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Sub-Riemannian Geometry and swimming at low Reynolds number

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with Piernicola Bettiol (LBMA, Brest) and Bernard Bonnard (INRIA, Sophia Antipolis).

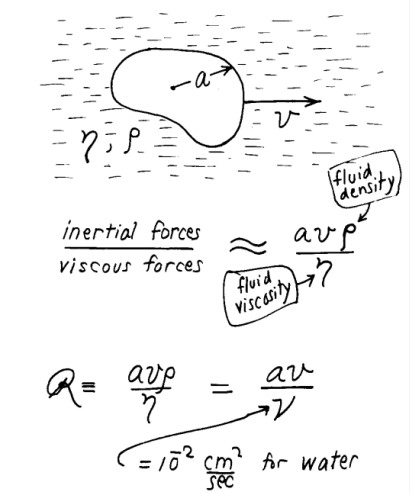
2017, 4th July

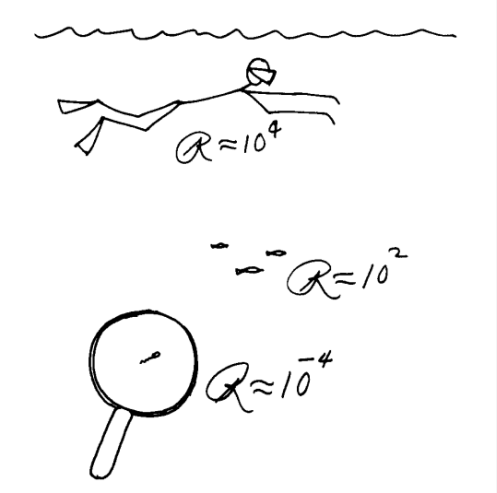
New Horizons in Optimal Control, Porto

A tribute to Helmut Maurer, Urszula Ledzewicz and Heinz Schaettler

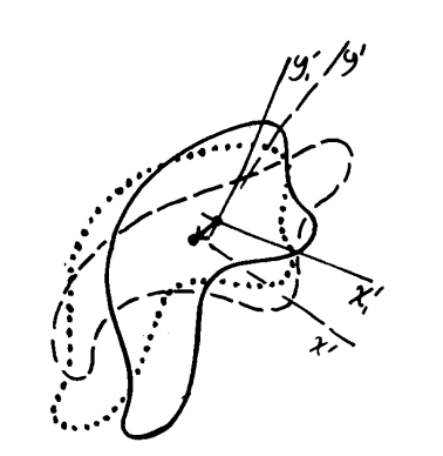
jrouot@laas.fr

Life at low Reynolds number - Purcell, 1977

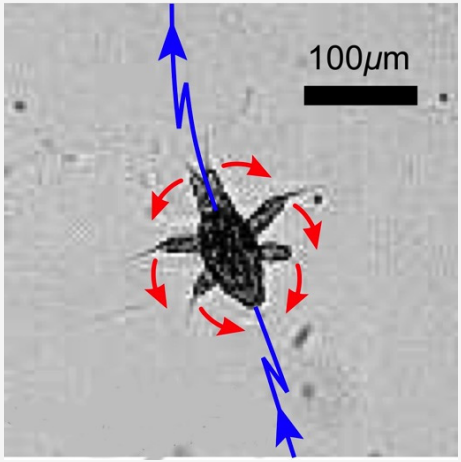




Reynolds number

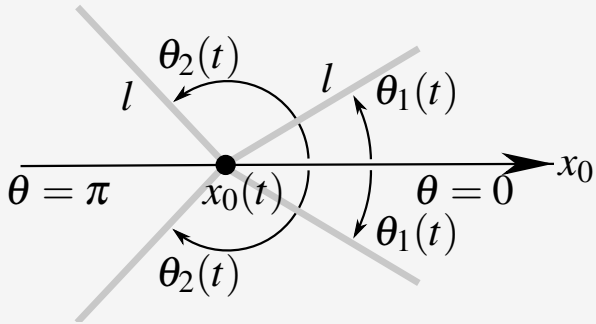


Shape deformations of a microswimmer



Zooplankton

2-link symmetric swimmer.

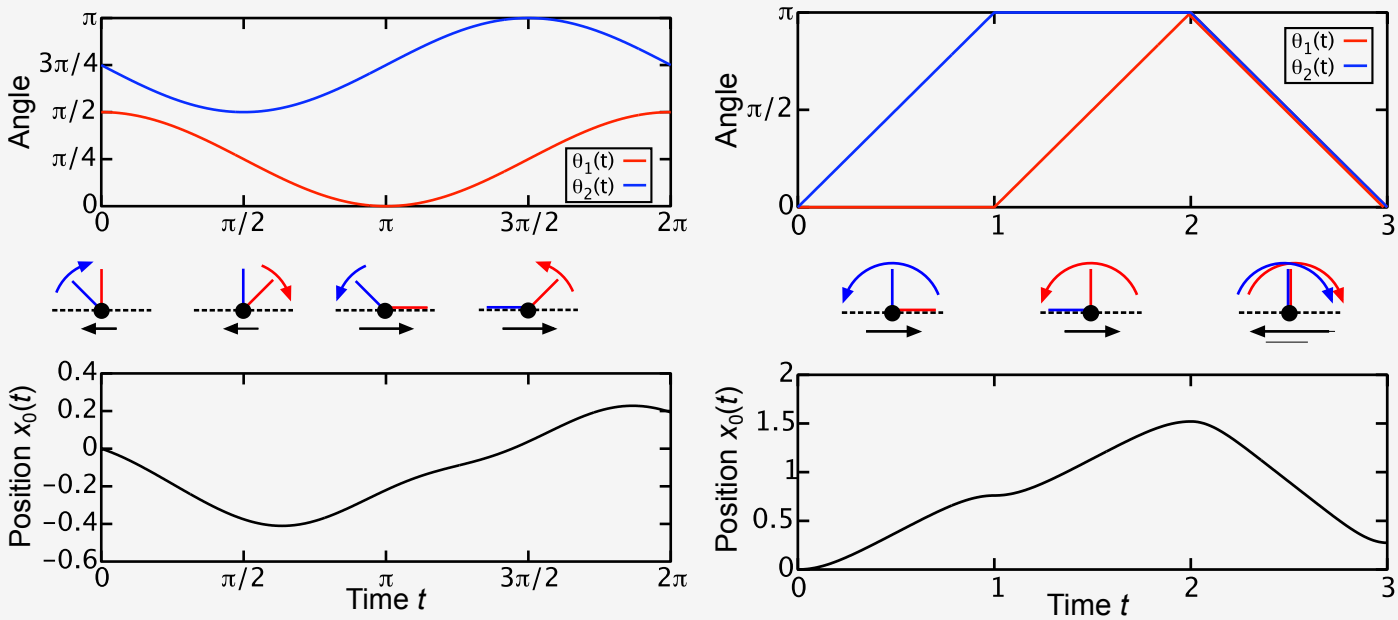


Dynamic.

