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Sub-Riemannian Geometry and swimming at low Reynolds number

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► **To cite this version:**

Jérémy Rouot, Piernicola Bettiol, Bernard Bonnard. Sub-Riemannian Geometry and swimming at low Reynolds number. New Horizons in Optimal Control, Jul 2017, Porto, Portugal. hal-02483357

HAL Id: hal-02483357

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Submitted on 18 Feb 2020

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Sub-Riemannian Geometry and swimming at low Reynolds number

Jérémy Rouot, LAAS-CNRS, Toulouse

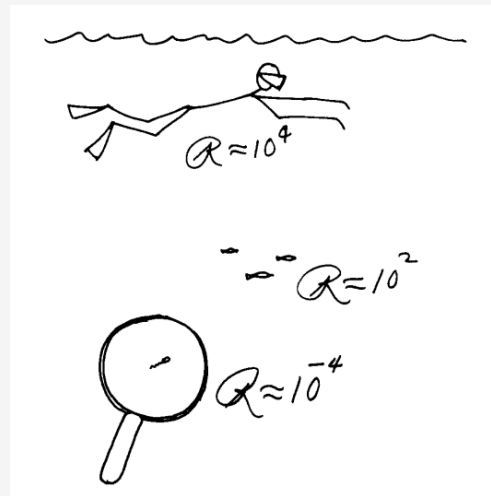
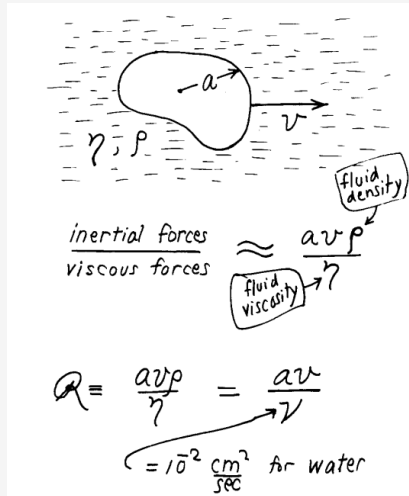
with Piernicola Bettiol (LBMA, Brest) and Bernard Bonnard (INRIA, Sophia Antipolis).

2017, 4th July

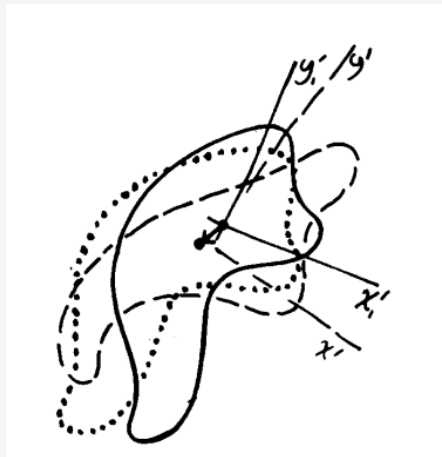
New Horizons in Optimal Control, Porto

A tribute to Helmut Maurer, Urszula Ledzewicz and Heinz Schaettler

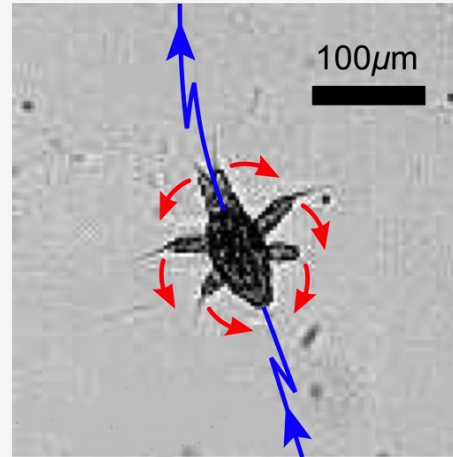
Life at low Reynolds number - Purcell, 1977



Reynolds number

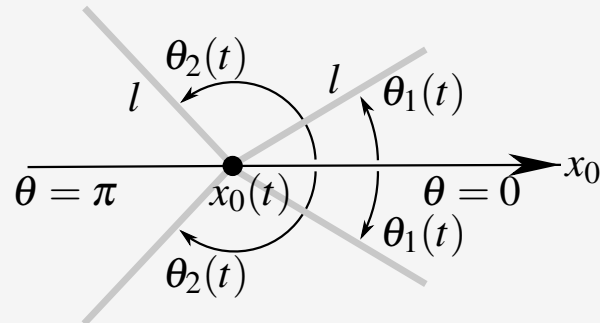


Shape deformations of a microswimmer



Zooplankton

2-link symmetric swimmer.

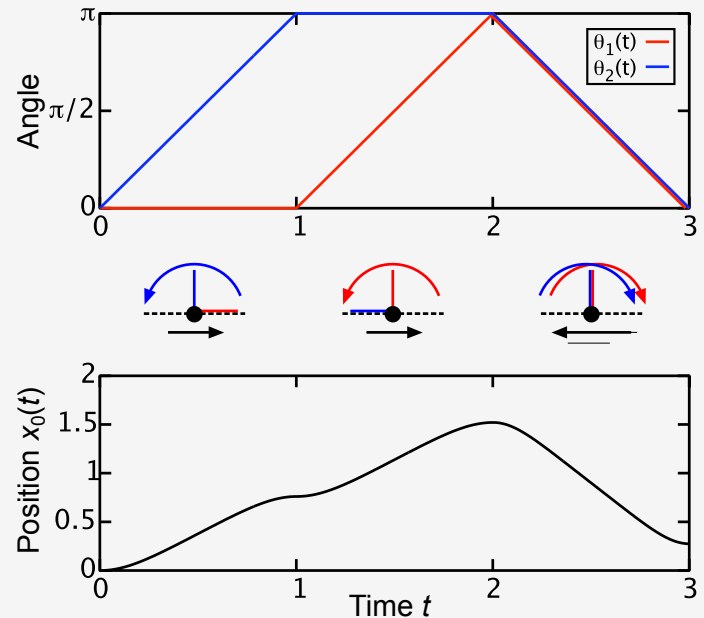
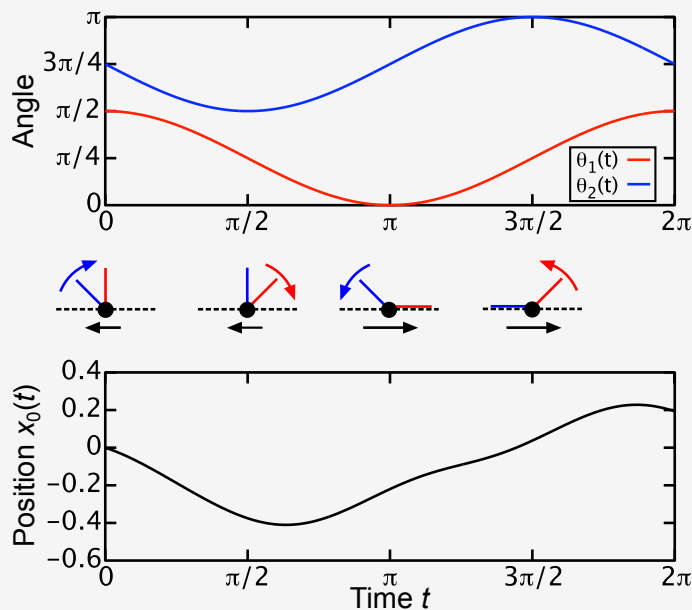


Dynamic.

$$\dot{x}_0 = \frac{\sum_{i=1}^2 l^2 \dot{\theta}_i \sin(\theta_i)}{2 + \sum_{i=1}^2 \sin^2(\theta_i)}, \quad \dot{\theta}_i = u_i, \quad i = 1, 2 \quad (\text{state constraint : } 0 \leq \theta_1 \leq \theta_2 \leq \pi).$$

Criterion. minimize drag forces : $\dot{q}S(q)\dot{q}^\top$, $\mathbf{q} = (\theta_1, \theta_2, x_0)$ and S is positive definite \implies quadratic form in (u_1, u_2) .

Definition 1. A stroke is a periodic motion of the shape variables (θ_1, θ_2) associated with a **periodic control** producing a net displacement of the position variables after one period T (we can fix $T = 2\pi$).



(left) Trigonometric controls. (right) Piecewise constant controls.

$$\mathcal{E}(q(\cdot)) = \mathbf{Geometric\ Efficiency} = x_0(T)/l_{SR}(q)$$

SR Length l_{SR} .

$$l_{SR}(q) = \int_0^T \sqrt{L(q, u)} dt, \quad L(q, u) = a(q) u_1^2 + 2b(q) u_1 u_2 + c(q) u_2^2$$

$$\min_{u(\cdot)} l_{SR}(q) \Leftrightarrow \min_{u(\cdot)} \int_0^T L(q, u) dt$$

Two steps.

$$\max_{u(\cdot), x_0(T)} \mathcal{E} \Leftrightarrow \left(\begin{array}{l} \dot{q} = \sum_{i=1}^2 u_i F_i \leftarrow \min_{u(\cdot)} \int_0^T L(q, u) dt, \\ \min_{u(\cdot)} \int_0^T L(q, u) dt \text{ with } \mathbf{x_0(T) fixed}, \text{ then } \underbrace{\text{select max}_{x_0(T)} \mathcal{E}}_{\text{Transversality cond.}} \end{array} \right)$$

Compute points on the Sub-Riemannian sphere \rightarrow
provide candidate solutions for the maximum of efficiency problem.

Introduce q^0 be the state s.t. : $\dot{q}^0 = L(q, u)$ and denote $\tilde{q} = (q, q^0)$,
 $c(\tilde{q}(0), \tilde{q}(T))$ is the cost function to minimize,
 B defines the boundary conditions : $(\tilde{q}(0), \tilde{q}(T)) \in B$.

Necessary conditions. (\tilde{q}, u) optimal $\Rightarrow \exists \tilde{p} = (p, p^0) : [0, T] \rightarrow (\mathbb{R}^n \times \mathbb{R})$ abs. continue and $\lambda \geq 0$ cst. with $(\tilde{p}, \lambda) \neq (0, 0)$ and s.t.

$$\dot{\tilde{q}} = \frac{\partial H}{\partial \tilde{p}}, \quad \dot{\tilde{p}} = -\frac{\partial H}{\partial \tilde{q}}, \quad \frac{\partial H}{\partial u} = 0$$

where $H(\tilde{q}, \tilde{p}, u) = p \cdot \dot{q} + p^0 \dot{q}^0$.

