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Optimal control theory, sub-Riemannian geometry and swimming of copepod

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Optimal control theory, sub-Riemannian geometry and swimming of copepod

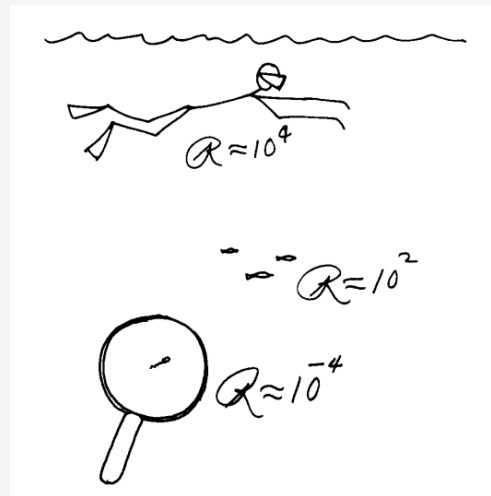
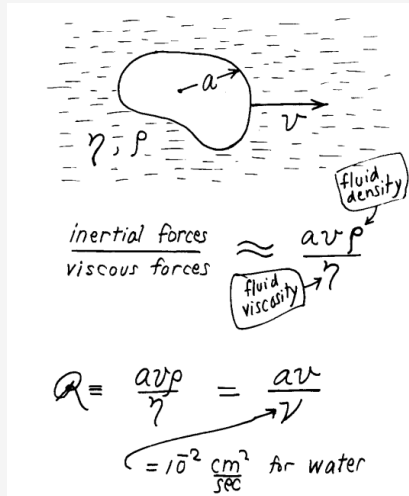
Jérémy Rouot, LAAS-CNRS, Toulouse

Congrès SMAI 2017

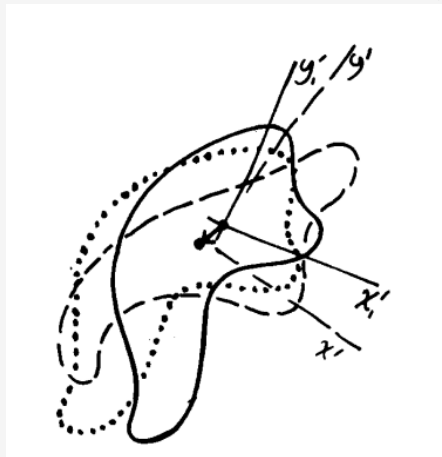
8e Biennale Française des Mathématiques Appliquées et Industrielles

La Tremblade

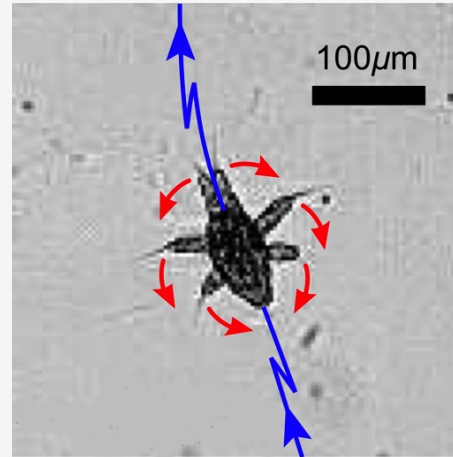
Life at low Reynolds number - Purcell, 1977



Reynolds number

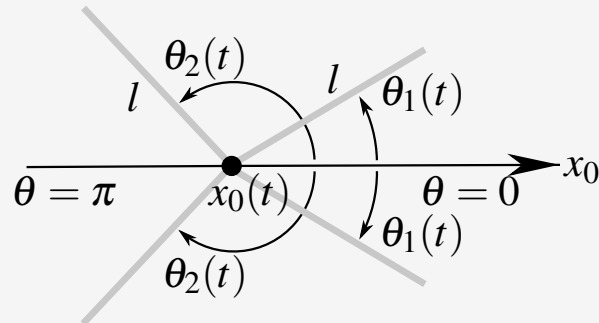


Shape deformations of a microswimmer



Zooplankton

2-link symmetric swimmer.

