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# Averaging techniques in the time minimal orbital transfer using low propulsion

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Joint work with B. Bonnard, J.B. Caillau and J.B. Pomet.

# Some historical points about optimal control and orbital transfer with low thrust

- $\approx$ 1970, electro-ionic propulsion where the thrust is very low (spiral transfer).  
*Missions:* Artemis-ESA (2001, GST), Smart-1-ESA (2006, GST to lunar orbit), BepiColombo-ESA (2016, Mercury).
- Epenoy-Geffroy (1997) use averaging techniques coupled with the Pontryagin Maximum Principle for low thrust transfer.
- Bonnard-Caillau (2007) computed explicitly the averaged Hamiltonian in the case of the energy cost.  
It defines a Riemannian metric which allows to compute extremal curves for coplanar transfers.

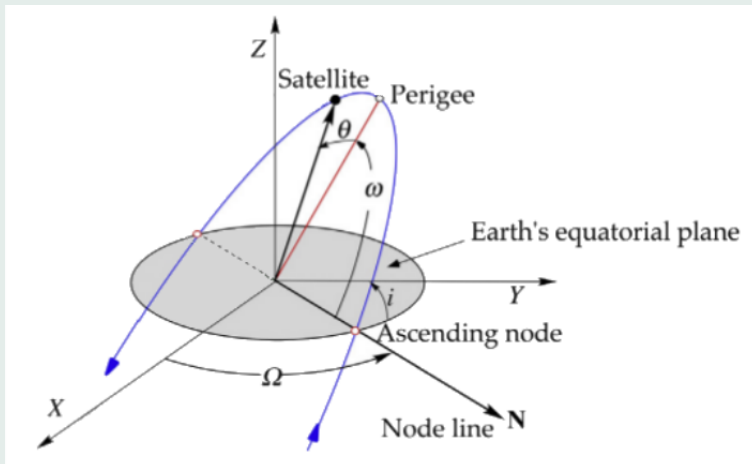
# Time minimal orbital transfer problem

$q \in \mathbb{R}^3$ : position vector of the satellite,

$$\ddot{q} + \nabla V(q) = 0, \quad V(q) = -\frac{1}{|q|}.$$

Elliptic orbits: restriction to  $\mathcal{E} = \{(q, \dot{q}) \mid q \times \dot{q} \neq 0 \text{ and } \frac{1}{2}|\dot{q}|^2 - V(q) < 0\}$ .

## Kepler's orbital elements



$(a, e, i, \omega, \Omega)$ : shape and orientation of the ellipse,

$\theta$ : angle locating the satellite on its orbit.

The uncontrolled motion can be described by  $(I, \varphi = F(I, \theta))$

$$\dot{I} = 0$$

$$\dot{\varphi} = \nu(I).$$

$0 < \varepsilon \ll 1$ ,  $(I(t), \varphi(t)) \in \mathbb{R}^5 \times S^1$ : state variables of the satellite,  $u(t) \in \mathbb{R}^3$ ,  $F_i(I, \varphi, \varepsilon), G_i(I, \varphi, \varepsilon)$   $i = 1..3$  are  **$2\pi$ -periodic** vector fields with respect to  $\varphi$ .

## Time minimal orbital transfer problem

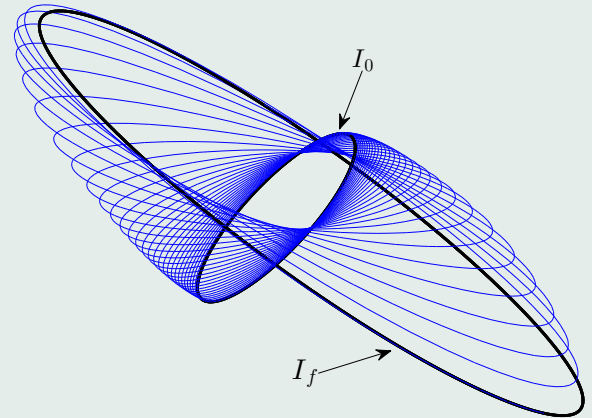
$$\min_{|u| \leq 1} t_f(u)$$

$$\text{subject to } \dot{I} = \varepsilon F_0(I, \varphi, \varepsilon) + \varepsilon \sum_{i=1}^3 u_i F_i(I, \varphi, \varepsilon), \quad \dot{\varphi} = \nu(I) + \varepsilon \sum_{i=1}^3 u_i G_i(I, \varphi, \varepsilon)$$

