  $K \in \mathcal{T}_\ell^{\mathbf{a}}$ , we have

$$\|\lambda_\ell^F\|_F = \sup_{\substack{\mathbf{v}_\ell \in \mathbf{RTN}_{p_{\mathbf{a}}^{\text{est}}}(K) \\ \mathbf{v}_\ell \cdot \mathbf{n}_K|_F \neq 0 \\ \mathbf{v}_\ell \cdot \mathbf{n}_K|_{F'} = 0 \ \forall F' \in \mathcal{F}_\ell, K, F' \neq F \\ (\mathbf{v}_\ell, \mathbf{r}_\ell)_K = 0 \ \forall \mathbf{r}_\ell \in [\mathbb{P}_{p_{\mathbf{a}}^{\text{est}}-1}(K)]^d}} \frac{(\mathbf{v}_\ell \cdot \mathbf{n}_K, \lambda_\ell^F)_F}{\|\mathbf{v}_\ell \cdot \mathbf{n}_K\|_F}. \quad (4.25)$$

Fix  $\mathbf{v}_\ell \in \mathbf{RTN}_{p_{\mathbf{a}}^{\text{est}}}(K)$  with the constraints as in (4.25). Using this  $\mathbf{v}_\ell$  as a test function in (4.19a) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} (\mathbf{v}_\ell \cdot \mathbf{n}_K, \lambda_\ell^F)_F &= (\nabla \cdot \mathbf{v}_\ell, \delta_\ell^{\mathbf{a}})_K - (\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}, \mathbf{v}_\ell)_K \\ &\leq \|\nabla \cdot \mathbf{v}_\ell\|_K \|\delta_\ell^{\mathbf{a}}\|_K + \|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_K \|\mathbf{v}_\ell\|_K. \end{aligned} \quad (4.26)$$

We now treat the two terms of (4.26) separately. For the first term, we start by employing the inverse inequality

$$\|\nabla \cdot \mathbf{v}_\ell\|_K \lesssim h_K^{-1} \|\mathbf{v}_\ell\|_K.$$

Furthermore, by scaling arguments, under the constraints of (4.25), we have

$$\|\mathbf{v}_\ell\|_K \lesssim h_F^{\frac{1}{2}} \|\mathbf{v}_\ell \cdot \mathbf{n}_K\|_F. \quad (4.27)$$

The first term of (4.26) can then be bounded as follows:

$$\begin{aligned} \|\nabla \cdot \mathbf{v}_\ell\|_K \|\delta_\ell^{\mathbf{a}}\|_K &\lesssim h_K^{-1} \|\mathbf{v}_\ell\|_K \|\delta_\ell^{\mathbf{a}}\|_K \lesssim h_F^{-\frac{1}{2}} \|\mathbf{v}_\ell \cdot \mathbf{n}_K\|_F \|\delta_\ell^{\mathbf{a}}\|_K \\ &\lesssim h_F^{\frac{1}{2}} \|\mathbf{v}_\ell \cdot \mathbf{n}_K\|_F \|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}, \end{aligned} \quad (4.28)$$

where we have also used the bound (4.22) and the mesh shape-regularity yielding  $h_{\omega_\ell^{\mathbf{a}}} \approx h_K \approx h_F$ . On the other hand, (4.27) leads to the following bound on the second term of (4.26)

$$\|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_K \|\mathbf{v}_\ell\|_K \lesssim h_F^{\frac{1}{2}} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_K \|\mathbf{v}_\ell \cdot \mathbf{n}_K\|_F. \quad (4.29)$$

In order to bound the supremum in (4.25), we combine (4.28), (4.29), and (4.26), to get

$$\|\lambda_\ell^F\|_F \lesssim h_F^{\frac{1}{2}} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}. \quad (4.30)$$

Afterwards,  $\|\psi_\ell^{\mathbf{a}}\|_F \leq 1$  and Lemma 4.3 yield

$$\|\psi_\ell^{\mathbf{a}}[\nabla u_\ell^{\text{ex}} \cdot \mathbf{n}_F]\|_F \leq \|[\nabla u_\ell^{\text{ex}}] \cdot \mathbf{n}_F\|_F \lesssim h_F^{-\frac{1}{2}} \|\nabla r^{\mathbf{a},hp}\|_{\omega_F}. \quad (4.31)$$

Using (4.30), (4.31), the shape-regularity yielding  $h_F \approx h_{F'}$ ,  $\forall F, F' \in \mathcal{F}_\ell^{\mathbf{a},\text{int}}$ , and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \sum_{F \in \mathcal{F}_\ell^{\mathbf{a},\text{int}}} \|\psi_\ell^{\mathbf{a}}[\nabla u_\ell^{\text{ex}} \cdot \mathbf{n}_F]\|_F \|\lambda_\ell^F\|_F &\lesssim \sum_{F \in \mathcal{F}_\ell^{\mathbf{a},\text{int}}} \|\psi_\ell^{\mathbf{a}}[\nabla u_\ell^{\text{ex}} \cdot \mathbf{n}_F]\|_F h_F^{\frac{1}{2}} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}} \\ &\lesssim \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}. \end{aligned} \quad (4.32)$$

Finally, combining the bounds (4.24) and (4.32) with (4.21) proves the estimate (4.1).  $\square$

## 5 Proof of guaranteed contraction with an exact solver

Our first main result is:

**Theorem 5.1** (Guaranteed contraction). *Let  $\ell \geq 0$ , let the pair  $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$  be obtained by the module REFINE of Section 3.3, and let  $V_{\ell+1} = \mathbb{P}_{\mathbf{p}_{\ell+1}}(\mathcal{T}_{\ell+1}) \cap H_0^1(\Omega)$  be the finite element space to be used on iteration  $(\ell+1)$  of the adaptive algorithm prescribed by Scheme 1. Let also  $\underline{\eta}_{\mathcal{M}_\ell^\#}$  be the computable discrete lower bound defined by (3.12). Then, two options arise. Either  $\eta(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta) = 0$ , in which case  $u_\ell = u$ , and the adaptive loop terminates. Or the new numerical solution  $u_{\ell+1}^{\text{ex}} \in V_{\ell+1}$  satisfies*

$$\|\nabla(u - u_{\ell+1}^{\text{ex}})\| \leq C_{\ell,\text{red}} \|\nabla(u - u_\ell^{\text{ex}})\| \quad (5.1)$$

with

$$0 \leq C_{\ell,\text{red}} := \sqrt{1 - \theta^2 \frac{\underline{\eta}_{\mathcal{M}_\ell^\#}^2}{\eta^2(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta)}} \leq C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}} < 1, \quad (5.2)$$

where  $C_{\ell,\text{red}}$  is a fully computable bound on the error reduction factor and  $C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}$  is a generic constant only depending on the marking parameter  $\theta$ , the space dimension  $d$ , the mesh shape-regularity  $\kappa_{\mathcal{T}}$ , and the maximal polynomial degree  $p_{\max}$ .

*Proof.* Let us assume that  $\eta(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta) \neq 0$ , the other case being trivial. The proof proceeds in two steps. In the first step, we follow the proof of [17, Theorem 5.2], see also proofs of [31, Theorem 3.1], [41, Theorem 5.3], hinging on the Pythagorean relation

$$\|\nabla(u - u_{\ell+1}^{\text{ex}})\|^2 = \|\nabla(u - u_\ell^{\text{ex}})\|^2 - \|\nabla(u_\ell^{\text{ex}} - u_{\ell+1}^{\text{ex}})\|^2. \quad (5.3)$$

By the computable lower bound on the incremental error (3.12) and the marking criterion (3.6), we have

$$\begin{aligned} \|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell^{\text{ex}})\| &\geq \|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell^{\text{ex}})\|_{\omega_\ell^\#} \geq \underline{\eta}_{\mathcal{M}_\ell^\#} \\ &= \frac{\underline{\eta}_{\mathcal{M}_\ell^\#}}{\eta(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta)} \eta(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta) \geq \theta \frac{\underline{\eta}_{\mathcal{M}_\ell^\#}}{\eta(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta)} \eta(u_\ell^{\text{ex}}, \mathcal{T}_\ell). \end{aligned} \quad (5.4)$$

Hence, combining (5.3) with (5.4) and using the error estimate (3.5), we infer

$$\|\nabla(u - u_{\ell+1}^{\text{ex}})\|^2 \leq \underbrace{\left(1 - \theta^2 \frac{\underline{\eta}_{\mathcal{M}_\ell^\#}^2}{\eta^2(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta)}\right)}_{=C_{\ell,\text{red}}^2} \|\nabla(u - u_\ell^{\text{ex}})\|^2. \quad (5.5)$$

In a second step, we show that the reduction factor  $C_{\ell,\text{red}}$ , defined in (5.2), is indeed bounded by a positive constant  $C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}$  strictly smaller than one. First, we verify that  $C_{\ell,\text{red}}$  from (5.2) is strictly smaller than one, i.e.

$$C_{\ell,\text{red}}^2 = 1 - \theta^2 \frac{\underline{\eta}_{\mathcal{M}_\ell^\#}^2}{\eta^2(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta)} < 1. \quad (5.6)$$