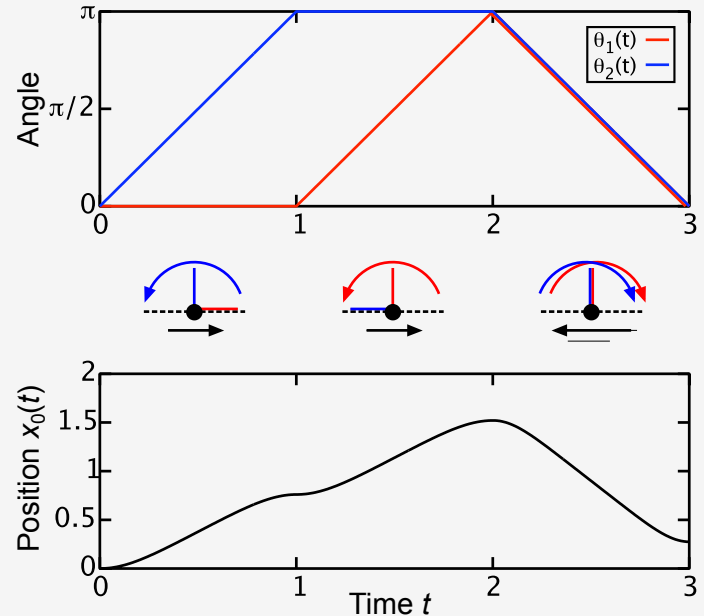
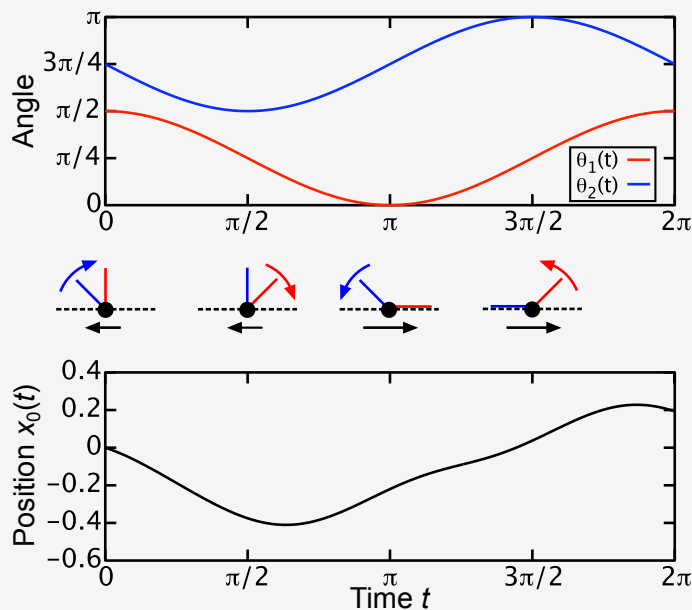


Dynamic.

$$\dot{x}_0 = \frac{\sum_{i=1}^2 l^2 \dot{\theta}_i \sin(\theta_i)}{2 + \sum_{i=1}^2 \sin^2(\theta_i)}, \quad \dot{\theta}_i = u_i, \quad i = 1, 2 \quad (\text{state constraint : } 0 \leq \theta_1 \leq \theta_2 \leq \pi).$$

Criterion. min drag forces : $\dot{q}S(q)\dot{q}^\top$, $\mathbf{q} = (\theta_1, \theta_2, x_0)$ and S is positive definite \implies quadratic form in (u_1, u_2) .

Definition 1. A stroke is a periodic motion of the shape variables (θ_1, θ_2) associated with a **periodic control** producing a net displacement of the position variables after one period T (we can fix $T = 2\pi$).



(left) Trigonometric controls. (right) Piecewise constant controls.

$$\mathcal{E}(q(\cdot)) = \text{Geometric Efficiency} = x_0(T)/l_{SR}(q)$$

SR Length l_{SR} .

$$l_{SR}(q) = \int_0^T \sqrt{L(q, u)} dt, \quad L(q, u) = a(q) u_1^2 + 2b(q) u_1 u_2 + c(q) u_2^2$$

$$\min_{u(\cdot)} l_{SR}(q) \Leftrightarrow \min_{u(\cdot)} \int_0^T L(q, u) dt$$

Two steps.

$$\max_{u(\cdot), x_0(T)} \mathcal{E} \Leftrightarrow \left(\begin{array}{l} \dot{q} = \sum_{i=1}^2 u_i F_i \leftarrow \min_{u(\cdot)} \int_0^T L(q, u) dt, \\ \min_{u(\cdot)} \int_0^T L(q, u) dt \text{ with } \mathbf{x}_0(\mathbf{T}) \text{ fixed, then } \underbrace{\text{select max}_{x_0(T)} \mathcal{E}}_{\text{Transversality cond.}} \end{array} \right)$$

Compute points on the Sub-Riemannian sphere \rightarrow
provide candidate solutions for the maximum of efficiency problem.

Introduce q^0 be the state s.t. : $\dot{q}^0 = L(q, u)$ and denote $\tilde{q} = (q, q^0)$,
 $c(\tilde{q}(0), \tilde{q}(T))$ is the cost function to minimize,
 B defines the boundary conditions : $(\tilde{q}(0), \tilde{q}(T)) \in B$.