$$I(0) = I_0, \quad I(t_f(u)) = I_f.$$

The vector field  $F_0$  takes into account conservative perturbations such as

- lunar and solar attraction,
- $J_2$ -effect.



# Averaging principle for dynamical system

Let  $f(X, \psi, \varepsilon), g(X, \psi, \varepsilon)$  be two smooth  **$2\pi$ -periodic** vector fields with respect to  $\psi \in S^1$ .

*Perturbed system*

$$\begin{aligned}\dot{X} &= \varepsilon f(X, \psi, \varepsilon) \\ \dot{\psi} &= \nu(X) + \varepsilon g(X, \psi, \varepsilon).\end{aligned}$$

*Averaged system*

$$\begin{aligned}\dot{Y} &= \varepsilon F(Y) \\ \text{where } F(Y) &= \frac{1}{2\pi} \int_0^{2\pi} f(Y, \psi, \varepsilon = 0) d\psi.\end{aligned}$$

**Theorem 1.** Assume  $|\nu(X)| > R > 0$  and let  $\tau > 0$  fixed.

If

- $(X_\varepsilon(\cdot), \psi_\varepsilon(\cdot))_\varepsilon$  is a family of solutions of the **perturbed system** on  $[0, \frac{\tau}{\varepsilon}]$ ,
- $X_\varepsilon(0) \xrightarrow{\varepsilon \rightarrow 0} X_0^0$ .

Then  $t \rightarrow X_\varepsilon(\varepsilon t)$  uniformly converges on  $[0, \tau]$  to  $t \rightarrow \bar{Y}(t)$  such that

- $\bar{Y}(\cdot)$  is a solution of the **averaged system**,
- $\bar{Y}(0) = X_0^0$ .

# Non averaged system

Applying the PMP, a minimizing trajectory is projection of an extremal  $(I(\cdot), \varphi(\cdot), p_I(\cdot), p_\varphi(\cdot))$  solution of the Hamiltonian system defined by

$$H(I, \varphi, p_I, p_\varphi, \varepsilon) = p_\varphi \nu(I) + \varepsilon K(I, \varphi, p_I, p_\varphi, \varepsilon),$$

$$K := \sqrt{\sum_{i=1}^m H_i^2}, \quad H_i(I, \varphi, p_I, p_\varphi, \varepsilon) = p_I F_i(I, \varphi, \varepsilon) + p_\varphi G_i(I, \varphi, \varepsilon), \quad i=0, \dots, m.$$

The extremal system is

$$\begin{aligned} \dot{I} &= \varepsilon \frac{\partial K}{\partial p_I}, & \dot{p}_I &= -p_\varphi \nu' - \varepsilon \frac{\partial K}{\partial I} \\ \dot{\varphi} &= \nu + \varepsilon \frac{\partial K}{\partial p_\varphi}, & \dot{p}_\varphi &= -\varepsilon \frac{\partial K}{\partial \varphi}. \end{aligned} \quad (*)$$

The adjoint vector  $p_I$  might be not a slow variable (  $\dot{p}_I = O(\varepsilon)$  ? ).

# Normalization of the non averaged system

We decide to normalize on the energy level  $\mathbf{H}(I, \varphi, \mathbf{p}_I, \mathbf{p}_\varphi, \varepsilon) = \varepsilon$ , then one can reduce our system by setting  $p_\varphi = \varepsilon \frac{K-1}{\nu}$ , and locally we can write

$$\mathbf{p}_\varphi = -\varepsilon \mathbf{h}(I, \varphi, \mathbf{p}_I, \varepsilon).$$

This yields to have  $p_I$  as a **slow variable**, so is  $I$ .

