



HAL
open science

Sub-Riemannian Geometry and swimming at low Reynolds number

Jérémy Rouot, Piernicola Bettiol, Bernard Bonnard

► **To cite this version:**

Jérémy Rouot, Piernicola Bettiol, Bernard Bonnard. Sub-Riemannian Geometry and swimming at low Reynolds number. New Horizons in Optimal Control, Jul 2017, Porto, Portugal. hal-02483357

HAL Id: hal-02483357

<https://inria.hal.science/hal-02483357v1>

Submitted on 18 Feb 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Sub-Riemannian Geometry and swimming at low Reynolds number

Jérémy Rouot, LAAS-CNRS, Toulouse

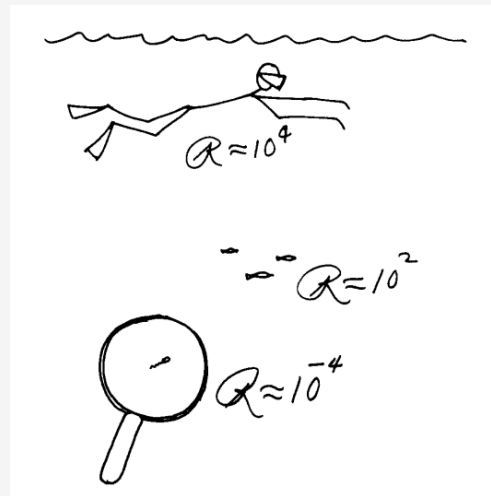
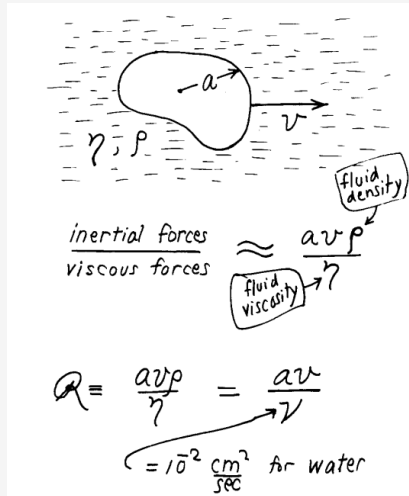
with Piernicola Bettiol (LBMA, Brest) and Bernard Bonnard (INRIA, Sophia Antipolis).

2017, 4th July

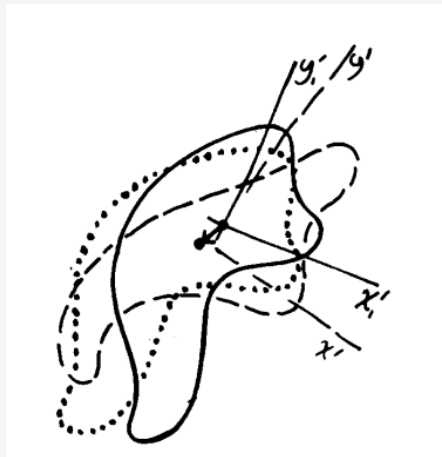
New Horizons in Optimal Control, Porto

A tribute to Helmut Maurer, Urszula Ledzewicz and Heinz Schaettler

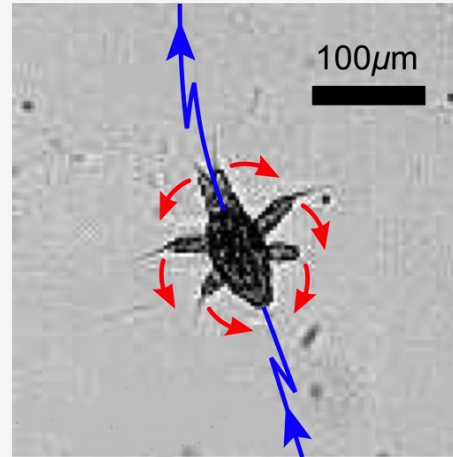
Life at low Reynolds number - Purcell, 1977



Reynolds number

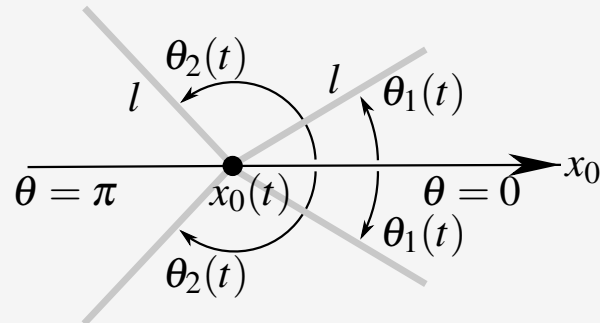


Shape deformations of a microswimmer



Zooplankton

2-link symmetric swimmer.

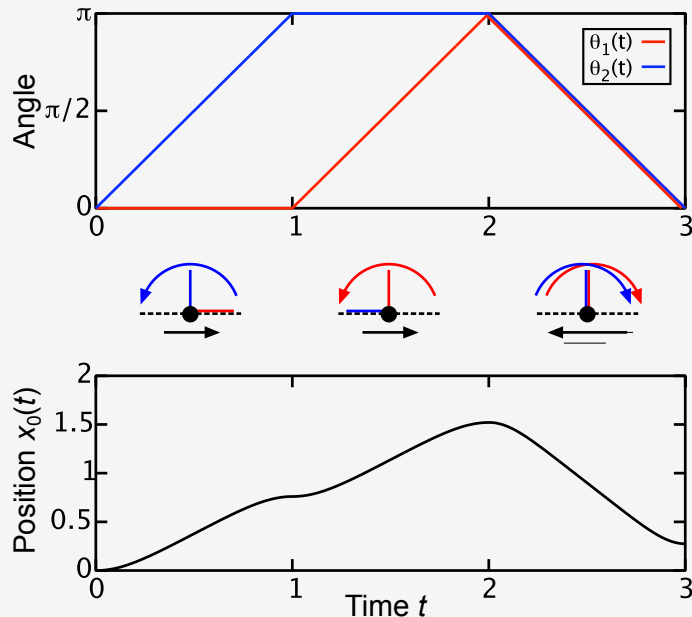
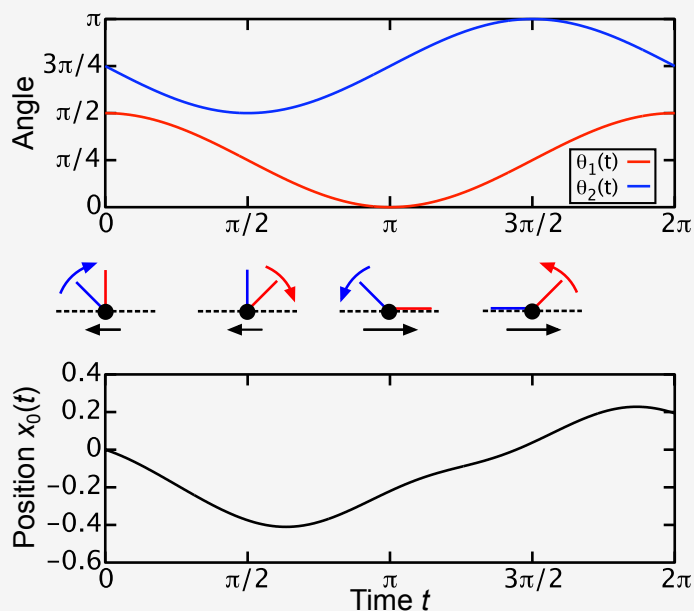


Dynamic.

$$\dot{x}_0 = \frac{\sum_{i=1}^2 l^2 \dot{\theta}_i \sin(\theta_i)}{2 + \sum_{i=1}^2 \sin^2(\theta_i)}, \quad \dot{\theta}_i = u_i, \quad i = 1, 2 \quad (\text{state constraint : } 0 \leq \theta_1 \leq \theta_2 \leq \pi).$$

Criterion. minimize drag forces : $\dot{q}S(q)\dot{q}^T$, $\mathbf{q} = (\theta_1, \theta_2, x_0)$ and S is positive definite \implies quadratic form in (u_1, u_2) .

Definition 1. A stroke is a periodic motion of the shape variables (θ_1, θ_2) associated with a **periodic control** producing a net displacement of the position variables after one period T (we can fix $T = 2\pi$).



(left) Trigonometric controls. (right) Piecewise constant controls.

$$\mathcal{E}(q(\cdot)) = \text{Geometric Efficiency} = x_0(T)/l_{SR}(q)$$

SR Length l_{SR} .

$$l_{SR}(q) = \int_0^T \sqrt{L(q, u)} dt, \quad L(q, u) = a(q) u_1^2 + 2b(q) u_1 u_2 + c(q) u_2^2$$

$$\min_{u(\cdot)} l_{SR}(q) \Leftrightarrow \min_{u(\cdot)} \int_0^T L(q, u) dt$$

Two steps.

$$\max_{u(\cdot), x_0(T)} \mathcal{E} \Leftrightarrow \left(\begin{array}{l} \dot{q} = \sum_{i=1}^2 u_i F_i \leftarrow \min_{u(\cdot)} \int_0^T L(q, u) dt, \\ \min_{u(\cdot)} \int_0^T L(q, u) dt \text{ with } \mathbf{x}_0(T) \text{ fixed, then } \underbrace{\text{select max}_{x_0(T)} \mathcal{E}}_{\text{Transversality cond.}} \end{array} \right)$$

Compute points on the Sub-Riemannian sphere \rightarrow
provide candidate solutions for the maximum of efficiency problem.

Introduce q^0 be the state s.t. : $\dot{q}^0 = L(q, u)$ and denote $\tilde{q} = (q, q^0)$,
 $c(\tilde{q}(0), \tilde{q}(T))$ is the cost function to minimize,
 B defines the boundary conditions : $(\tilde{q}(0), \tilde{q}(T)) \in B$.

Necessary conditions. (\tilde{q}, u) optimal $\Rightarrow \exists \tilde{p} = (p, p^0) : [0, T] \rightarrow (\mathbb{R}^n \times \mathbb{R})$ abs. continue and $\lambda \geq 0$ cst. with $(\tilde{p}, \lambda) \neq (0, 0)$ and s.t.

$$\dot{\tilde{q}} = \frac{\partial H}{\partial \tilde{p}}, \quad \dot{\tilde{p}} = -\frac{\partial H}{\partial \tilde{q}}, \quad \frac{\partial H}{\partial u} = 0$$

where $H(\tilde{q}, \tilde{p}, u) = p \cdot \dot{q} + p^0 \dot{q}^0$.