Transversality Conditions

$$(\tilde{p}(0), -\tilde{p}(T)) - \lambda \left(\frac{\partial c}{\partial \tilde{q}(0)}(\tilde{q}(0), \tilde{q}(T)), \frac{\partial c}{\partial \tilde{q}(T)}(\tilde{q}(0), \tilde{q}(T)) \right) \in (T_{(\tilde{q}(0), \tilde{q}(T))} B)^\perp$$

Definition 2. $(\tilde{q}, \tilde{p}, \lambda, u)$ is called a **normal** extremal if $\lambda > 0$, **abnormal** if $\lambda = 0$.
 An abnormal trajectory $q(\cdot)$ which is not the projection of a normal extremal is called *strict*.

$$\dot{q} = \sum_{i=1}^2 u_i F_i(q) \leftarrow \min_{u(\cdot)} q^0(T), \quad q = (\theta_1, \theta_2, x_0), \quad \mathbf{x}_0(\mathbf{T}) \text{ fixed .}$$

$$F_i = \frac{\sin(\theta_i)}{\Delta} \frac{\partial}{\partial x_0} + \frac{\partial}{\partial \theta_i}, \quad \Delta = 2 + \sin^2(\theta_1) + \sin^2(\theta_2)$$

PMP :

$$H(z, u) = u_1 H_1(z) + u_2 H_2(z) + p^0 L(q, u), \quad z = (q, p)$$

with $H_i = p \cdot F_i$, $i = 1, 2$.

— normal case : $p^0 = -1/2$

$$H_n = 1/2 (a(q) H_1^2 + 2b(q) H_1 H_2 + c(q) H_2^2)$$

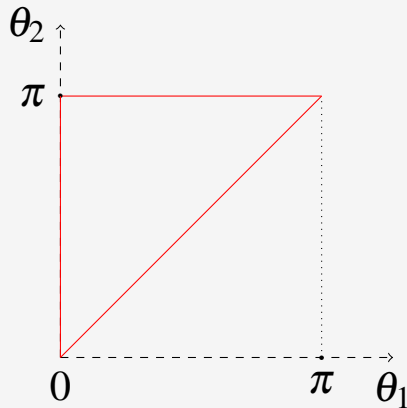
— abnormal case : $p^0 = 0$ hence $H_1 = H_2 = \{H_1, H_2\} = 0$, then

$$u_1 \{ \{H_1, H_2\}, H_1 \} + u_2 \{ \{H_1, H_2\}, H_2 \} = 0.$$

Proposition 3. *The surface $\Sigma : \{q; \det(F_1(q), F_2(q), [F_1, F_2](q)) = 0\}$ contained abnormal trajectories and is defined by*

- $\theta_i = 0$ or π , for $i = 1, 2$,
- $\theta_1 = \theta_2$.

*It is formed by the **boundaries of the physical domain** : $\theta_i \in [0, \pi]$, $i = 1, 2$, $\theta_1 \leq \theta_2$.*



: abnormal trajectories
in the plane (θ_1, θ_2)

Boundary conditions

$$x_0(0) = 0, \quad x_0(T) = x_f, \quad \theta_i(0) = \theta_i(T), \quad i = 1, 2$$

Transversality conditions

$$p_{\theta_i}(0) = p_{\theta_i}(T), \quad i = 1, 2$$

Boundary value problem

$$\begin{cases} \dot{z} = \vec{H}_n(z) \\ x_0(0) = 0, \quad x_0(T) = x_f \text{ (fixed) } , \\ \theta_i(0) = \theta_i(T), \quad i = 1, 2 \\ p_{\theta_i}(0) = p_{\theta_i}(T), \quad i = 1, 2 \end{cases}$$

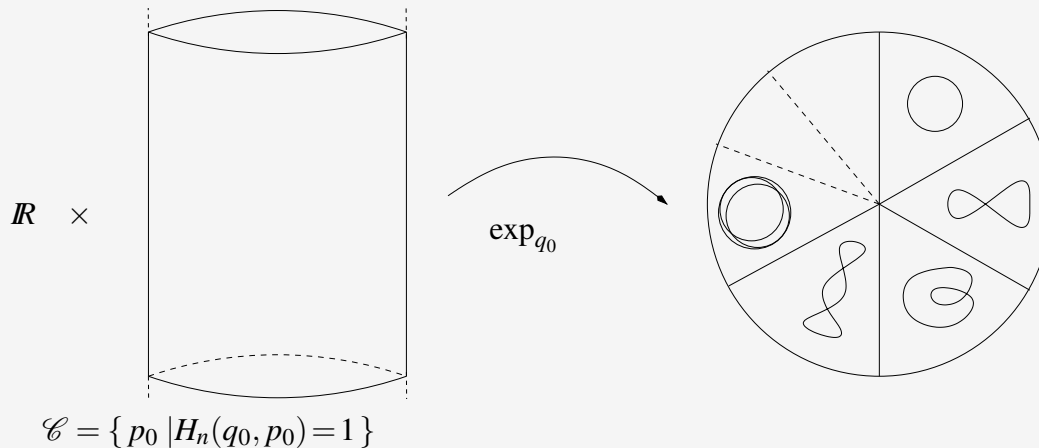
- HamPath, *indirect method* : based on Maximum Principle, *Newton* type algorithm (sensitive to initialization) and computation of **second order necessary optimality conditions** .
- Bocop, *direct method* : gives an initialization for the shooting algorithm of HamPath.

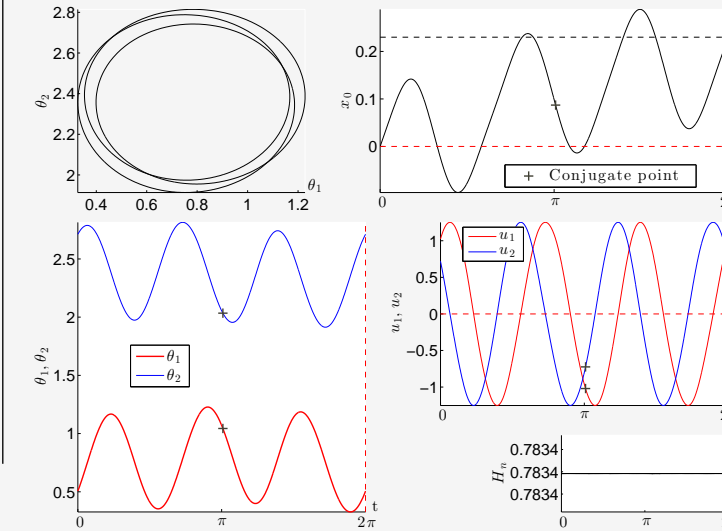
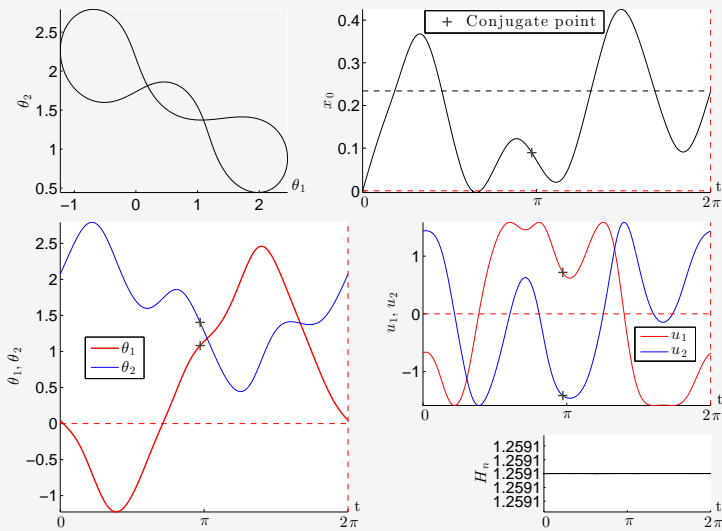
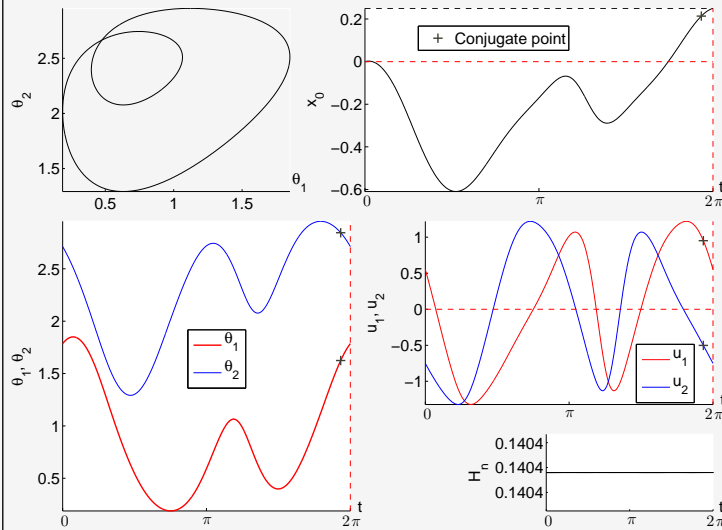
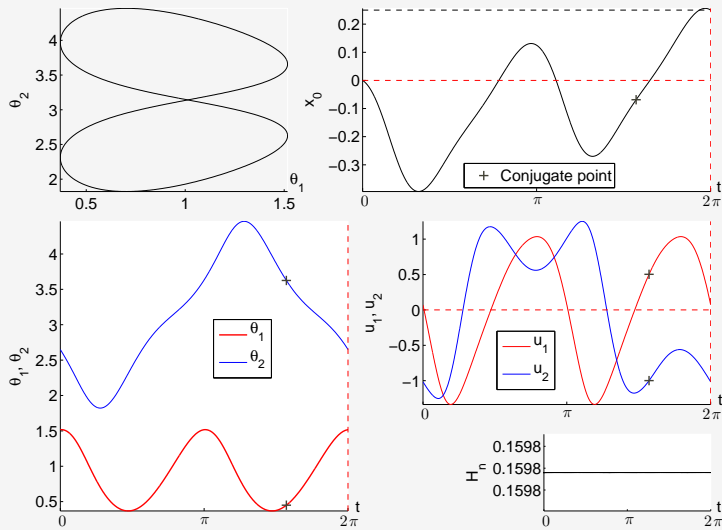
First conjugate time t_c : The first time t_c when the exponential mapping