→ The system can be averaged classically with respect to  $\varphi$ .

## Boundary conditions and transversality conditions

$$I(0) = I_0, \quad p_\varphi(0) = 0, \quad I(t_f(\varepsilon)) = I_f, \quad p_\varphi(t_f(\varepsilon)) = 0, \quad H = \varepsilon.$$

**Remark.** *Parameterizing by  $\varphi$ , the non averaged extremal system (\*) is hamiltonian*

$$\frac{dI}{d\varphi} = \varepsilon \frac{\partial h}{\partial p_I}, \quad \frac{dp_I}{d\varphi} = -\varepsilon \frac{\partial h}{\partial I}. \quad (\text{non averaged})$$



# Averaged of the orbital transfer problem

The averaged dynamic is defined by

$$\begin{aligned}\bar{h}(I, p_I) &:= \frac{1}{2\pi} \int_0^{2\pi} h(I, \varphi, p_I, \varepsilon = 0) d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{K-1}{\nu} (I, \varphi, p_I, p_\varphi = 0, \varepsilon = 0) d\varphi.\end{aligned}$$

Parameterizing by  $s := \varepsilon(\varphi - \varphi_0)$  leads to the **averaged boundary value problem**

$$\begin{aligned}\frac{dI}{ds} &= \frac{\partial \bar{h}}{\partial p_I}, & \frac{dp_I}{ds} &= -\frac{\partial \bar{h}}{\partial I}, & & (\text{averaged}) \\ I(0) &= I_0, & I(s_f) &= I_f, & \bar{h} &= 0.\end{aligned}$$

This system defines a distance on the set of elliptic Keplerian orbits

$$d(I_0, I_f) = s_f.$$

# Singularity

The Hamiltonian  $H$  is non smooth on the set

$$\mathcal{S}_\varepsilon = \{(I, \varphi, p_I, p_\varphi) \mid H(I, \varphi, p_I, p_\varphi) - \varepsilon = 0 \text{ and } H_i(I, \varphi, p_I, p_\varphi, \varepsilon) = 0, i = 1, 2, 3\}.$$

$$\text{codim}(\mathcal{S}_\varepsilon) = 3 \text{ in } \{(I, \varphi, p_I, p_\varphi) \mid H(I, \varphi, p_I, p_\varphi) - \varepsilon = 0\}.$$

We define

$$\mathcal{S}_0 = \{(I, \varphi, p_I, p_\varphi) \mid p_\varphi = 0 \text{ and } H_i(I, \varphi, p_I, p_\varphi = 0, \varepsilon = 0) = 0, i = 1, 2, 3\}.$$

Then, the averaged Hamiltonian is non smooth on

$$\bar{\mathcal{S}} = \{(I, p_I) \mid \exists \varphi, (I, \varphi, p_I, 0) \in \mathcal{S}_0\}$$

which is of **codimension 2**.

# Convergence theorem

Let  $V$  an open subset of the lifted Keplerian elliptic orbits set such that  $\bar{\mathcal{S}} \cap V = \emptyset$ .