Transversality Conditions

$$(\tilde{p}(0), -\tilde{p}(T)) - \lambda \left(\frac{\partial c}{\partial \tilde{q}(0)}(\tilde{q}(0), \tilde{q}(T)), \frac{\partial c}{\partial \tilde{q}(T)}(\tilde{q}(0), \tilde{q}(T)) \right) \in (T_{(\tilde{q}(0), \tilde{q}(T))} B)^\perp$$

Definition 2. $(\tilde{q}, \tilde{p}, \lambda, u)$ is called a **normal** extremal if $\lambda > 0$, **abnormal** if $\lambda = 0$.
 An abnormal trajectory $q(\cdot)$ which is not the projection of a normal extremal is called *strict*.

$$\dot{q} = \sum_{i=1}^2 u_i F_i(q) \leftarrow \min_{u(\cdot)} q^0(T), \quad q = (\theta_1, \theta_2, x_0), \quad \mathbf{x}_0(\mathbf{T}) \text{ fixed .}$$

$$F_i = \frac{\sin(\theta_i)}{\Delta} \frac{\partial}{\partial x_0} + \frac{\partial}{\partial \theta_i}, \quad \Delta = 2 + \sin^2(\theta_1) + \sin^2(\theta_2)$$

PMP :

$$H(z, u) = u_1 H_1(z) + u_2 H_2(z) + p^0 L(q, u), \quad z = (q, p)$$

with $H_i = p \cdot F_i$, $i = 1, 2$.

— normal case : $p^0 = -1/2$

$$H_n = 1/2 (a(q) H_1^2 + 2b(q) H_1 H_2 + c(q) H_2^2)$$

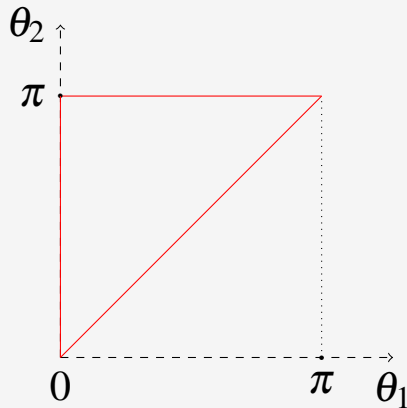
— abnormal case : $p^0 = 0$ hence $H_1 = H_2 = \{H_1, H_2\} = 0$, then

$$u_1 \{ \{H_1, H_2\}, H_1 \} + u_2 \{ \{H_1, H_2\}, H_2 \} = 0.$$

Proposition 3. *The surface $\Sigma : \{q; \det(F_1(q), F_2(q), [F_1, F_2](q)) = 0\}$ contained abnormal trajectories and is defined by*

- $\theta_i = 0$ or π , for $i = 1, 2$,
- $\theta_1 = \theta_2$.

*It is formed by the **boundaries of the physical domain** : $\theta_i \in [0, \pi]$, $i = 1, 2$, $\theta_1 \leq \theta_2$.*



: abnormal trajectories
in the plane (θ_1, θ_2)

Boundary conditions

$$x_0(0) = 0, \quad x_0(T) = x_f, \quad \theta_i(0) = \theta_i(T), \quad i = 1, 2$$

Transversality conditions

$$p_{\theta_i}(0) = p_{\theta_i}(T), \quad i = 1, 2$$

Boundary value problem

$$\begin{cases} \dot{z} = \vec{H}_n(z) \\ x_0(0) = 0, \quad x_0(T) = x_f \text{ (fixed) } , \\ \theta_i(0) = \theta_i(T), \quad i = 1, 2 \\ p_{\theta_i}(0) = p_{\theta_i}(T), \quad i = 1, 2 \end{cases}$$

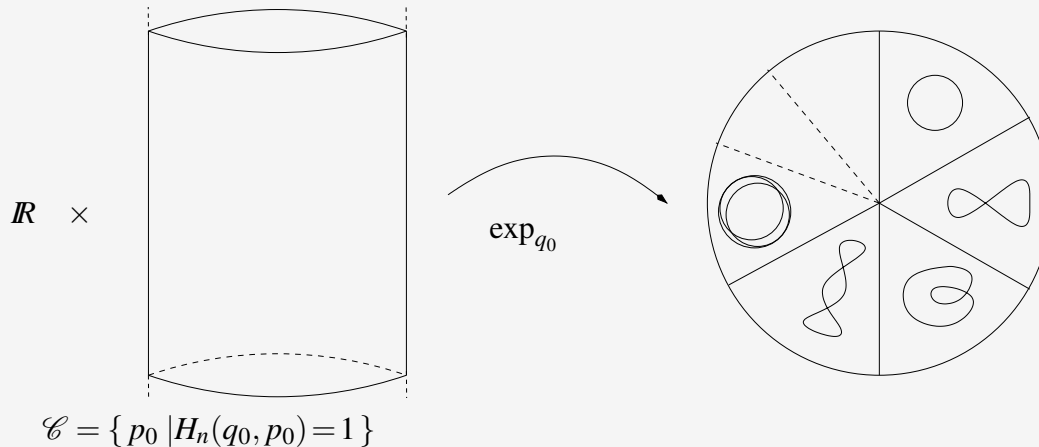
- HamPath, *indirect method* : based on Maximum Principle, *Newton* type algorithm (sensitive to initialization) and computation of **second order necessary optimality conditions** .
- Bocop, *direct method* : gives an initialization for the shooting algorithm of HamPath.

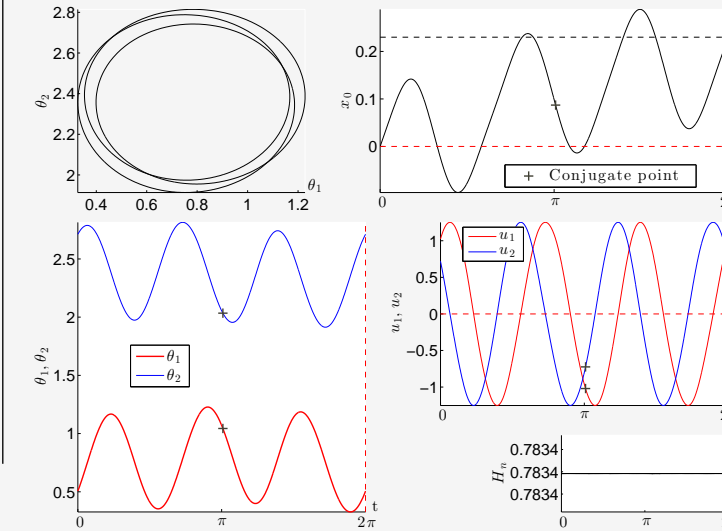
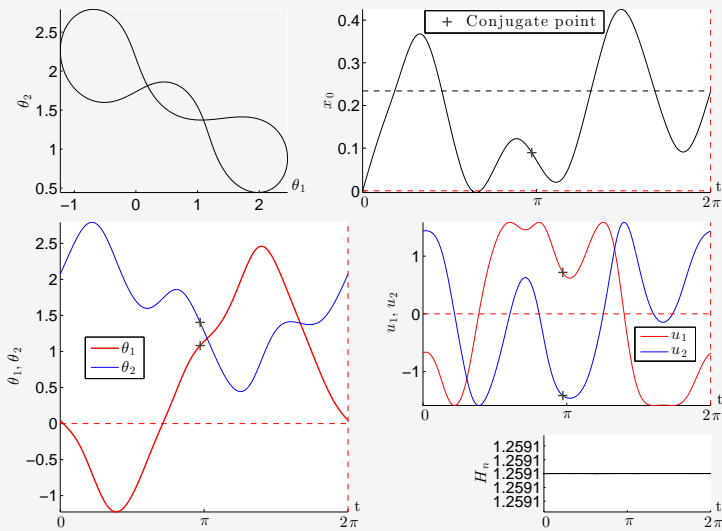
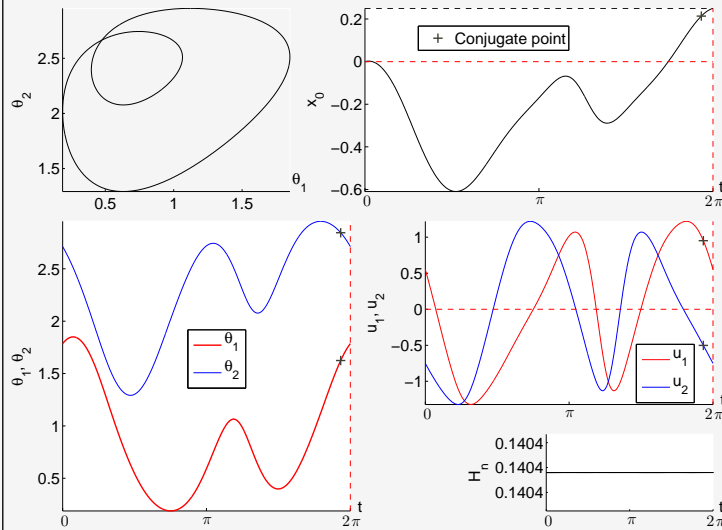
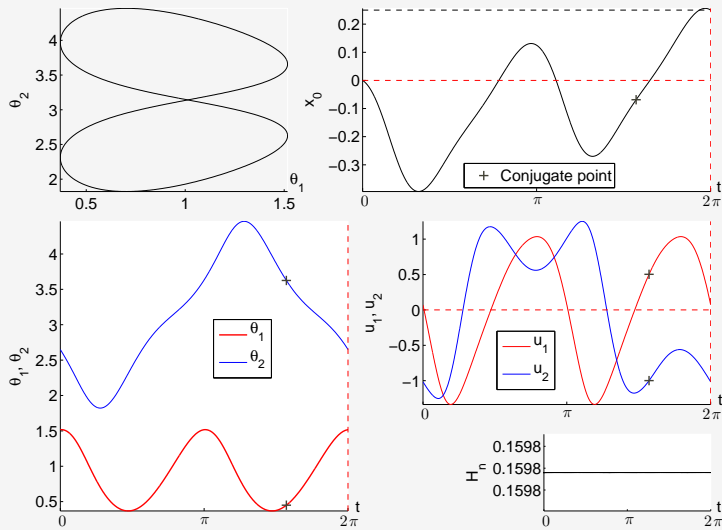
First conjugate time t_c : The first time t_c when the exponential mapping