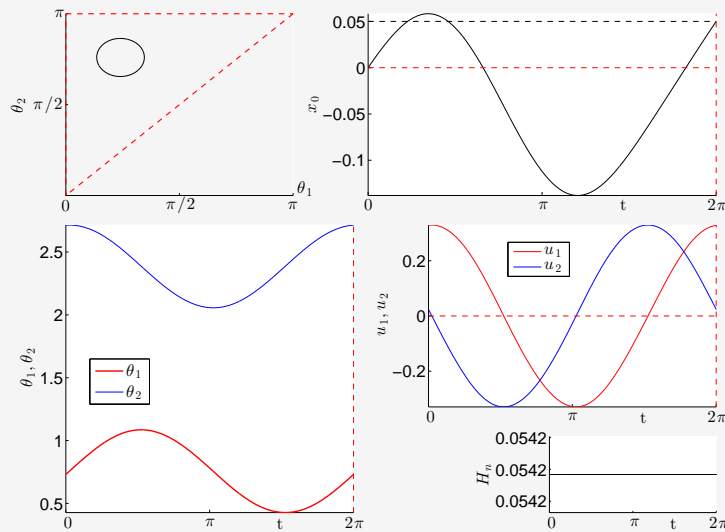
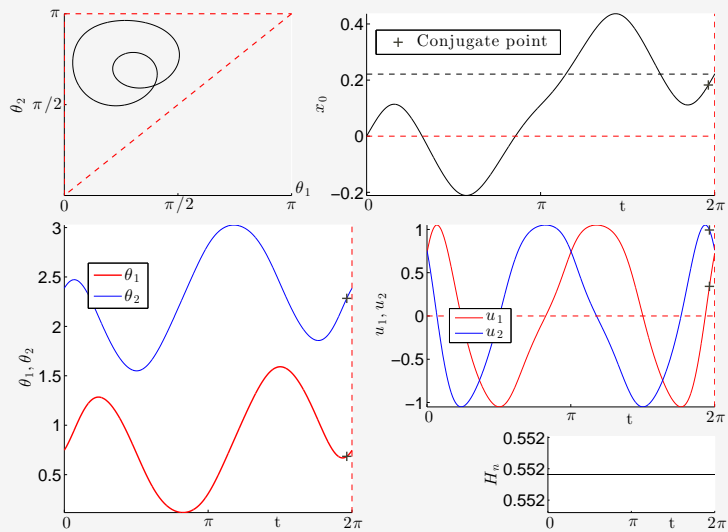
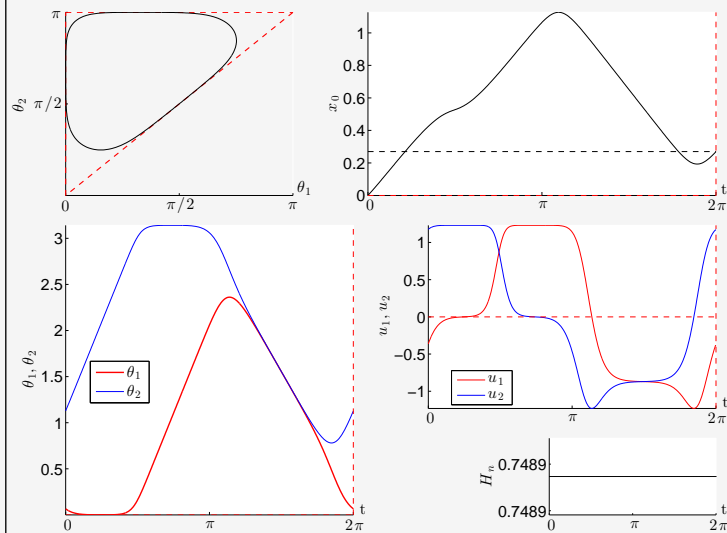
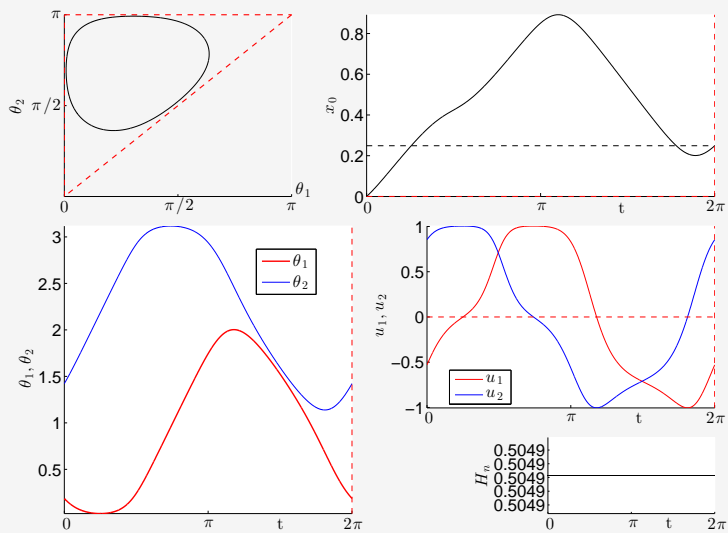
$$\exp_{q_0} : \mathbb{R} \times \mathcal{C} \rightarrow M, \quad (t, p_0) \mapsto q(t, q_0, p_0)$$

is not an immersion at (t_c, p_0) .

Theorem 4. *Let $q : [0, T] \rightarrow \mathbb{R}^n$ be a strict normal stroke. If $q(\cdot)$ has at least one conjugate point on $]0, T[$, then $q(\cdot)$ is not a local minimizer for the L^∞ topology on the controls considering the optimal control problem with **fixed extremities**.*







Stokes theorem

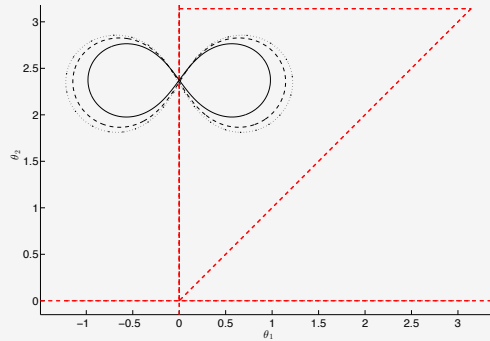
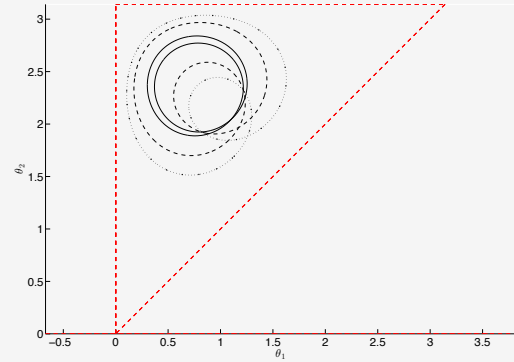
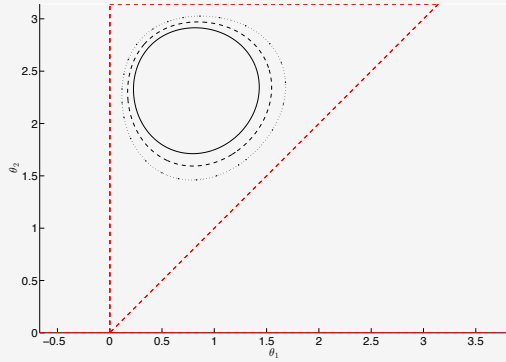
Proposition 5. Take the one-form on \mathbb{R}^2 : $\omega := \sum_{i=1}^2 \frac{\sin \theta_i}{\Delta(\theta)} d\theta_i$, $\left(F_i = \frac{\sin \theta_i}{\Delta(\theta)} \frac{\partial}{\partial \theta_i}\right)$.

Then

1. $d\omega = -f(\theta_1, \theta_2) d\theta_1 \wedge d\theta_2$, $f > 0$ in the interior of the abnormal triangle and $f = 0$ on the edges of the abnormal triangle.
2. for any bounded Stokes domain $D \subset \mathbb{R}^2$, we have

$$x_0(T) = \oint_{\partial D} \sum_{i=1}^2 \frac{\sin(\theta_i)}{\Delta} d\theta_i = \int_D -\frac{f(\theta_1, \theta_2)}{\Delta^2} d\theta_1 \wedge d\theta_2 = \int_D d\omega.$$

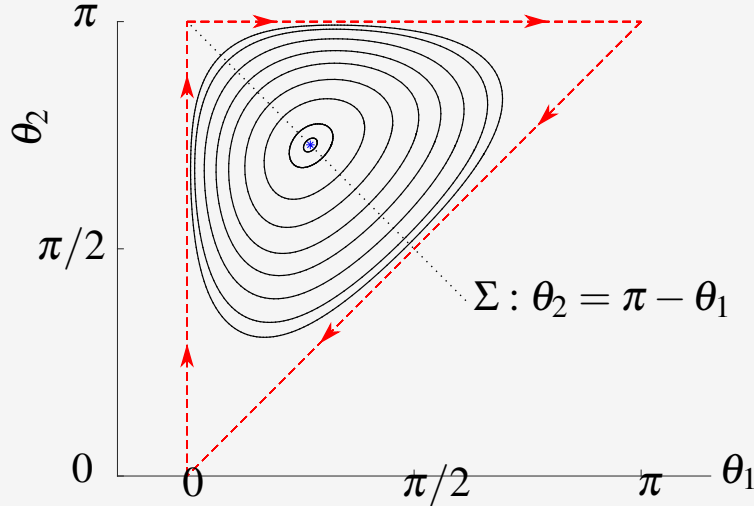
Generate normal strokes



Numerical fact :

Only the simple loops don't have conjugate points on $]0, T[$. They are the only candidates for optimality.