Necessary conditions. (\tilde{q}, u) optimal $\Rightarrow \exists \tilde{p} = (p, p^0) : [0, T] \rightarrow (\mathbb{R}^n \times \mathbb{R})$ abs. continue and $\lambda \geq 0$ cst. with $(\tilde{p}, \lambda) \neq (0, 0)$ and s.t.

$$\dot{\tilde{q}} = \frac{\partial H}{\partial \tilde{p}}, \quad \dot{\tilde{p}} = -\frac{\partial H}{\partial \tilde{q}}, \quad \frac{\partial H}{\partial u} = 0$$

where $H(\tilde{q}, \tilde{p}, u) = p \cdot \dot{q} + p^0 \dot{q}^0$.

Transversality Conditions

$$(\tilde{p}(0), -\tilde{p}(T)) - \lambda \left(\frac{\partial c}{\partial \tilde{q}(0)}(\tilde{q}(0), \tilde{q}(T)), \frac{\partial c}{\partial \tilde{q}(T)}(\tilde{q}(0), \tilde{q}(T)) \right) \in (T_{(\tilde{q}(0), \tilde{q}(T))} B)^\perp$$

Definition 2. $(\tilde{q}, \tilde{p}, \lambda, u)$ is called a **normal** extremal if $\lambda > 0$, **abnormal** if $\lambda = 0$.
 An abnormal trajectory $q(\cdot)$ which is not the projection of a normal extremal is called *strict*.

$$\dot{q} = \sum_{i=1}^2 u_i F_i(q) \leftarrow \min_{u(\cdot)} q^0(T), \quad q = (\theta_1, \theta_2, x_0), \quad \mathbf{x}_0(\mathbf{T}) \text{ fixed .}$$

$$F_i = \frac{\sin(\theta_i)}{\Delta} \frac{\partial}{\partial x_0} + \frac{\partial}{\partial \theta_i}, \quad \Delta = 2 + \sin^2(\theta_1) + \sin^2(\theta_2)$$

PMP :

$$H(z, u) = u_1 H_1(z) + u_2 H_2(z) + p^0 L(q, u), \quad z = (q, p)$$

with $H_i = p \cdot F_i$, $i = 1, 2$.

— normal case : $p^0 = -1/2$

$$H_n = 1/2 (a(q) H_1^2 + 2b(q) H_1 H_2 + c(q) H_2^2)$$

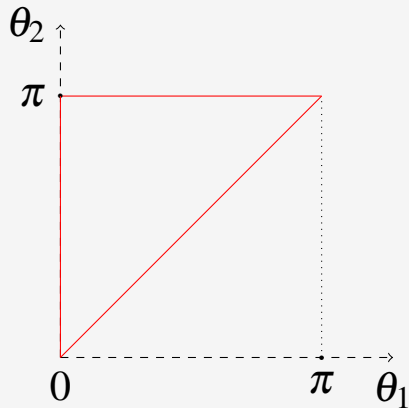
— abnormal case : $p^0 = 0$ hence $H_1 = H_2 = \{H_1, H_2\} = 0$, then

$$u_1 \{ \{H_1, H_2\}, H_1 \} + u_2 \{ \{H_1, H_2\}, H_2 \} = 0.$$

Proposition 3. *The surface $\Sigma : \{q; \det(F_1(q), F_2(q), [F_1, F_2](q)) = 0\}$ contained abnormal trajectories and is defined by*

- $\theta_i = 0$ or π , for $i = 1, 2$,
- $\theta_1 = \theta_2$.

*It is formed by the **boundaries of the physical domain** : $\theta_i \in [0, \pi]$, $i = 1, 2$, $\theta_1 \leq \theta_2$.*



: abnormal trajectories
in the plane (θ_1, θ_2)

Boundary conditions

$$x_0(0) = 0, \quad x_0(T) = x_f, \quad \theta_i(0) = \theta_i(T), \quad i = 1, 2$$

Transversality conditions

$$p_{\theta_i}(0) = p_{\theta_i}(T), \quad i = 1, 2$$

Boundary value problem

$$\begin{cases} \dot{z} = \vec{H}_n(z) \\ x_0(0) = 0, \quad x_0(T) = x_f \text{ (fixed) } , \\ \theta_i(0) = \theta_i(T), \quad i = 1, 2 \\ p_{\theta_i}(0) = p_{\theta_i}(T), \quad i = 1, 2 \end{cases}$$