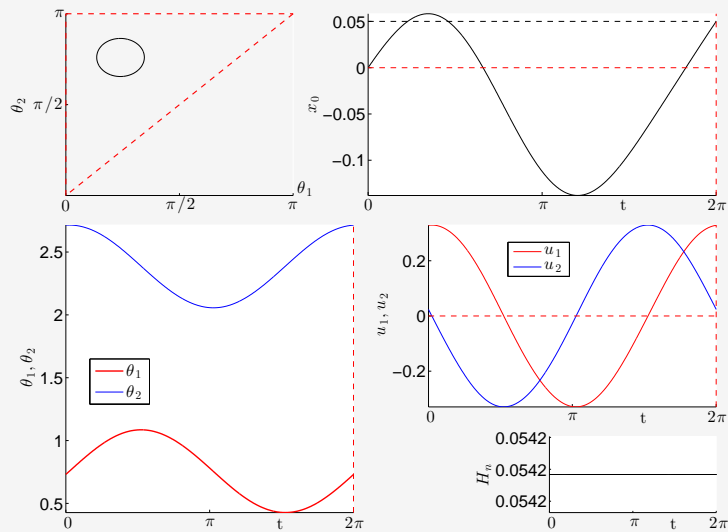
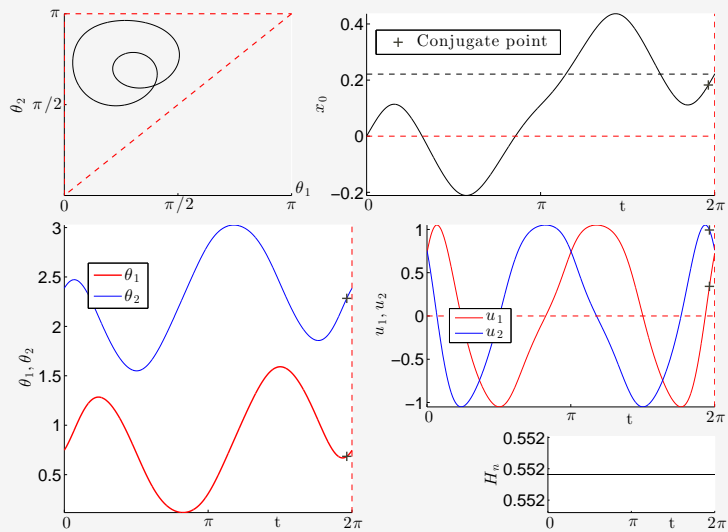
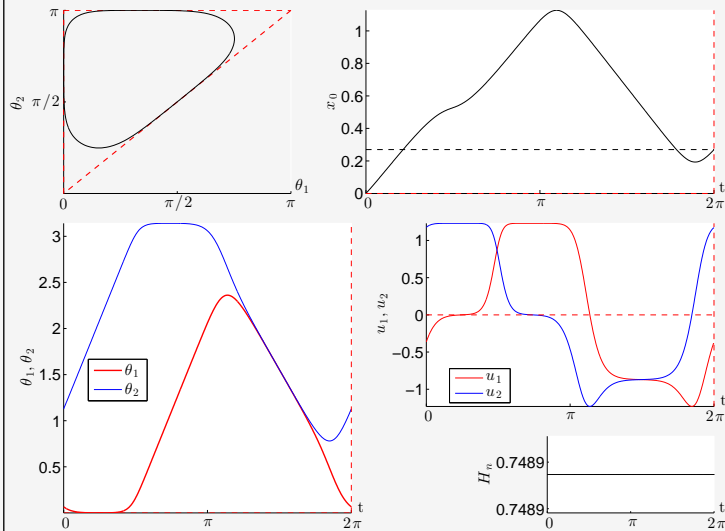
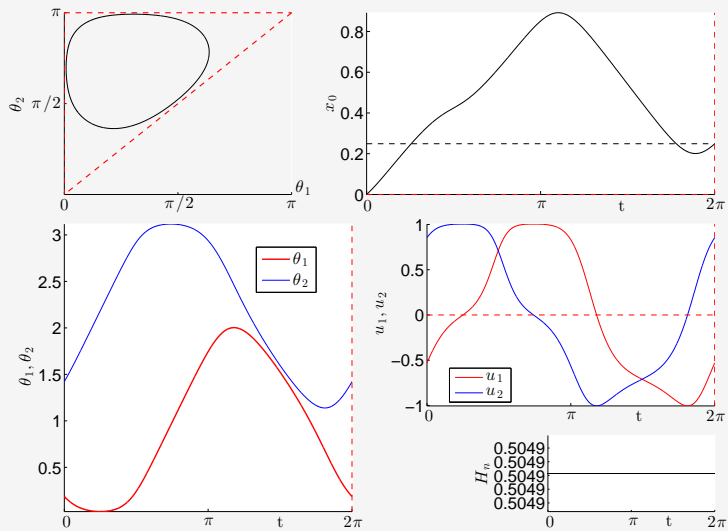
$$\exp_{q_0} : \mathbb{R} \times \mathcal{C} \rightarrow M, \quad (t, p_0) \mapsto q(t, q_0, p_0)$$

is not an immersion at (t_c, p_0) .

Theorem 4. *Let $q : [0, T] \rightarrow \mathbb{R}^n$ be a strict normal stroke. If $q(\cdot)$ has at least one conjugate point on $]0, T[$, then $q(\cdot)$ is not a local minimizer for the L^∞ topology on the controls considering the optimal control problem with **fixed extremities**.*







Stokes theorem

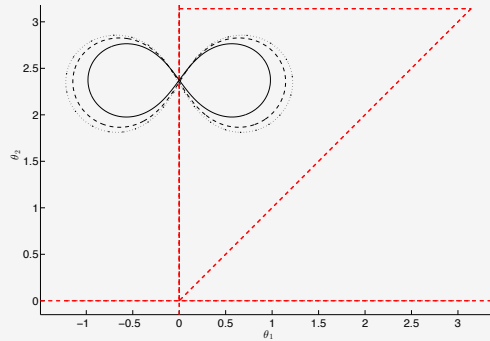
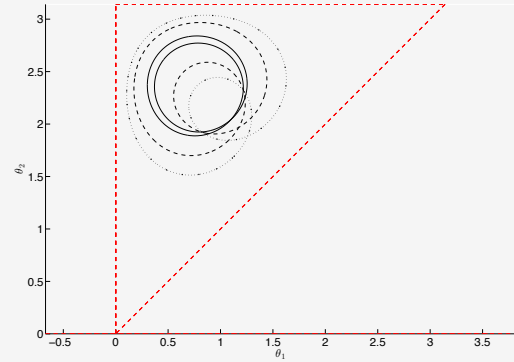
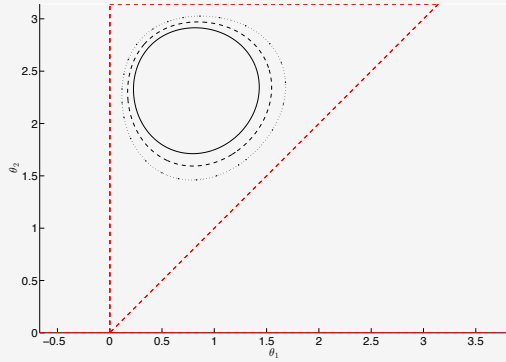
Proposition 5. Take the one-form on \mathbb{R}^2 : $\omega := \sum_{i=1}^2 \frac{\sin \theta_i}{\Delta(\theta)} d\theta_i$, $\left(F_i = \frac{\sin \theta_i}{\Delta(\theta)} \frac{\partial}{\partial \theta_i}\right)$.

Then

1. $d\omega = -f(\theta_1, \theta_2) d\theta_1 \wedge d\theta_2$, $f > 0$ in the interior of the abnormal triangle and $f = 0$ on the edges of the abnormal triangle.
2. for any bounded Stokes domain $D \subset \mathbb{R}^2$, we have

$$x_0(T) = \oint_{\partial D} \sum_{i=1}^2 \frac{\sin(\theta_i)}{\Delta} d\theta_i = \int_D -\frac{f(\theta_1, \theta_2)}{\Delta^2} d\theta_1 \wedge d\theta_2 = \int_D d\omega.$$

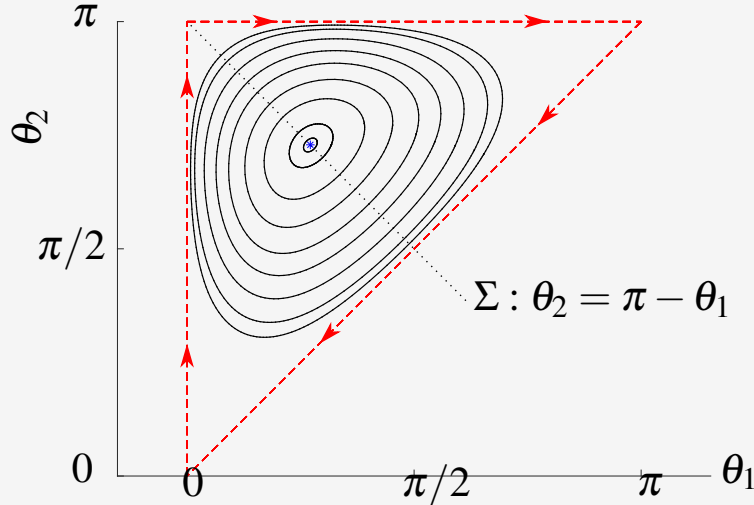
Generate normal strokes



Numerical fact :

Only the simple loops don't have conjugate points on $]0, T[$. They are the only candidates for optimality.