Theorem 6. *There is a one parameter family of simple loop strokes parameterized by the displacement $x_0(T)$ or the energy level $H_n(q_0, p_0)$.*

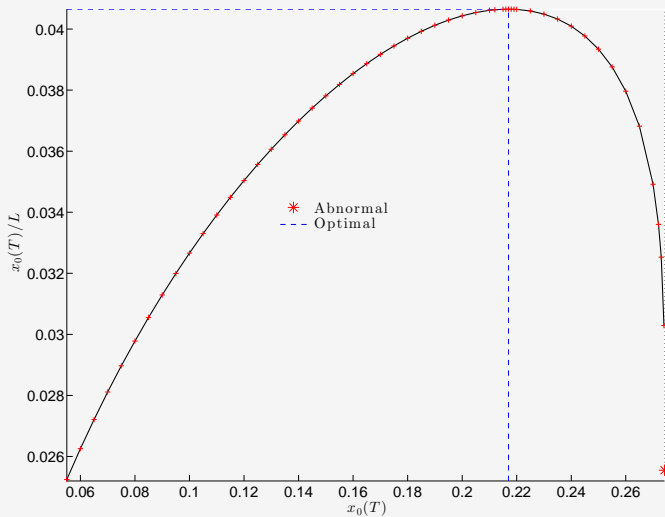


One parameter family of geodesic strokes.

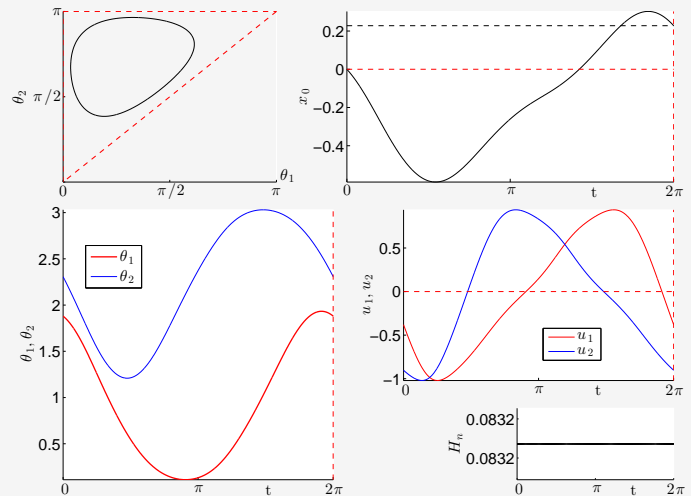
Geometric efficiency : $\mathcal{E} = x_0(T)/l_{SR}(q)$, $(\mathbf{x}_0(T) \text{ free})$

Transversality conditions

$$p_{x_0}(T) = q^0(T)/x_0(T), \quad p^0(T) = -1/2$$



Efficiency curve with continuation on $x_0(T)$.



Optimal normal stroke

Theorem 7. *The abnormal stroke is not minimizing for both costs (length with fixed displacement and efficiency).*

$$\begin{aligned}
& \min_{u(\cdot)} \mathcal{E}(\tilde{q}(\cdot)), \quad \tilde{q} = (q, q^0) = (\theta_1, \theta_2, x_0, q^0) \\
& \dot{q} = u_1 F_1(q) + u_2 F_2(q), \\
& \dot{q}^0 = L(q, u), \\
& q^0(0) = 0, \\
& x_0(0) = 0, \quad x_0(T) \in [x_T, x_T + d] \\
& \theta_1(0) = \theta_1(T), \quad \theta_2(0) = \theta_2(T).
\end{aligned}$$

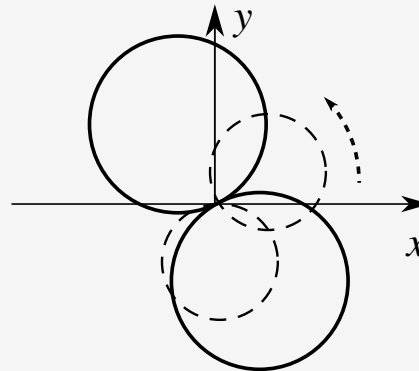
Proposition 8. *If a minimizer (\bar{q}, \bar{u}) is abnormal, then necessarily $\bar{x}_0(T) = x_T + d$.*

Normal form of order -1 Nilpotent model of order -1 , the Brockett-Heisenberg model :

$$\hat{F} = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad \hat{G} = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}$$

where (weight of x, y)=1 and (weight of z)=2 (privileged coordinates).

— model is **not generic** (families of simple loops geodesics)



Families of simple loops for the model of order -1 .

Theorem 9 (Brockett, Chakir et al). *The model of order -1 is equivalent to the model of order 0.*

Application to the Copepod z cannot be identified to the displacement x_0 .
 $x = \theta_1 - \theta_{10}$, $y = \theta_2 - \theta_{20}$.

Perturbation of the model of order -1 by terms of order 0 :

$$F_1(x, y, z) = \frac{\partial}{\partial x} + (a_{11}(\theta_{10})xy + a_{02}(\theta_{10})y^2 + y) \frac{\partial}{\partial z}$$

$$F_2(x, y, z) = \frac{\partial}{\partial y} + (a_{11}(\theta_{10})xy + a_{02}(\theta_{10})x^2 - x) \frac{\partial}{\partial z}$$

$$(F_1, F_2) \underset{\varphi}{\iff} (\hat{F}, \hat{G})$$

Lemma 10. *The only value of θ_{10} such that the transformation φ doesn't mix up the shape variables (x, y) and the displacement z corresponds to the center of swimming of the family.*

$$\varphi(x, y, z) = (x + c_{011}(\theta_{10})yz + c_{001}(\theta_{10})z, y - c_{011}(\theta_{10})xz + c_{001}(\theta_{10})z, z + P(x, y, z, \theta_{10})),$$

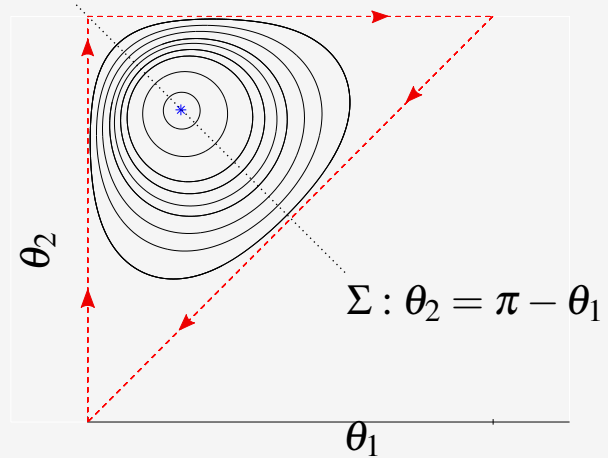
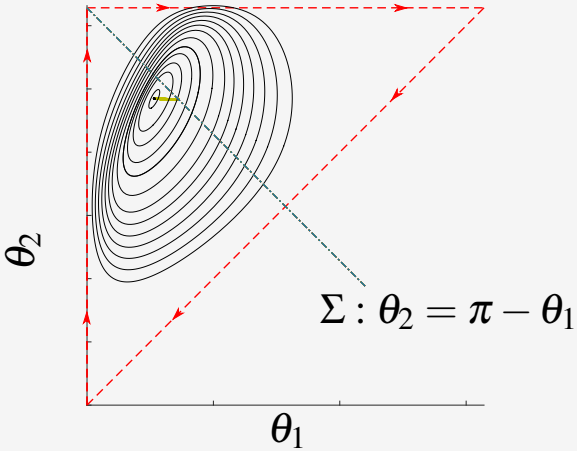
$$\Rightarrow c_{011}(\theta_{10}) = c_{001}(\theta_{10}) = 0.$$

Theorem 11 (Chakir, Gauthier, Kupka, 1996). *The generic model is given by the normal form of order 1*

$$F = \hat{F} + yQ(x,y)\frac{\partial}{\partial z}, \quad G = \hat{G} - xQ(x,y)\frac{\partial}{\partial z},$$

Q quadratic in (x,y) .

Remark 12. *This normal form can be used to approximate the one parameter family of simple strokes.*

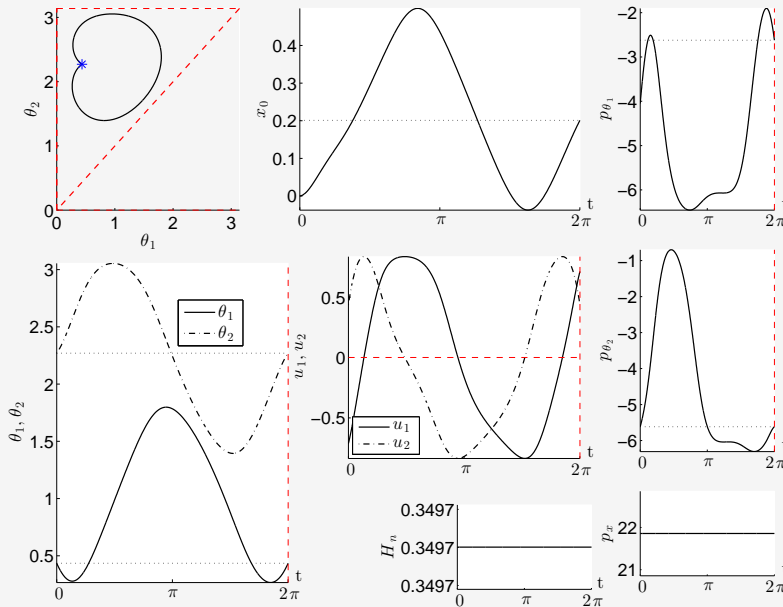


One parameter family of simple loops with different metrics.

$$\mathcal{E}^l = \frac{x_0(T) \mathbf{m}(\boldsymbol{\theta}(0))}{l(q)}, \quad (m \text{ smooth})$$

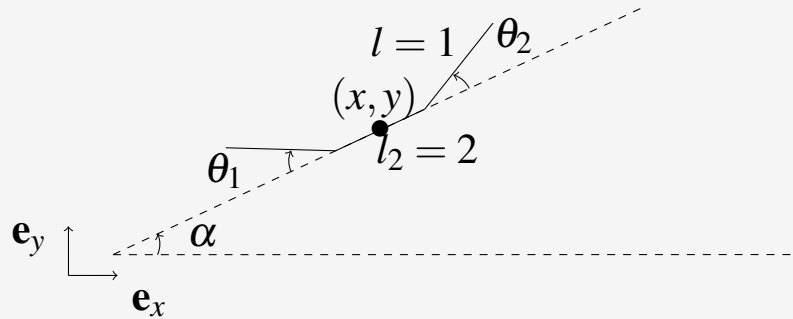
Transversality condition

$$p_{\theta}(0) - p_{\theta}(2\pi) = \lambda \frac{\partial \mathcal{E}^l}{\partial \theta(0)}$$



- *Contact point* : expression of the generic normal form at any point inside the triangle \rightarrow unique family of simple loops.
- *Martinet point* : compute the normal form for a point on the edges \rightarrow locate the eight loops.
- estimation of the first conjugate time using normal forms.
- swimmer model with more than 2 pairs of links.