$$\dot{x}_0 = \frac{\sum_{i=1}^{2} l^2\, \dot{\theta}_i \sin(\theta_i)}{2 + \sum_{i=1}^{2} \sin^2(\theta_i)}, \quad \dot{\theta}_i = u_i,\ i = 1,2 \quad (\text{state constraint} : 0 \leq \theta_1 \leq \theta_2 \leq \pi).$$

Criterion. minimize drag forces : $\dot{q} S(q) \dot{q}^{\mathsf{T}}$, $\boldsymbol{q} = (\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \boldsymbol{x_0})$ and S is positive definite $\Longrightarrow$ quadratic form in (u_1, u_2).

Definition 1. *A stroke is a periodic motion of the shape variables* (θ_1, θ_2) *associated with a* **periodic control** *producing a net displacement of the position variables after one period* T *(we can fix* $T = 2\pi$*).*



(left) Trigonometric controls. (right) Piecewise constant controls.

$$\mathscr{E}(q(\cdot)) = \textbf{Geometric Efficiency} = x_0(T)/l_{SR}(q)$$

SR Length l_{SR}.

$$l_{SR}(q) = \int_0^T \sqrt{L(q,u)}\,\mathrm{d}t, \qquad L(q,u) = a(q)\,u_1^2 + 2b(q)\,u_1 u_2 + c(q)\,u_2^2$$

$$\min_{u(\cdot)} l_{SR}(q) \Leftrightarrow \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t$$

Two steps.

$$\dot{q} = \sum_{i=1}^{2} u_i F_i \leftarrow \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t,$$

$$\max_{u(\cdot),\, x_0(T)} \mathscr{E} \Leftrightarrow \left(\min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t \quad \text{with } \boldsymbol{x_0(T)} \textbf{ fixed}, \text{ then } \underbrace{\text{select} \max_{x_0(T)} \mathscr{E}}_{\text{Transversality cond.}} \right)$$

Compute points on the Sub-Riemannian sphere $\rightarrow$
provide candidate solutions for the maximum of efficiency problem.

Introduce q^0 be the state s.t. : $\dot{q}^0 = L(q,u)$ and denote $\widetilde{q} = (q,q^0)$,
$c(\widetilde{q}(0),\widetilde{q}(T))$ is the cost function to minimize,
B defines the boundary conditions : $(\widetilde{q}(0),\widetilde{q}(T)) \in B$.

Necessary conditions. $(\widetilde{q},u)$ optimal $\Rightarrow \exists\ \widetilde{p} = (p,p^0) : [0,T] \to (\mathbb{R}^n \times \mathbb{R})$ abs. continue and $\lambda \geq 0$ cst. with $(\widetilde{p},\lambda) \neq (0,0)$ and s.t.

$$\dot{\widetilde{q}} = \frac{\partial H}{\partial \widetilde{p}}, \quad \dot{\widetilde{p}} = -\frac{\partial H}{\partial \widetilde{q}}, \quad \frac{\partial H}{\partial u} = 0$$

where $H(\widetilde{q},\widetilde{p},u) = p \cdot \dot{q} + p^0 \dot{q}^0$.
Transversality Conditions

$$(\widetilde{p}(0),-\widetilde{p}(T)) - \lambda \left(\frac{\partial c}{\partial \widetilde{q}(0)}(\widetilde{q}(0),\widetilde{q}(T)), \frac{\partial c}{\partial \widetilde{q}(T)}(\widetilde{q}(0),\widetilde{q}(T)) \right) \in (T_{(\widetilde{q}(0),\widetilde{q}(T))}B)^{\perp}$$

Definition 2. *$(\widetilde{q},\widetilde{p},\lambda,u)$ is called a* **normal** *extremal if $\lambda > 0$,* **abnormal** *if $\lambda = 0$. An abnormal trajectory $q(\cdot)$ which is not the projection of a normal extremal is called strict.*

$$\dot{q} = \sum_{i=1}^{2} u_i F_i(q) \leftarrow \min_{u(.)} q^0(T), \quad q = (\theta_1, \theta_2, x_0), \quad \boldsymbol{x_0(T)} \textbf{ fixed } .$$

$$F_i = \frac{\sin(\theta_i)}{\Delta}\frac{\partial}{\partial x_0} + \frac{\partial}{\partial \theta_i}, \qquad \Delta = 2 + \sin^2(\theta_1) + \sin^2(\theta_2)$$

PMP :

$$H(z,u) = u_1 H_1(z) + u_2 H_2(z) + p^0 L(q,u), \qquad z = (q,p)$$

with $H_i = p \cdot F_i$, $i = 1,2$.

— normal case : $p^0 = -1/2$

$$H_n = 1/2\,(a(q)H_1^2 + 2b(q)H_1H_2 + c(q)H_2^2)$$

— abnormal case : $p^0 = 0$ hence $H_1 = H_2 = \{H_1, H_2\} = 0$, then

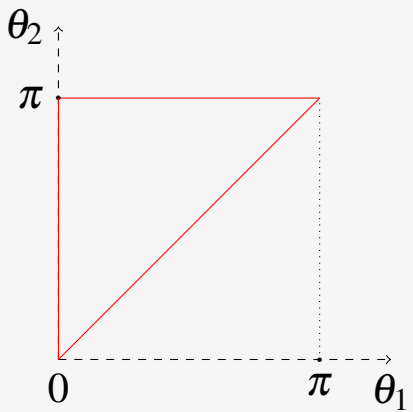
$$u_1\{\{H_1,H_2\},H_1\} + u_2\{\{H_1,H_2\},H_2\} = 0.$$

Proposition 3. *The surface* $\Sigma : \{q;\ \det(F_1(q), F_2(q), [F_1, F_2](q)) = 0\}$ *contained abnormal trajectories and is defined by*

— $\theta_i = 0$ *or* π, *for* $i = 1, 2$,

— $\theta_1 = \theta_2$.

