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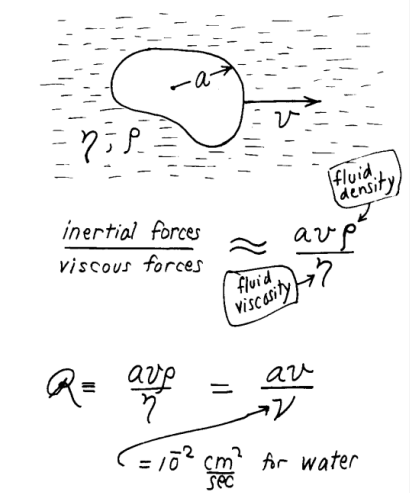
Optimal control theory, sub-Riemannian geometry and swimming of copepod

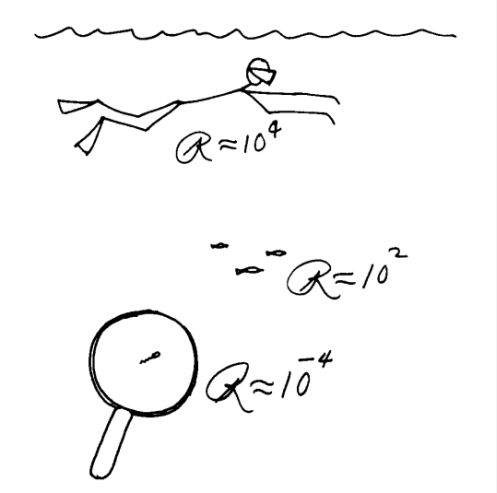
Jérémy Rouot, LAAS-CNRS, Toulouse

Congrès SMAI 2017
8e Biennale Française des Mathématiques Appliquées et Industrielles

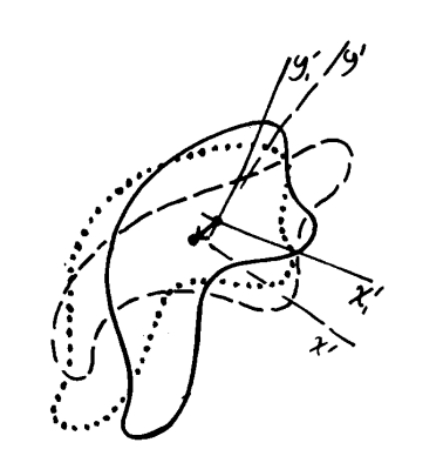
La Tremblade

Joint work with P. Bettiol, B. Bonnard
jrouot@laas.fr

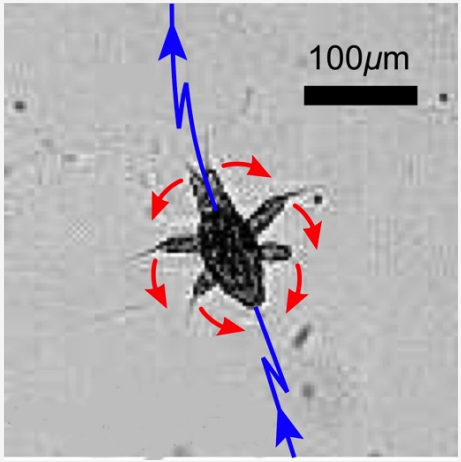




Reynolds number

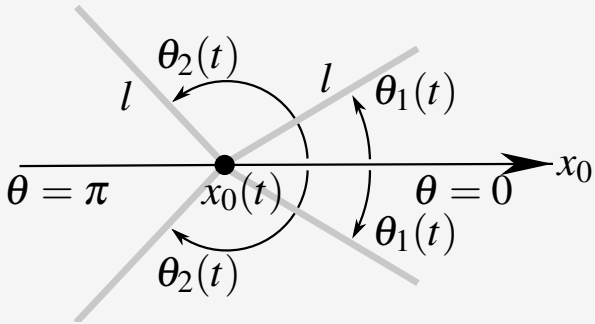


Shape deformations of a microswimmer



Zooplankton

2-link symmetric swimmer.