The Purcell swimmer



Dynamic.

$$\dot{q} = D(\alpha)G(\theta)\dot{\theta}, \quad D(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\dot{\theta} = H(\theta)\tau, \quad \tau : \text{torque}, \quad \theta = (\theta_1, \theta_2), \quad q = (x, y, \alpha).$$

(G, H algebraic fractions of trigonometric polynomials).

Min. work of the drag forces $E(u) = \int_0^T (uH^{-1}u)dt.$

Controls $u = \dot{\theta}$

$$\dot{X} = u_1F_1(X) + u_2F_2(X), \quad X = (\theta_1, \theta_2, x, y, \alpha)$$

Theorem 13. Nilpotent approx. for (F_1, F_2)

$$\hat{F}_1 = \frac{\partial}{\partial x_1}, \quad \hat{F}_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} + x_1^2 \frac{\partial}{\partial x_5}.$$

$$\varphi(\theta_1, \theta_2, x, y, \alpha) = (x_1, x_2, x_3, x_4, x_5) = \mathbf{X}, \quad \varphi \text{ t.q. } \varphi_* F_i = \hat{F}_i + O(|\mathbf{X}|^3)$$

Cartan flat model

$$\dot{X} = \sum_{i=1}^2 u_i \hat{F}_i(X), \quad \min_u \int_0^T (u_1^2 + u_2^2) dt.$$

PMP (x, u) optimal \implies projection of a solution of

$$H_n = 1/2 (H_1^2 + H_2^2), \quad H_i = p \cdot \hat{F}_i, \quad i = 1, 2$$

Hamiltonians lifts

$$\begin{aligned} H_1 &= p \cdot \hat{F}_1(X) = p_1, & H_2 &= p \cdot \hat{F}_2(X) = p_2 + p_3 x_1 + p_4 x_3 + p_5 x_1^2, \\ H_3 &= p \cdot [\hat{F}_1, \hat{F}_2](X) = -p_3 - 2x_1 p_5, & H_4 &= p \cdot [[\hat{F}_1, \hat{F}_2], \hat{F}_1](X) = -2p_5, \\ H_5 &= p \cdot [[\hat{F}_1, \hat{F}_2], \hat{F}_2](X) = p_4. \end{aligned}$$

Poincaré coordinates : $(H_1, H_2, H_3, H_4, H_5)$

$$\begin{aligned}\dot{H}_1 &= dH_1(\vec{H}_n) = \{H_1, H_2\}H_2 = p \cdot [\hat{F}_1, \hat{F}_2] H_2 = H_2H_3, \\ \dot{H}_2 &= -H_3H_1, \quad \dot{H}_3 = H_1H_4 + H_2H_5, \\ \dot{H}_4 &= 0 \quad \text{i.e.} \quad H_4 = c_4, \quad \dot{H}_5 = 0 \quad \text{i.e.} \quad H_5 = c_5.\end{aligned}$$

Fix the energy level : $H_1^2 + H_2^2 = 1$, and let $H_1 = \cos(\vartheta)$ and $H_2 = \sin(\vartheta)$.

$$\dot{H}_1 = -\sin(\vartheta)\dot{\vartheta} = H_2H_3 = \sin(\vartheta)H_3.$$

Hence $\dot{\vartheta} = -H_3$ and

$$\ddot{\vartheta} = -(H_1c_4 + H_2c_5) = -c_4\cos(\vartheta) - c_5\sin(\vartheta) = -\omega^2\sin(\vartheta + \phi)$$

where ω, ϕ are constants and satisfy $\omega^2\sin(\phi) = c_4$ and $\omega^2\cos(\phi) = c_5$.

Then, let $\psi = \vartheta + \phi$, we have

$$1/2\dot{\psi}^2 - \omega^2\cos(\psi) = B, \quad B \text{ is constant.}$$

Theorem 14. *The normal extremals associated to the Cartan flat model is **integrable** and solutions can be expressed as polynomials of elliptic functions of the 1st and 2nd kind $(u, \text{sn}(u), \text{cn}(u), \text{dn}(u), E(u))$.*

Oscillating case

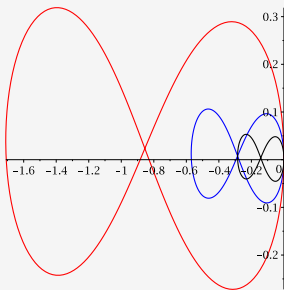
$$x_1(u) = 1/\omega \left[x_1(\varphi_0) - 2k \sin(\phi) \operatorname{cn}(u, k) + (-u + 2E(u, k)) \cos(\phi) \right],$$

$$x_2(u) = 1/\omega \left[x_2(\varphi_0) - 2k \cos(\phi) \operatorname{cn}(u, k) + (u - 2E(u, k)) \sin(\phi) \right]$$

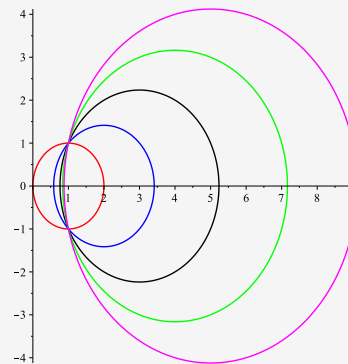
Rotating case

$$x_1(u) = (-2 \cos(\phi) u + 2 \cos(\phi) E(u/k, k) k - 2 \sin(\phi) \operatorname{dn}(u/k, k) k + \cos(\phi) u k^2 + x_1(\varphi_0) k^2) \omega^{-1} k^{-2},$$

$$x_2(u) = (2 \sin(\phi) u - 2 \sin(\phi) E(u/k, k) k - 2 \cos(\phi) \operatorname{dn}(u/k, k) k - \sin(\phi) u k^2 + x_2(\varphi_0) k^2) \omega^{-1} k^{-2}$$

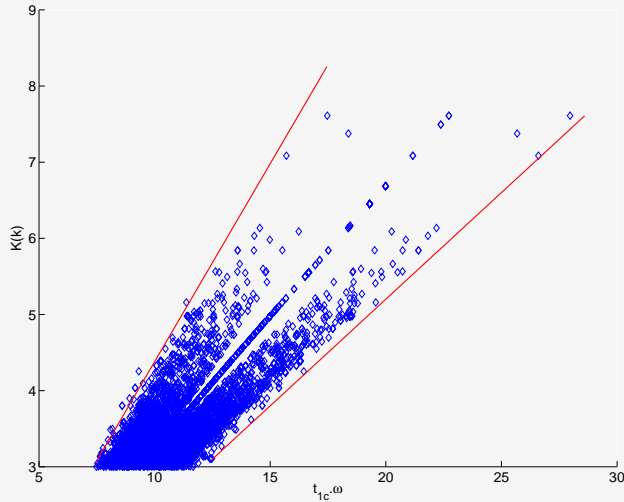


(left) Oscillating case,

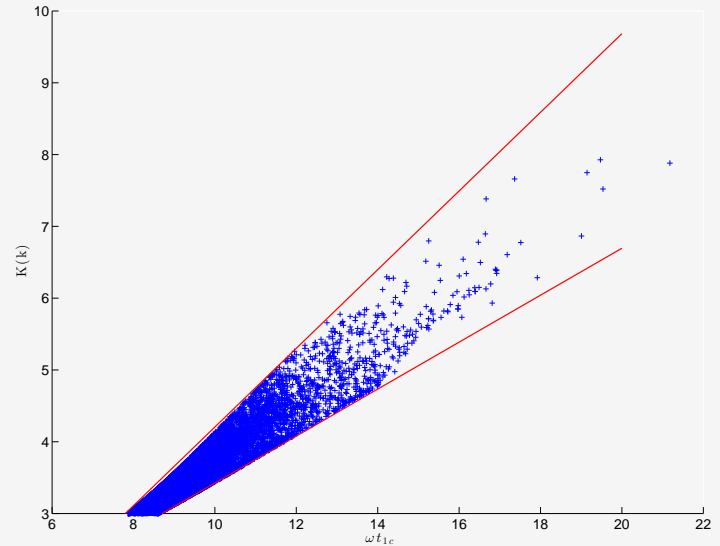


(right) Rotating case

Consider a family of normal extremals parameterized by $\hat{p}(0)$, we compute numerically the first conjugate time.



Oscillating case



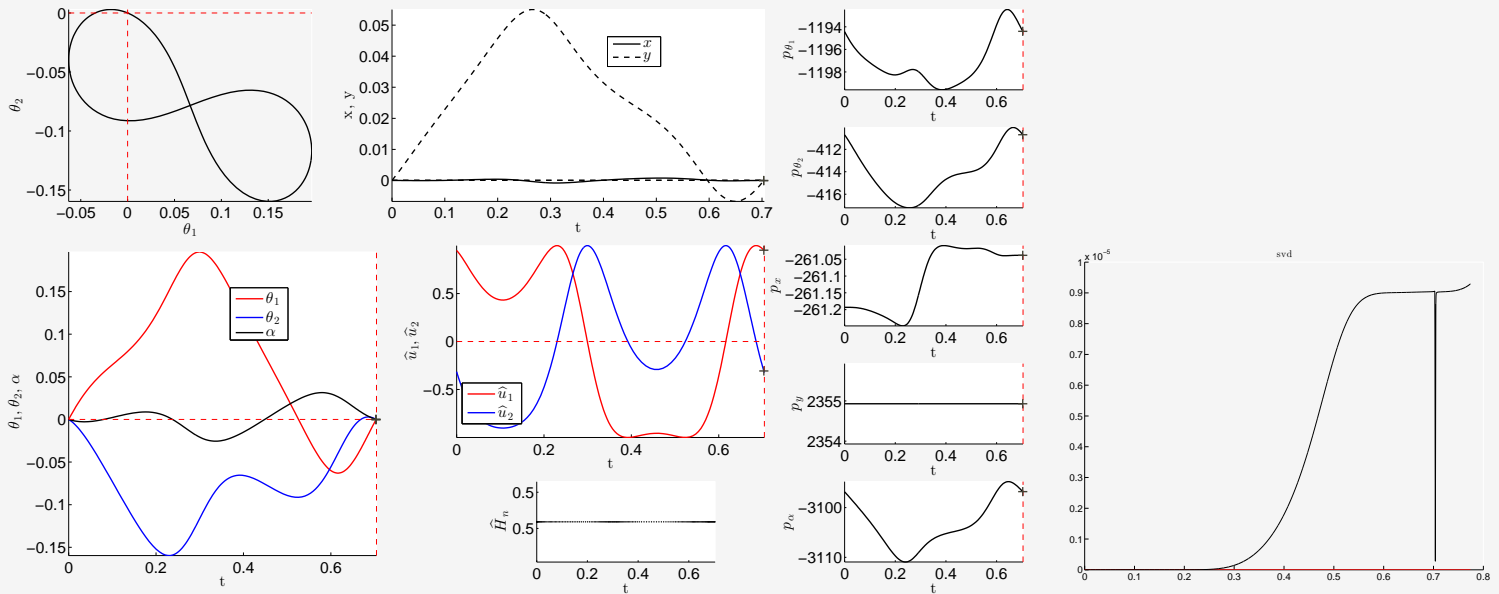
Rotating case

Numerical fact : Conjugate times are located in a sector.