It is formed by the **boundaries of the physical domain** *:* $\theta_i \in [0, \pi]$, $i = 1, 2, \theta_1 \leq \theta_2$.



: abnormal trajectories

in the plane (θ_1, θ_2)

Boundary conditions

$$x_0(0) = 0, \quad x_0(T) = x_f, \quad \theta_i(0) = \theta_i(T),\ i = 1,2$$

Transversality conditions

$$p_{\theta_i}(0) = p_{\theta_i}(T),\ i = 1,2$$

Boundary value problem

$$\begin{cases} \dot{z} = \overrightarrow{H}_n(z) \\ x_0(0) = 0, \quad x_0(T) = x_f \ \textbf{(fixed)} \\ \theta_i(0) = \theta_i(T),\ i = 1,2 \\ p_{\theta_i}(0) = p_{\theta_i}(T),\ i = 1,2 \end{cases},$$

— `HamPath`, *indirect method :* based on Maximum Principle, *Newton* type algorithm (sensitive to initialization) and computation of **second order necessary optimality conditions** .

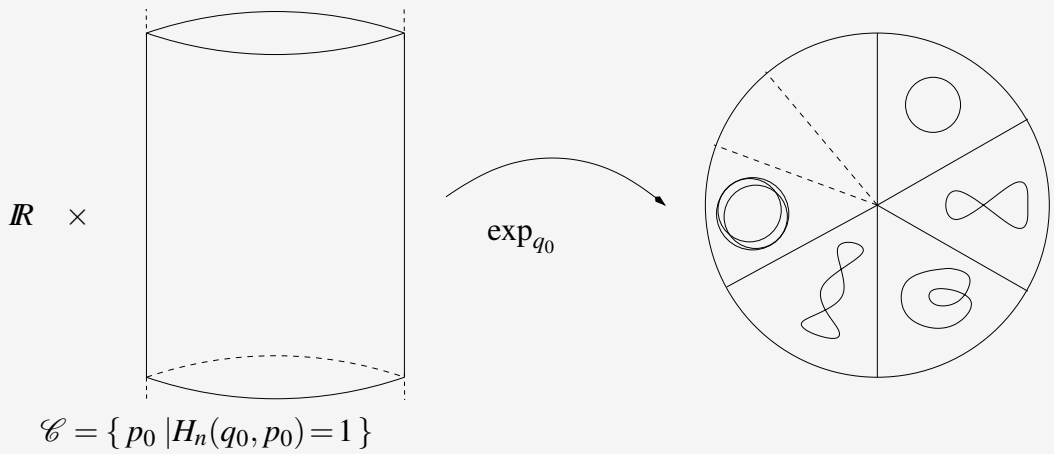
— `Bocop`, *direct method :* gives an initialization for the shooting algorithm of `HamPath`.

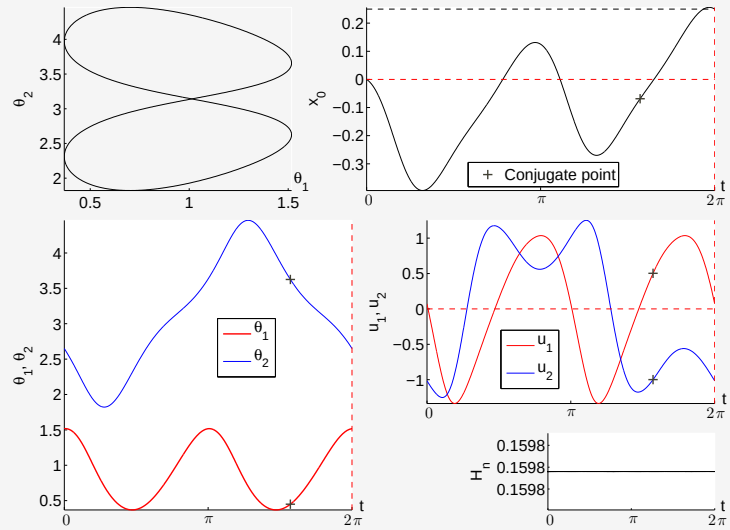
First conjugate time t_c : The first time t_c when the exponential mapping

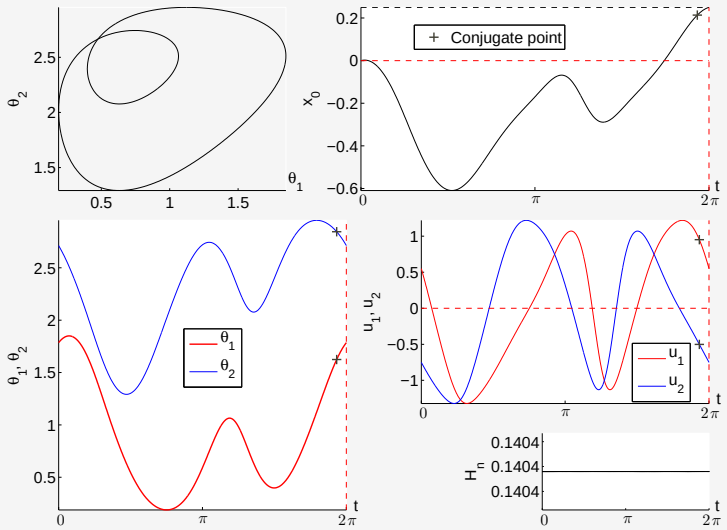
$$\exp_{q_0} : \mathbb{R} \times \mathscr{C} \to M, \quad (t, p_0) \mapsto q(t, q_0, p_0)$$