Theorem 6. *There is a one parameter family of simple loop strokes parameterized by the displacement $x_0(T)$ or the energy level $H_n(q_0, p_0)$.*



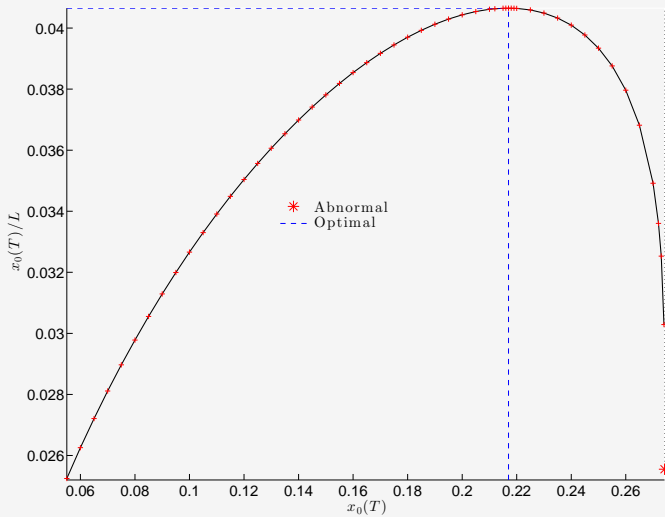
One parameter family of geodesic strokes.

Select on the point on the SR sphere with maximum of efficiency

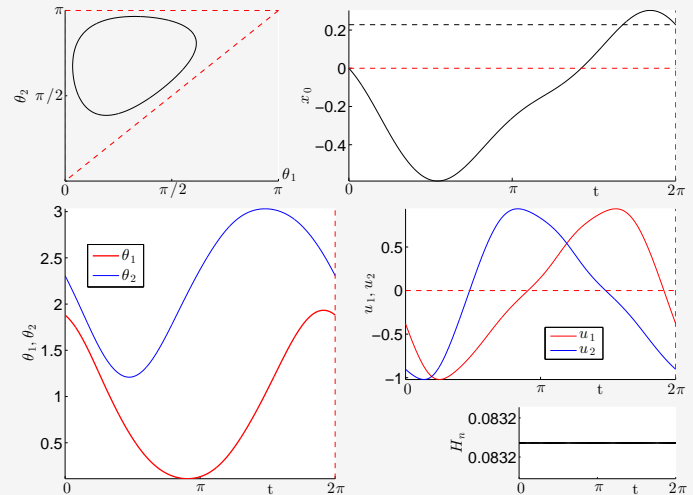
Geometric efficiency : $\mathcal{E} = x_0(T)/l_{SR}(q)$, $(\mathbf{x}_0(T) \text{ free})$

Transversality conditions

$$p_{x_0}(T) = q^0(T)/x_0(T), \quad p^0(T) = -1/2$$



Efficiency curve with continuation on $x_0(T)$.



Optimal normal stroke

Theorem 7. *The abnormal stroke is not minimizing for both costs (length with fixed displacement and efficiency).*

$$\begin{aligned}
& \min_{u(\cdot)} \mathcal{E}(\tilde{q}(\cdot)), \quad \tilde{q} = (q, q^0) = (\theta_1, \theta_2, x_0, q^0) \\
& \dot{q} = u_1 F_1(q) + u_2 F_2(q), \\
& \dot{q}^0 = L(q, u), \\
& q^0(0) = 0, \\
& x_0(0) = 0, \quad x_0(T) \in [x_T, x_T + d] \\
& \theta_1(0) = \theta_1(T), \quad \theta_2(0) = \theta_2(T).
\end{aligned}$$

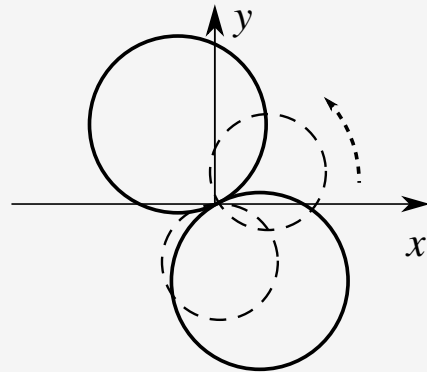
Proposition 8. *If a minimizer (\bar{q}, \bar{u}) is abnormal, then necessarily $\bar{x}_0(T) = x_T + d$.*

Normal form of order -1 Nilpotent model of order -1 , the Brockett-Heisenberg model :

$$\hat{F} = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad \hat{G} = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}$$

where (weight of x, y)=1 and (weight of z)=2 (privileged coordinates).

— model is **not generic** (families of simple loops geodesics)



Families of simple loops for the model of order -1 .

Theorem 9 (Brockett, Chakir et al). *The model of order -1 is equivalent to the model of order 0.*

Application to the Copepod z cannot be identified to the displacement x_0 .
 $x = \theta_1 - \theta_{10}$, $y = \theta_2 - \theta_{20}$.

Perturbation of the model of order -1 by terms of order 0 :

$$F_1(x, y, z) = \frac{\partial}{\partial x} + (a_{11}(\theta_{10})xy + a_{02}(\theta_{10})y^2 + y) \frac{\partial}{\partial z}$$

$$F_2(x, y, z) = \frac{\partial}{\partial y} + (a_{11}(\theta_{10})xy + a_{02}(\theta_{10})x^2 - x) \frac{\partial}{\partial z}$$

$$(F_1, F_2) \underset{\varphi}{\iff} (\hat{F}, \hat{G})$$

Lemma 10. *The only value of θ_{10} such that the transformation φ doesn't mix up the shape variables (x, y) and the displacement z corresponds to the center of swimming of the family.*

$$\varphi(x, y, z) = (x + c_{011}(\theta_{10})yz + c_{001}(\theta_{10})z, y - c_{011}(\theta_{10})xz + c_{001}(\theta_{10})z, z + P(x, y, z, \theta_{10})),$$

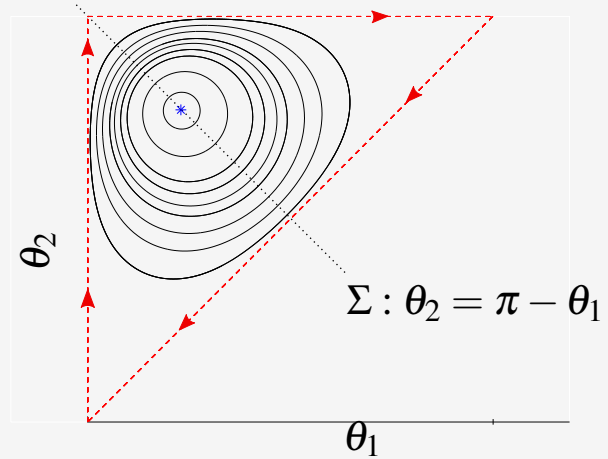
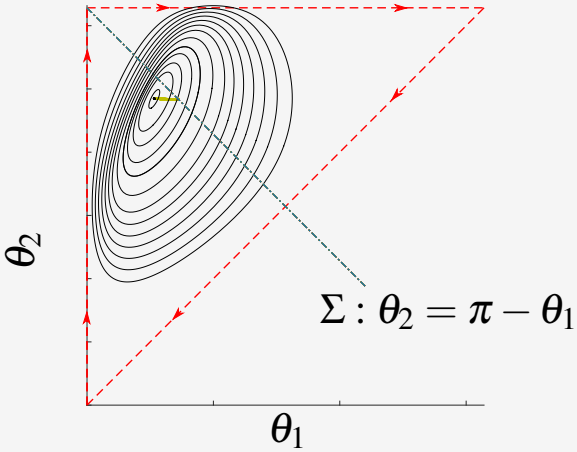
$$\Rightarrow c_{011}(\theta_{10}) = c_{001}(\theta_{10}) = 0.$$

Theorem 11 (Chakir, Gauthier, Kupka, 1996). *The generic model is given by the normal form of order 1*

$$F = \hat{F} + yQ(x,y)\frac{\partial}{\partial z}, \quad G = \hat{G} - xQ(x,y)\frac{\partial}{\partial z},$$

Q quadratic in (x,y) .

Remark 12. *This normal form can be used to approximate the one parameter family of simple strokes.*

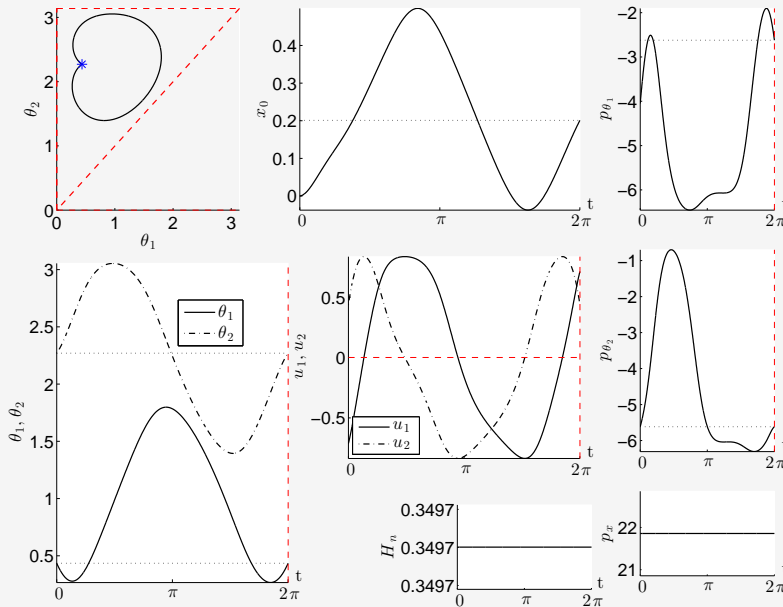


One parameter family of simple loops with different metrics.

$$\mathcal{E}^l = \frac{x_0(T) \mathbf{m}(\boldsymbol{\theta}(0))}{l(q)}, \quad (m \text{ smooth})$$

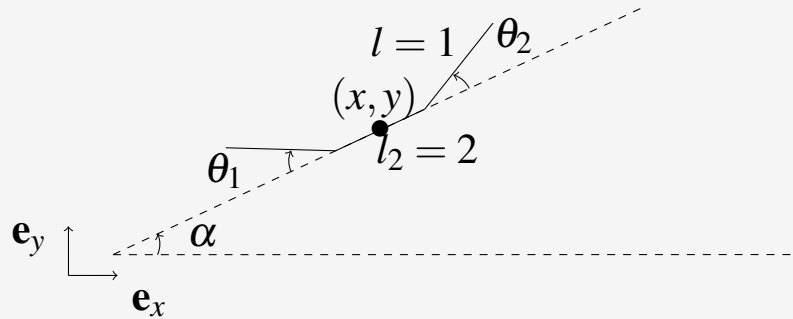
Transversality condition

$$p_{\theta}(0) - p_{\theta}(2\pi) = \lambda \frac{\partial \mathcal{E}^l}{\partial \theta(0)}$$



- *Contact point* : expression of the generic normal form at any point inside the triangle \rightarrow unique family of simple loops.
- *Martinet point* : compute the normal form for a point on the edges \rightarrow locate the eight loops.
- estimation of the first conjugate time using normal forms.
- swimmer model with more than 2 pairs of links.