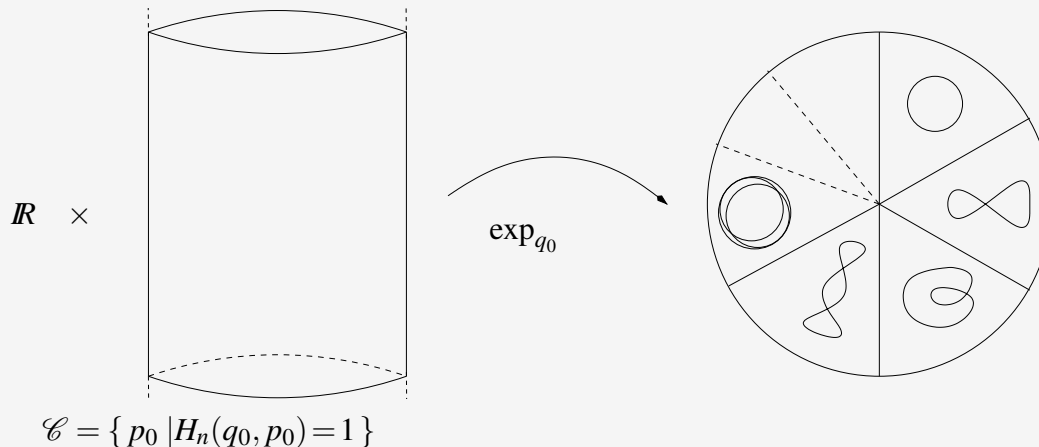
- HamPath, *indirect method* : based on Maximum Principle, *Newton* type algorithm (sensitive to initialization) and computation of **second order necessary optimality conditions** .
- Bocop, *direct method* : gives an initialization for the shooting algorithm of HamPath.

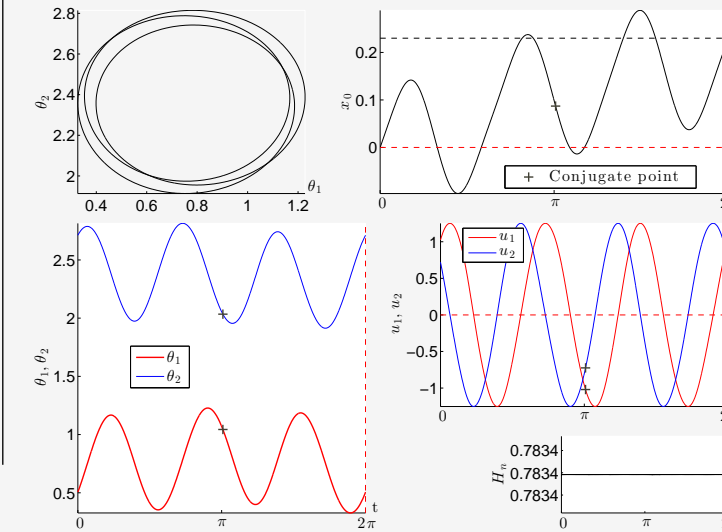
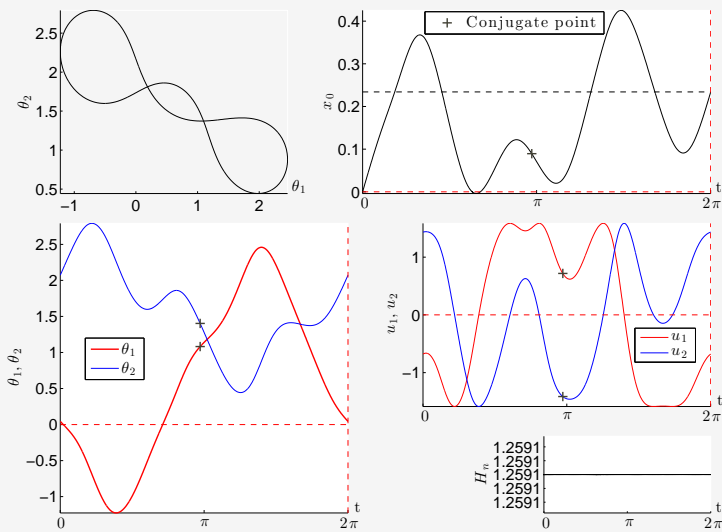
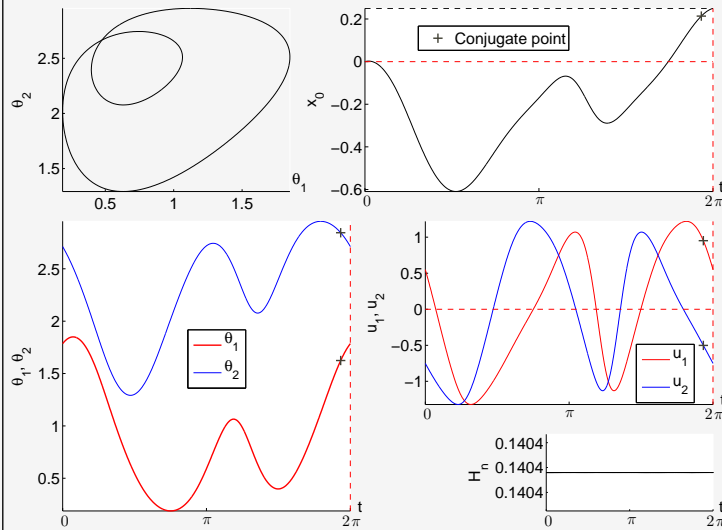
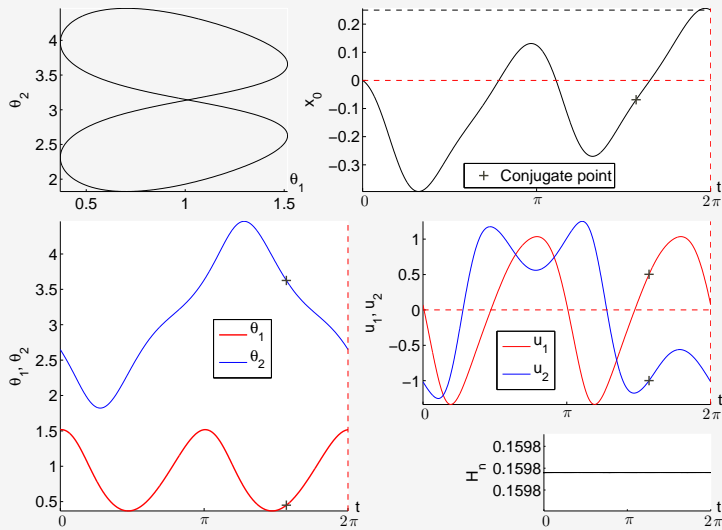
First conjugate time t_c : The first time t_c when the exponential mapping