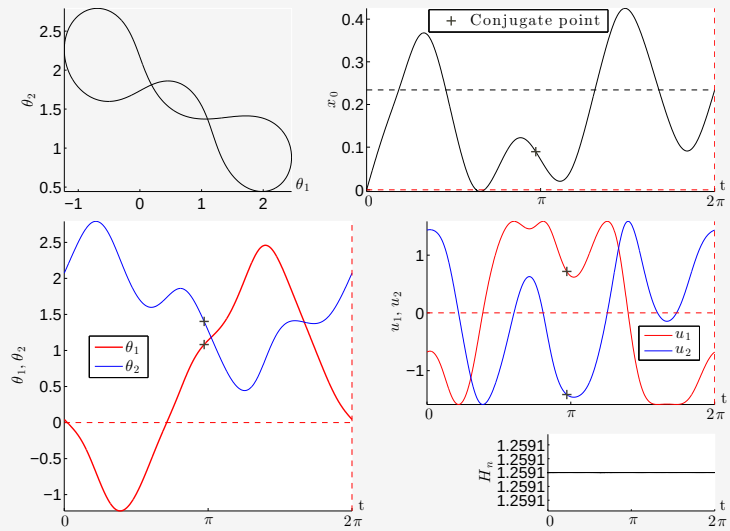
is not an immersion at (t_c, p_0).

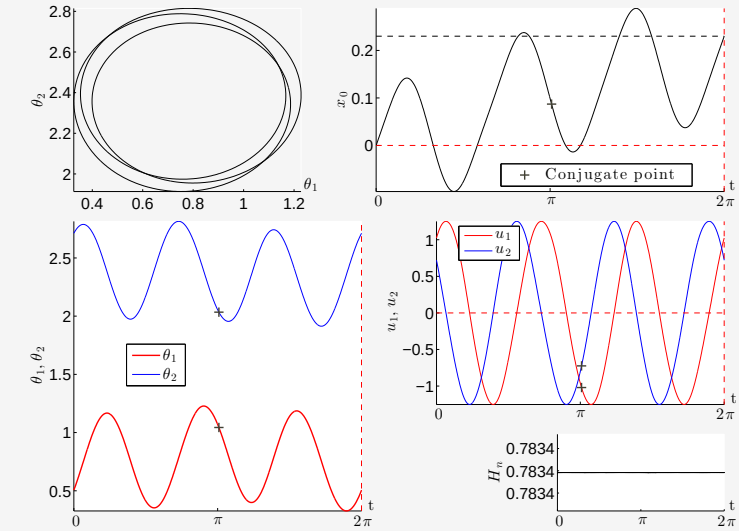
Theorem 4. *Let $q : [0,T] \longrightarrow \mathbb{R}^n$ be a strict normal stroke. If $q(.)$ has at least one conjugate point on $]0,T[$, then $q(\cdot)$* **is not a local minimizer** *for the L^∞ topology on the controls considering the optimal control problem with* **fixed extremities**.