The Purcell swimmer



Dynamic.

$$\dot{q} = D(\alpha)G(\theta)\dot{\theta}, \quad D(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\dot{\theta} = H(\theta)\tau, \quad \tau : \text{torque}, \quad \theta = (\theta_1, \theta_2), \quad q = (x, y, \alpha).$$

(G, H algebraic fractions of trigonometric polynomials).

Min. work of the drag forces $E(u) = \int_0^T (uH^{-1}u)dt.$

Controls $u = \dot{\theta}$

$$\dot{X} = u_1F_1(X) + u_2F_2(X), \quad X = (\theta_1, \theta_2, x, y, \alpha)$$

Theorem 13. Nilpotent approx. for (F_1, F_2)

$$\hat{F}_1 = \frac{\partial}{\partial x_1}, \quad \hat{F}_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} + x_1^2 \frac{\partial}{\partial x_5}.$$

$$\varphi(\theta_1, \theta_2, x, y, \alpha) = (x_1, x_2, x_3, x_4, x_5) = \mathbf{X}, \quad \varphi \text{ t.q. } \varphi_* F_i = \hat{F}_i + O(|\mathbf{X}|^3)$$

Cartan flat model

$$\dot{X} = \sum_{i=1}^2 u_i \hat{F}_i(X), \quad \min_u \int_0^T (u_1^2 + u_2^2) dt.$$

PMP (x, u) optimal \implies projection of a solution of

$$H_n = 1/2 (H_1^2 + H_2^2), \quad H_i = p \cdot \hat{F}_i, \quad i = 1, 2$$

Hamiltonians lifts

$$\begin{aligned} H_1 &= p \cdot \hat{F}_1(X) = p_1, & H_2 &= p \cdot \hat{F}_2(X) = p_2 + p_3 x_1 + p_4 x_3 + p_5 x_1^2, \\ H_3 &= p \cdot [\hat{F}_1, \hat{F}_2](X) = -p_3 - 2x_1 p_5, & H_4 &= p \cdot [[\hat{F}_1, \hat{F}_2], \hat{F}_1](X) = -2p_5, \\ H_5 &= p \cdot [[\hat{F}_1, \hat{F}_2], \hat{F}_2](X) = p_4. \end{aligned}$$

Poincaré coordinates : $(H_1, H_2, H_3, H_4, H_5)$

$$\begin{aligned}\dot{H}_1 &= dH_1(\vec{H}_n) = \{H_1, H_2\}H_2 = p \cdot [\hat{F}_1, \hat{F}_2] H_2 = H_2H_3, \\ \dot{H}_2 &= -H_3H_1, \quad \dot{H}_3 = H_1H_4 + H_2H_5, \\ \dot{H}_4 &= 0 \quad \text{i.e.} \quad H_4 = c_4, \quad \dot{H}_5 = 0 \quad \text{i.e.} \quad H_5 = c_5.\end{aligned}$$

Fix the energy level : $H_1^2 + H_2^2 = 1$, and let $H_1 = \cos(\vartheta)$ and $H_2 = \sin(\vartheta)$.

$$\dot{H}_1 = -\sin(\vartheta)\dot{\vartheta} = H_2H_3 = \sin(\vartheta)H_3.$$

Hence $\dot{\vartheta} = -H_3$ and

$$\ddot{\vartheta} = -(H_1c_4 + H_2c_5) = -c_4\cos(\vartheta) - c_5\sin(\vartheta) = -\omega^2\sin(\vartheta + \phi)$$

where ω, ϕ are constants and satisfy $\omega^2\sin(\phi) = c_4$ and $\omega^2\cos(\phi) = c_5$.

Then, let $\psi = \vartheta + \phi$, we have

$$1/2\dot{\psi}^2 - \omega^2\cos(\psi) = B, \quad B \text{ is constant.}$$

Theorem 14. *The normal extremals associated to the Cartan flat model is **integrable** and solutions can be expressed as polynomials of elliptic functions of the 1st and 2nd kind $(u, \text{sn}(u), \text{cn}(u), \text{dn}(u), E(u))$.*

Oscillating case

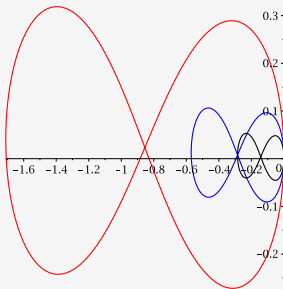
$$x_1(u) = 1/\omega \left[x_1(\varphi_0) - 2k \sin(\phi) \operatorname{cn}(u, k) + (-u + 2E(u, k)) \cos(\phi) \right],$$

$$x_2(u) = 1/\omega \left[x_2(\varphi_0) - 2k \cos(\phi) \operatorname{cn}(u, k) + (u - 2E(u, k)) \sin(\phi) \right]$$

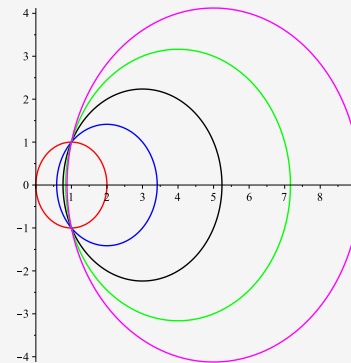
Rotating case

$$x_1(u) = (-2 \cos(\phi) u + 2 \cos(\phi) E(u/k, k) k - 2 \sin(\phi) \operatorname{dn}(u/k, k) k + \cos(\phi) u k^2 + x_1(\varphi_0) k^2) \omega^{-1} k^{-2},$$

$$x_2(u) = (2 \sin(\phi) u - 2 \sin(\phi) E(u/k, k) k - 2 \cos(\phi) \operatorname{dn}(u/k, k) k - \sin(\phi) u k^2 + x_2(\varphi_0) k^2) \omega^{-1} k^{-2}$$

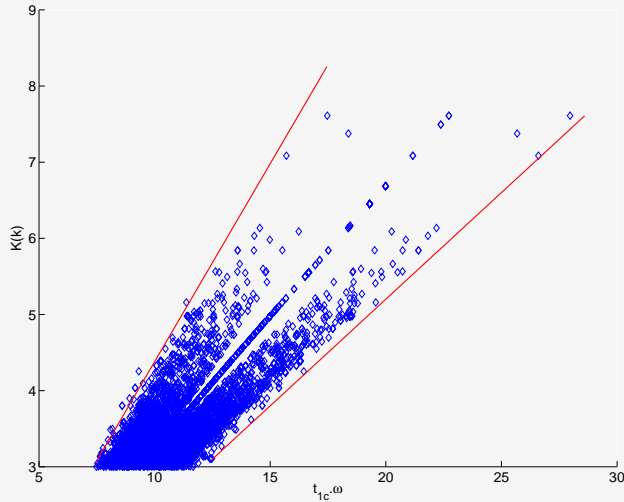


(left) Oscillating case,

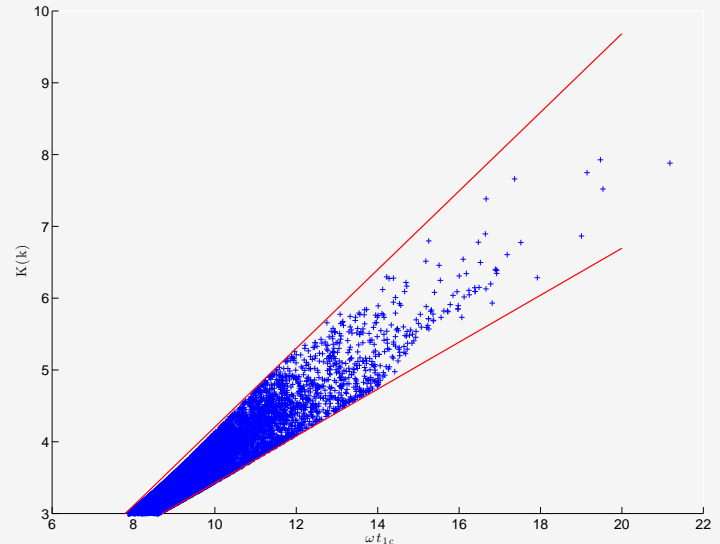


(right) Rotating case

Consider a family of normal extremals parameterized by $\hat{p}(0)$, we compute numerically the first conjugate time.



Oscillating case



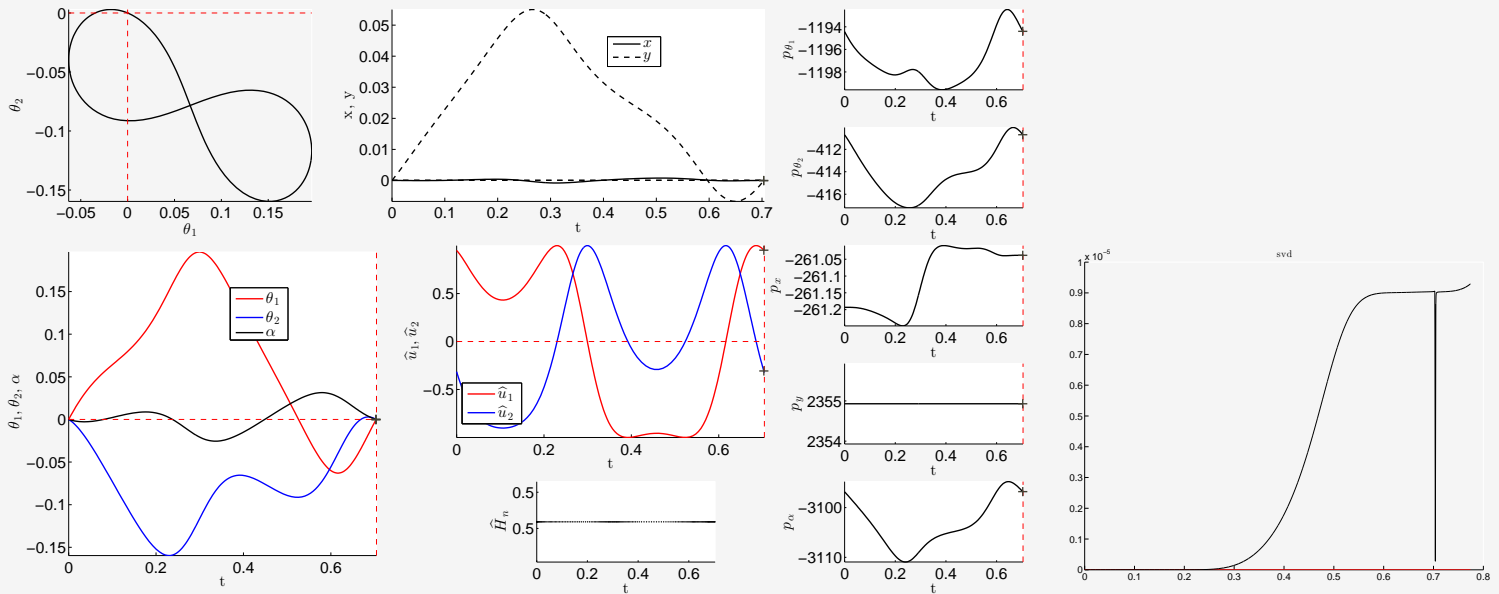
Rotating case

Numerical fact : Conjugate times are located in a sector.