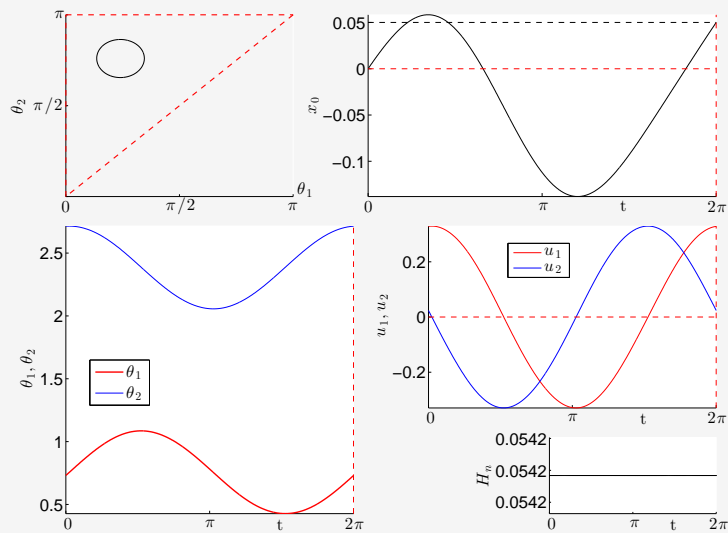
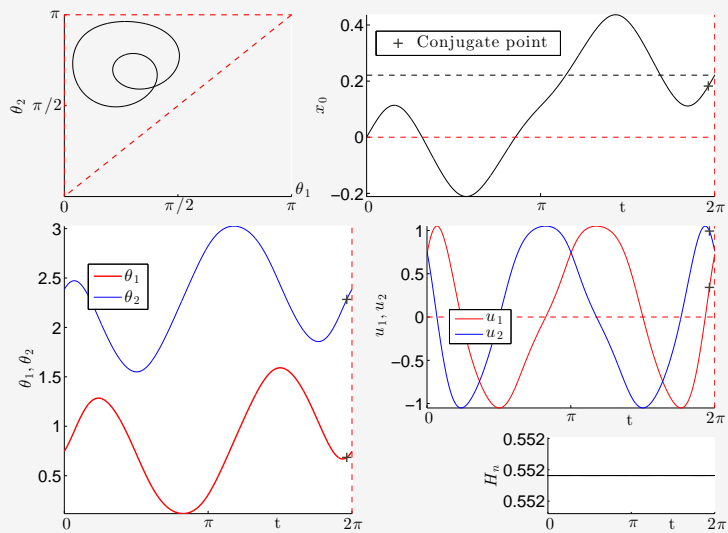
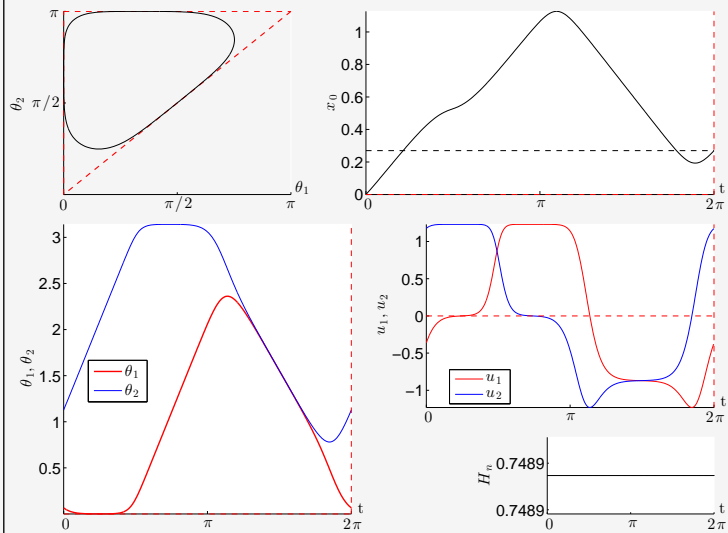
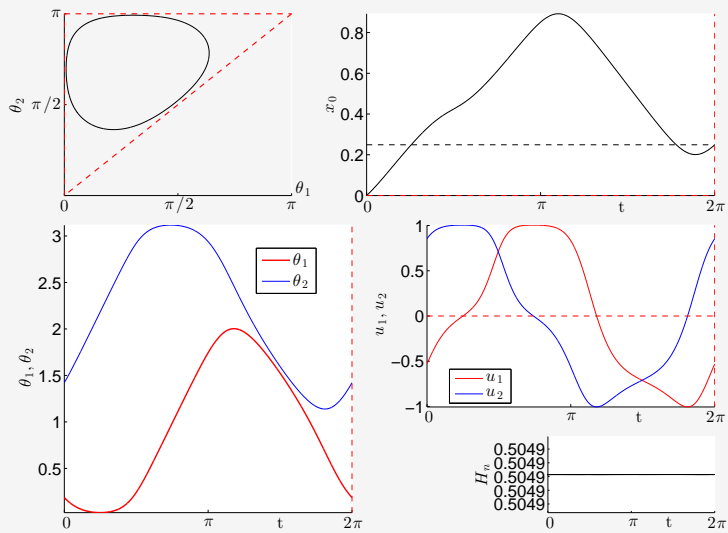
$$\exp_{q_0} : \mathbb{R} \times \mathcal{C} \rightarrow M, \quad (t, p_0) \mapsto q(t, q_0, p_0)$$