**Theorem 2.** *If*

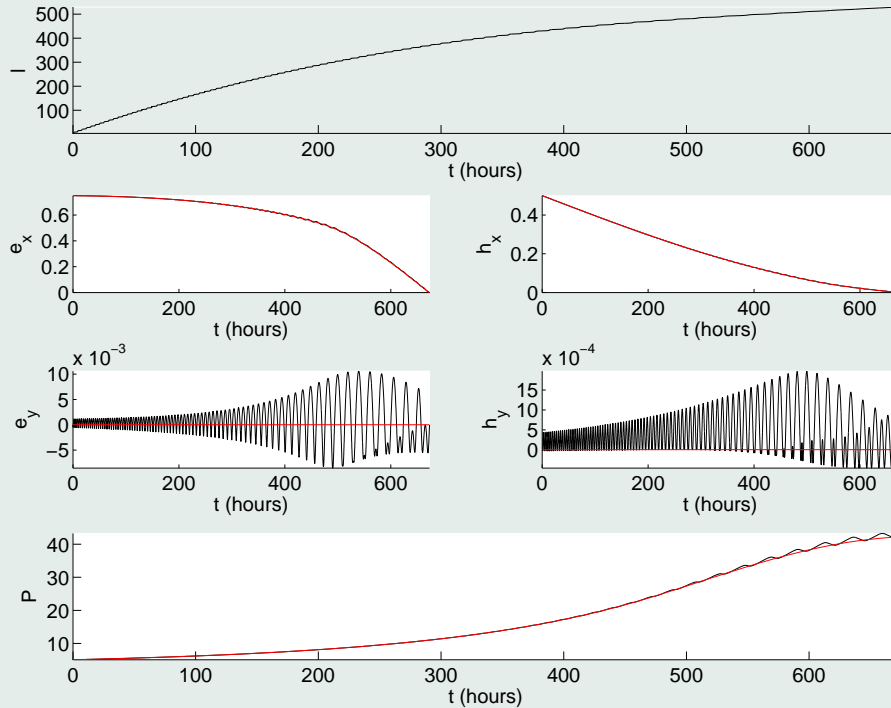
- $(I^\varepsilon(\cdot), \varphi^\varepsilon(\cdot), p_{I^\varepsilon}(\cdot), p_{\varphi^\varepsilon}(\cdot))_\varepsilon$  is a family of extremals defined on  $[0, t_f(\varepsilon)]$  of the non averaged boundary value problem,
- $\forall t \in [0, t_f(\varepsilon)]$ ,  $(I^\varepsilon(t), p_{I^\varepsilon}(t)) \in K \subset V$ ,  $K$  compact, so that, for  $\varepsilon$  small enough  $(I^\varepsilon(t), \varphi^\varepsilon(t), p_{I^\varepsilon}(t), p_{\varphi^\varepsilon}(t)) \notin \mathcal{S}_\varepsilon$ .

Then  $t \rightarrow (I^\varepsilon(\varepsilon t), p_{I^\varepsilon}(\varepsilon t))$  converges uniformly to  $t \rightarrow (\bar{I}(t), \bar{p}_I(t))$  on  $[0, \bar{t}_f]$  which is an extremal of the averaged boundary values problem with conditions

$$\bar{I}(0) = I_0, \quad \bar{I}(\bar{t}_f) = I_f.$$

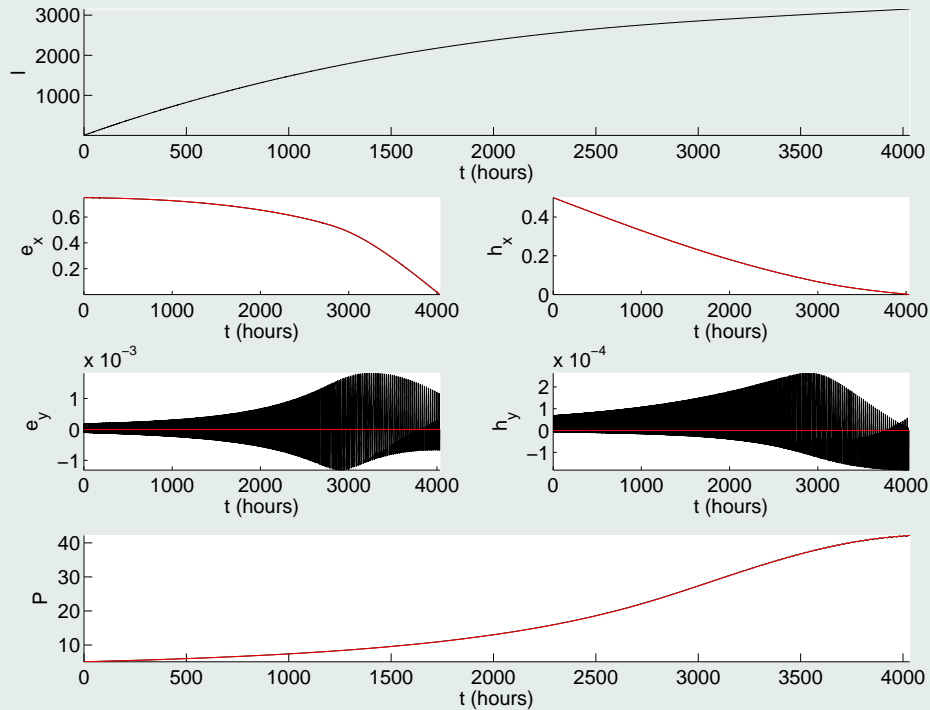
*Dowód.* Using the uniform convergence on  $[0, T]$  of the previous averaging theorem and the equicontinuity of the family  $(I^\varepsilon(\cdot))_\varepsilon$ , the uniform convergence and the boundary conditions are satisfied. □