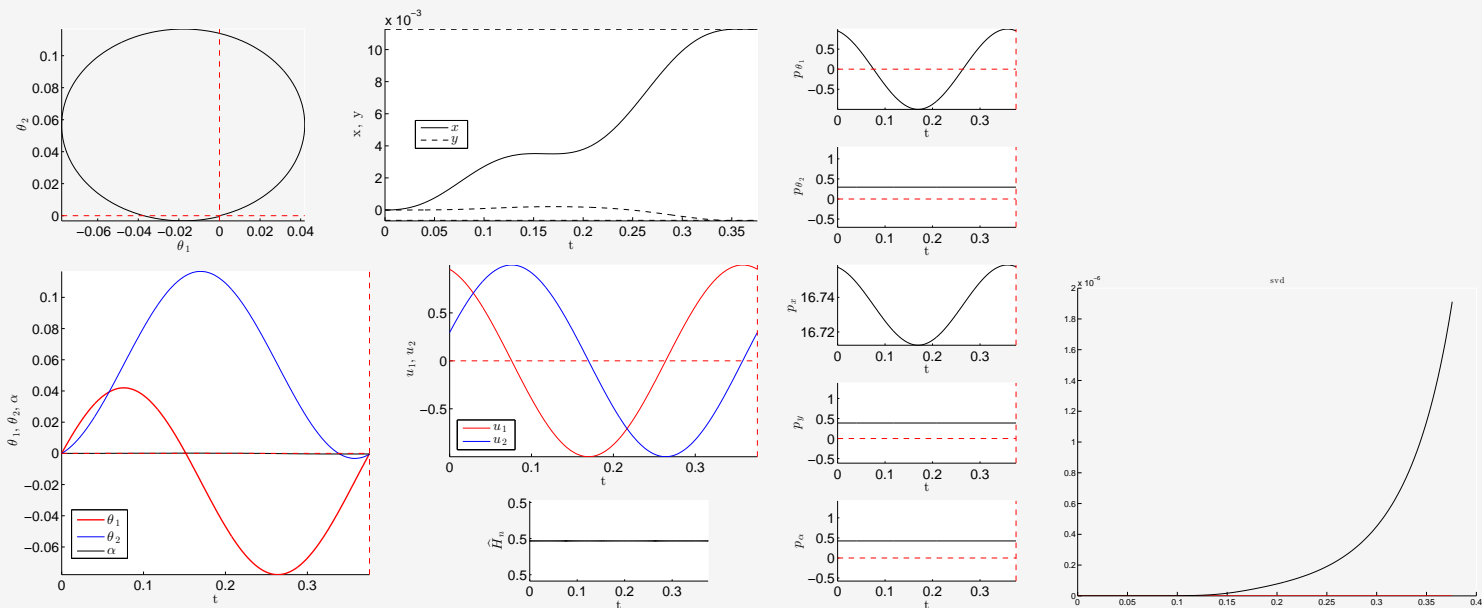
$$0.34 \omega t_{1c} - 0.4 < K(k) < 0.53 \omega t_{1c} - 0.8 \quad : \text{Oscillating case,}$$

$$0.33 \omega t_{1c} + 0.16 < K(k) < 0.55 \omega t_{1c} - 1.27 \quad : \text{Rotating case.}$$

Initialization of algorithms from the microlocal analytical solutions



(left) Control, state and adjoint of the physical variables (θ, x, y, α) for the **oscillating case** and for the nilpotent system : stroke with **eight shape**,
 (right) SVD of the rank condition to compute conjugate points.



(left) Control, state and adjoint of the physical variables (θ, x, y, α) for the **rotating case** and for the nilpotent system : **simple loop stroke**,
(right) SVD of the rank condition to compute conjugate points.

Proposition 15 (Numerical). *For the Cartan problem, **simple loops** (rotating case) are the only strokes without conjugate points.*

- nilpotent model \rightarrow give an initial adjoint vector $\hat{p}(0)$ to initialize strokes with small amplitudes associated with the true dynamic and to the cost $\int_0^T (u_1^2 + u_2^2) dt$.

Boundary value problem

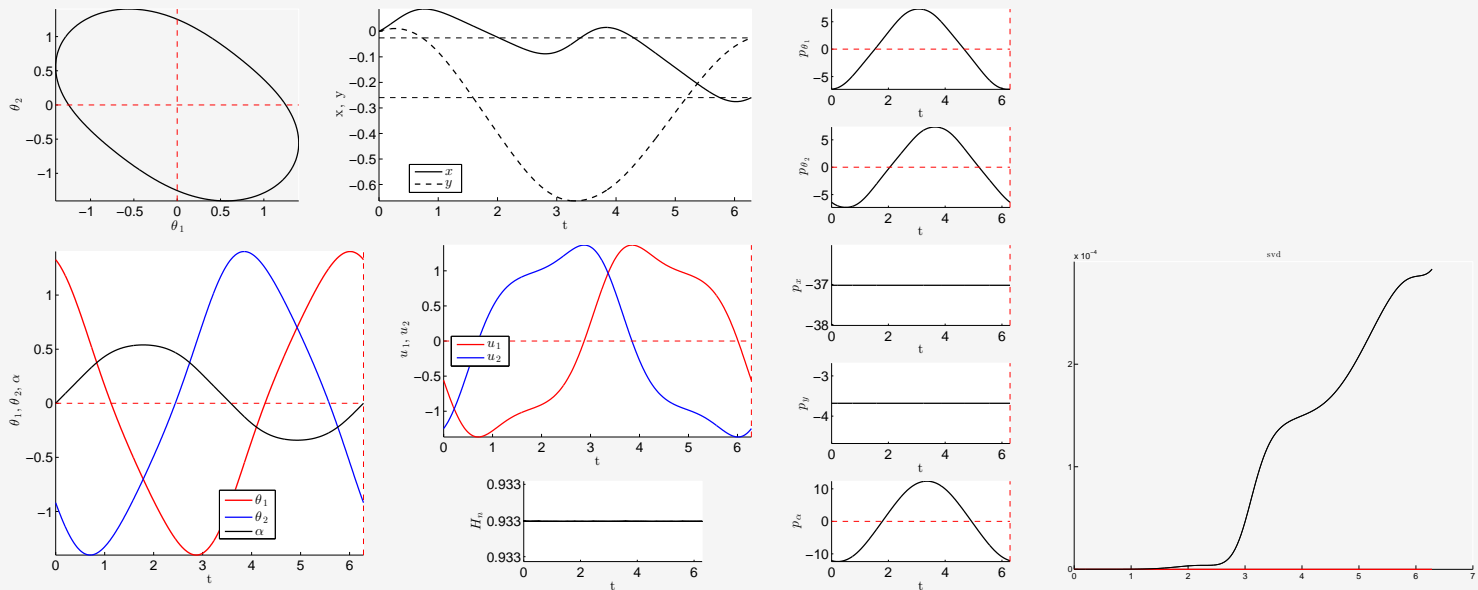
$$\left\{ \begin{array}{l} \dot{q} = \frac{\partial \tilde{H}_n}{\partial p}, \quad \dot{p} = -\frac{\partial \tilde{H}_n}{\partial q}, \\ \theta_j(T) = \theta_j(0) \quad j = 1, 2, \\ x(0) = y(0) = \alpha(0) = 0, \\ x(T)^2 + y(T)^2 = r_1, \quad \alpha(T) = 0, \\ p_{\theta_j}(T) = p_{\theta_j}(0), \quad j = 1, 2, \quad p_\alpha(0) = p_\alpha(T) \end{array} \right.$$

\tilde{H}_n : Hamiltonian associated with the cost $\int_0^T (u_1^2 + u_2^2) dt$.