As  $\theta \in (0, 1]$ , (5.6) is equivalent to showing that  $\frac{\eta^2_{\mathcal{M}_\ell^\#}}{\eta^2(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta)} > 0$ . In order to verify this, we show that  $\frac{\eta^2_{\mathcal{M}_\ell^\#}}{\eta^2(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta)} \geq \frac{1}{C^2}$ , with a constant  $C$  strictly greater than 1, i.e.  $\eta^2(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta) \leq C^2 \eta^2_{\mathcal{M}_\ell^\#}$ . For this, we decompose the error estimate  $\eta^2(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta)$ , using the partition of unity  $\sum_{\mathbf{a} \in \mathcal{V}_K} \psi_\ell^\mathbf{a}|_K = 1$ , the Cauchy–Schwarz inequality, and the fact that each simplex has  $(d+1)$  vertices, as

$$\begin{aligned}
\eta^2(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta) &= \sum_{K \in \mathcal{M}_\ell^\theta} \|\nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell\|_K^2 \\
&= \sum_{K \in \mathcal{M}_\ell^\theta} \left\| \sum_{\mathbf{a} \in \mathcal{V}_{\ell, K}} (\psi_\ell^\mathbf{a} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^\mathbf{a}) \right\|_K^2 \\
&\leq \sum_{K \in \mathcal{M}_\ell^\theta} (d+1) \sum_{\mathbf{a} \in \mathcal{V}_{\ell, K}} \|\psi_\ell^\mathbf{a} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^\mathbf{a}\|_K^2 \\
&= (d+1) \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} \|\psi_\ell^\mathbf{a} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^\mathbf{a}\|_{\omega_\ell^\mathbf{a} \cap \omega_\ell}^2 \\
&\leq (d+1) \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} \|\psi_\ell^\mathbf{a} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^\mathbf{a}\|_{\omega_\ell^\mathbf{a}}^2. \tag{5.7}
\end{aligned}$$

We note that in (5.7), we have crucially employed the extended set of marked vertices  $\tilde{\mathcal{V}}_\ell^\#$  of Section 3.2. Next, Proposition 4.1 applied on each patchwise contribution  $\|\psi_\ell^\mathbf{a} \nabla u_\ell^{\text{ex}} + \boldsymbol{\sigma}_\ell^\mathbf{a}\|_{\omega_\ell^\mathbf{a}}$  of the sum in (5.7) and the localization (3.13) of  $\eta_{\mathcal{M}_\ell^\#}$  given by (3.12) yield

$$\eta^2(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta) \leq (d+1)^2 C_{\text{st}}^2 \eta_{\mathcal{M}_\ell^\#}^2 \tag{5.8}$$

with the constant  $(d+1)C_{\text{st}} > 1$ , depending only on the space dimension  $d$ , the mesh shape-regularity  $\kappa_{\mathcal{T}}$ , and the maximal polynomial degree  $p_{\text{max}}$ . Finally, employing (5.8) to bound  $C_{\ell, \text{red}}$  in (5.5) from above and defining the constant  $C_{\theta, d, \kappa_{\mathcal{T}}, p_{\text{max}}}$  as

$$C_{\theta, d, \kappa_{\mathcal{T}}, p_{\text{max}}} := \sqrt{1 - \frac{\theta^2}{(d+1)^2 C_{\text{st}}^2}} < 1$$

finishes the proof.  $\square$

Assuming the linear systems (3.2) being solved exactly and all the algebraic computations being performed in exact arithmetic, Theorem 5.1, in particular, implies:

**Corollary 5.2** (Convergence of the adaptive algorithm). *Let the Assumptions of Theorem 5.1 be satisfied. Then*

$$\lim_{\ell \rightarrow \infty} \|\nabla(u - u_\ell^{\text{ex}})\| = 0.$$

**Remark 5.3** (Comparison with previous works). *In contrast to the other results, see e.g. the reduction properties in [31, 41, 15, 13, 11, 12], the reduction factor  $C_{\ell, \text{red}}$  in (5.1) remains fully computable as in [17]. Notably, in [12], the authors have showed a reduction property which is actually  $p$ -robust, i.e. with a constant independent of the polynomial degrees  $\mathbf{p}_\ell$ , under additional assumption of a local saturation property on each marked patch. This property has been, so far, only observed numerically. It requests  $p$ -refinement of form  $p_{\ell+1, K} = p_{\ell, K} + \lceil \lambda p_{\ell, K} \rceil$ , and not the increment of the local polynomial degree by a constant factor which we employ in this work. We also note that the present reduction factor  $C_{\ell, \text{red}}$  of Theorem 5.1 has the same structure as the one of [17, Theorem 5.2].*

## 6 The $hp$ -adaptive algorithm with inexact solver

We now recall the building blocks of an inexact  $hp$ -adaptive algorithm of Scheme 2, by summing up the necessary ingredients and notation of modules ONE\_SOLVER\_STEP and ESTIMATE from [18, Sections 4.1 and 4.2].



A new version of the adaptive stopping criterion controlling the adaptive sub-loop in Scheme 2 is defined in Section 6.2. In the present inexact setting, we crucially employ slightly modified modules MARK and REFINER, described in Section 6.3, and formulate in Section 6.5 the necessary theoretical conditions on the error balancing parameter of the adaptive stopping criterion ensuring convergence of the adaptive algorithm also in this case.

### 6.1 Adaptive sub-loop of ONE\_SOLVER\_STEP and ESTIMATE

On the initial state  $\ell = 0$ , let the exact algebraic solver and  $hp$ -refinement criteria from Section 3 be used. Let henceforth  $\ell \geq 1$  be the current iteration number. We employ the module ONE\_SOLVER\_STEP exactly as in [18], with its output being an approximation  $u_\ell := \sum_{n=1}^{N_\ell} (U_\ell)_n \psi_\ell^n \in V_\ell$  of the unavailable exact Galerkin approximation  $u_\ell^{\text{ex}}$  given by (3.1). We recall that the algebraic residual vector  $R_\ell$  associated with  $U_\ell$  is given by

$$R_\ell := F_\ell - A_\ell U_\ell. \quad (6.1)$$

Moreover, we employ its functional representation  $\mathbf{r}_\ell \in \mathbb{P}_{\mathbf{p}_\ell}(\mathcal{T}_\ell)$ ,  $\mathbf{r}_\ell|_{\partial\Omega} = 0$ , satisfying

$$(\mathbf{r}_\ell, \psi_\ell^n) = (R_\ell)_n \quad 1 \leq n \leq N_\ell. \quad (6.2)$$

For details regarding the construction of such  $\mathbf{r}_\ell$ , we refer to Papež *et al.* [35, Section 5.1]. Note that the property (6.2) together with the algebraic system (3.2) yield the functional equivalent of the algebraic relation (6.1)

$$(\mathbf{r}_\ell, v_\ell) = (f, v_\ell) - (\nabla u_\ell, \nabla v_\ell) \quad v_\ell \in V_\ell. \quad (6.3)$$

The quality of the current approximate solution  $u_\ell$  obtained by ONE\_SOLVER\_STEP is subsequently assessed within the ESTIMATE module identical to the one from [18, Section 4.2] based on equilibrated flux reconstructions by local problems on patches. The module ESTIMATE in the inexact setting outputs a collection of local error indicators  $\{\eta_{\text{alg},K}(u_\ell), \eta_{\text{dis},K}(u_\ell)\}_{K \in \mathcal{T}_\ell}$  together with the total error lower bound  $\mu(u_\ell)$ , see (6.6) and (6.7) below, respectively. Let us first recall the definition of the discretization flux reconstruction:

**Definition 6.1** (Discretization flux reconstruction  $\sigma_{\ell,\text{dis}}$  by local minimization). *Let  $u_\ell$  be the current inexact approximation obtained by the module ONE\_SOLVER\_STEP. For each  $\mathbf{a} \in \mathcal{V}_\ell$ , let the local discretization flux reconstruction  $\sigma_{\ell,\text{dis}}^{\mathbf{a}} \in \mathbf{V}_\ell^{\mathbf{a}}$  be given by the following local minimization problem*

$$\sigma_{\ell,\text{dis}}^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_\ell \in \mathbf{V}_\ell^{\mathbf{a}}, \\ \nabla \cdot \mathbf{v}_\ell = \psi_\ell^{\mathbf{a}} f - \nabla \psi_\ell^{\mathbf{a}} \cdot \nabla u_\ell - \psi_\ell^{\mathbf{a}} \mathbf{r}_\ell}} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell + \mathbf{v}_\ell\|_{\omega_\ell^{\mathbf{a}}}. \quad (6.4)$$

Then, the discretization flux reconstruction is defined as  $\sigma_{\ell,\text{dis}} := \sum_{\mathbf{a} \in \mathcal{V}_\ell} \sigma_{\ell,\text{dis}}^{\mathbf{a}}$ , with each local contribution  $\sigma_{\ell,\text{dis}}^{\mathbf{a}}$  being extended by zero outside of  $\omega_\ell^{\mathbf{a}}$ .