States :  $I(t) = (P, e_x, e_y, h_x, h_y)(t)$  et  $\varphi(t) = l(t)$



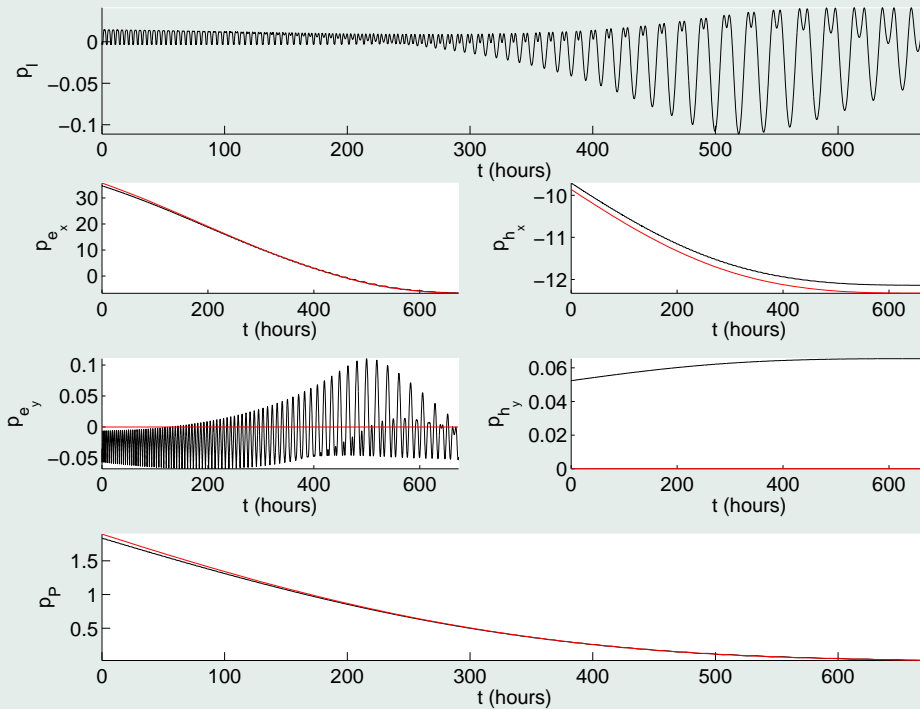
*Minimizing trajectories - Thrust 3N,  $\varepsilon \approx 2e-2$   
black: non averaged, red: averaged.*