is not an immersion at (t_c, p_0) .

Theorem 4. *Let $q : [0, T] \rightarrow \mathbb{R}^n$ be a strict normal stroke. If $q(\cdot)$ has at least one conjugate point on $]0, T[$, then $q(\cdot)$ is not a local minimizer for the L^∞ topology on the controls considering the optimal control problem with **fixed extremities**.*



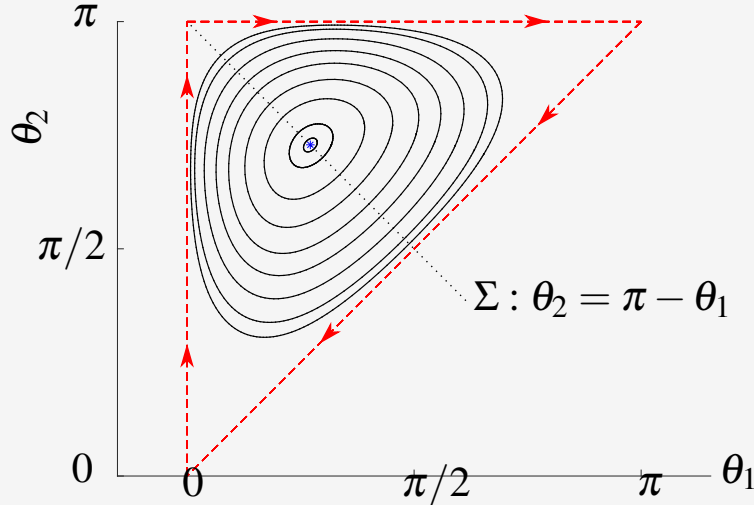




Numerical fact :

Only the simple loops don't have conjugate points on $]0, T[$. They are the only candidates for optimality.