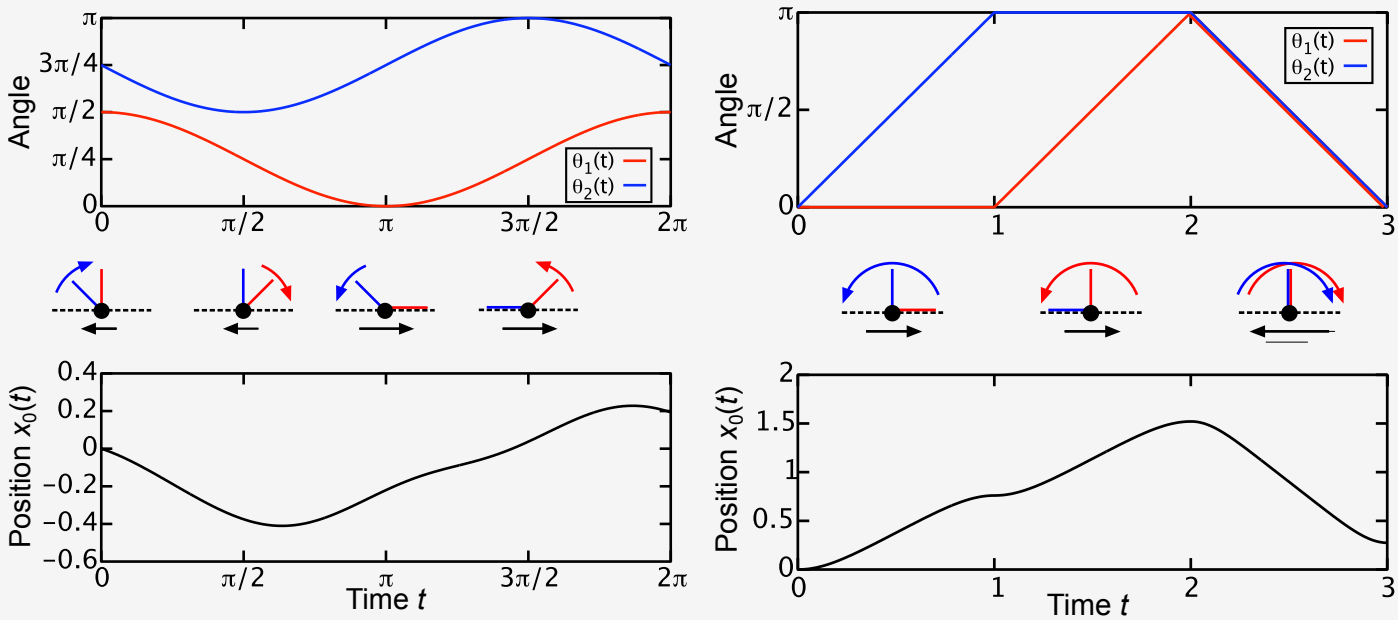


Dynamic.

$$\dot{x}_0 = \frac{\sum_{i=1}^{2} l^2\, \dot{\theta}_i \sin(\theta_i)}{2 + \sum_{i=1}^{2} \sin^2(\theta_i)}, \quad \dot{\theta}_i = u_i,\ i = 1,2 \quad (\text{state constraint} : 0 \leq \theta_1 \leq \theta_2 \leq \pi).$$

Criterion. min drag forces : $\dot{q} S(q) \dot{q}^{\mathsf{T}}$, $\boldsymbol{q} = (\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \boldsymbol{x_0})$ and S is positive definite $\Longrightarrow$ quadratic form in (u_1, u_2).

Definition 1. *A stroke is a periodic motion of the shape variables* (θ_1, θ_2) *associated with a* **periodic control** *producing a net displacement of the position variables after one period* T *(we can fix* $T = 2\pi$*).*



(left) Trigonometric controls. (right) Piecewise constant controls.

$$\mathscr{E}(q(\cdot)) = \textbf{Geometric Efficiency} = x_0(T)/l_{SR}(q)$$

SR Length l_{SR}.

$$l_{SR}(q) = \int_0^T \sqrt{L(q,u)}\,\mathrm{d}t, \qquad L(q,u) = a(q)\,u_1^2 + 2b(q)\,u_1 u_2 + c(q)\,u_2^2$$

$$\min_{u(\cdot)} l_{SR}(q) \Leftrightarrow \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t$$

Two steps.

$$\dot{q} = \sum_{i=1}^{2} u_i F_i \leftarrow \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t,$$

$$\max_{u(\cdot),\, x_0(T)} \mathscr{E} \Leftrightarrow \left(\min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t \quad \text{with } \boldsymbol{x_0(T)} \textbf{ fixed}, \text{ then } \underbrace{\text{select} \max_{x_0(T)} \mathscr{E}}_{\text{Transversality cond.}} \right)$$

Compute points on the Sub-Riemannian sphere $\rightarrow$
provide candidate solutions for the maximum of efficiency problem.

Introduce q^0 be the state s.t. : $\dot{q}^0 = L(q,u)$ and denote $\widetilde{q} = (q, q^0)$,
$c(\widetilde{q}(0), \widetilde{q}(T))$ is the cost function to minimize,
B defines the boundary conditions : $(\widetilde{q}(0), \widetilde{q}(T)) \in B$.