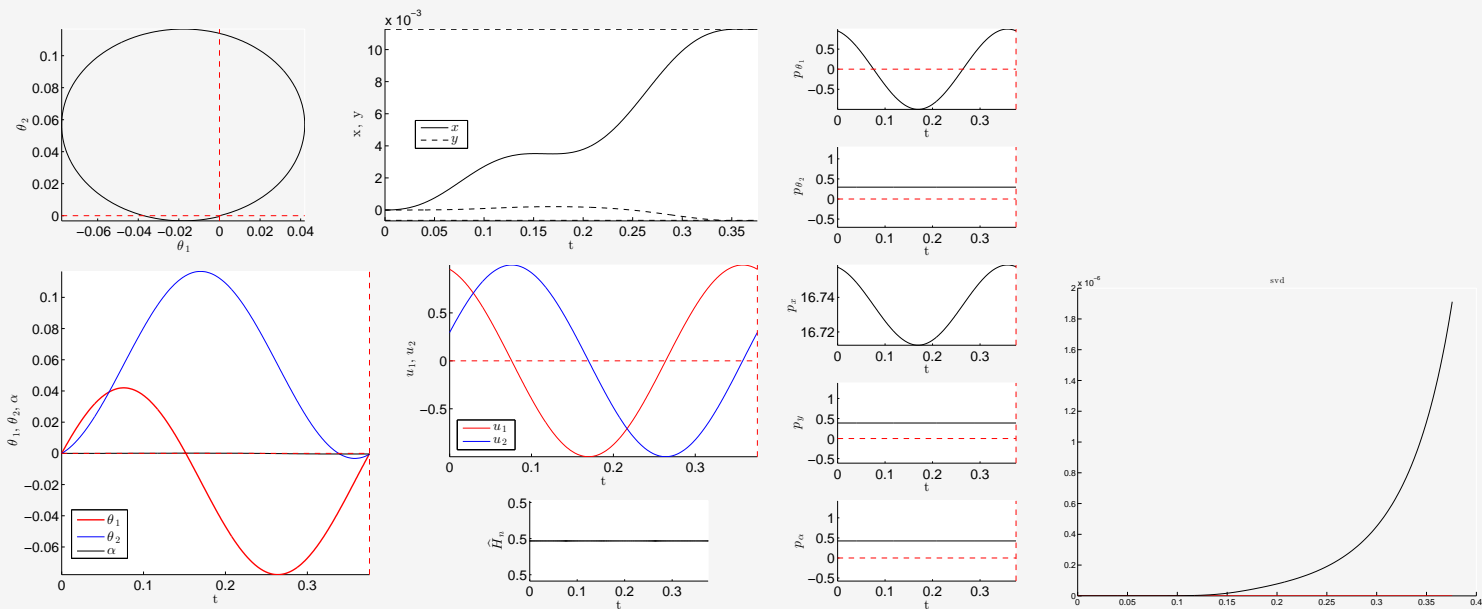
$$0.34 \omega t_{1c} - 0.4 < K(k) < 0.53 \omega t_{1c} - 0.8 \quad : \text{Oscillating case,}$$

$$0.33 \omega t_{1c} + 0.16 < K(k) < 0.55 \omega t_{1c} - 1.27 \quad : \text{Rotating case.}$$

Initialization of algorithms from the microlocal analytical solutions



(left) Control, state and adjoint of the physical variables (θ, x, y, α) for the **oscillating case** and for the nilpotent system : stroke with **eight shape**,
 (right) SVD of the rank condition to compute conjugate points.



(left) Control, state and adjoint of the physical variables (θ, x, y, α) for the **rotating case** and for the nilpotent system : **simple loop stroke**,
(right) SVD of the rank condition to compute conjugate points.

Proposition 15 (Numerical). *For the Cartan problem, **simple loops** (rotating case) are the only strokes without conjugate points.*

- nilpotent model \rightarrow give an initial adjoint vector $\hat{p}(0)$ to initialize strokes with small amplitudes associated with the true dynamic and to the cost $\int_0^T (u_1^2 + u_2^2) dt$.

Boundary value problem

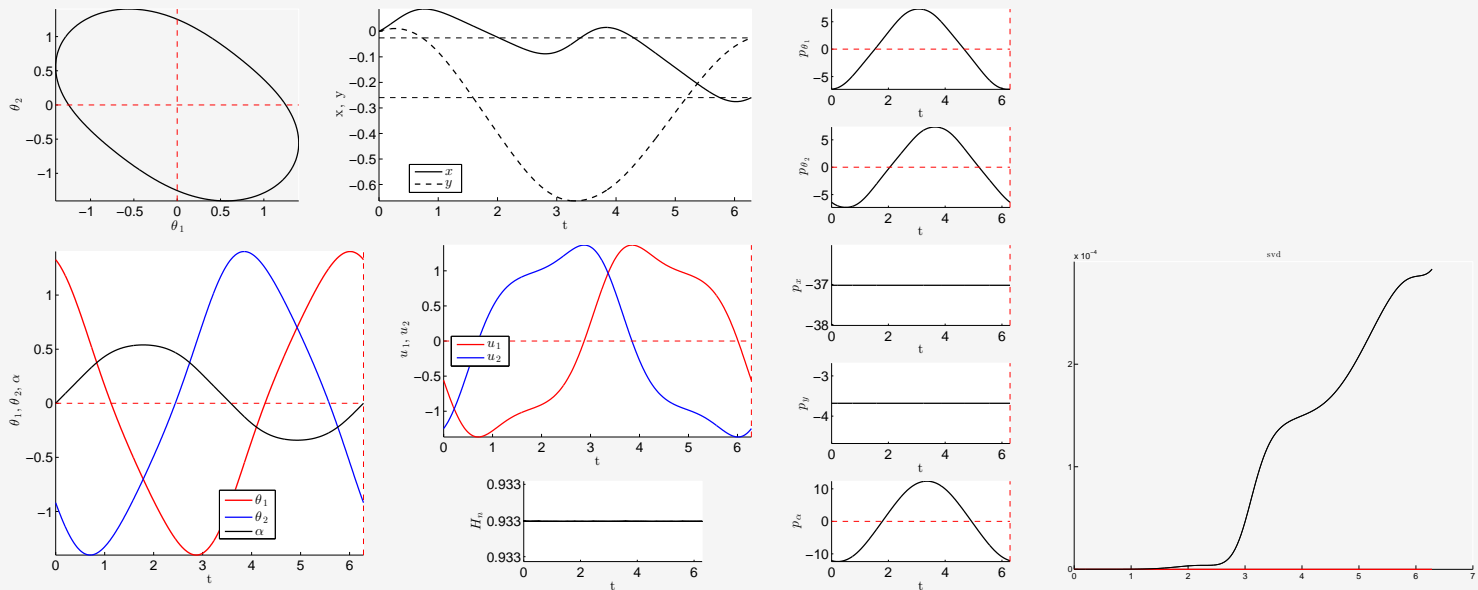
$$\left\{ \begin{array}{l} \dot{q} = \frac{\partial \tilde{H}_n}{\partial p}, \quad \dot{p} = -\frac{\partial \tilde{H}_n}{\partial q}, \\ \theta_j(T) = \theta_j(0) \quad j = 1, 2, \\ x(0) = y(0) = \alpha(0) = 0, \\ x(T)^2 + y(T)^2 = r_1, \quad \alpha(T) = 0, \\ p_{\theta_j}(T) = p_{\theta_j}(0), \quad j = 1, 2, \quad p_\alpha(0) = p_\alpha(T) \end{array} \right.$$

\tilde{H}_n : Hamiltonian associated with the cost $\int_0^T (u_1^2 + u_2^2) dt$.