Theorem 5. *There is a one parameter family of simple loop strokes parameterized by the displacement $x_0(T)$ or the energy level $H_n(q_0, p_0)$.*



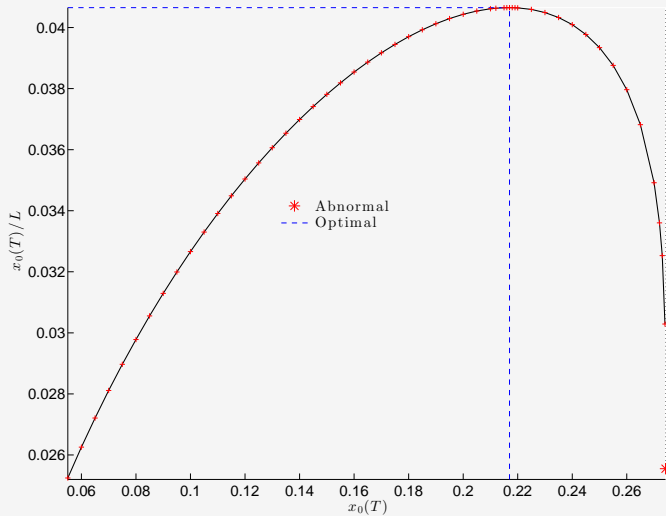
One parameter family of geodesic strokes.

Select on the point on the SR sphere with maximum of efficiency

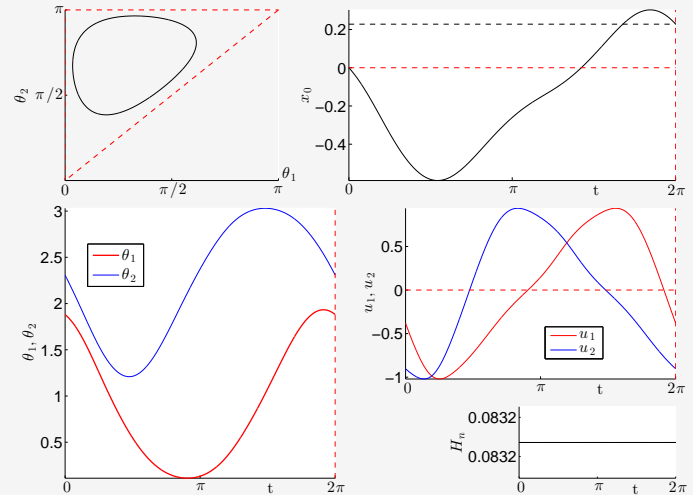
Geometric efficiency : $\mathcal{E} = x_0(T)/l_{SR}(q)$, $(\mathbf{x}_0(T) \text{ free})$

Transversality conditions

$$p_{x_0}(T) = q^0(T)/x_0(T), \quad p^0(T) = -1/2$$



Efficiency curve with continuation on $x_0(T)$.



Optimal normal stroke

Theorem 6. *The abnormal stroke is not minimizing for both costs (length with fixed displacement and efficiency).*

SR problem (D, g) where $D = \text{span}\{F, G\}$ is a 2-dimensional distribution in \mathbb{R}^3 and g is a SR metric.

Normal form of order -1 Nilpotent model of order -1 , the Brockett model :

$$\hat{F} = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad \hat{G} = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}$$

where (weight of x, y)=1 and (weight of z)=2 (privileged coordinates).

Application to the Copepod (x, y, z) are related to the physical coordinates $(\theta_1, \theta_2, x_0)$ by change of coordinates $(x = \theta_1, y = \theta_2)$.

\Rightarrow this model is **not generic** and z (always increasing) cannot be identified with the displacement x_0 .

Normal form of order 1

Theorem 7 (Chakir, Gauthier, Kupka, 1996). *The generic model is given by the normal form of order 1*

$$F = \hat{F} + yQ(x, y) \frac{\partial}{\partial z}, \quad G = \hat{G} - xQ(x, y) \frac{\partial}{\partial z},$$

Q quadratic in (x, y) .