Necessary conditions. $(\widetilde{q}, u)$ optimal $\Rightarrow \exists\ \widetilde{p} = (p, p^0) : [0,T] \to (\mathbb{R}^n \times \mathbb{R})$ abs. continue and $\lambda \geq 0$ cst. with $(\widetilde{p}, \lambda) \neq (0,0)$ and s.t.

$$\dot{\widetilde{q}} = \frac{\partial H}{\partial \widetilde{p}}, \quad \dot{\widetilde{p}} = -\frac{\partial H}{\partial \widetilde{q}}, \quad \frac{\partial H}{\partial u} = 0$$

where $H(\widetilde{q}, \widetilde{p}, u) = p \cdot \dot{q} + p^0 \dot{q}^0$.

Transversality Conditions

$$(\widetilde{p}(0), -\widetilde{p}(T)) - \lambda \left(\frac{\partial c}{\partial \widetilde{q}(0)}(\widetilde{q}(0), \widetilde{q}(T)), \frac{\partial c}{\partial \widetilde{q}(T)}(\widetilde{q}(0), \widetilde{q}(T)) \right) \in (T_{(\widetilde{q}(0), \widetilde{q}(T))} B)^{\perp}$$

Definition 2. *$(\widetilde{q}, \widetilde{p}, \lambda, u)$ is called a* **normal** *extremal if $\lambda > 0$,* **abnormal** *if $\lambda = 0$. An abnormal trajectory $q(\cdot)$ which is not the projection of a normal extremal is called strict.*

$$\dot{q} = \sum_{i=1}^{2} u_i F_i(q) \leftarrow \min_{u(.)} q^0(T), \quad q = (\theta_1, \theta_2, x_0), \quad \boldsymbol{x_0(T)} \textbf{ fixed } .$$

$$F_i = \frac{\sin(\theta_i)}{\Delta}\frac{\partial}{\partial x_0} + \frac{\partial}{\partial \theta_i}, \qquad \Delta = 2 + \sin^2(\theta_1) + \sin^2(\theta_2)$$

PMP :

$$H(z,u) = u_1 H_1(z) + u_2 H_2(z) + p^0 L(q,u), \qquad z = (q,p)$$

with $H_i = p \cdot F_i$, $i = 1,2$.

— normal case : $p^0 = -1/2$

$$H_n = 1/2\,(a(q)H_1^2 + 2b(q)H_1H_2 + c(q)H_2^2)$$

— abnormal case : $p^0 = 0$ hence $H_1 = H_2 = \{H_1, H_2\} = 0$, then

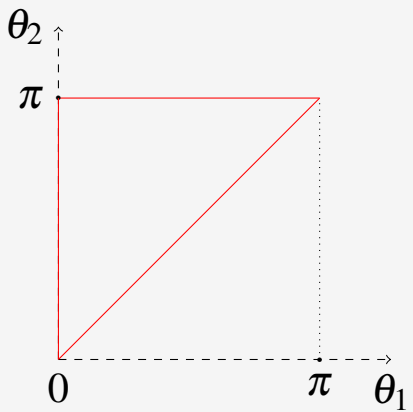
$$u_1\{\{H_1,H_2\},H_1\} + u_2\{\{H_1,H_2\},H_2\} = 0.$$

Proposition 3. *The surface* $\Sigma : \{q;\ \det(F_1(q), F_2(q), [F_1, F_2](q)) = 0\}$ *contained abnormal trajectories and is defined by*

— $\theta_i = 0$ *or* π*, for* $i = 1, 2$*,*

— $\theta_1 = \theta_2$*.*

It is formed by the **boundaries of the physical domain** *:* $\theta_i \in [0, \pi]$*,* $i = 1, 2, \theta_1 \leq \theta_2$*.*



: abnormal trajectories

in the plane (θ_1, θ_2)

Boundary conditions

$$x_0(0) = 0, \quad x_0(T) = x_f, \quad \theta_i(0) = \theta_i(T),\ i = 1,2$$

Transversality conditions

$$p_{\theta_i}(0) = p_{\theta_i}(T),\ i = 1,2$$

Boundary value problem

$$\begin{cases} \dot{z} = \overrightarrow{H}_n(z) \\ x_0(0) = 0, \quad x_0(T) = x_f \ \textbf{(fixed)} \\ \theta_i(0) = \theta_i(T),\ i = 1,2 \\ p_{\theta_i}(0) = p_{\theta_i}(T),\ i = 1,2 \end{cases},$$

— `HamPath`, *indirect method :* based on Maximum Principle, *Newton* type algorithm (sensitive to initialization) and computation of **second order necessary optimality conditions** .