Note that the Neumann compatibility condition for the problem (6.4) is a direct consequence of (6.3). Moreover, using the multilevel construction described in [34, Section 4.1] and [18, Section 4.2], we build the algebraic error flux reconstruction  $\sigma_{\ell,\text{alg}} \in \mathbf{H}(\text{div}, \Omega)$  such that

$$(\nabla \cdot \sigma_{\ell,\text{alg}}, q_\ell)_K = (\mathbf{r}_\ell, q_\ell)_K \quad \forall K \in \mathcal{T}_\ell, \quad \forall q_\ell \in \mathbb{P}_{p_\ell,K}(K). \quad (6.5)$$

Then, as established in [18], cf. also [24, 34], the following upper bounds on the energy norm of the total and algebraic error hold true:

$$\underbrace{\|\nabla(u - u_\ell)\|}_{\text{total error}} \leq \eta(u_\ell, \mathcal{T}_\ell) := \left\{ \sum_{K \in \mathcal{T}_\ell} \eta_K^2(u_\ell) \right\}^{\frac{1}{2}}, \quad \text{with} \quad \eta_K(u_\ell) := \underbrace{\|\nabla u_\ell + \sigma_{\ell,\text{dis}}\|_K}_{\eta_{\text{dis},K}(u_\ell)} + \underbrace{\|\sigma_{\ell,\text{alg}}\|_K}_{\eta_{\text{alg},K}(u_\ell)}, \quad (6.6a)$$

$$\underbrace{\|\nabla(u_\ell^{\text{ex}} - u_\ell)\|}_{\text{algebraic error}} \leq \eta_{\text{alg}}(u_\ell, \mathcal{T}_\ell) := \left\{ \sum_{K \in \mathcal{T}_\ell} \eta_{\text{alg},K}^2(u_\ell) \right\}^{\frac{1}{2}}. \quad (6.6b)$$

Next, for each vertex  $\mathbf{a} \in \mathcal{V}_\ell$ , let  $V_\ell^\mathbf{a}$  be a suitable finite-dimensional subspace of  $H_*^1(\omega_\ell^\mathbf{a})$

$$H_*^1(\omega_\ell^\mathbf{a}) := \begin{cases} \{v \in H^1(\omega_\ell^\mathbf{a}), (v, 1)_{\omega_\ell^\mathbf{a}} = 0\} & \text{if } \mathbf{a} \in \mathcal{V}_\ell^{\text{int}}, \\ \{v \in H^1(\omega_\ell^\mathbf{a}), v = 0 \text{ on } \partial\omega_\ell^\mathbf{a} \cap \partial\Omega\} & \text{if } \mathbf{a} \in \mathcal{V}_\ell^{\text{ext}}. \end{cases}$$

Following [35, Theorem 2], cf. also [18], we construct the total residual lifting  $\rho_{\ell, \text{tot}} := \sum_{\mathbf{a} \in \mathcal{V}_\ell} \psi_\ell^\mathbf{a} \rho_{\ell, \text{tot}}^\mathbf{a} \in H_0^1(\Omega)$ , where each vertex contribution  $\rho_{\ell, \text{tot}}^\mathbf{a} \in V_\ell^\mathbf{a}$  solves the local primal finite element problem

$$(\nabla \rho_{\ell, \text{tot}}^\mathbf{a}, \nabla v_\ell)_{\omega_\ell^\mathbf{a}} = (f, \psi_\ell^\mathbf{a} v_\ell)_{\omega_\ell^\mathbf{a}} - (\nabla u_\ell, \nabla(\psi_\ell^\mathbf{a} v_\ell))_{\omega_\ell^\mathbf{a}} \quad \forall v_\ell \in V_\ell^\mathbf{a}.$$

Then, besides the upper bounds (6.6a) and (6.6b), the following lower bound on the total error is at our disposal, supposing  $\rho_{\ell, \text{tot}} \neq 0$ ,

$$\|\nabla(u - u_\ell)\| \geq \frac{\sum_{\mathbf{a} \in \mathcal{V}_\ell} \|\nabla \rho_{\ell, \text{tot}}^\mathbf{a}\|_{\omega_\ell^\mathbf{a}}^2}{\|\nabla \rho_{\ell, \text{tot}}\|} =: \mu(u_\ell). \quad (6.7)$$

## 6.2 Adaptive stopping criterion for the algebraic solver

The above error estimators computed within module **ESTIMATE** enable us to stop the adaptive sub-loop between the modules **ONE-SOLVER-STEP** and **ESTIMATE** once the current approximation  $u_\ell$  is such that

$$\|\nabla(u_\ell^{\text{ex}} - u_\ell)\| \leq \tilde{\gamma}_\ell \|\nabla(u - u_\ell^{\text{ex}})\|, \quad (6.8)$$

i.e. when the algebraic error is  $\tilde{\gamma}_\ell$ -times smaller than the discretization error (typically  $\tilde{\gamma}_\ell > 0$  is of order 0.05). Relying on the output of the module **ESTIMATE**, the requirement (6.8) is ensured due to Galerkin orthogonality of  $u_\ell^{\text{ex}}$

$$\|\nabla(u - u_\ell)\|^2 = \|\nabla(u - u_\ell^{\text{ex}})\|^2 + \|\nabla(u_\ell^{\text{ex}} - u_\ell)\|^2$$

via (6.6b), (6.7) by the following criterion

$$\eta_{\text{alg}}(u_\ell, \mathcal{T}_\ell) \leq \tilde{\gamma}_\ell \underbrace{\sqrt{\mu^2(u_\ell) - \eta_{\text{alg}}^2(u_\ell, \mathcal{T}_\ell)}}_{\text{discretization error lower bound}}, \quad (6.9)$$

under the condition

$$\eta_{\text{alg}}(u_\ell, \mathcal{T}_\ell) \leq \mu(u_\ell). \quad (6.10)$$

As already used in [35, Section 6.3] and [18, Section 4.3], (6.9) is equivalent to criterion

$$\eta_{\text{alg}}(u_\ell, \mathcal{T}_\ell) \leq \gamma_\ell \mu(u_\ell) \quad (6.11)$$

with

$$\gamma_\ell = \frac{\tilde{\gamma}_\ell}{(1 + \tilde{\gamma}_\ell^2)^{\frac{1}{2}}} < 1.$$

The choice of the parameter  $\gamma_\ell$  in (6.11) will follow from the theoretical analysis to be carried out in Section 8 below.

## 6.3 Modules MARK and REFINE

For the sake of brevity, we keep the notation introduced in Section 3.2 unchanged. The module **MARK** in the inexact setting takes as input an inexact approximation  $u_\ell$  and the corresponding error indicators computed within module **ESTIMATE**. It proceeds again in two phases. However, in the marking criterion (3.6) we employ in place of the total error estimator  $\eta(u_\ell, \cdot)$  only its component corresponding to the discretization error, so the modified bulk chasing criterion to select a subset of marked vertices  $\mathcal{V}_\ell^\theta \subset \mathcal{V}_\ell$  reads

$$\eta_{\text{dis}}\left(u_\ell, \bigcup_{\mathbf{a} \in \mathcal{V}_\ell^\theta} \mathcal{T}_\ell^\mathbf{a}\right) \geq \theta \eta_{\text{dis}}(u_\ell, \mathcal{T}_\ell), \quad (6.12)$$

where  $\eta_{\text{dis}}(u_\ell, \mathcal{S}) := \{\sum_{K \in \mathcal{S}} \eta_{\text{dis},K}(u_\ell)^2\}^{\frac{1}{2}}$ , for any subset  $\mathcal{S} \subset \mathcal{T}_\ell$ , and  $\theta \in (0, 1]$  is a fixed threshold parameter. The rest of the module follows straightforwardly the steps described in Section 3.2 leading to the extended set of marked vertices  $\tilde{\mathcal{V}}_\ell^\#$ . In the module **REFINE**, the new pair  $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$  is determined on the basis of the two local problems (3.9), with an approximate solution  $u_\ell$  in place of the exact Galerkin approximation  $u_\ell^{\text{ex}}$  (recall that  $u_\ell^{\text{ex}}$  is not at our disposal here).

#### 6.4 Discrete lower bound on the incremental error on marked simplices

Proceeding as in the exact setting, for each vertex from the extended set of marked vertices  $\tilde{\mathcal{V}}_\ell^\#$ , let the residual lifting  $r^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}$  be defined as the solution of the local problem (3.11), with  $u_\ell^{\text{ex}}$  replaced by the inexact approximation  $u_\ell$ :

$$(\nabla r^{\mathbf{a},hp}, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}, \quad (6.13)$$

with the local space  $V_\ell^{\mathbf{a},hp}$  given by (3.10). Extending the residual liftings  $r^{\mathbf{a},hp}$  by zero outside  $\omega_\ell^{\mathbf{a}}$ , let  $u_{\ell+1}^{\text{ex}} \in V_{\ell+1}$  be the (unavailable) exact Galerkin approximation on the next level. By [18, Lemma 5.1] used with the extended set  $\tilde{\mathcal{V}}_\ell^\#$  in place of  $\mathcal{V}_\ell^\theta$ , cf. (3.12), the following lower bound holds true

$$\|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell)\|_{\omega_\ell^\#} \geq \underline{\eta}_{\mathcal{M}_\ell^\#}^*, \quad \underline{\eta}_{\mathcal{M}_\ell^\#}^* := \begin{cases} \frac{\sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}}^2}{\left\| \nabla \left( \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} r^{\mathbf{a},hp} \right) \right\|_{\omega_\ell^\#}^2} & \text{if } \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} r^{\mathbf{a},hp} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6.14)$$

We note that the localization in the spirit of (3.13) is again available here.

#### 6.5 Conditions on the adaptive stopping criterion parameter $\gamma_\ell$

We now specify the required conditions on the parameter  $\gamma_\ell$  in the adaptive stopping criterion (6.11) that will ensure error reduction on each adaptive loop step and consequently convergence of the inexact  $hp$ -adaptive algorithm of Scheme 2.