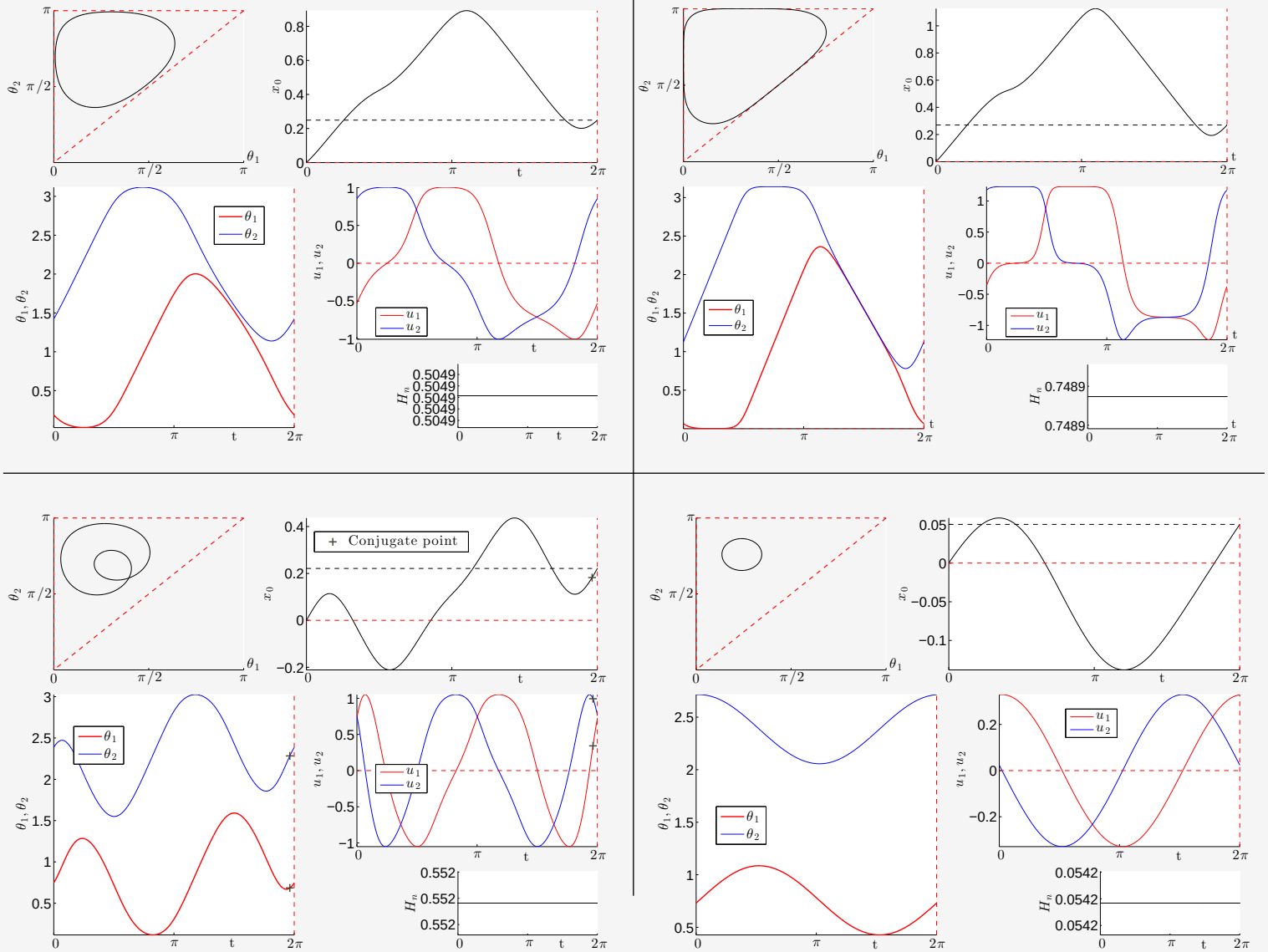



θ2
θ1
x0
t
+ Conjugate point
θ1, θ2
u1, u2
Hn
0.1598
0.1598
0.1598


θ2
θ1
x0
t
+ Conjugate point
θ1, θ2
u1, u2
Hn
0.1404
0.1404
0.1404


θ2
θ1
x0
t
+ Conjugate point
θ1, θ2
u1, u2
Hn
1.2591
1.2591
1.2591
1.2591
1.2591


θ2
θ1
x0
t
+ Conjugate point
θ1, θ2
u1, u2
Hn
0.7834
0.7834
0.7834


θ_2
θ_1
x_0
t
θ_1, θ_2
u_1, u_2
H_n
u_1
u_2
+ Conjugate point

Stokes theorem

Proposition 5. *Take the one-form on* $\mathbb{R}^2$ *:* $\omega := \sum_{i=1}^{2} \frac{\sin\theta_i}{\Delta(\theta)} \mathrm{d}\theta_i, \quad \left(F_i = \frac{\sin\theta_i}{\Delta(\theta)} \frac{\partial}{\partial\theta_i}\right).$

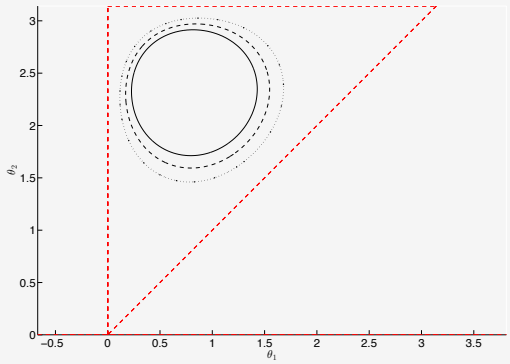
Then

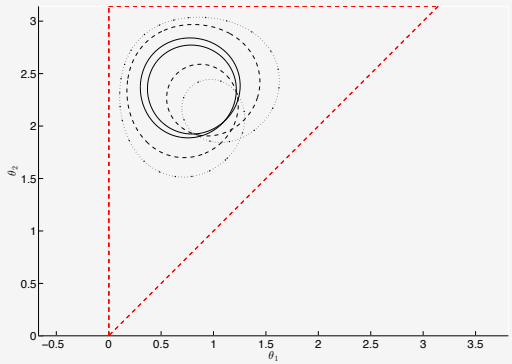
1. $\mathrm{d}\omega = -f(\theta_1, \theta_2)\,\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$, $f > 0$ *in the interior of the abnormal triangle and* $f = 0$ *on the edges of the abnormal triangle.*

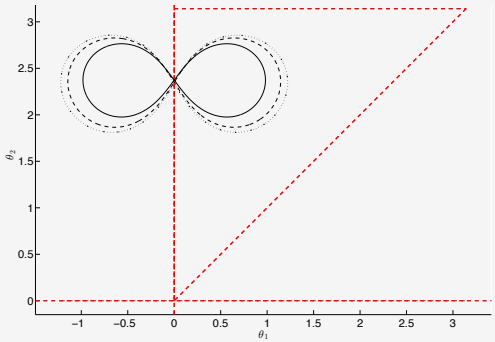
2. *for any bounded Stokes domain* $D \subset \mathbb{R}^2$*, we have*

$$x_0(T) = \oint_{\partial D} \sum_{i=1}^{2} \frac{\sin(\theta_i)}{\Delta} \mathrm{d}\theta_i = \int_D -\frac{f(\theta_1, \theta_2)}{\Delta^2} \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 = \int_D \mathrm{d}\omega.$$

Generate normal strokes

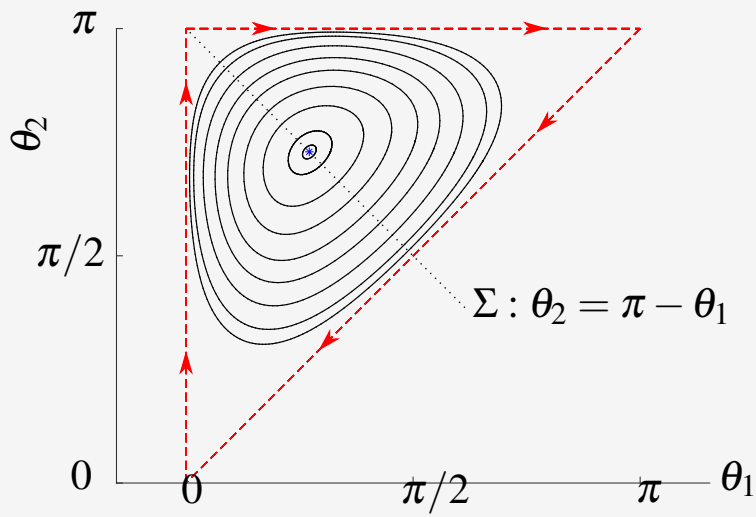






Numerical fact :
Only the simple loops don't have conjugate points on $]0, T[$. They are the only candidates for optimality.