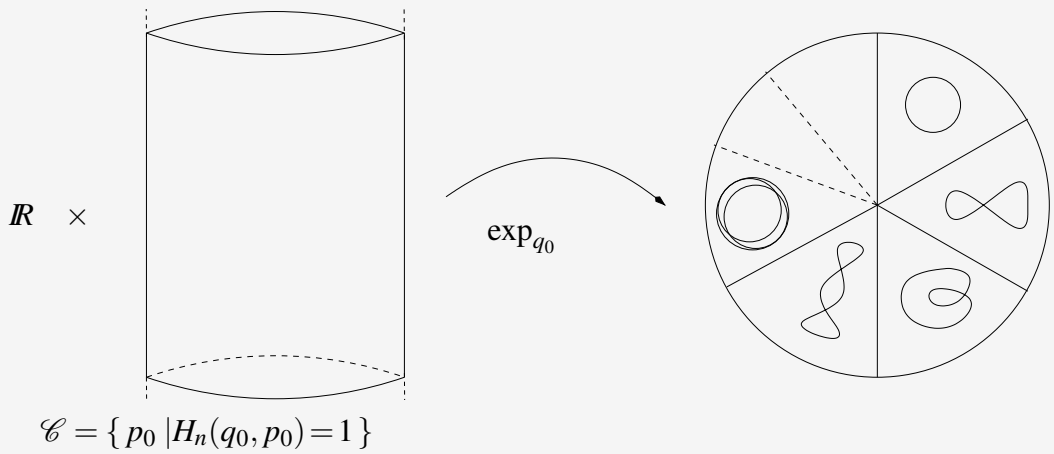
— `Bocop`, *direct method :* gives an initialization for the shooting algorithm of `HamPath`.

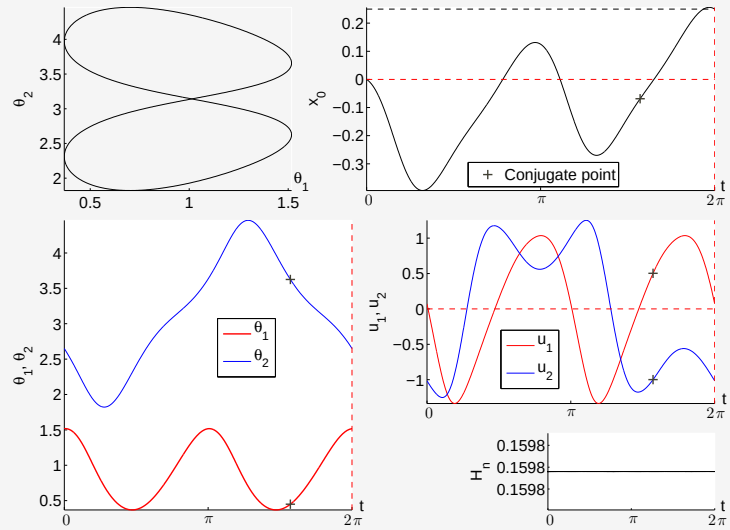
First conjugate time t_c : The first time t_c when the exponential mapping

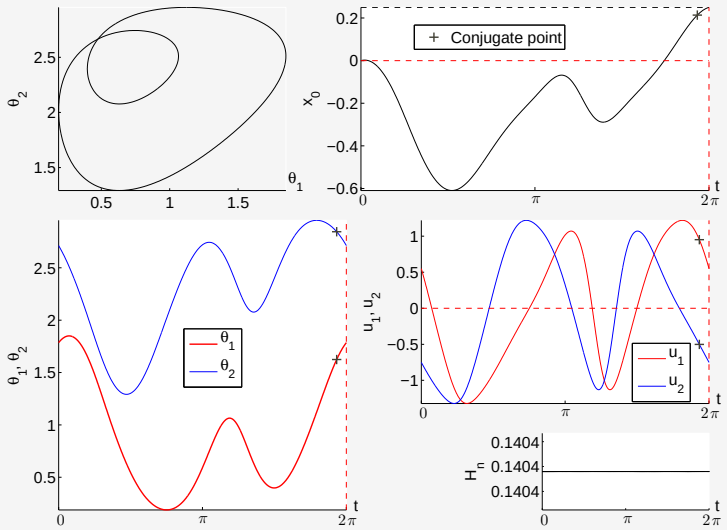
$$\exp_{q_0} : \mathbb{R} \times \mathscr{C} \to M, \quad (t, p_0) \mapsto q(t, q_0, p_0)$$