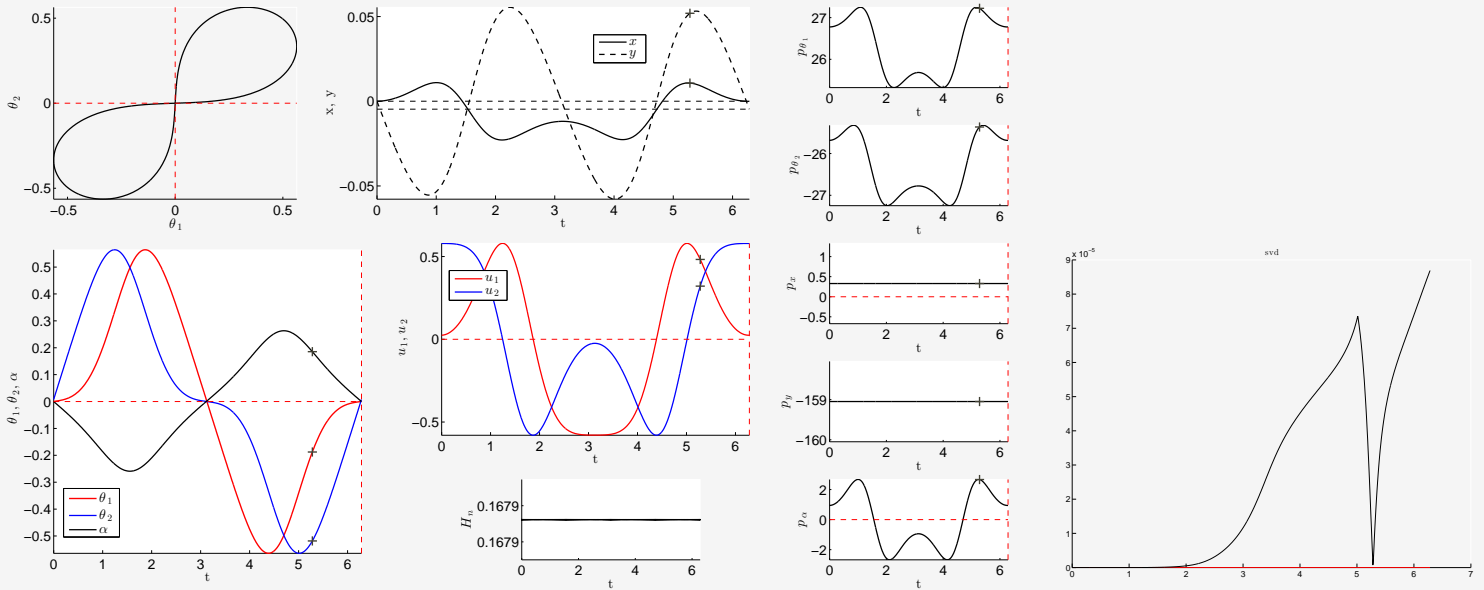
— 2 discrete continuations :

- on the radius r_1
- from the cost $\int_0^T (u_1^2 + u_2^2) dt$ to the mechanical cost.

Discrete continuation on the amplitudes of the strokes



(left) Normal stroke with simple loop for the true dynamic, minimizing the cost $\int_0^T (u_1^2 + u_2^2) dt$ ($T = 2\pi$, $r_1 = 0.068$) (right) Test for the conjugate points : no conjugate points on $[0, 2\pi]$.

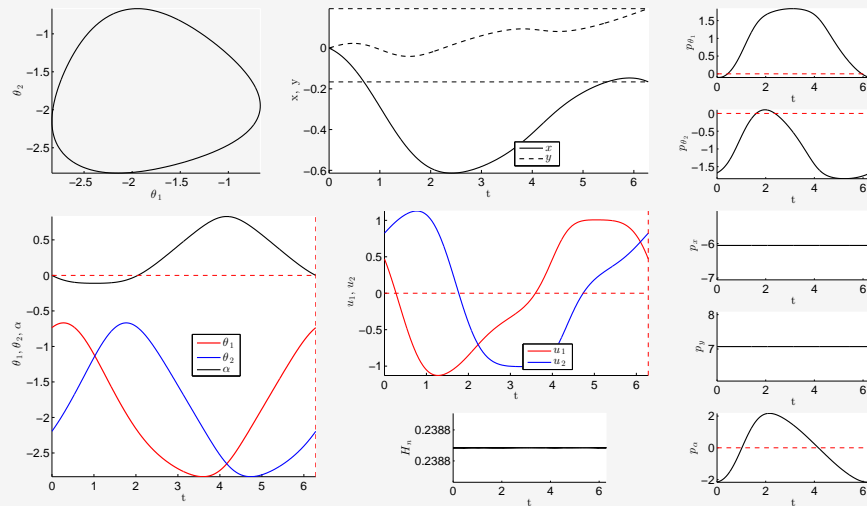


(left) Normal stroke with eight shape loop for the true dynamic, minimizing the cost $\int_0^T (u_1^2 + u_2^2) dt$ ($T = 2\pi$, $r_1 = 4.6e-4$) (right) Test for the conjugate points.

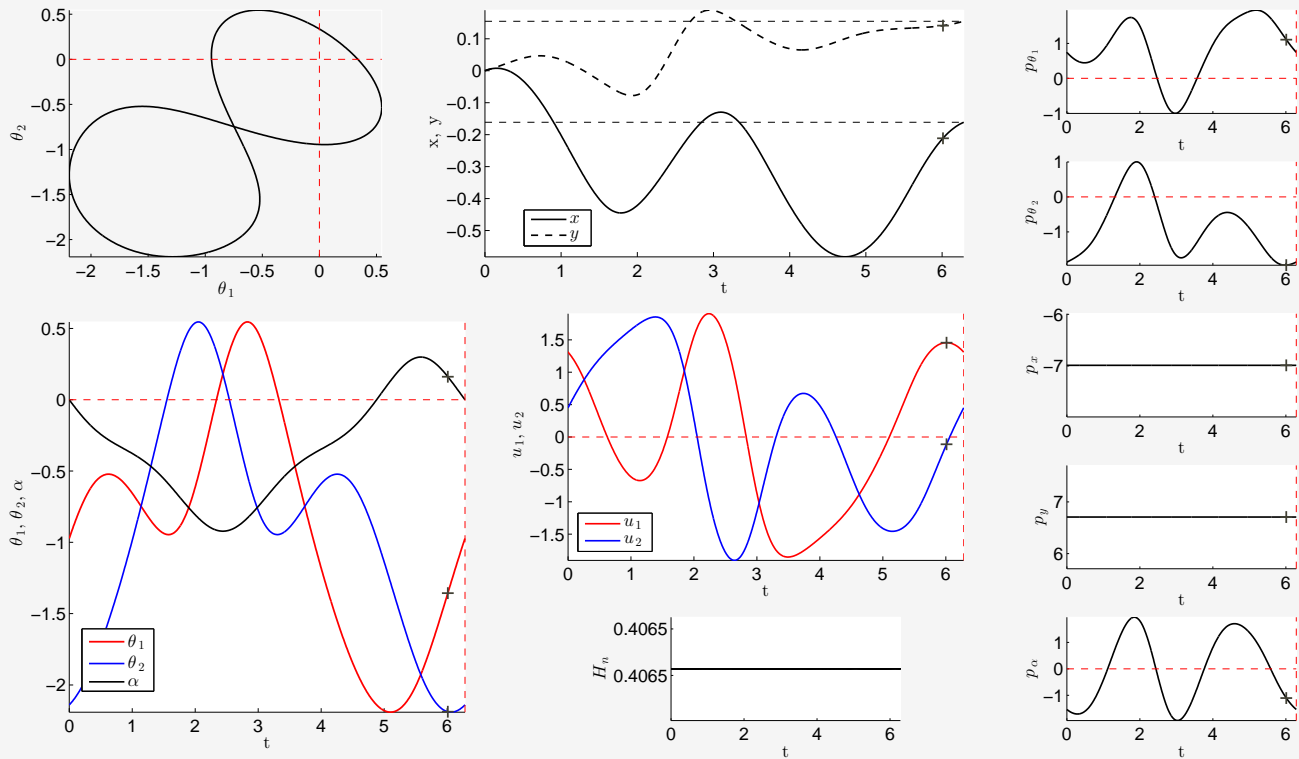
Discrete continuation on the cost

- \tilde{H}_n : normal Hamiltonian for the cost $\int_0^T u_1^2 + u_2^2 dt$ cost,
- H_n : normal Hamiltonian for the mechanical cost.

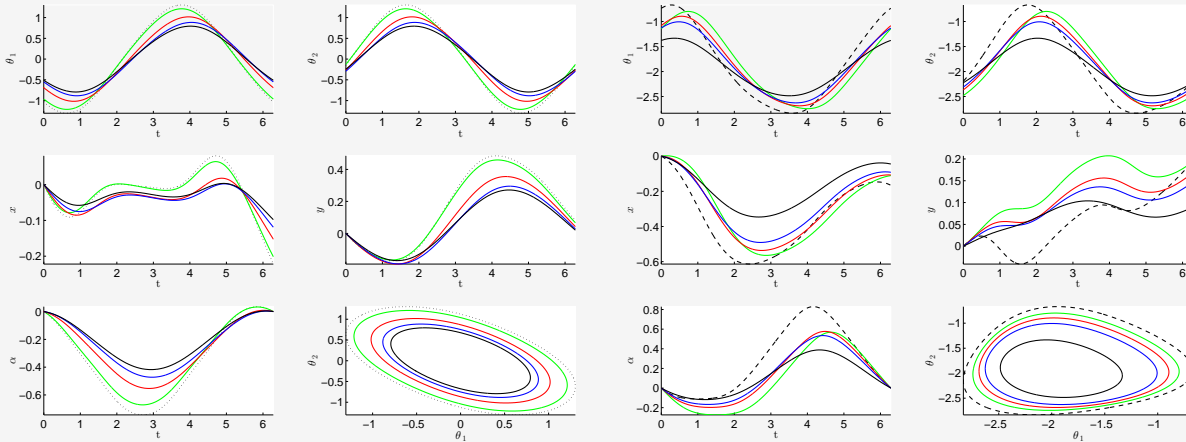
$$H = \lambda H_n + (1 - \lambda) \tilde{H}_n, \quad \lambda \in [0, 1].$$



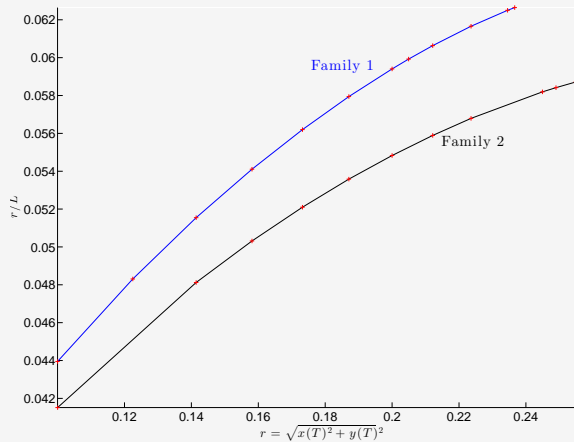
Stroke for the Purcell swimmer minimizing the mechanical energy ($T = 2\pi$, $r_1 = 0.065$).



States and adjoints for the Purcell swimmer (mechanical cost)



Two families of normal strokes for the Purcell swimmer.