Let the iteration counter  $\ell \geq 1$  be given. First, it turns out to be necessary to emulate the  $hp$ -refinement decision for the current inexact approximation  $u_\ell$ . More precisely, after computing the error estimators associated with the current approximation  $u_\ell$  within the **ESTIMATE** module, we actually perform a supplementary call of the modules **MARK** and **REFINE**, defined in Section 6.3, in order to obtain the auxiliary set of marked vertices and the auxiliary refinement suggestions. Thus, the corresponding auxiliary local patchwise spaces  $V_\ell^{\mathbf{a},hp}$  given by (3.10) can be constructed. Furthermore, for each vertex  $\mathbf{a}$  from the auxiliary set of marked vertices, we construct the residual lifting  $r^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}$  by solving the local problem (6.13) and we define the following fully computable constants

$$\xi_\ell^2 := \frac{\underline{\eta}_{\mathcal{M}_\ell^\#}^2}{\eta^2(u_\ell^{\text{ex}}, \mathcal{M}_\ell^\theta)}, \quad \ell = 0, \quad \xi_\ell^2 := \frac{\left(\underline{\eta}_{\mathcal{M}_\ell^\#}^*\right)^2 + \|\sigma_{\ell,\text{alg}}\|^2}{\eta_{\text{dis}}^2(u_\ell, \mathcal{M}_\ell^\theta)}, \quad \ell \geq 1. \quad (6.15)$$

We require that the parameter  $\gamma_\ell$  in the adaptive stopping criterion (6.11) is such that

$$0 < \gamma_\ell^2 < \min_{\ell'=\ell-1, \ell} \left\{ \frac{\theta^2 \xi_{\ell'}^2}{4(1 + 3\theta^2 \xi_{\ell'}^2)} \right\}, \quad \ell \geq 1. \quad (6.16)$$

Let

$$C_{\text{st}}^{\mathbf{a}} := \max \left\{ \frac{\|\psi_\ell^{\mathbf{a}} \nabla u_\ell + \sigma_{\ell,\text{dis}}^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}}{\|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}} + \|\sigma_{\ell,\text{alg}}\|_{\omega_\ell^{\mathbf{a}}}}, 1 \right\} \quad \forall \mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#, \quad \text{and} \quad C_{\text{st}}^\ell := \max_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} C_{\text{st}}^{\mathbf{a}}; \quad (6.17)$$

the values of the constants  $C_{\text{st}}^{\mathbf{a}}$  are computationally typically in the range 1–10. Let also  $C_{\text{st}} \geq 1$  be the constant from Proposition 7.1 below that only depends on the space dimension  $d$ , the mesh shape-regularity  $\kappa_{\mathcal{T}}$ , and the maximal polynomial degree  $p_{\text{max}}$ ; then  $C_{\text{st}}^{\mathbf{a}}$  and  $C_{\text{st}}^\ell$  are uniformly bounded from

above by  $C_{\text{st}}$  for all  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#$  and all  $\ell$ . We have already shown in (5.8) that the ratio  $\xi_0$  is bounded from below, and, using  $C_{\text{st}}^\ell$  and  $C_{\text{st}}$ , such a bound for  $\ell \geq 1$  is the content of Lemma 8.1 below. Consequently, a sufficient theoretical condition for the practical requirement (6.16) to hold is

$$0 < \gamma_\ell^2 < \frac{\theta^2 \xi^2}{4(1 + 3\theta^2 \xi^2)}, \quad \xi^2 := \frac{1}{2(d+1)^2 (C_{\text{st}})^2}, \quad \ell \geq 1, \quad (6.18)$$

so that conditions (6.16) are uniform in the iteration counter  $\ell$ . Moreover, they do not request excessively small values of the parameters  $\gamma_\ell$  and allow us to use the reasonable choice of  $\gamma_\ell \approx 0.05$  in most cases, in contrast to (to our knowledge) all results available in the literature so far. Remark that we automatically have

$$\gamma_\ell < 1. \quad (6.19)$$

Finally, we also observe that unless  $\eta(u_\ell, \mathcal{T}_\ell) = 0$ , in which case  $u_\ell = u$  and the adaptive loop terminates, (6.12) together with (8.9) below implies  $\eta_{\text{dis}}(u_\ell, \mathcal{M}_\ell^\theta) \neq 0$  and thus  $\xi_\ell \neq +\infty$ ; actually (8.11) below gives  $\theta^2 \xi_\ell^2 < 2$ .

## 7 Discrete stability of equilibrated fluxes in an inexact setting

In this section, we extend the result of Section 4 to the inexact setting. Recall that  $x_1 \lesssim x_2$  means that there exists a generic positive constant  $C$  that only depends on the space dimension  $d$ , the shape-regularity  $\kappa_{\mathcal{T}}$  of the underlying mesh, and the polynomial degree of approximation employed locally such that  $x_1 \leq Cx_2$ . In particular, we show the following counterpart of Proposition 4.1:

**Proposition 7.1** (Discrete stability of the local flux equilibration in the inexact setting). *Let the approximate solution  $u_\ell$  satisfy*

$$(f, \psi_\ell^\mathbf{a})_{\omega_\ell^\mathbf{a}} - (\nabla u_\ell, \nabla \psi_\ell^\mathbf{a})_{\omega_\ell^\mathbf{a}} - (\mathbf{r}_\ell, \psi_\ell^\mathbf{a})_{\omega_\ell^\mathbf{a}} = 0 \quad \forall \mathbf{a} \in \mathcal{V}_\ell^{\text{int}}.$$

*Let the local discretization flux reconstruction  $\boldsymbol{\sigma}_{\ell, \text{dis}}^\mathbf{a}$  be constructed by (6.4), let the algebraic error flux reconstruction  $\boldsymbol{\sigma}_{\ell, \text{alg}} \in \mathbf{H}(\text{div}, \Omega)$  satisfy (6.5), and let  $r^{\mathbf{a}, hp}$  be given by (6.13). Then there exists a constant  $C_{\text{st}} \geq 1$  only depending on the space dimension  $d$ , the mesh shape-regularity  $\kappa_{\mathcal{T}}$ , and the maximal polynomial degree  $p_{\text{max}}$  such that*

$$\|\psi_\ell^\mathbf{a} \nabla u_\ell + \boldsymbol{\sigma}_{\ell, \text{dis}}^\mathbf{a}\|_{\omega_\ell^\mathbf{a}} \leq C_{\text{st}} (\|\nabla r^{\mathbf{a}, hp}\|_{\omega_\ell^\mathbf{a}} + \|\boldsymbol{\sigma}_{\ell, \text{alg}}\|_{\omega_\ell^\mathbf{a}}) \quad \forall \mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#. \quad (7.1)$$

*Thus, in particular, the constants  $C_{\text{st}}^\mathbf{a}$  and  $C_{\text{st}}^\ell$  from (6.17) are uniformly bounded from above by  $C_{\text{st}}$ .*

*Proof.* We follow the proof of Proposition 4.1 and the proof of [34, Theorem 7.7]. For a fixed vertex  $\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#$ , let us recall from the proof of Proposition 4.1 the two reformulations of the local minimization problem (6.4). The first reads: find the pair  $(\boldsymbol{\sigma}_{\ell, \text{dis}}^\mathbf{a}, \delta_\ell^\mathbf{a}) \in \mathbf{V}_\ell^\mathbf{a} \times Q_\ell^\mathbf{a}$  such that

$$(\boldsymbol{\sigma}_{\ell, \text{dis}}^\mathbf{a}, \mathbf{v}_\ell)_{\omega_\ell^\mathbf{a}} - (\delta_\ell^\mathbf{a}, \nabla \cdot \mathbf{v}_\ell)_{\omega_\ell^\mathbf{a}} = -(\psi_\ell^\mathbf{a} \nabla u_\ell, \mathbf{v}_\ell)_{\omega_\ell^\mathbf{a}} \quad \forall \mathbf{v}_\ell \in \mathbf{V}_\ell^\mathbf{a}, \quad (7.2a)$$

$$(\nabla \cdot \boldsymbol{\sigma}_{\ell, \text{dis}}^\mathbf{a}, q_\ell)_{\omega_\ell^\mathbf{a}} = (\psi_\ell^\mathbf{a} f - \nabla u_\ell \cdot \nabla \psi_\ell^\mathbf{a} - \psi_\ell^\mathbf{a} \mathbf{r}_\ell, q_\ell)_{\omega_\ell^\mathbf{a}} \quad \forall q_\ell \in Q_\ell^\mathbf{a}, \quad (7.2b)$$

with the test and trial spaces  $\mathbf{V}_\ell^\mathbf{a}$ ,  $Q_\ell^\mathbf{a}$  defined by (3.3) and (4.18), respectively. On the other hand, the hybridized formulation reads: find  $\boldsymbol{\sigma}_{\ell, \text{dis}}^\mathbf{a}$  in the broken space  $\mathbf{RTN}_{p_\mathbf{a}^{\text{est}}}(\mathcal{T}_\ell^\mathbf{a})$ ,  $\delta_\ell^\mathbf{a} \in Q_\ell^\mathbf{a}$ , and  $\lambda_\ell^F \in \mathbb{P}_{p_\mathbf{a}^{\text{est}}}(F)$  for all  $F \in \mathcal{F}_\ell^\mathbf{a} \setminus \mathcal{F}_\ell^{\mathbf{a}, \partial\Omega}$  such that