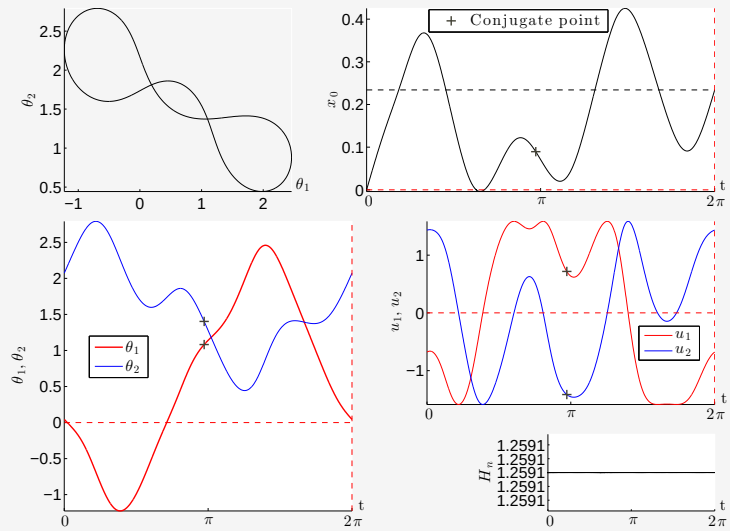
is not an immersion at (t_c, p_0).

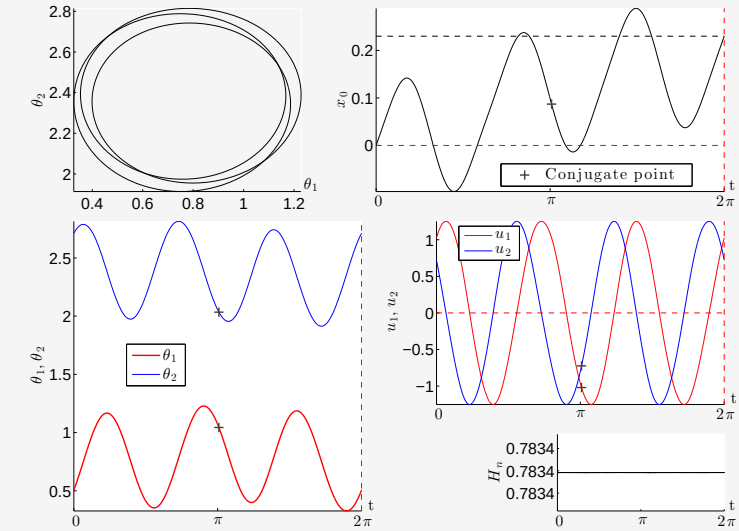
Theorem 4. *Let $q : [0,T] \longrightarrow \mathbb{R}^n$ be a strict normal stroke. If $q(.)$ has at least one conjugate point on $]0,T[$, then $q(\cdot)$* **is not a local minimizer** *for the L^∞ topology on the controls considering the optimal control problem with* **fixed extremities**.