Theorem 6. *There is a one paramater family of simple loop strokes parameterized by the displacement $x_0(T)$ or the energy level $H_n(q_0, p_0)$.*



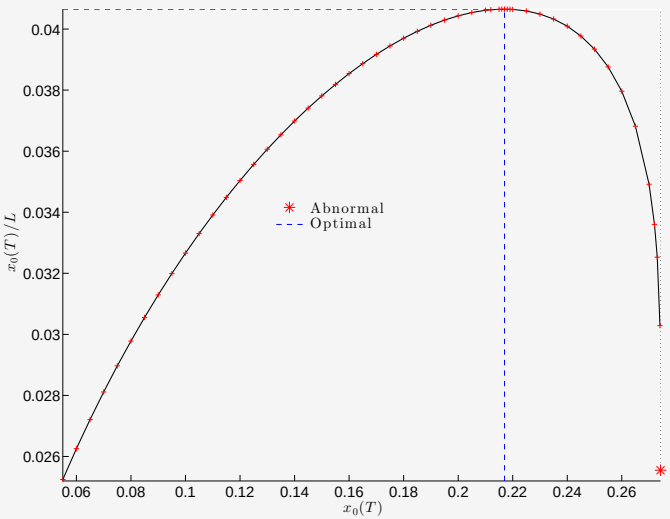
One parameter family of geodesic strokes.

Select on the point on the SR sphere with maximum of efficiency

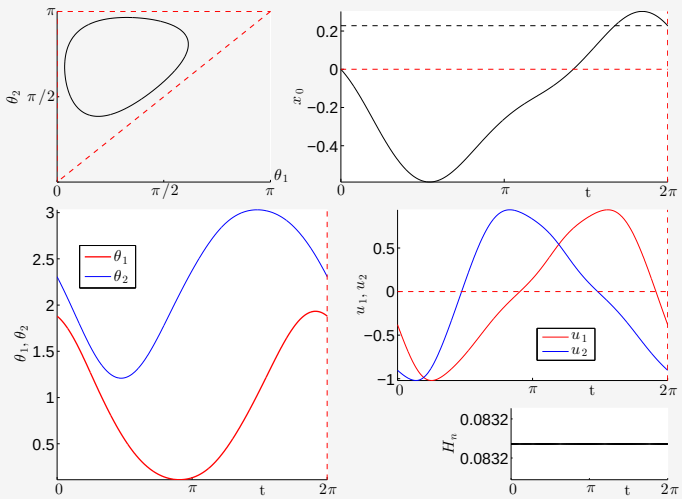
Geometric efficiency : $\mathscr{E} = x_0(T)/l_{SR}(q)$, $\quad$ **($x_0(T)$ free)**

Transversality conditions

$$p_{x_0}(T) = q^0(T)/x_0(T), \qquad p^0(T) = -1/2$$



Efficiency curve with continuation on $x_0(T)$.



Optimal normal stroke

Theorem 7. *The abnormal stroke is not minimizing for both costs (length with fixed displacement and efficiency).*

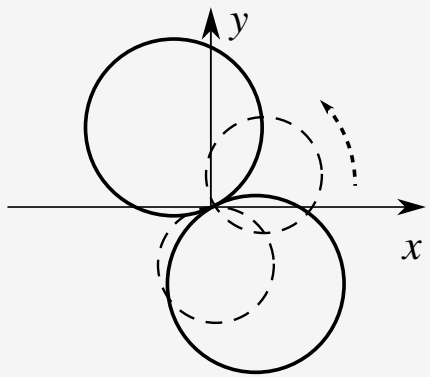
$$
\begin{aligned}
&\min_{u(\cdot)} \quad \mathscr{E}(\widetilde{q}(\cdot)), \qquad \widetilde{q} = (q, q^0) = (\theta_1, \theta_2, x_0, q^0) \\
&\dot{q} = u_1 F_1(q) + u_2 F_2(q), \\
&\dot{q}^0 = L(q, u), \\
&q^0(0) = 0, \\
&x_0(0) = 0, \quad x_0(T) \in [x_T, x_T + d] \\
&\theta_1(0) = \theta_1(T), \quad \theta_2(0) = \theta_2(T).
\end{aligned}
$$

Proposition 8. *If a minimizer* $(\overline{q}, \overline{u})$ *is abnormal, then necessarily* $\overline{x}_0(T) = x_T + d$.

Normal form of order -1 Nilpotent model of order -1, the Brockett-Heisenberg model :

$$\hat{F} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad \hat{G} = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$$

where (weight of x, y)=1 and (weight of z)=2 (privileged coordinates).
— model is **not generic** (families of simple loops geodesics)



Families of simples loops for the model of order -1.

Theorem 9 (Brockett, Chakir et al). *The model of order -1 is equivalent to the model of order 0.*

Application to the Copepod z cannot be identified to the displacement x_0.
$x = \theta_1 - \theta_{10}$, $y = \theta_2 - \theta_{20}$.

Perturbation of the model of order -1 by terms of order 0 :

$$F_1(x,y,z) = \frac{\partial}{\partial x} + (a_{11}(\theta_{10})\,xy + a_{02}(\theta_{10})\,y^2 + y)\,\frac{\partial}{\partial z}$$
$$F_2(x,y,z) = \frac{\partial}{\partial y} + (a_{11}(\theta_{10})\,xy + a_{02}(\theta_{10})\,x^2 - x)\,\frac{\partial}{\partial z}$$
$$(F_1, F_2) \quad \underset{\varphi}{\Longleftrightarrow} \quad (\hat{F}, \hat{G})$$

Lemma 10. *The only value of θ_{10} such that the transformation φ doesn't mix up the* **shape variables (x,y) and the displacement z** *corresponds to the center of swimming of the family.*

$$\varphi(x,y,z) = (x + c_{011}(\theta_{10})\,\boldsymbol{yz} + c_{001}(\theta_{10})\,\boldsymbol{z},\ y - c_{011}(\theta_{10})\,\boldsymbol{xz} + c_{001}(\theta_{10})\,\boldsymbol{z},\ z + P(x,y,z,\theta_{10}))\,,$$