$$\sum_{K \in \mathcal{T}_\ell^\mathbf{a}} (\psi_\ell^\mathbf{a} \nabla u_\ell + \boldsymbol{\sigma}_{\ell, \text{dis}}^\mathbf{a}, \mathbf{v}_\ell)_K - \sum_{K \in \mathcal{T}_\ell^\mathbf{a}} (\nabla \cdot \mathbf{v}_\ell, \delta_\ell^\mathbf{a})_K + \sum_{K \in \mathcal{T}_\ell^\mathbf{a}} \sum_{F \in \mathcal{F}_{\ell, K} \setminus \mathcal{F}_\ell^{\mathbf{a}, \partial\Omega}} (\mathbf{v}_\ell \cdot \mathbf{n}_K, \lambda_\ell^F)_F = 0 \quad \forall \mathbf{v}_\ell \in \mathbf{RTN}_{p_\mathbf{a}^{\text{est}}}(\mathcal{T}_\ell^\mathbf{a}), \quad (7.3a)$$

$$\sum_{K \in \mathcal{T}_\ell^\mathbf{a}} (\nabla \cdot \boldsymbol{\sigma}_{\ell, \text{dis}}^\mathbf{a}, q_\ell)_K = \sum_{K \in \mathcal{T}_\ell^\mathbf{a}} (\psi_\ell^\mathbf{a} f - \nabla u_\ell \cdot \nabla \psi_\ell^\mathbf{a} - \psi_\ell^\mathbf{a} \mathbf{r}_\ell, q_\ell)_K \quad \forall q_\ell \in Q_\ell^\mathbf{a}, \quad (7.3b)$$

$$- \sum_{K \in \mathcal{T}_F} (\boldsymbol{\sigma}_{\ell, \text{dis}}^\mathbf{a} \cdot \mathbf{n}_K, \xi_\ell)_F = 0 \quad \forall \xi_\ell \in \mathbb{P}_{p_\mathbf{a}^{\text{est}}}(F), \forall F \in \mathcal{F}_\ell^\mathbf{a} \setminus \mathcal{F}_\ell^{\mathbf{a}, \partial\Omega}, \quad (7.3c)$$

with  $\mathcal{F}_\ell^{\mathbf{a},\partial\Omega}$  given by (4.20). In this hybridized formulation, we are allowed to employ  $\psi_\ell^{\mathbf{a}}\nabla u_\ell + \boldsymbol{\sigma}_{\ell,\text{dis}}^{\mathbf{a}} \in \mathbf{RTN}_{p_{\mathbf{a}}^{\text{est}}}(\mathcal{T}_\ell^{\mathbf{a}})$  as a test function  $\mathbf{v}_\ell$  in (7.3a). Furthermore, employing  $\delta_\ell^{\mathbf{a}}$  as  $q_\ell$  in (7.3b),  $\lambda_\ell^F$  as  $\xi_\ell$  in (7.3c), followed by summing (7.3a)–(7.3c), we obtain, analogously to (4.21),

$$\|\psi_\ell^{\mathbf{a}}\nabla u_\ell + \boldsymbol{\sigma}_{\ell,\text{dis}}^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}^2 \leq \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} \|\psi_\ell^{\mathbf{a}}(f + \Delta u_\ell - \mathbf{r}_\ell)\|_K \|\delta_\ell^{\mathbf{a}}\|_K + \sum_{F \in \mathcal{F}_\ell^{\mathbf{a},\text{int}}} \|\psi_\ell^{\mathbf{a}}[\nabla u_\ell \cdot \mathbf{n}_F]\|_F \|\lambda_\ell^F\|_F. \quad (7.4)$$

We have also used the fact that the hat function  $\psi_\ell^{\mathbf{a}}|_F = 0 \ \forall F \in \mathcal{F}_\ell^{\mathbf{a},\text{ext}}$  for vertices  $\mathbf{a} \in \mathcal{V}_\ell^{\text{int}}$ , whereas the boundary faces are excluded from the sum and  $\psi_\ell^{\mathbf{a}}|_F = 0 \ \forall F \in \mathcal{F}_\ell^{\mathbf{a},\text{ext}} \setminus \mathcal{F}_\ell^{\mathbf{a},\partial\Omega}$  for  $\mathbf{a} \in \mathcal{V}_\ell^{\text{ext}}$ .

We proceed by treating the first term of (7.4). First, to bound  $\|\psi_\ell^{\mathbf{a}}(f + \Delta u_\ell - \mathbf{r}_\ell)\|_K$  from above, we employ the triangle inequality

$$\|\psi_\ell^{\mathbf{a}}(f + \Delta u_\ell - \mathbf{r}_\ell)\|_K \leq \|\psi_\ell^{\mathbf{a}}\|_{\infty,K} \|f + \Delta u_\ell - \mathbf{r}_\ell\|_K \leq \|f + \Delta u_\ell\|_K + \|\mathbf{r}_\ell\|_K \quad \forall K \in \mathcal{T}_\ell^{\mathbf{a}}. \quad (7.5)$$

We note that the arguments used in proof of Lemma 4.2 stay valid also in the inexact setting with only approximate solution  $u_\ell$  and  $r^{\mathbf{a},hp}$  now solving (6.13) instead of (3.11). Hence, the element residual  $\|f + \Delta u_\ell\|_K$  in (7.5) satisfies the property analogous to (4.2) from Lemma 4.2,

$$h_K \|f + \Delta u_\ell\|_K \lesssim \|\nabla r^{\mathbf{a},hp}\|_K. \quad (7.6)$$

To treat  $\|\mathbf{r}_\ell\|_K$ , we use property (6.5) of the algebraic error flux reconstruction  $\boldsymbol{\sigma}_{\ell,\text{alg}}$  and an inverse inequality. We thus obtain

$$\|\mathbf{r}_\ell\|_K \leq \|\nabla \cdot \boldsymbol{\sigma}_{\ell,\text{alg}}\|_K \lesssim h_K^{-1} \|\boldsymbol{\sigma}_{\ell,\text{alg}}\|_K. \quad (7.7)$$

The inf-sup stability of discretization (7.2), similarly to (4.22), yields

$$\|\delta_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}} \lesssim h_{\omega_\ell^{\mathbf{a}}} \|\psi_\ell^{\mathbf{a}}\nabla u_\ell + \boldsymbol{\sigma}_{\ell,\text{dis}}^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}. \quad (7.8)$$

Combining (7.5) with (7.6) and (7.7), using the Cauchy–Schwarz inequality, the shape-regularity yielding  $h_K \approx h_{\omega_\ell^{\mathbf{a}}}$ , and estimate (7.8), we arrive at

$$\begin{aligned} \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} \|\psi_\ell^{\mathbf{a}}(f + \Delta u_\ell - \mathbf{r}_\ell)\|_K \|\delta_\ell^{\mathbf{a}}\|_K &\lesssim \sum_{K \in \mathcal{T}_\ell^{\mathbf{a}}} h_K^{-1} (\|\nabla r^{\mathbf{a},hp}\|_K + \|\boldsymbol{\sigma}_{\ell,\text{alg}}\|_K) \|\delta_\ell^{\mathbf{a}}\|_K \\ &\lesssim h_{\omega_\ell^{\mathbf{a}}}^{-1} (\|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}} + \|\boldsymbol{\sigma}_{\ell,\text{alg}}\|_{\omega_\ell^{\mathbf{a}}}) \|\delta_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}} \\ &\lesssim (\|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}} + \|\boldsymbol{\sigma}_{\ell,\text{alg}}\|_{\omega_\ell^{\mathbf{a}}}) \|\psi_\ell^{\mathbf{a}}\nabla u_\ell + \boldsymbol{\sigma}_{\ell,\text{dis}}^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}. \end{aligned} \quad (7.9)$$

On the other hand, for bounding the second term of (7.4), the arguments used to obtain (4.32) in the proof of Proposition 4.1 remain valid also in the inexact setting, so that

$$\sum_{F \in \mathcal{F}_\ell^{\mathbf{a},\text{int}}} \|\psi_\ell^{\mathbf{a}}[\nabla u_\ell \cdot \mathbf{n}_F]\|_F \|\lambda_\ell^F\|_F \lesssim \|\nabla r^{\mathbf{a},hp}\|_{\omega_\ell^{\mathbf{a}}} \|\psi_\ell^{\mathbf{a}}\nabla u_\ell + \boldsymbol{\sigma}_{\ell,\text{dis}}^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}. \quad (7.10)$$

Finally, employing the bounds (7.9) and (7.10) in (7.4) finishes the proof.  $\square$

## 8 Proof of guaranteed contraction with an inexact solver

We start with a useful auxiliary bound for the ratio  $\xi_\ell$  defined in (6.15).