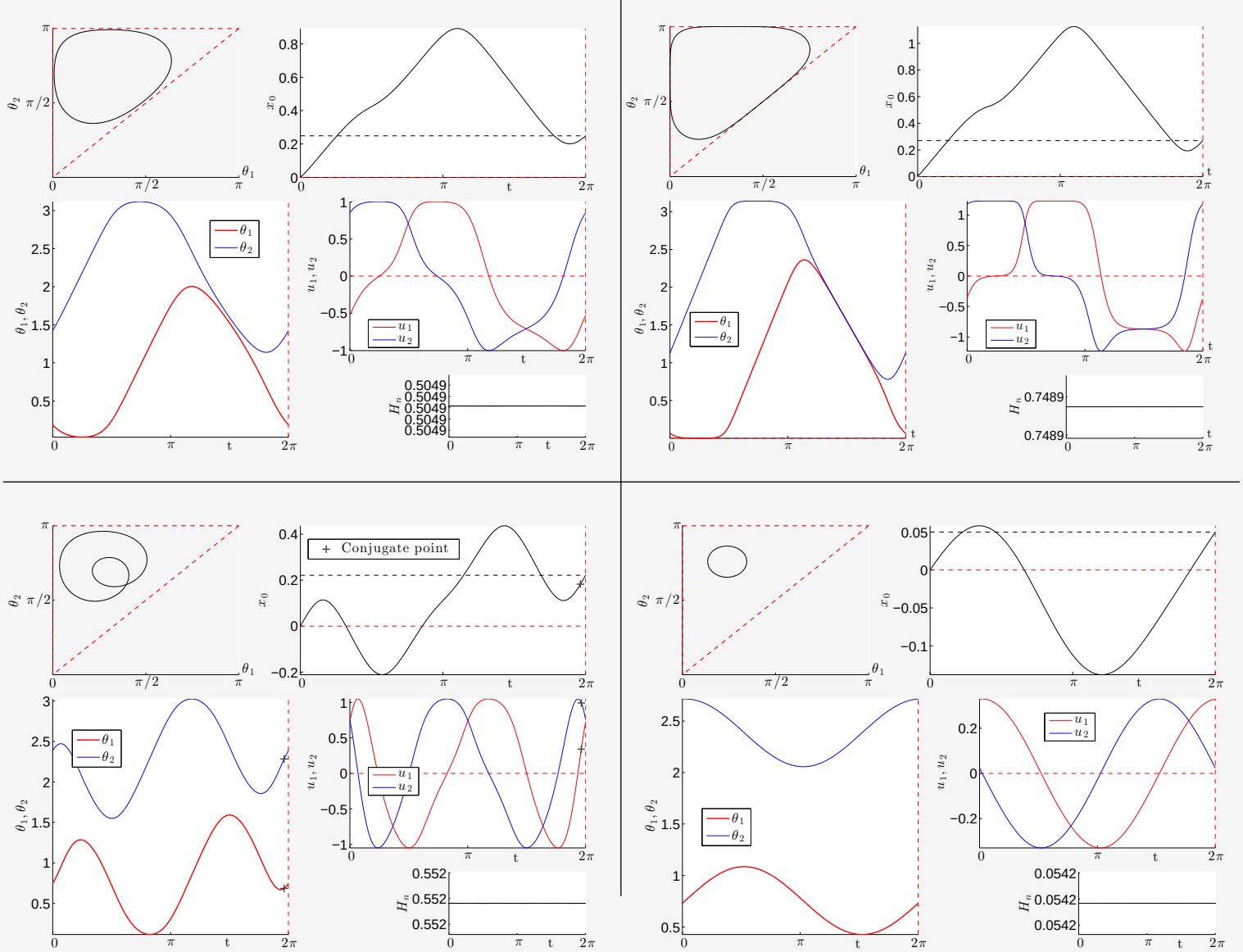



θ2
θ1
x0
t
+ Conjugate point
θ1, θ2
u1, u2
Hn
0.1598
0.1598
0.1598


θ2
θ1
x0
t
+ Conjugate point
θ1, θ2
u1, u2
Hn
0.1404
0.1404
0.1404

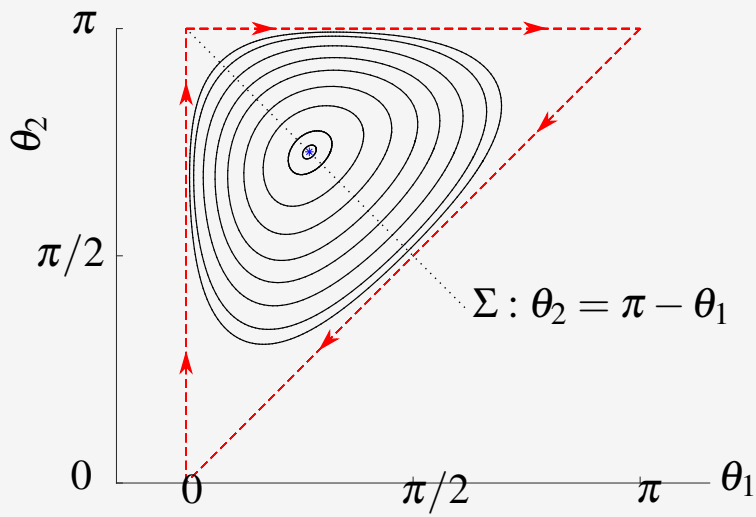

θ2
θ1
x0
t
+ Conjugate point
θ1, θ2
u1, u2
Hn
1.2591
1.2591
1.2591
1.2591
1.2591


θ2
θ1
x0
t
+ Conjugate point
θ1, θ2
u1, u2
Hn
0.7834
0.7834
0.7834


θ_2
θ_1
x_0
t
θ_1, θ_2
u_1, u_2
H_α
u_1
u_2
+ Conjugate point

Numerical fact :
Only the simple loops don't have conjugate points on $]0, T[$. They are the only candidates for optimality.