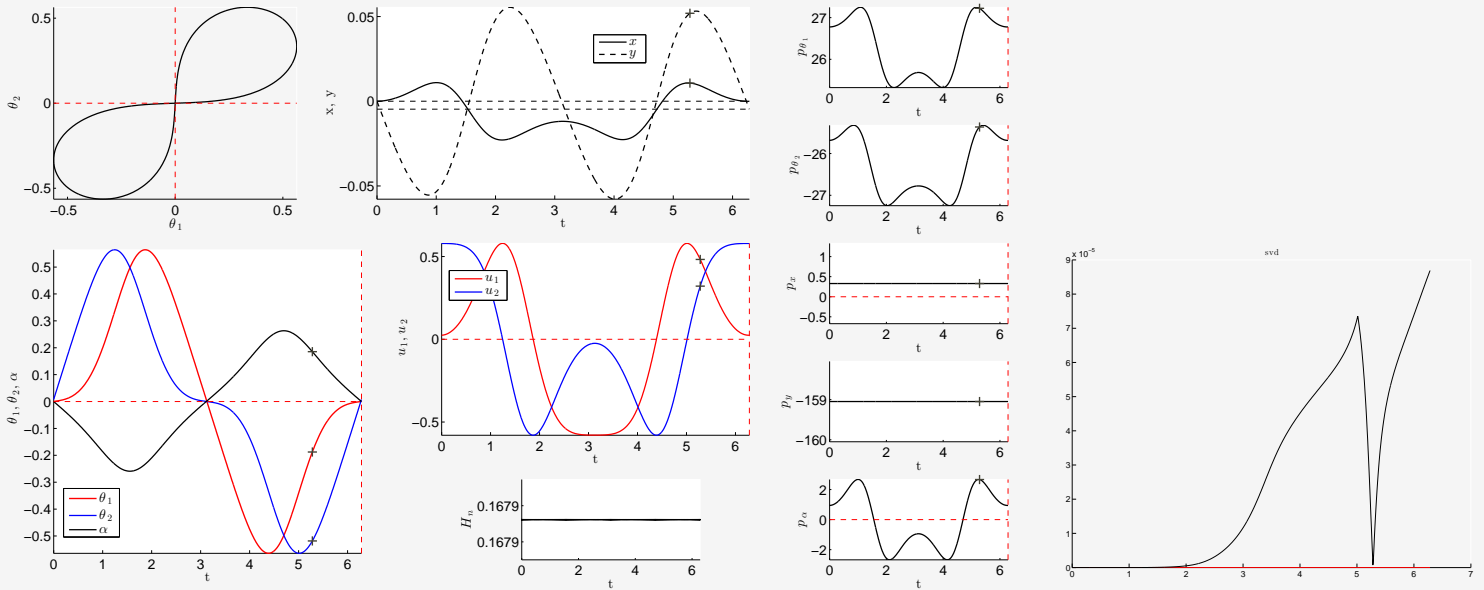
— 2 discrete continuations :

- on the radius r_1
- from the cost $\int_0^T (u_1^2 + u_2^2) dt$ to the mechanical cost.

Discrete continuation on the amplitudes of the strokes



(left) Normal stroke with simple loop for the true dynamic, minimizing the cost $\int_0^T (u_1^2 + u_2^2) dt$ ($T = 2\pi$, $r_1 = 0.068$) (right) Test for the conjugate points : no conjugate points on $[0, 2\pi]$.

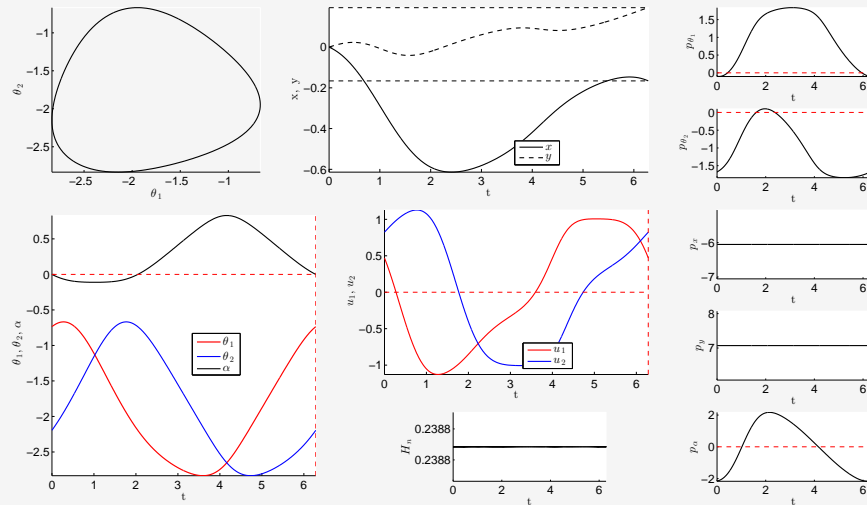


(left) Normal stroke with eight shape loop for the true dynamic, minimizing the cost $\int_0^T (u_1^2 + u_2^2) dt$ ($T = 2\pi$, $r_1 = 4.6e-4$) (right) Test for the conjugate points.

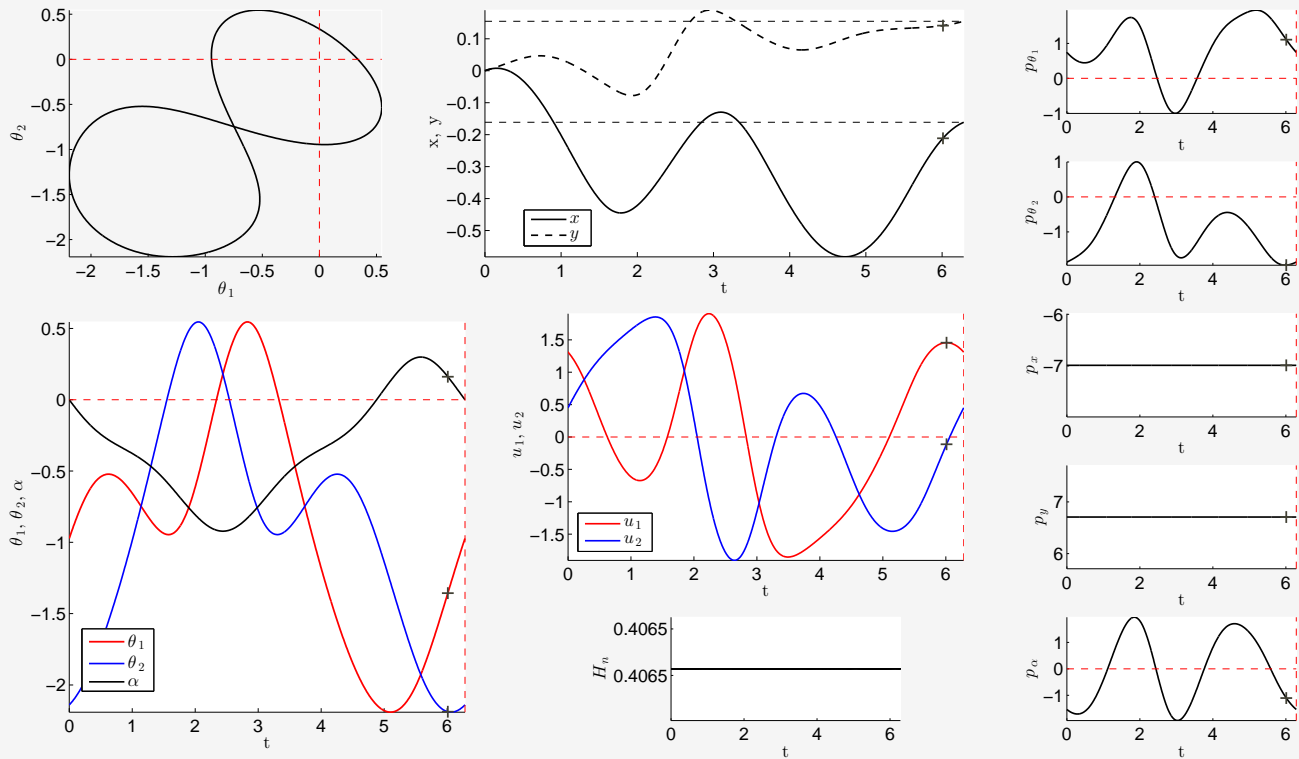
Discrete continuation on the cost

- \tilde{H}_n : normal Hamiltonian for the cost $\int_0^T u_1^2 + u_2^2 dt$ cost,
- H_n : normal Hamiltonian for the mechanical cost.

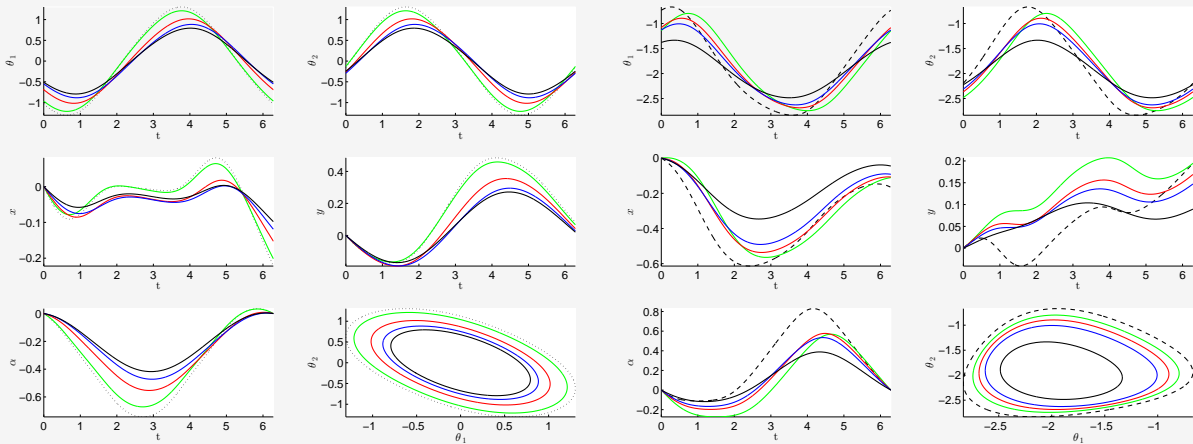
$$H = \lambda H_n + (1 - \lambda)\tilde{H}_n, \quad \lambda \in [0, 1].$$



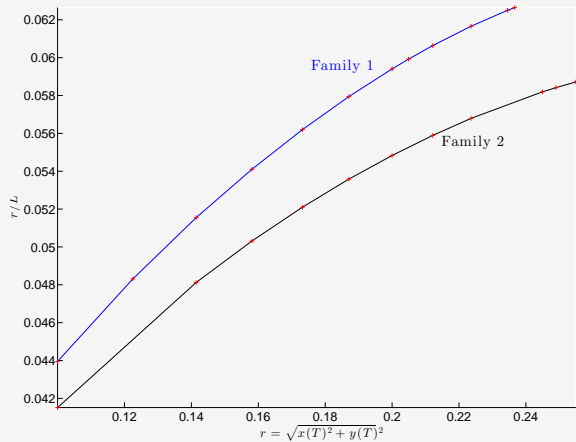
Stroke for the Purcell swimmer minimizing the mechanical energy ($T = 2\pi$, $r_1 = 0.065$).



States and adjoints for the Purcell swimmer (mechanical cost)



Two families of normal strokes for the Purcell swimmer.