**Lemma 8.1** (Bound on  $\xi_\ell$ ). *Let  $\ell \geq 1$ , let  $\underline{\eta}_{\mathcal{M}_\ell^\sharp}^*$  be the computable discrete lower bound defined by (6.14), and let  $\eta_{\text{dis}}(u_\ell, \mathcal{M}_\ell^\theta)$  be given by (6.12). Then, there holds*

$$\xi_\ell^2 = \frac{\left(\underline{\eta}_{\mathcal{M}_\ell^\sharp}^*\right)^2 + \|\boldsymbol{\sigma}_{\ell,\text{alg}}\|^2}{\eta_{\text{dis}}^2(u_\ell, \mathcal{M}_\ell^\theta)} \geq \frac{1}{2(d+1)^2(C_{\text{st}}^\ell)^2} \geq \frac{1}{2(d+1)^2(C_{\text{st}})^2} = \xi^2. \quad (8.1)$$

*Proof.* We proceed similarly as in the proof of Theorem 5.1 while relying on Proposition 7.1. By decomposition of the error estimate  $\eta_{\text{dis}}(u_\ell, \mathcal{T}_\ell)$ , using the partition of unity  $\sum_{\mathbf{a} \in \mathcal{V}_K} \psi_\ell^{\mathbf{a}}|_K = 1$ , the Cauchy–Schwarz inequality, the fact that each simplex has  $(d+1)$  vertices, and the extended set of marked vertices  $\tilde{\mathcal{V}}_\ell^\#$ , we can derive

$$\begin{aligned} \eta_{\text{dis}}^2(u_\ell, \mathcal{M}_\ell^\theta) &= \sum_{K \in \mathcal{M}_\ell^\theta} \|\nabla u_\ell + \boldsymbol{\sigma}_{\ell, \text{dis}}\|_K^2 = \sum_{K \in \mathcal{M}_\ell^\theta} \left\| \sum_{\mathbf{a} \in \mathcal{V}_{\ell, K}} (\psi_\ell^{\mathbf{a}} \nabla u_\ell + \boldsymbol{\sigma}_{\ell, \text{dis}}^{\mathbf{a}}) \right\|_K^2 \\ &\leq \sum_{K \in \mathcal{M}_\ell^\theta} (d+1) \sum_{\mathbf{a} \in \mathcal{V}_{\ell, K}} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell + \boldsymbol{\sigma}_{\ell, \text{dis}}^{\mathbf{a}}\|_K^2 \\ &= (d+1) \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell + \boldsymbol{\sigma}_{\ell, \text{dis}}^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}} \cap \omega_\ell}^2 \\ &\leq (d+1) \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} \|\psi_\ell^{\mathbf{a}} \nabla u_\ell + \boldsymbol{\sigma}_{\ell, \text{dis}}^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}^2. \end{aligned} \quad (8.2)$$

Furthermore (6.17) together with Proposition 7.1 give

$$\|\psi_\ell^{\mathbf{a}} \nabla u_\ell + \boldsymbol{\sigma}_{\ell, \text{dis}}^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}} \leq C_{\text{st}}^\ell (\|\nabla r^{\mathbf{a}, hp}\|_{\omega_\ell^{\mathbf{a}}} + \|\boldsymbol{\sigma}_{\ell, \text{alg}}\|_{\omega_\ell^{\mathbf{a}}}) \quad \forall \mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#, \quad C_{\text{st}}^\ell \leq C_{\text{st}}. \quad (8.3)$$

Applying (8.3) on each patchwise contribution  $\|\psi_\ell^{\mathbf{a}} \nabla u_\ell + \boldsymbol{\sigma}_{\ell, \text{dis}}^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}^2$  in (8.2) and using the inequality  $(a+b)^2 \leq 2a^2 + 2b^2$  for  $a, b \geq 0$  together with the localization of the lower bound  $\underline{\eta}_{\mathcal{M}_\ell^\#}^*$  in the spirit of (3.13) leads to

$$\begin{aligned} \eta_{\text{dis}}^2(u_\ell, \mathcal{M}_\ell^\theta) &\leq (d+1)(C_{\text{st}}^\ell)^2 \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} (\|\nabla r^{\mathbf{a}, hp}\|_{\omega_\ell^{\mathbf{a}}} + \|\boldsymbol{\sigma}_{\ell, \text{alg}}\|_{\omega_\ell^{\mathbf{a}}})^2 \\ &\leq 2(d+1)(C_{\text{st}}^\ell)^2 \left( \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} \|\nabla r^{\mathbf{a}, hp}\|_{\omega_\ell^{\mathbf{a}}}^2 + \sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} \|\boldsymbol{\sigma}_{\ell, \text{alg}}\|_{\omega_\ell^{\mathbf{a}}}^2 \right) \\ &\leq 2(d+1)^2 (C_{\text{st}}^\ell)^2 \left( \frac{\sum_{\mathbf{a} \in \tilde{\mathcal{V}}_\ell^\#} \|\nabla r^{\mathbf{a}, hp}\|_{\omega_\ell^{\mathbf{a}}}^2}{d+1} + \|\boldsymbol{\sigma}_{\ell, \text{alg}}\|_{\omega_\ell^\#}^2 \right) \\ &\leq 2(d+1)^2 (C_{\text{st}}^\ell)^2 \left( \left( \underline{\eta}_{\mathcal{M}_\ell^\#}^* \right)^2 + \|\boldsymbol{\sigma}_{\ell, \text{alg}}\|^2 \right), \end{aligned} \quad (8.4)$$

which concludes the proof.  $\square$

We now give an important intermediate result giving a guaranteed bound on the error reduction factor between the current inexact approximation  $u_\ell$  and the (unavailable) next level exact solution  $u_{\ell+1}^{\text{ex}}$ . Namely, the following lemma improves the result of [18, Lemma 5.3] in showing that the reduction factor is strictly smaller than one:

**Lemma 8.2** (Guaranteed bound on the auxiliary energy error reduction factor). *Let the current inexact approximation  $u_\ell$ ,  $\ell \geq 0$ , be obtained by the adaptive sub-loop of Section 6.1 employing the stopping criterion (6.11) with the parameter  $\gamma_\ell$  satisfying (6.16) for  $\ell \geq 1$ . Let the pair  $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$  be obtained by the module REFINe of Section 6.3 and let  $\underline{\eta}_{\mathcal{M}_\ell^\#}^*$  be the computable discrete lower bound defined by (6.14). Then, unless  $\eta(u_\ell, \mathcal{T}_\ell) = 0$ , in which case  $u_\ell = u$ , and the adaptive loop terminates, the next (unavailable) exact finite element solution  $u_{\ell+1}^{\text{ex}} \in V_{\ell+1}$  satisfies*

$$\|\nabla(u - u_{\ell+1}^{\text{ex}})\| \leq C_{\ell, \text{red}}^* \|\nabla(u - u_\ell)\| \quad (8.5)$$

with

$$C_{\ell, \text{red}}^* := C_{\ell, \text{red}} \text{ from (5.2)} \quad \text{if } \ell = 0, \quad (8.6a)$$

$$0 \leq C_{\ell, \text{red}}^* := \sqrt{1 - \frac{\left( \underline{\eta}_{\mathcal{M}_\ell^\#}^* \right)^2 + \|\boldsymbol{\sigma}_{\ell, \text{alg}}\|^2}{\eta^2(u_\ell, \mathcal{T}_\ell)}} + \gamma_\ell^2 \leq C_{\theta, d, \kappa_{\mathcal{T}}, p_{\max}}^* < 1 \quad \text{if } \ell \geq 1, \quad (8.6b)$$

where  $C_{\ell,\text{red}}^*$  is a fully computable bound on the auxiliary error reduction factor and  $C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}^*$  is a generic constant only depending on the marking parameter  $\theta$ , the space dimension  $d$ , the mesh shape-regularity  $\kappa_{\mathcal{T}}$ , and the maximal polynomial degree  $p_{\max}$ .

*Proof.* As we suppose an exact solve on the initial mesh, the claim (8.5) for  $\ell = 0$  coincides with that of Theorem 5.1. Let thus  $\ell \geq 1$  and consider only the non trivial case when  $\eta(u_\ell, \mathcal{T}_\ell) \neq 0$ . We follow the proofs of Theorem 5.1 and [18, Lemma 5.3] and proceed in two steps. In the first step, we note that the Pythagorean relation between the current (available) inexact approximation  $u_\ell$  and the (unavailable) next level exact finite element solution  $u_{\ell+1}^{\text{ex}} \in V_{\ell+1}$  holds true. We thus have

$$\|\nabla(u - u_{\ell+1}^{\text{ex}})\|^2 = \|\nabla(u - u_\ell)\|^2 - \|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell)\|^2. \quad (8.7)$$

Employing the discrete lower bound (6.14) to bound the second term of on the right-hand side of (8.7), adding and subtracting  $\|\sigma_{\ell,\text{alg}}\|^2$ , using the adaptive stopping criterion (6.11) with parameter  $\gamma_\ell$  satisfying the second condition in (6.16), and the total error bounds (6.7) and (6.6a), we obtain the following chain of inequalities leading to the computable bound on the auxiliary reduction factor  $C_{\ell,\text{red}}^*$  of (8.6b):

$$\begin{aligned} \|\nabla(u - u_{\ell+1}^{\text{ex}})\|^2 &\leq \|\nabla(u - u_\ell)\|^2 - \left(\eta_{\mathcal{M}_\ell^\#}^*\right)^2 - \|\sigma_{\ell,\text{alg}}\|^2 + \|\sigma_{\ell,\text{alg}}\|^2 \\ &= \|\nabla(u - u_\ell)\|^2 - \frac{\left(\eta_{\mathcal{M}_\ell^\#}^*\right)^2 + \|\sigma_{\ell,\text{alg}}\|^2}{\eta^2(u_\ell, \mathcal{T}_\ell)} \eta^2(u_\ell, \mathcal{T}_\ell) + \|\sigma_{\ell,\text{alg}}\|^2 \\ &\leq \|\nabla(u - u_\ell)\|^2 - \frac{\left(\eta_{\mathcal{M}_\ell^\#}^*\right)^2 + \|\sigma_{\ell,\text{alg}}\|^2}{\eta^2(u_\ell, \mathcal{T}_\ell)} \eta^2(u_\ell, \mathcal{T}_\ell) + \gamma_\ell^2 \mu^2(u_\ell) \\ &\leq \underbrace{\left(1 - \frac{\left(\eta_{\mathcal{M}_\ell^\#}^*\right)^2 + \|\sigma_{\ell,\text{alg}}\|^2}{\eta^2(u_\ell, \mathcal{T}_\ell)} + \gamma_\ell^2\right)}_{=(C_{\ell,\text{red}}^*)^2} \|\nabla(u - u_\ell)\|^2. \end{aligned} \quad (8.8)$$