Theorem 5. *There is a one paramater family of simple loop strokes parameterized by the displacement $x_0(T)$ or the energy level $H_n(q_0, p_0)$.*



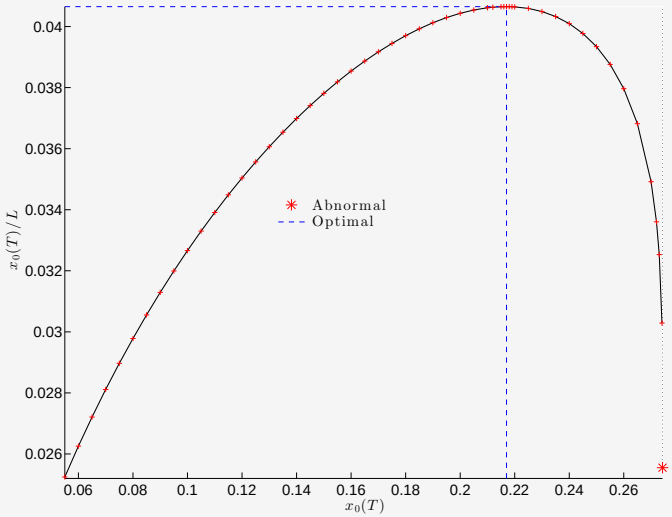
One parameter family of geodesic strokes.

Select on the point on the SR sphere with maximum of efficiency

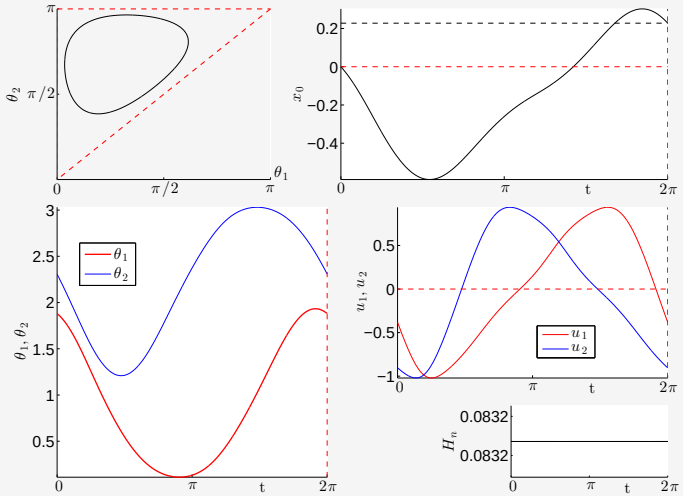
Geometric efficiency : $\mathscr{E} = x_0(T)/l_{SR}(q)$, $\quad$ **($x_0(T)$ free)**

Transversality conditions

$$p_{x_0}(T) = q^0(T)/x_0(T), \qquad p^0(T) = -1/2$$



Efficiency curve with continuation on $x_0(T)$.



Optimal normal stroke

Theorem 6. *The abnormal stroke is not minimizing for both costs (length with fixed displacement and efficiency).*

SR problem (D,g) where $D = \mathrm{span}\{F,G\}$ is a 2-dimensional distribution in $\mathbb{R}^3$ and g is a SR metric.

Normal form of order -1 Nilpotent model of order -1, the Brockett model :

$$\hat{F} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},\ \hat{G} = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$$

where (weight of x,y)=1 and (weight of z)=2 (privileged coordinates).

Application to the Copepod (x,y,z) are related to the physical coordinates (θ_1,θ_2,x_0) by change of coordinates $(x=\theta_1,\, y=\theta_2)$.

$\Rightarrow$ this model is **not generic** and z (always increasing) cannot be identified with the displacement x_0.

Normal form of order 1

Theorem 7 (Chakir, Gauthier, Kupka, 1996)**.** *The generic model is given by the normal form of order* 1

$$F = \hat{F} + yQ(x,y)\frac{\partial}{\partial z},\ G = \hat{G} - xQ(x,y)\frac{\partial}{\partial z},$$

Q *quadratic in* (x,y).

Application to the Copepod At a contact point $(\beta, \pi-\beta)$, $\beta \in]0,\pi/2[$, truncating the Taylor expansion of F_1, F_2, we consider :

$$\begin{aligned}
\hat{F}_1(x,y,z) = \frac{\partial}{\partial x} + (&a_{30}(\beta)x^3 + a_{21}(\beta)x^2y + a_{12}(\beta)xy^2 + a_{03}(\beta)y^3 + a_{20}(\beta)x^2 \\
&+ a_{11}(\beta)xy + a_{02}(\beta)y^2 + a_{10}(\beta)x + a_{01}(\beta)y + a_{00}(\beta))\frac{\partial}{\partial z} \\
\hat{F}_2(x,y,z) = \frac{\partial}{\partial y} + (&-a_{30}(\beta)y^3 - a_{21}(\beta)xy^2 - a_{12}(\beta)x^2y - a_{03}(\beta)x^3 + a_{20}(\beta)y^2 \\
&+ a_{11}(\beta)xy + a_{02}(\beta)x^2 - a_{10}(\beta)y - a_{01}(\beta)x + a_{00}(\beta))\frac{\partial}{\partial z}.
\end{aligned}$$