Application to the Copepod At a contact point $(\beta, \pi - \beta)$, $\beta \in]0, \pi/2[$, truncating the Taylor expansion of F_1, F_2 , we consider :

$$\begin{aligned} \hat{F}_1(x, y, z) &= \frac{\partial}{\partial x} + (a_{30}(\beta)x^3 + a_{21}(\beta)x^2y + a_{12}(\beta)xy^2 + a_{03}(\beta)y^3 + a_{20}(\beta)x^2 \\ &\quad + a_{11}(\beta)xy + a_{02}(\beta)y^2 + a_{10}(\beta)x + a_{01}(\beta)y + a_{00}(\beta)) \frac{\partial}{\partial z} \\ \hat{F}_2(x, y, z) &= \frac{\partial}{\partial y} + (-a_{30}(\beta)y^3 - a_{21}(\beta)xy^2 - a_{12}(\beta)x^2y - a_{03}(\beta)x^3 + a_{20}(\beta)y^2 \\ &\quad + a_{11}(\beta)xy + a_{02}(\beta)x^2 - a_{10}(\beta)y - a_{01}(\beta)x + a_{00}(\beta)) \frac{\partial}{\partial z}. \end{aligned}$$

Gauge group \mathcal{D} : set of formal diffeomorphisms preserving the origin,

\mathcal{F} : set of formal power series in x, y, z .

— action of $\varphi \in \mathcal{D}$, $\varphi.(A, B) = (\varphi_*(A), \varphi_*(B))$,

— action of $\alpha \in \mathcal{F}$, $\alpha.(A, B) = (A \cos \alpha + B \sin \alpha, -A \sin \alpha + B \cos \alpha)$.

Under the action of the gauge group, (\hat{F}_1, \hat{F}_2) can be put under the form given in Theorem 8 with

$$Q(x, y) = 1/8 \frac{(-11 \cos^6 \beta + 17 \cos^4 \beta - 8 \cos^2 \beta + 4) x^2}{(\cos^6 \beta - 5 \cos^4 \beta + 8 \cos^2 \beta - 4) \cos^2 \beta} + 1/8 \frac{(5 \cos^8 \beta - 33 \cos^6 \beta + 2 \cos^4 \beta + 28 \cos^2 \beta - 8) y^2}{(\cos^6 \beta - 5 \cos^4 \beta + 8 \cos^2 \beta - 4) \cos^2 \beta}.$$

Periodicity equations

$$\theta_i(0) = \theta_i(T), \quad \dot{\theta}_i(0) = \dot{\theta}_i(T), \quad i = 1, 2.$$

Admissible transformations

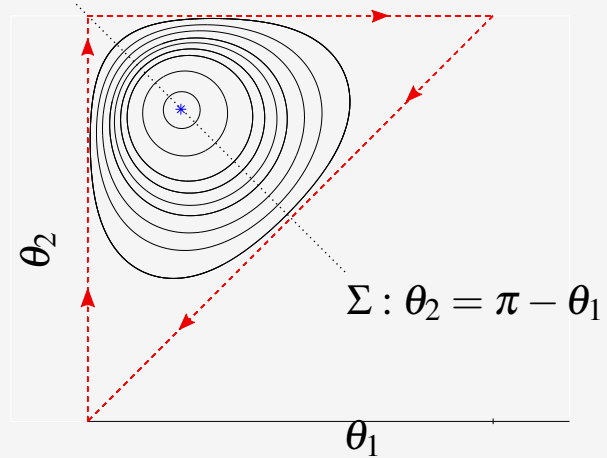
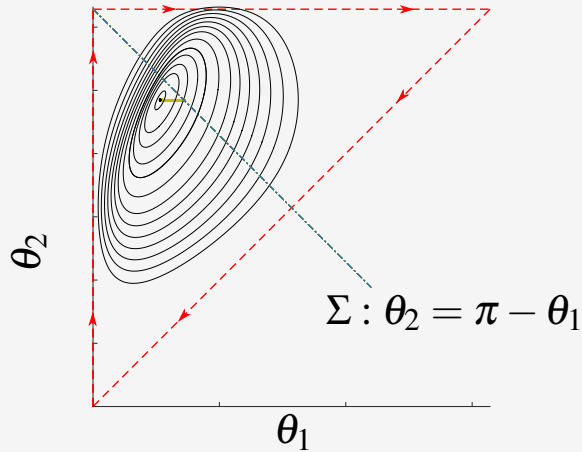
$$\mathcal{G} = \{ \varphi \in \mathcal{D} \mid \varphi(\theta, x_0) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta, x_0)) \}.$$

Transformation mixing up shape variables (x, y) and the displacement z

$$\varphi(x, y, z) = (x + c_{011}(\beta)yz + c_{001}(\beta)z, y - c_{011}(\beta)xz + c_{001}(\beta)z, z + P(x, y, z, \beta)),$$

$$\Rightarrow c_{011}(\beta) = c_{001}(\beta) = 0.$$

Theorem 8. *The only stroke that can be computed with transformations of \mathcal{G} is reduced to the **center of the family** of simple loop strokes (obtained by the true system).*



One parameter family of simple loops corresponding to different metrics for the true system.

The center of family = SR invariant : related to the inverse optimal control problem.