$$\Rightarrow c_{011}(\theta_{10}) = c_{001}(\theta_{10}) = 0.$$

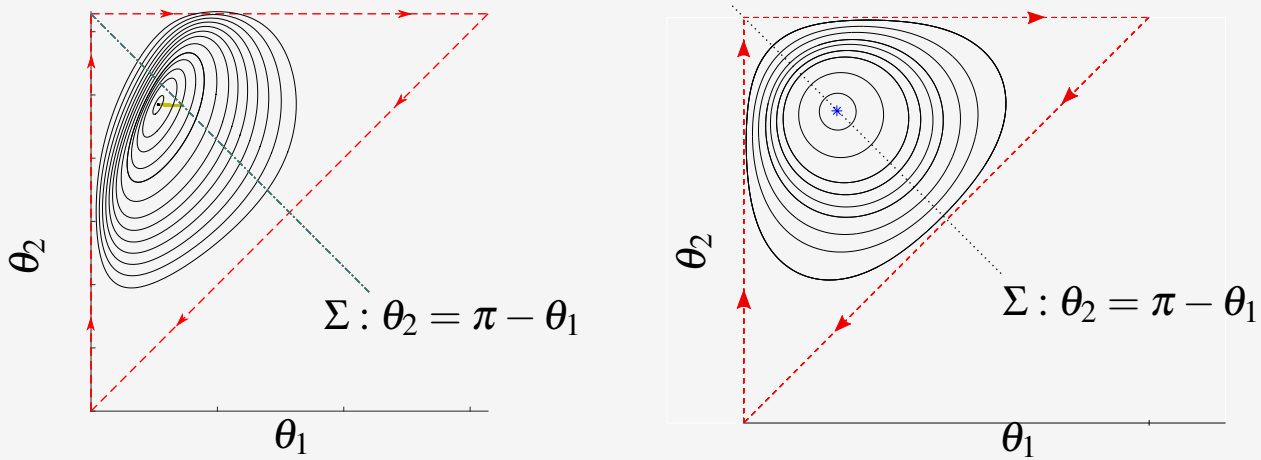
Theorem 11 (Chakir, Gauthier, Kupka, 1996)**.** *The generic model is given by the normal form of order* 1

$$F = \hat{F} + yQ(x,y)\frac{\partial}{\partial z},\ G = \hat{G} - xQ(x,y)\frac{\partial}{\partial z},$$

Q *quadratic in* (x,y).

Remark 12. *This normal form can be used to approximate the one parameter family of simple strokes.*

Different costs to minimize

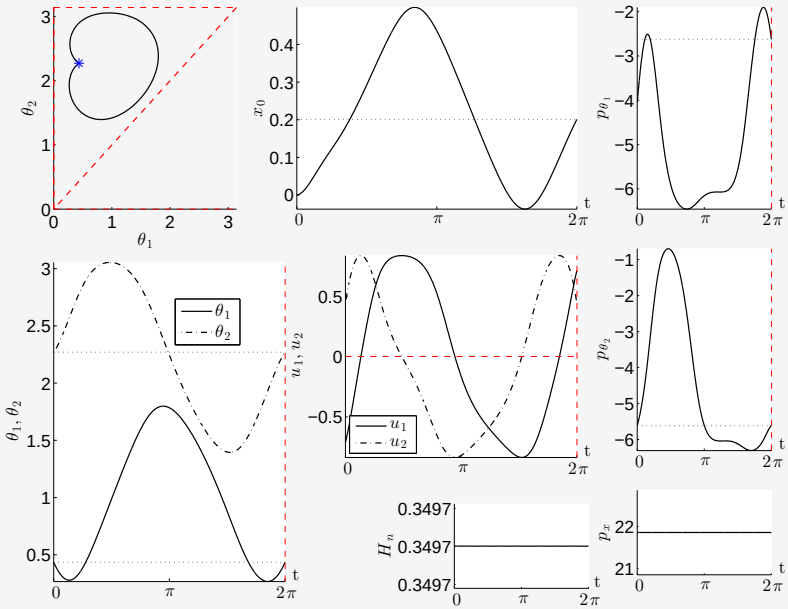


One parameter family of simple loops with different metrics.

$$\mathscr{E}' = \frac{x_0(T)\,\boldsymbol{m}(\boldsymbol{\theta}(\mathbf{0}))}{l(q)}, \quad (m \text{ smooth})$$

Transversality condition

$$p_\theta(0) - p_\theta(2\pi) = \lambda \frac{\partial \mathscr{E}'}{\partial \theta(0)}$$



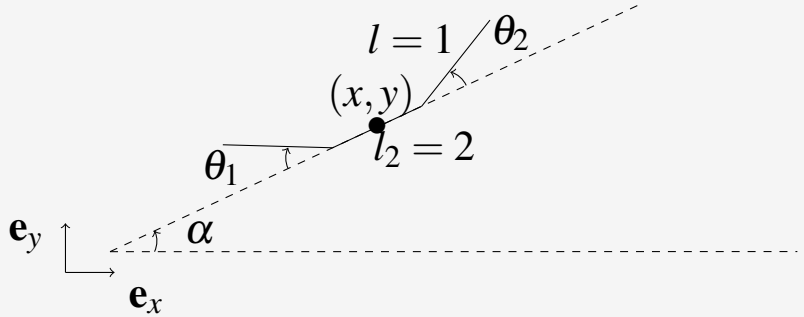
— *Contact point* : expression of the generic normal form at any point inside the triangle $\rightarrow$ unique family of simple loops.

— *Martinet point* : compute the normal form for a point on the edges $\rightarrow$ locate the eight loops.

— estimation of the first conjugate time using normal forms.

— swimmer model with more than 2 pairs of links.

The Purcell swimmer



Dynamic.

$$\dot{q} = D(\alpha)G(\theta)\dot{\theta}, \qquad D(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\dot{\theta} = H(\theta)\tau, \qquad \tau : \text{torque}, \quad \theta = (\theta_1, \theta_2), \quad q = (x, y, \alpha).$$

(G, H algebraic fractions of trigonometric polynomials).

Min. work of the drag forces $E(u) = \int_0^T (uH^{-1}u)\mathrm{d}t$.