Geometric efficiency :

$$\mathcal{E}(\gamma) = \sqrt{x(T)^2 + y(T)^2} / l_{SR}(\gamma)$$

Used to compare family of simple loops.

$$\left\{ \begin{array}{l} \min \quad J(q(\cdot), u(\cdot)) := c(q(0), q(T)) \\ \text{sb. to} \\ \quad \dot{q}(t) = F(q(t), u(t)) \\ \quad u(t) \in U \\ \quad m(q(0), q(T)) = 0, \end{array} \right.$$

$H(q, u, p) := p \cdot F(q, u)$ and $h(q_0, q_T) := c(q_0, q_T) + \mathbf{v} \cdot m(q_0, q_T)$.

Consider a **normal extremal** (\bar{q}, \bar{p}) associated to u .

Second variation

$$\begin{aligned} \delta^2 J(\delta q(\cdot), \delta u(\cdot)) &:= 1/2 [\delta q(0)^\top \quad \delta q(T)^\top] \mathbf{C} [\delta q(0) \quad \delta q(T)]^\top \\ &+ 1/2 \int_0^T (\delta q(t)^\top \partial_{qq} \mathbf{H}(t) \delta q(t) + 2 \delta q(t)^\top \partial_{qu} \mathbf{H}(t) \delta u(t) + \delta u(t)^\top \partial_{uu} \mathbf{H}(t) \delta u(t)) dt \end{aligned}$$

where $\mathbf{C} := \text{Hessian}_{q_0, q_T}(h)$.

$$\left\{ \begin{array}{l} \min \\ \text{sous} \end{array} \right. \quad \begin{array}{l} \delta^2 J(\delta q(\cdot), \delta u(\cdot)) \\ \\ \dot{\delta} q := \partial_q \mathbf{F}(t) \delta q(t) + \partial_u \mathbf{F}(t) \delta u(t) \\ \nabla_{q_0} m(\bar{q}(0), \bar{q}(T)) \delta q(0) + \nabla_{q_T} m(\bar{q}(0), \bar{q}(T)) \delta q(T) = 0, \end{array}$$

Monodromy matrix. $\Phi(.,.)$ associated with the linearized Hamiltonian system :

$$\left\{ \begin{array}{l} \frac{d}{dt} \Phi(t, s) = \mathbf{Z} \Phi(t, s) \\ \Phi(s, s) = \text{Id}, \end{array} \right.$$

where

$$\mathbf{Z} := \begin{bmatrix} \partial_q \mathbf{F} - \partial_u \mathbf{F} [\partial_{uu} \mathbf{H}]^{-1} \partial_{qu} \mathbf{H}^T & -\partial_u \mathbf{F} [\partial_{uu} \mathbf{H}]^{-1} \partial_u \mathbf{F}^T \\ -\partial_{qq} \mathbf{H} + \partial_{qu} \mathbf{H} [\partial_{uu} \mathbf{H}]^{-1} \partial_{qu} \mathbf{H}^T & -\partial_q \mathbf{F} [\partial_{uu} \mathbf{H}]^{-1} \partial_u \mathbf{F}^T \end{bmatrix}.$$

Define

$$\mathcal{W} := \begin{bmatrix} \phi_{22} \phi_{12}^{-1} & \phi_{21} - \phi_{22} \phi_{12}^{-1} \phi_{11} \\ -\phi_{12} & \phi_{12}^{-1} \phi_{11} \end{bmatrix}, \quad \Phi(0, T) =: \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

Theorem 16 (Standard conditions). *Assume*

- (i) $\partial_{uu}\mathbf{H}(t) \leq -\varepsilon Id$ on $[0, T]$, $\bar{u}(\cdot)$ bounded and $(\partial_q \mathbf{F}(\cdot), \partial_u \mathbf{F}(\cdot))$ is controllable on $[0, T]$,
- (ii) : the extremal $(\bar{q}(\cdot), \bar{u}(\cdot), p(\cdot))$ **doesn't have conjugate points** on $[0, T]$,
- (iii) : there exists $\gamma > 0$ t.q.

$$\begin{bmatrix} \xi_0^T & \xi_1^T \end{bmatrix} \mathcal{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right|^2,$$

for all vectors $\xi_0, \xi_1 \in \mathbb{R}^n \setminus \{0\}$ s.t.

$$\nabla_{q_0} m((\bar{q}(0), \bar{q}(T))) \xi_0 + \nabla_{q_T} m((\bar{q}(0), \bar{q}(T))) \xi_1 = 0.$$

Then $(\bar{q}(\cdot), \bar{u}(\cdot))$ is a $W^{1,\infty}$ -minimizer and locally **unique**.

Boundary values

$$\begin{aligned}\theta_j(0) &= \theta_j(T) \quad j = 1, 2, \quad \alpha(0) = \alpha(T), \\ x(0) &= 0, \quad y(0) = 0, \quad x(T)^2 + y(T)^2 = r \quad (r \text{ is fixed}).\end{aligned}$$

Proposition 17. Take $a = (\phi, \sigma) \in I = (-\varepsilon, \varepsilon)^2$ for $\varepsilon > 0$. ($\bar{q} = (\bar{\theta}_1, \bar{\theta}_2, \bar{x}, \bar{y}, \bar{\alpha})$ and $\bar{u} = (\bar{u}_1, \bar{u}_2)$) a normal extremal.

For all $a \in I$ and $t \in [0, T]$, define $q^a(\cdot) = (\theta_1^a(\cdot), \theta_2^a(\cdot), x^a(\cdot), y^a(\cdot), \alpha^a(\cdot))$ where

$$\begin{aligned}u_j^a(t) &= \bar{u}_j(t + \sigma), \quad \theta_j^a(t) = \bar{\theta}_j(t + \sigma) \quad j = 1, 2, \\ x^a(t) &= \cos(\phi) (\bar{x}(t + \sigma) - \bar{x}(\sigma)) - \sin(\phi) (\bar{y}(t + \sigma) - \bar{y}(\sigma)), \\ y^a(t) &= \sin(\phi) (\bar{x}(t + \sigma) - \bar{x}(\sigma)) + \cos(\phi) (\bar{y}(t + \sigma) - \bar{y}(\sigma)), \\ \alpha^a(t) &= \bar{\alpha}(t + \sigma) + \phi.\end{aligned}$$

Then $(\bar{q}(\cdot), p(\cdot), \bar{u}(\cdot))$ is continuously embedded in a family of extremals $(q^a(\cdot), p^a(\cdot), u^a(\cdot))_{a \in I}$ where $p^a(\cdot)$ is the adjoint vector associated with $(q^a(\cdot), u^a(\cdot))$.

\implies **Standard conditions fail**

Theorem 18 (Refined Conditions, Gavriel, Vinter (2014)). *Assume the reference normal extremal $(\bar{q}(\cdot), \bar{u}(\cdot), p(\cdot))$ is embedded in a family of extremals and*

(i) $\partial_{uu}\mathbf{H} \leq -\varepsilon Id$ on $[0, T]$, $\bar{u}(\cdot)$ is bounded and $(\partial_q \mathbf{F}(\cdot), \partial_u \mathbf{F}(\cdot))$ is controllable on $[0, T]$,

(ii) : the extremal $(\bar{q}(\cdot), \bar{u}(\cdot), p(\cdot))$ **doesn't have conjugate points** on $[0, T]$,

(iii) : there exists $\gamma > 0$ s.t.

$$\begin{bmatrix} \varepsilon T & \varepsilon T \\ \xi_0 & \xi_1 \end{bmatrix} \mathcal{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left\| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right\|^2,$$

for all vectors $\xi_0, \xi_1 \in \mathbb{R}^n \setminus \{0\}$ s.t.

$$\nabla_{q_0} m((\bar{q}(0), \bar{q}(T))) \xi_0 + \nabla_{q_T} m((\bar{q}(0), \bar{q}(T))) \xi_1 = 0 \quad \text{and} \quad \Gamma^T \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = 0.$$

$$\text{where } \Gamma := \left[\begin{array}{c} D_\alpha q^\alpha(0) \\ D_\alpha q^\alpha(T) \end{array} \right] \Big|_{\alpha=0}.$$

Then $(\bar{q}(\cdot), \bar{u}(\cdot))$ is a **local** $W^{1,\infty}$ -**minimizer**.

Define the matrix N_s from the subspace \mathcal{L}_s s.t.

$$\mathcal{L}_s = \{ (\xi_0, \xi_T) \in \mathbb{R}^5 \times \mathbb{R}^5 \mid \nabla_{q_0, q_T} m(q_0, q_T) (\xi_0 \quad \xi_T)^\top = 0 \} =: \text{Im}(N_s)$$

Standard conditions. Does the matrix $\mathcal{W}_s := \underbrace{N_s^\top (\mathcal{W}^\top + \mathcal{W}) N_s}_{4 \times 4}$ is positive-definite?

Consider

$$\Gamma_r = \begin{pmatrix} \nabla_\phi q^a(0) & \nabla_\phi q^a(T) \\ \nabla_\sigma q^a(0) & \nabla_\sigma q^a(T) \end{pmatrix}_{\phi=0, \sigma=0}$$

and the linear subspace \mathcal{L}_r s.t.

$$\mathcal{L}_r := \mathcal{L}_s \cap \{ (\xi_0, \xi_T) \in \mathbb{R}^5 \times \mathbb{R}^5 \mid \Gamma_r^\top (\xi_0 \quad \xi_T)^\top = 0 \} =: \text{Im}(N_r)$$

Refined conditions. Does the matrix $\mathcal{W}_r := \underbrace{N_r^\top (\mathcal{W}^\top + \mathcal{W}) N_r}_{2 \times 2}$ is positive-definite?

Relative tolerance	(Standard Conditions) $\text{Spec}(\mathcal{W}_s)$	(Refined Conditions) $\text{Spec}(\mathcal{W}_r)$
10^{-4}	1319.91 3.44629 −2.61575e-5 −4.17860e-3	36179.7 13.8018
10^{-7}	1320.17 3.44676 9.81190e-6 −5.40128e-6	36186.9 13.8037

- Standard conditions fail : \mathcal{W}_s has zero eigenvalues.
- Refined conditions are satisfied : \mathcal{W}_r is positive-definite.

Theorem 19 (Numerical). *The simple loop normal stroke (\bar{q}, \bar{u}) is $W^{1,\infty}$ – optimal.*