In the second step, we show that the reduction factor  $C_{\ell,\text{red}}^*$  defined in (8.6b) is indeed bounded by a positive constant  $C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}^*$  strictly smaller than one. The bound on the total error estimate  $\eta(u_\ell, \mathcal{T}_\ell) \leq \eta_{\text{alg}}(u_\ell, \mathcal{T}_\ell) + \eta_{\text{dis}}(u_\ell, \mathcal{T}_\ell)$ , with the estimators  $\eta_\bullet(u_\ell, \mathcal{S}) := \{\sum_{K \in \mathcal{S}} \eta_{\bullet,K}(u_\ell)^2\}^{\frac{1}{2}}$  defined for any subset  $\mathcal{S} \subseteq \mathcal{T}_\ell$ , together with the stopping criterion (6.11) and the total error lower bound (6.7) and upper bound (6.6a), yield

$$\begin{aligned} \eta(u_\ell, \mathcal{T}_\ell) &\leq \eta_{\text{alg}}(u_\ell, \mathcal{T}_\ell) + \eta_{\text{dis}}(u_\ell, \mathcal{T}_\ell) \\ &\leq \gamma_\ell \mu(u_\ell) + \eta_{\text{dis}}(u_\ell, \mathcal{T}_\ell) \leq \gamma_\ell \eta(u_\ell, \mathcal{T}_\ell) + \eta_{\text{dis}}(u_\ell, \mathcal{T}_\ell). \end{aligned}$$

Thus, since  $\gamma_\ell < 1$  from (6.19), we can bound the total error estimate  $\eta(u_\ell, \mathcal{T}_\ell)$  only by means of its component corresponding to the discretization error only,

$$\eta(u_\ell, \mathcal{T}_\ell) \leq \frac{\eta_{\text{dis}}(u_\ell, \mathcal{T}_\ell)}{(1 - \gamma_\ell)}. \quad (8.9)$$

Employing (8.9) and the marking criterion (6.12) to bound  $(C_{\ell,\text{red}}^*)^2$  leads to

$$\begin{aligned} (C_{\ell,\text{red}}^*)^2 &= 1 - \frac{\left(\eta_{\mathcal{M}_\ell^\#}^*\right)^2 + \|\sigma_{\ell,\text{alg}}\|^2}{\eta^2(u_\ell, \mathcal{T}_\ell)} + \gamma_\ell^2 \leq 1 - (1 - \gamma_\ell)^2 \frac{\left(\eta_{\mathcal{M}_\ell^\#}^*\right)^2 + \|\sigma_{\ell,\text{alg}}\|^2}{\eta_{\text{dis}}^2(u_\ell, \mathcal{T}_\ell)} + \gamma_\ell^2 \\ &\leq 1 - (1 - \gamma_\ell)^2 \theta^2 \frac{\left(\eta_{\mathcal{M}_\ell^\#}^*\right)^2 + \|\sigma_{\ell,\text{alg}}\|^2}{\eta_{\text{dis}}^2(u_\ell, \mathcal{M}_\ell^\theta)} + \gamma_\ell^2. \end{aligned} \quad (8.10)$$



Using the notation (6.15) and developing by the Young inequality  $2\gamma_\ell \leq 4\gamma_\ell^2 + 1/4$ , we see

$$(C_{\ell,\text{red}}^*)^2 \leq 1 - (1 - \gamma_\ell)^2 \theta^2 \xi_\ell^2 + \gamma_\ell^2 \leq 1 - \frac{3}{4} \theta^2 \xi_\ell^2 + 3\gamma_\ell^2 \theta^2 \xi_\ell^2 + \gamma_\ell^2.$$

Thus, using the second condition in (6.16), we obtain

$$(C_{\ell,\text{red}}^*)^2 < 1 - \frac{1}{2} \theta^2 \xi_\ell^2, \quad (8.11)$$

and, finally, employing the bound (8.1) from Lemma 8.1 leads to

$$C_{\ell,\text{red}}^* \leq C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}^*, \quad C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}^* := \sqrt{1 - \frac{\theta^2}{4(d+1)^2 C_{\text{st}}^2}} < 1,$$

which finishes the proof.  $\square$

The second main result of this paper, completing Theorem 5.1 in the inexact solver setting, is summarized in the following. In contrast to [18, Theorem 5.4], the reduction factor  $C_{\ell,\text{red}}$  is strictly and uniformly smaller than one.

**Theorem 8.3** (Guaranteed contraction for inexact solutions). *Let the current inexact approximation  $u_\ell$ ,  $\ell \geq 0$ , be obtained by the adaptive sub-loop of Section 6.1 employing the stopping criterion (6.11) with the parameter  $\gamma_\ell$  satisfying (6.16) for  $\ell \geq 1$ . Let the pair  $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$  be obtained by the module REFINE of Section 6.3 and let  $\underline{\eta}_{\mathcal{M}_\ell^\sharp}^*$  be the computable discrete lower bound defined by (6.14). Let also  $C_{\ell,\text{red}}^* \leq C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}^* < 1$  be given by (8.6). Moreover, let  $u_{\ell+1} \in V_{\ell+1}$  be the inexact finite element approximation on iteration  $(\ell + 1)$  of the adaptive loop prescribed by Scheme 2, satisfying the stopping criterion*

$$\eta_{\text{alg}}(u_{\ell+1}, \mathcal{T}_{\ell+1}) \leq \gamma_{\ell+1} \mu(u_{\ell+1}) \quad (8.12)$$

*with the parameter  $\gamma_{\ell+1}$  satisfying the conditions (6.16) (with the iteration counter  $\ell$  set to  $\ell + 1$ ). Then the resulting error reduction between the current inexact approximation  $u_\ell$  and the next inexact approximation  $u_{\ell+1}$  verifies*

$$\|\nabla(u - u_{\ell+1})\| \leq C_{\ell,\text{red}} \|\nabla(u - u_\ell)\| \quad \text{with} \quad 0 \leq C_{\ell,\text{red}} := \frac{C_{\ell,\text{red}}^*}{\sqrt{1 - \gamma_{\ell+1}^2}} \leq C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}^\# < 1, \quad (8.13)$$

where  $C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}^\#$  is a generic constant only depending on the marking parameter  $\theta$ , the space dimension  $d$ , the mesh shape-regularity  $\kappa_{\mathcal{T}}$ , and the maximal polynomial degree  $p_{\max}$ .

*Proof.* Similarly the proof of [18, Theorem 5.4] but using the Galerkin orthogonality, we have

$$\|\nabla(u - u_{\ell+1})\|^2 = \|\nabla(u - u_{\ell+1}^{\text{ex}})\|^2 + \|\nabla(u_{\ell+1}^{\text{ex}} - u_{\ell+1})\|^2. \quad (8.14)$$

For bounding the first term, we employ the auxiliary estimate (8.5). We bound the second term of (8.14), i.e. the algebraic error on the iteration  $(\ell + 1)$ , by the algebraic error estimate  $\eta_{\text{alg}}(u_{\ell+1}, \mathcal{T}_{\ell+1})$  defined in (6.6b). Then, the stopping criterion (8.12) with the parameter  $\gamma_{\ell+1}$  and the total error lower bound (6.7) lead to

$$\|\nabla(u_{\ell+1}^{\text{ex}} - u_{\ell+1})\| \leq \eta_{\text{alg}}(u_{\ell+1}, \mathcal{T}_{\ell+1}) \leq \gamma_{\ell+1} \mu(u_{\ell+1}) \leq \gamma_{\ell+1} \|\nabla(u - u_{\ell+1})\|.$$

The error reduction property of (8.13) then follows from

$$\|\nabla(u - u_{\ell+1})\|^2 \leq (C_{\ell,\text{red}}^*)^2 \|\nabla(u - u_\ell)\|^2 + \gamma_{\ell+1}^2 \|\nabla(u - u_{\ell+1})\|^2.$$

Indeed, the first condition in (6.16) gives

$$0 < \gamma_{\ell+1}^2 \leq \frac{\theta^2 \xi_\ell^2}{4(1 + 3\theta^2 \xi_\ell^2)} \leq \frac{\theta^2 \xi_\ell^2}{4 - \theta^2 \xi_\ell^2}$$

(recall that (5.6) and (8.11) give  $\theta^2 \xi_\ell^2 < 2 < 4$ ) and we have

$$\gamma_{\ell+1}^2 \leq \frac{\theta^2 \xi_\ell^2}{4 - \theta^2 \xi_\ell^2} \iff \frac{1}{1 - \gamma_{\ell+1}^2} \leq \frac{1 - \frac{\theta^2 \xi_\ell^2}{4}}{1 - \frac{\theta^2 \xi_\ell^2}{2}}.$$

Thus, from (8.11), we infer

$$\frac{(C_{\ell,\text{red}}^*)^2}{1 - \gamma_{\ell+1}^2} \leq 1 - \frac{\theta^2 \xi_\ell^2}{2} \frac{1 - \frac{\theta^2 \xi_\ell^2}{4}}{1 - \frac{\theta^2 \xi_\ell^2}{2}} = 1 - \frac{\theta^2 \xi_\ell^2}{4},$$

so that the bound (8.1) from Lemma 8.1 leads to

$$C_{\ell,\text{red}} \leq C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}^\#, \quad C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}}^\# := \sqrt{1 - \frac{\theta^2}{8(d+1)^2 C_{\text{st}}^2}} < 1,$$

and the proof is finished.  $\square$

In extension to Corollary 5.2, Theorem 8.3 implies:

**Corollary 8.4** (Convergence of the adaptive algorithm with inexact solvers). *Let the assumptions of Theorem 8.3 be satisfied. Then*

$$\lim_{\ell \rightarrow \infty} \|\nabla(u - u_\ell)\| = 0.$$

## 9 Numerical illustration

We quickly illustrate here the proposed exact and inexact  $hp$ -adaptive algorithms on a two-dimensional test case with a singular weak solution. The setting is taken from [17, 18], where ample numerical illustrations were supplied for the  $hp$ -adaptive strategies that we call henceforth “practical”. Our goal is to show that their “theory-prone” modifications developed here lead to structurally the same behavior, though with typically worse ratio price/outcome.