States :  $I(t) = (P, e_x, e_y, h_x, h_y)(t)$  et  $\varphi(t) = l(t)$



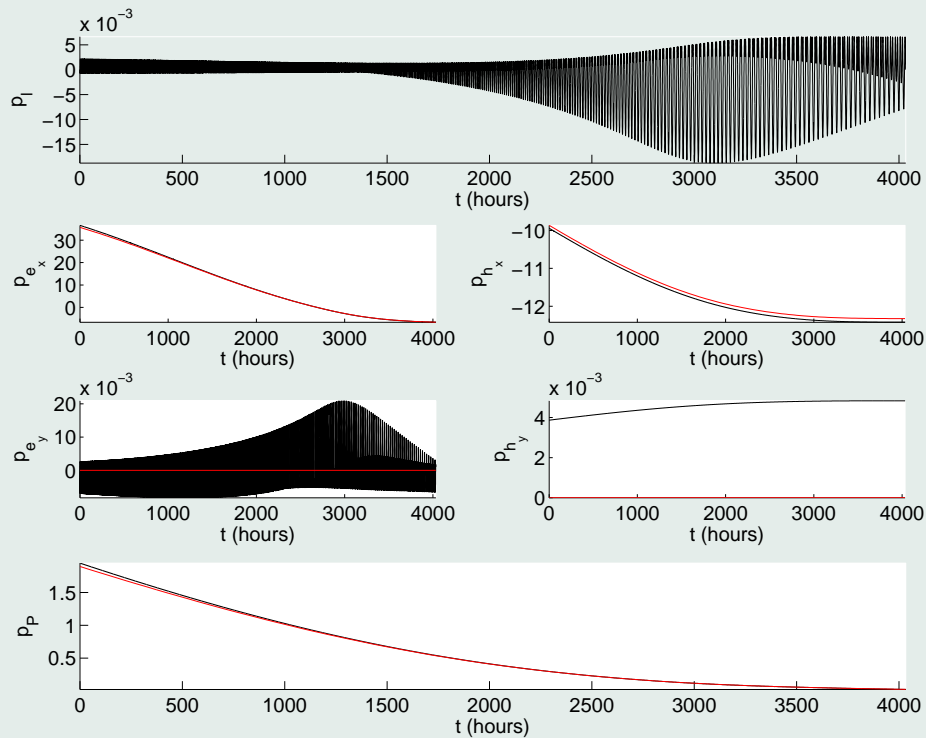
Minimizing trajectories - **Thrust 0.5N**,  $\varepsilon \approx 4e-3$   
**black: non averaged, red: averaged.**

# Adjoint $p_I(t) = (p_P, p_{e_x}, p_{e_y}, p_{h_x}, p_{h_y})(t)$ et $p_\varphi(t) = p_l(t)$



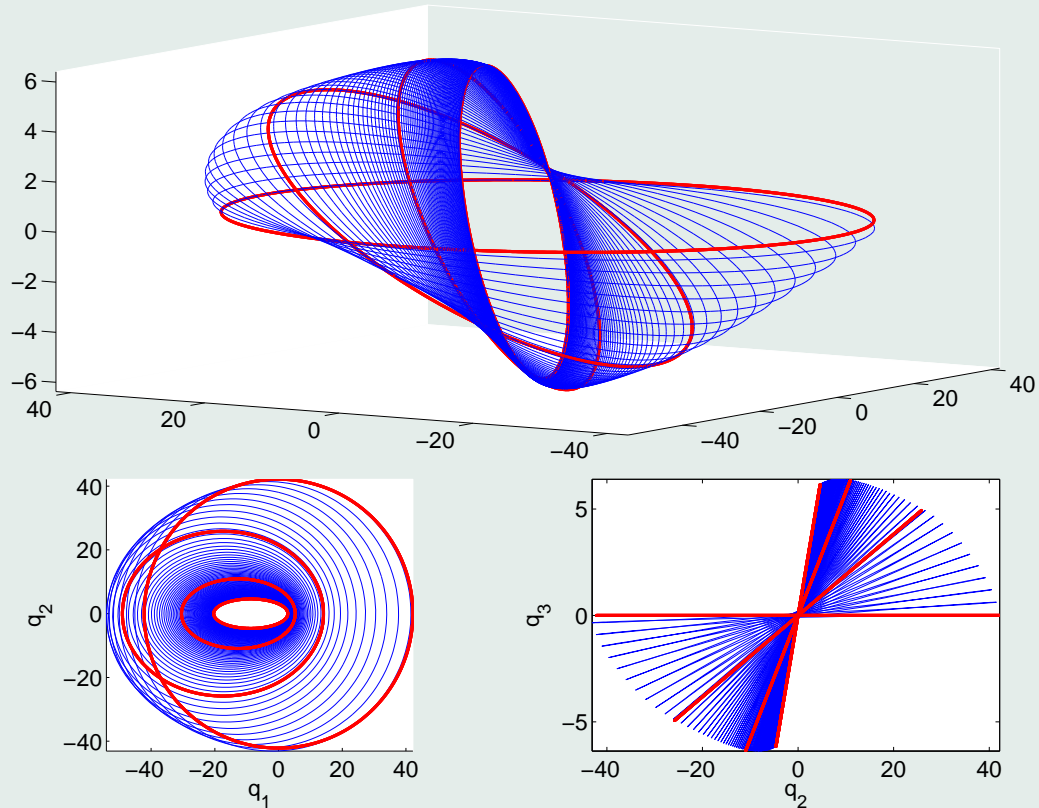
Minimizing trajectories - Thrust  $3N$ ,  $\varepsilon \approx 2e-2$   
*black: non averaged, red: averaged.*