Gauge group $\mathscr{D}$: set of formal diffeomorphisms preserving the origin,
$\mathscr{F}$: set of formal power series in x, y, z.
— action of $\varphi \in \mathscr{D}$, $\varphi.(A,B) = (\varphi_*(A), \varphi_*(B))$,
— action of $\alpha \in \mathscr{F}$, $\alpha.(A,B) = (A\cos\alpha + B\sin\alpha, -A\sin\alpha + B\cos\alpha)$.

Under the action of the gauge group, $(\hat{F}_1, \hat{F}_2)$ can be put under the form given in Theorem 8 with

$$Q(x,y) = 1/8\,\frac{\left(-11\cos^6\beta + 17\cos^4\beta - 8\cos^2\beta + 4\right)x^2}{(\cos^6\beta - 5\cos^4\beta + 8\cos^2\beta - 4)\cos^2\beta} + 1/8\,\frac{\left(5\cos^8\beta - 33\cos^6\beta + 2\cos^4\beta + 28\cos^2\beta - 8\right)y^2}{(\cos^6\beta - 5\cos^4\beta + 8\cos^2\beta - 4)\cos^2\beta}.$$

Periodicity equations

$$\theta_i(0) = \theta_i(T), \qquad \dot{\theta}_i(0) = \dot{\theta}_i(T),\, i = 1,2.$$

Admissible transformations

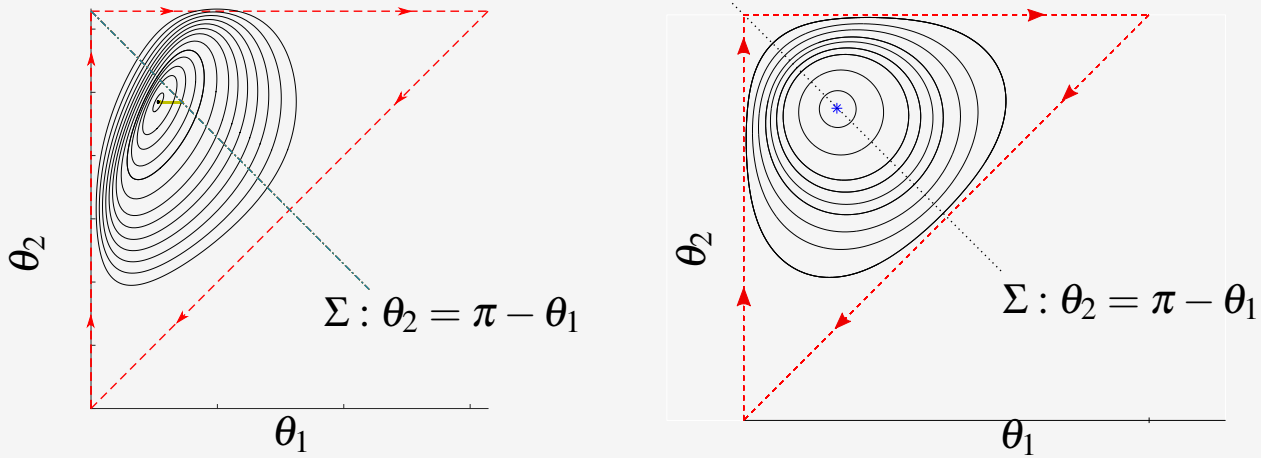
$$\mathscr{G} = \{\varphi \in \mathscr{D} \mid \varphi(\theta, x_0) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta, x_0))\}.$$

Transformation mixing up shape variables (x,y) and the displacement z

$$\varphi(x,y,z) = (x + c_{011}(\beta)\,\boldsymbol{yz} + c_{001}(\beta)\,\boldsymbol{z},\; y - c_{011}(\beta)\,\boldsymbol{xz} + c_{001}(\beta)\,\boldsymbol{z},\; z + P(x,y,z,\beta))\,,$$

$\Rightarrow c_{011}(\beta) = c_{001}(\beta) = 0.$

Theorem 8. *The only stroke that can be computed with transformations of $\mathscr{G}$ is reduced to the* **center of the family** *of simple loop strokes (obtained by the true system).*



One parameter family of simple loops corresponding to different metrics for the true system.

The center of family = SR invariant : related to the inverse optimal control problem.