Geometric efficiency :

$$\mathcal{E}(\gamma) = \sqrt{x(T)^2 + y(T)^2} / l_{SR}(\gamma)$$

Used to compare family of simple loops.

$$\left\{ \begin{array}{l} \min \quad J(q(\cdot), u(\cdot)) := c(q(0), q(T)) \\ \text{sb. to} \\ \quad \dot{q}(t) = F(q(t), u(t)) \\ \quad u(t) \in U \\ \quad m(q(0), q(T)) = 0, \end{array} \right.$$

$H(q, u, p) := p \cdot F(q, u)$ and $h(q_0, q_T) := c(q_0, q_T) + \mathbf{v} \cdot m(q_0, q_T)$.

Consider a **normal extremal** (\bar{q}, \bar{p}) associated to u .

Second variation

$$\begin{aligned} \delta^2 J(\delta q(\cdot), \delta u(\cdot)) &:= 1/2 [\delta q(0)^\top \quad \delta q(T)^\top] \mathbf{C} [\delta q(0) \quad \delta q(T)]^\top \\ &+ 1/2 \int_0^T (\delta q(t)^\top \partial_{qq} \mathbf{H}(t) \delta q(t) + 2 \delta q(t)^\top \partial_{qu} \mathbf{H}(t) \delta u(t) + \delta u(t)^\top \partial_{uu} \mathbf{H}(t) \delta u(t)) dt \end{aligned}$$

where $\mathbf{C} := \text{Hessian}_{q_0, q_T}(h)$.

$$\left\{ \begin{array}{l} \min \\ \text{sous} \end{array} \right. \quad \begin{array}{l} \delta^2 J(\delta q(\cdot), \delta u(\cdot)) \\ \dot{\delta} q := \partial_q \mathbf{F}(t) \delta q(t) + \partial_u \mathbf{F}(t) \delta u(t) \\ \nabla_{q_0} m(\bar{q}(0), \bar{q}(T)) \delta q(0) + \nabla_{q_T} m(\bar{q}(0), \bar{q}(T)) \delta q(T) = 0, \end{array}$$

Monodromy matrix. $\Phi(.,.)$ associated with the linearized Hamiltonian system :

$$\left\{ \begin{array}{l} \frac{d}{dt} \Phi(t, s) = \mathbf{Z} \Phi(t, s) \\ \Phi(s, s) = \text{Id}, \end{array} \right.$$

where

$$\mathbf{Z} := \begin{bmatrix} \partial_q \mathbf{F} - \partial_u \mathbf{F} [\partial_{uu} \mathbf{H}]^{-1} \partial_{qu} \mathbf{H}^T & -\partial_u \mathbf{F} [\partial_{uu} \mathbf{H}]^{-1} \partial_u \mathbf{F}^T \\ -\partial_{qq} \mathbf{H} + \partial_{qu} \mathbf{H} [\partial_{uu} \mathbf{H}]^{-1} \partial_{qu} \mathbf{H}^T & -\partial_q \mathbf{F} [\partial_{uu} \mathbf{H}]^{-1} \partial_u \mathbf{F}^T \end{bmatrix}.$$

Define

$$\mathcal{W} := \begin{bmatrix} \phi_{22} \phi_{12}^{-1} & \phi_{21} - \phi_{22} \phi_{12}^{-1} \phi_{11} \\ -\phi_{12} & \phi_{12}^{-1} \phi_{11} \end{bmatrix}, \quad \Phi(0, T) =: \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

Theorem 16 (Standard conditions). *Assume*

- (i) $\partial_{uu}\mathbf{H}(t) \leq -\varepsilon Id$ on $[0, T]$, $\bar{u}(\cdot)$ bounded and $(\partial_q \mathbf{F}(\cdot), \partial_u \mathbf{F}(\cdot))$ is controllable on $[0, T]$,
- (ii) : the extremal $(\bar{q}(\cdot), \bar{u}(\cdot), p(\cdot))$ **doesn't have conjugate points** on $[0, T]$,
- (iii) : there exists $\gamma > 0$ t.q.

$$\begin{bmatrix} \xi_0^T & \xi_1^T \end{bmatrix} \mathcal{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left\| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right\|^2,$$

for all vectors $\xi_0, \xi_1 \in \mathbb{R}^n \setminus \{0\}$ s.t.

$$\nabla_{q_0} m((\bar{q}(0), \bar{q}(T))) \xi_0 + \nabla_{q_T} m((\bar{q}(0), \bar{q}(T))) \xi_1 = 0.$$

Then $(\bar{q}(\cdot), \bar{u}(\cdot))$ is a $W^{1,\infty}$ -minimizer and locally **unique**.

Boundary values

$$\begin{aligned}\theta_j(0) &= \theta_j(T) \quad j = 1, 2, \quad \alpha(0) = \alpha(T), \\ x(0) &= 0, \quad y(0) = 0, \quad x(T)^2 + y(T)^2 = r \quad (r \text{ is fixed}).\end{aligned}$$

Proposition 17. Take $a = (\phi, \sigma) \in I = (-\varepsilon, \varepsilon)^2$ for $\varepsilon > 0$. $(\bar{q} = (\bar{\theta}_1, \bar{\theta}_2, \bar{x}, \bar{y}, \bar{\alpha}))$ and $\bar{u} = (\bar{u}_1, \bar{u}_2)$ a normal extremal.

For all $a \in I$ and $t \in [0, T]$, define $q^a(\cdot) = (\theta_1^a(\cdot), \theta_2^a(\cdot), x^a(\cdot), y^a(\cdot), \alpha^a(\cdot))$ where

$$\begin{aligned}u_j^a(t) &= \bar{u}_j(t + \sigma), \quad \theta_j^a(t) = \bar{\theta}_j(t + \sigma) \quad j = 1, 2, \\ x^a(t) &= \cos(\phi) (\bar{x}(t + \sigma) - \bar{x}(\sigma)) - \sin(\phi) (\bar{y}(t + \sigma) - \bar{y}(\sigma)), \\ y^a(t) &= \sin(\phi) (\bar{x}(t + \sigma) - \bar{x}(\sigma)) + \cos(\phi) (\bar{y}(t + \sigma) - \bar{y}(\sigma)), \\ \alpha^a(t) &= \bar{\alpha}(t + \sigma) + \phi.\end{aligned}$$

Then $(\bar{q}(\cdot), p(\cdot), \bar{u}(\cdot))$ is continuously embedded in a family of extremals $(q^a(\cdot), p^a(\cdot), u^a(\cdot))_{a \in I}$ where $p^a(\cdot)$ is the adjoint vector associated with $(q^a(\cdot), u^a(\cdot))$.

\implies **Standard conditions fail**

Theorem 18 (Refined Conditions, Gavriel, Vinter (2014)). *Assume the reference normal extremal $(\bar{q}(\cdot), \bar{u}(\cdot), p(\cdot))$ is embedded in a family of extremals and*

(i) $\partial_{uu}\mathbf{H} \leq -\varepsilon Id$ on $[0, T]$, $\bar{u}(\cdot)$ is bounded and $(\partial_q \mathbf{F}(\cdot), \partial_u \mathbf{F}(\cdot))$ is controllable on $[0, T]$,

(ii) : the extremal $(\bar{q}(\cdot), \bar{u}(\cdot), p(\cdot))$ **doesn't have conjugate points** on $[0, T]$,

(iii) : there exists $\gamma > 0$ s.t.

$$\begin{bmatrix} \varepsilon T & \varepsilon T \\ \xi_0 & \xi_1 \end{bmatrix} \mathcal{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left\| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right\|^2,$$

for all vectors $\xi_0, \xi_1 \in \mathbb{R}^n \setminus \{0\}$ s.t.

$$\nabla_{q_0} m((\bar{q}(0), \bar{q}(T))) \xi_0 + \nabla_{q_T} m((\bar{q}(0), \bar{q}(T))) \xi_1 = 0 \quad \text{and} \quad \Gamma^T \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = 0.$$

$$\text{where } \Gamma := \left[\begin{array}{c} D_\alpha q^\alpha(0) \\ D_\alpha q^\alpha(T) \end{array} \right] \Big|_{\alpha=0}.$$

Then $(\bar{q}(\cdot), \bar{u}(\cdot))$ is a **local $W^{1,\infty}$ -minimizer**.

Define the matrix N_s from the subspace \mathcal{L}_s s.t.

$$\mathcal{L}_s = \{ (\xi_0, \xi_T) \in \mathbb{R}^5 \times \mathbb{R}^5 \mid \nabla_{q_0, q_T} m(q_0, q_T) (\xi_0 \quad \xi_T)^\top = 0 \} =: \text{Im}(N_s)$$

Standard conditions. Does the matrix $\mathcal{W}_s := \underbrace{N_s^\top (\mathcal{W}^\top + \mathcal{W}) N_s}_{4 \times 4}$ is positive-definite?

Consider

$$\Gamma_r = \begin{pmatrix} \nabla_\phi q^a(0) & \nabla_\phi q^a(T) \\ \nabla_\sigma q^a(0) & \nabla_\sigma q^a(T) \end{pmatrix}_{\phi=0, \sigma=0}$$

and the linear subspace \mathcal{L}_r s.t.

$$\mathcal{L}_r := \mathcal{L}_s \cap \{ (\xi_0, \xi_T) \in \mathbb{R}^5 \times \mathbb{R}^5 \mid \Gamma_r^\top (\xi_0 \quad \xi_T)^\top = 0 \} =: \text{Im}(N_r)$$

Refined conditions. Does the matrix $\mathcal{W}_r := \underbrace{N_r^\top (\mathcal{W}^\top + \mathcal{W}) N_r}_{2 \times 2}$ is positive-definite?

Relative tolerance	(Standard Conditions) $\text{Spec}(\mathcal{W}_s)$	(Refined Conditions) $\text{Spec}(\mathcal{W}_r)$
10^{-4}	1319.91 3.44629 −2.61575e-5 −4.17860e-3	36179.7 13.8018
10^{-7}	1320.17 3.44676 9.81190e-6 −5.40128e-6	36186.9 13.8037

- Standard conditions fail : \mathcal{W}_s has zero eigenvalues.
- Refined conditions are satisfied : \mathcal{W}_r is positive-definite.

Theorem 19 (Numerical). *The simple loop normal stroke (\bar{q}, \bar{u}) is $W^{1,\infty}$ – optimal.*