# Adjoint $p_I(t) = (p_P, p_{e_x}, p_{e_y}, p_{h_x}, p_{h_y})(t)$ et $p_\varphi(t) = p_l(t)$



Minimizing trajectories - Thrust  $0.5N$ ,  $\varepsilon \approx 4e-3$   
black: non averaged, red: averaged.

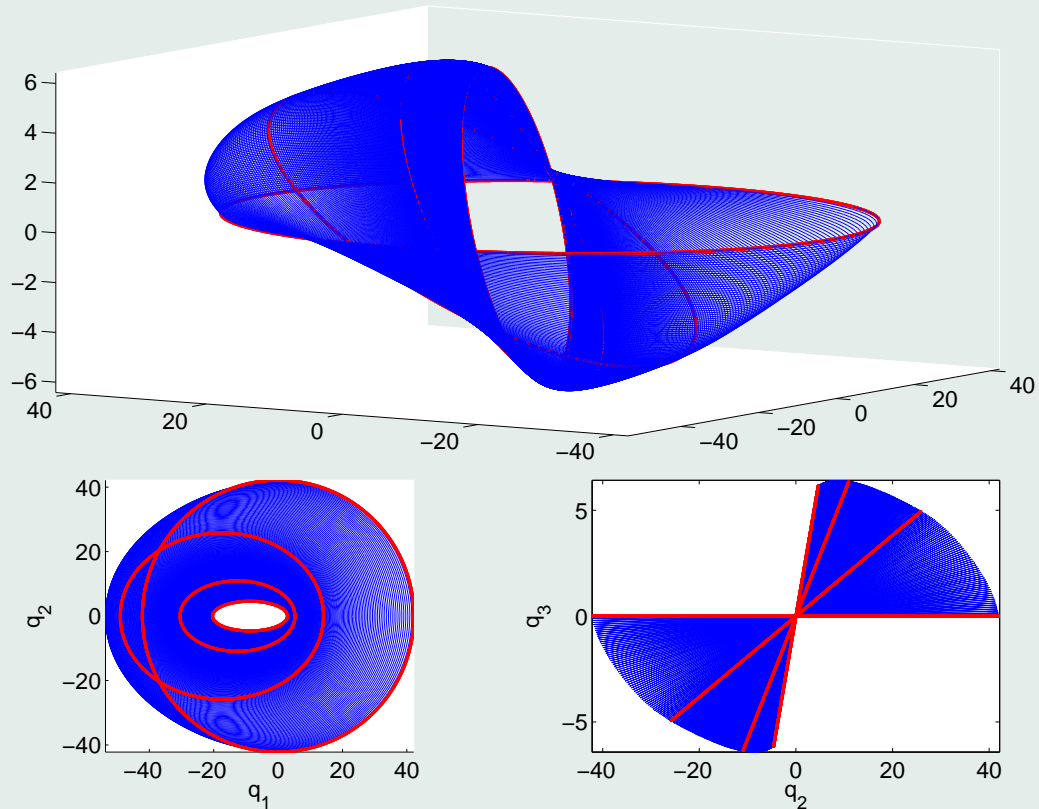
# Cartesian coordinates



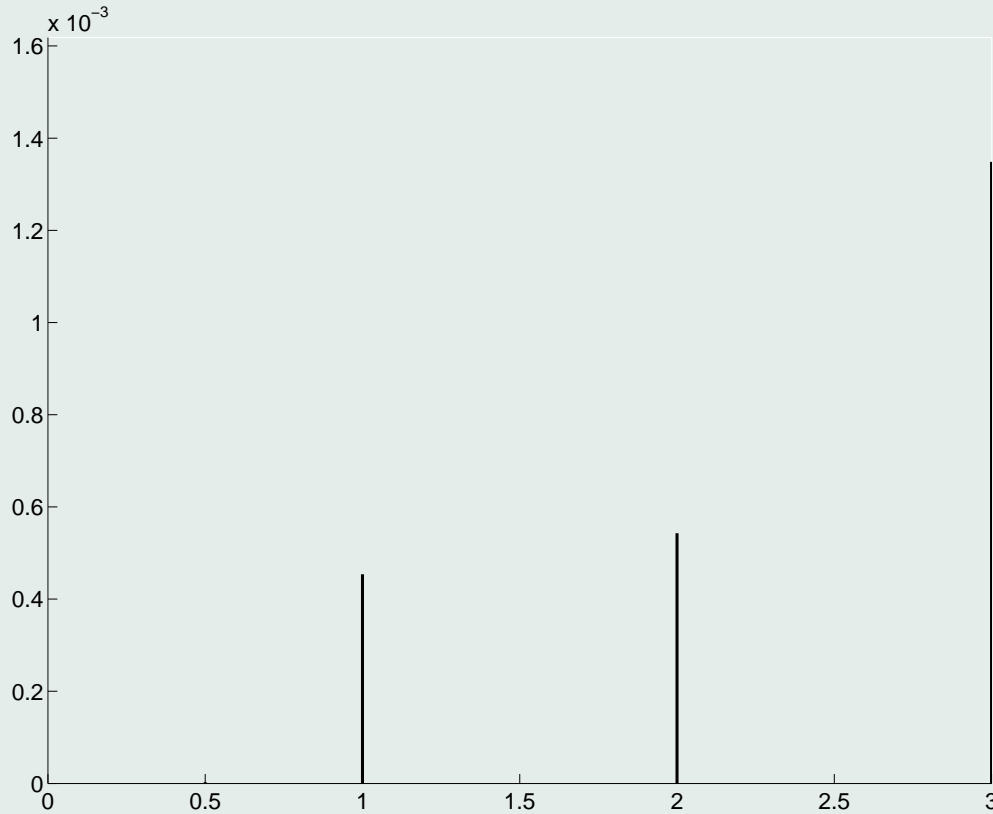
*Thrust 3N - some orbits of the averaged trajectories are represented in red.*



# Cartesian coordinates



*Thrust  $0.5N$  - some orbits of the averaged trajectories are represented in red.*



*Relative error between  $t_{f_{optimal}}$  (non averaged optimal time) et  $t_{f_{moy}}$  (estimated from normalized optimal angle  $s_f$  of the averaged system).*

# References

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- [3] Bonnard, Bernard; Caillau, Jean-Baptiste. *Riemannian metric of the averaged energy minimization problem in orbital transfer with low thrust*. Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), no. 3, 395–411.