Consider the classical re-entrant corner problem, cf. [30, 17, 18], posed on the L-shape domain  $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1] \times [-1, 0]$  with  $f = 0$  and the weak solution, in polar coordinates,

$$u(r, \varphi) = r^{\frac{2}{3}} \sin\left(\frac{2\varphi}{3}\right).$$

We start the computation on a coarse criss-cross grid  $\mathcal{T}_0$  with  $\max_{K \in \mathcal{T}_0} h_K = 0.25$  and all the polynomial degrees set uniformly to 1. We first assess the quality of the guaranteed bound on the error reduction factor  $C_{\ell,\text{red}}$  from Theorem 5.1 throughout the exact  $hp$ -adaptive algorithm in terms of the effectivity index defined as

$$I_{\text{red}}^{\text{eff}} := \frac{C_{\ell,\text{red}}}{\frac{\|\nabla(u - u_{\ell+1}^{\text{ex}})\|}{\|\nabla(u - u_\ell^{\text{ex}})\|}}. \quad (9.1)$$

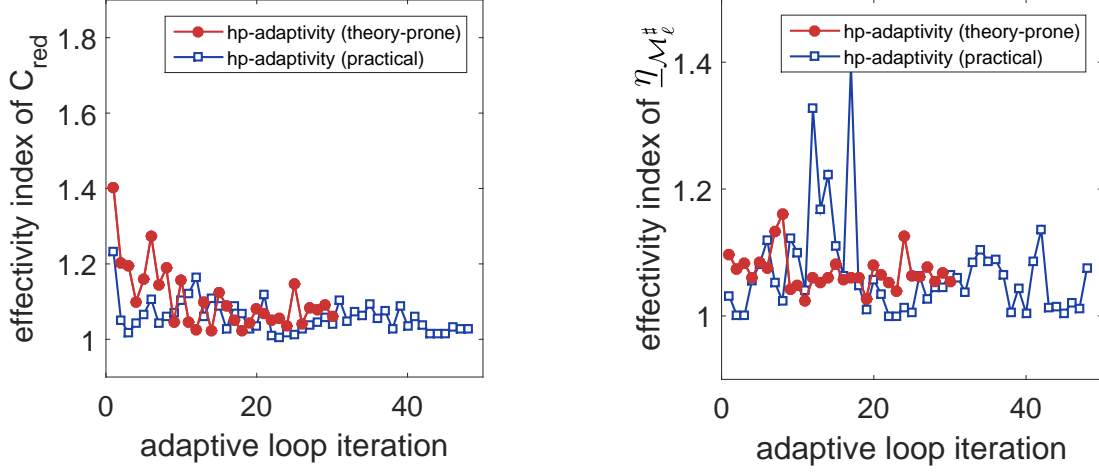
We also verify the sharpness of the underlying discrete lower bound  $\underline{\eta}_{\mathcal{M}_\ell^\#}$  given by (3.12) in terms of the effectivity index defined as

$$I_{\text{LB}}^{\text{eff}} := \frac{\|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell^{\text{ex}})\|_{\omega_\ell^\#}}{\underline{\eta}_{\mathcal{M}_\ell^\#}}. \quad (9.2)$$

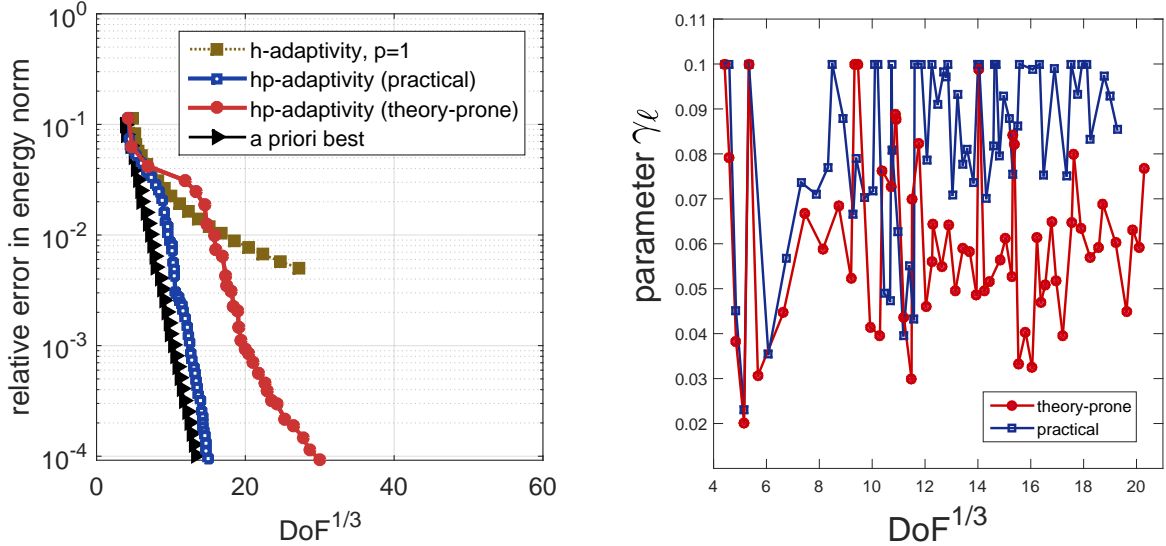
The results are reported in Figure 3, where we also plot the corresponding “practical” effectivity indices taken from [17, Figure 15]. Very similar and close-to-optimal results can be observed in all cases.

Figure 4, left, then illustrates that the generated sequences of meshes and polynomial degree distributions in the exact solver case still lead to asymptotic exponential convergence, albeit with worse constants and a worse ratio price/outcome. In particular, in the “practical” case of [17, Section 6.2], there holds  $\|\nabla(u - u_\ell^{\text{ex}})\| \leq C_1 \exp\left(-C_2 \text{DoF}_\ell^{\frac{1}{3}}\right)$  with  $C_1 = 4.73$  and  $C_2 = 0.69$ , whereas in the present theory-prone setting, the (second) constant(s) worsen(s) to  $C_1 = 0.641799$  and  $C_2 = 0.305573$ . This is explained by the present

additional assumptions on the  $h$ - and  $p$ -refinements, namely the interior node property, the extension of the marked region, and the steeper polynomial degree increase. Similar exponential convergence results are observed in the inexact solver case. Finally, Figure 4, right, plots the evolution of the stopping parameter  $\gamma_\ell$  satisfying the theoretical assumption (6.16) during the  $hp$  adaptation iterations. Though the theoretical requirement (6.16) is more stringent than the “practical” requirement of [18, equation (5.12)], illustrated in [18, Figure 20 (right)], comparable results in both cases are observed. In particular,  $\gamma_\ell$  satisfying (6.16) still takes the very-reasonable-in-practice value around 0.05 (at most 5% algebraic error in the total error, recalling (6.11)).



**Figure 3:** Effectivity index for the error reduction factor estimate  $C_{\ell,\text{red}}$  of Theorem 5.1 given by (9.1) (left); effectivity index for the discrete lower bound  $\eta_{\mathcal{M}_\ell^\#}$  of (3.12) given by (9.2) (right). Both marked as “theory-prone”. Corresponding “practical” effectivity indices from [17, Figure 15].



**Figure 4:** Relative energy error  $\|\nabla(u - u_\ell^{\text{ex}})\|/\|\nabla u\|$  as a function of  $\text{DoF}_\ell^{\frac{1}{3}}$ , obtained with the present “theory-prone” exact  $hp$ -adaptive algorithm of Section 3, the “practical” exact  $hp$ -adaptive algorithm of [17], its purely  $h$ -adaptive version, and  $hp$ -adaptation exploiting the a priori knowledge of the weak solution (left); values of the parameter  $\gamma_\ell$  in the present “theory-prone” inexact  $hp$ -adaptive algorithm of Section 6 compared to the parameter  $\gamma_\ell$  from [18] (right).

## 10 Conclusions and outlook

The contribution of this work is the theoretical study of the adaptive refinement strategies for conforming *hp*-adaptive finite element algorithm proposed in [17] for exact solvers and in [18] for inexact solvers. We described in detail all the necessary modifications of these two methods in order to guarantee strict error reduction on each step of the adaptive loop and consequent convergence of the resulting adaptive algorithms. The numerical tests confirm that the obtained meshes and polynomial degree distributions still lead to asymptotic exponential convergence and that the values of stopping coefficient  $\gamma_\ell$  from (6.11), which needs to satisfy the conditions (6.16), remain reasonable (around 0.05). Further theoretical analysis concerning the computational (quasi-)optimality of the proposed adaptive algorithms will be a subject of some forthcoming work.

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