Controls $u = \dot{\theta}$

$$\dot{X} = u_1 F_1(X) + u_2 F_2(X), \quad X = (\theta_1, \theta_2, x, y, \alpha)$$

Theorem 13. *Nilpotent approx. for* (F_1, F_2)

$$\hat{F}_1 = \frac{\partial}{\partial x_1}, \quad \hat{F}_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} + x_1^2 \frac{\partial}{\partial x_5}.$$

$$\boldsymbol{\varphi(\theta_1, \theta_2, x, y, \alpha) = (x_1, x_2, x_3, x_4, x_5) = X}, \quad \boldsymbol{\varphi} \ t.q. \ \varphi_* F_i = \hat{F}_i + O(|X|^3)$$

Cartan flat model

$$\dot{X} = \sum_{i=1}^{2} u_i \hat{F}_i(X), \quad \min_u \int_0^T (u_1^2 + u_2^2)\, \mathrm{d}t.$$

PMP (x, u) optimal $\Longrightarrow$ projection of a solution of

$$H_n = 1/2\,(H_1^2 + H_2^2), \quad H_i = p \cdot \hat{F}_i, \ i = 1, 2$$

Hamiltonians lifts

$$H_1 = p \cdot \hat{F}_1(X) = p_1, \qquad H_2 = p \cdot \hat{F}_2(X) = p_2 + p_3 x_1 + p_4 x_3 + p_5 x_1^2,$$
$$H_3 = p \cdot [\hat{F}_1, \hat{F}_2](X) = -p_3 - 2x_1 p_5, \quad H_4 = p \cdot [[\hat{F}_1, \hat{F}_2], \hat{F}_1](X) = -2p_5,$$
$$H_5 = p \cdot [[\hat{F}_1, \hat{F}_2], \hat{F}_2](X) = p_4.$$

Poincaré coordinates : $(H_1, H_2, H_3, H_4, H_5)$

$$\begin{aligned}
&\dot{H}_1 = \mathrm{d}H_1(\overrightarrow{H}_n) = \{H_1, H_2\}H_2 = p \cdot [\hat{F}_1, \hat{F}_2]\, H_2 = H_2 H_3, \\
&\dot{H}_2 = -H_3 H_1, \qquad \dot{H}_3 = H_1 H_4 + H_2 H_5, \\
&\dot{H}_4 = 0 \quad \text{i.e.} \quad H_4 = c_4, \quad \dot{H}_5 = 0 \quad \text{i.e.} \quad H_5 = c_5.
\end{aligned}$$

Fix the energy level : $H_1^2 + H_2^2 = 1$, and let $H_1 = \cos(\vartheta)$ and $H_2 = \sin(\vartheta)$.

$$\dot{H}_1 = -\sin(\vartheta)\dot{\vartheta} = H_2 H_3 = \sin(\vartheta) H_3.$$

Hence $\dot{\vartheta} = -H_3$ and

$$\ddot{\vartheta} = -(H_1 c_4 + H_2 c_5) = -c_4 \cos(\vartheta) - c_5 \sin(\vartheta) = -\omega^2 \sin(\vartheta + \phi)$$

where ω, ϕ are constants and satisfy $\omega^2 \sin(\phi) = c_4$ and $\omega^2 \cos(\phi) = c_5$.
Then, let $\psi = \vartheta + \phi$, we have

$$1/2\dot{\psi}^2 - \omega^2 \cos(\psi) = B, \quad B \text{ is constant.}$$

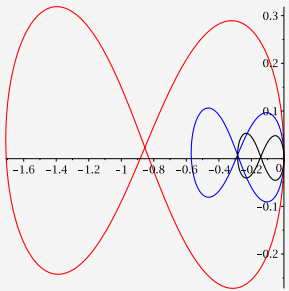
Theorem 14. *The normal extremals associated to the Cartan flat model is* **integrable** *and solutions can be expressed as polynomials of elliptic functions of the 1st and 2nd kind* $(u, \mathrm{sn}(u), \mathrm{cn}(u), \mathrm{dn}(u), E(u))$.

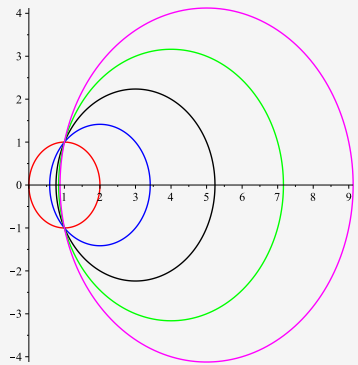
Oscillating case

$$x_1(u) = 1/\omega\left[x_1(\varphi_0) - 2k\sin(\phi)\,\mathrm{cn}(u,k) + (-u + 2E(u,k))\cos(\phi)\right],$$
$$x_2(u) = 1/\omega\left[x_2(\varphi_0) - 2k\cos(\phi)\,\mathrm{cn}(u,k) + (u - 2E(u,k))\sin(\phi)\right]$$

Rotating case

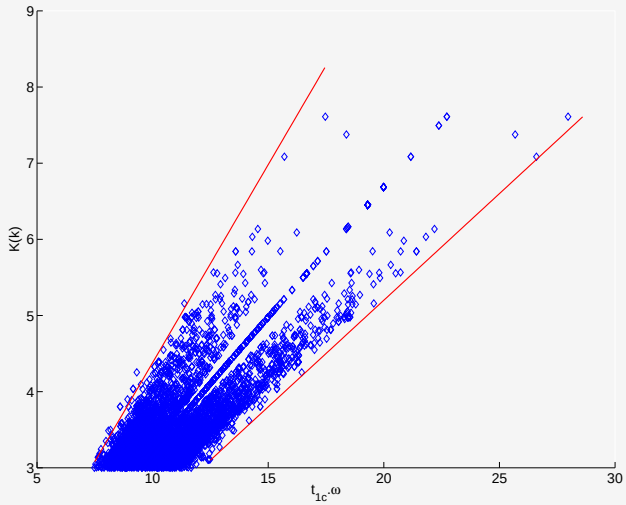
$$x_1(u) = \left(-2\cos(\phi)\,u + 2\cos(\phi)\,\mathrm{E}(u/k,k)\,k - 2\sin(\phi)\,\mathrm{dn}(u/k,k)\,k + \cos(\phi)\,uk^2 + x_1(\varphi_0)\,k^2\right)\omega^{-1}k^{-2},$$
$$x_2(u) = \left(2\sin(\phi)\,u - 2\sin(\phi)\,\mathrm{E}(u/k,k)\,k - 2\cos(\phi)\,\mathrm{dn}(u/k,k)\,k - \sin(\phi)\,uk^2 + x_2(\varphi_0)\,k^2\right)\omega^{-1}k^{-2}$$



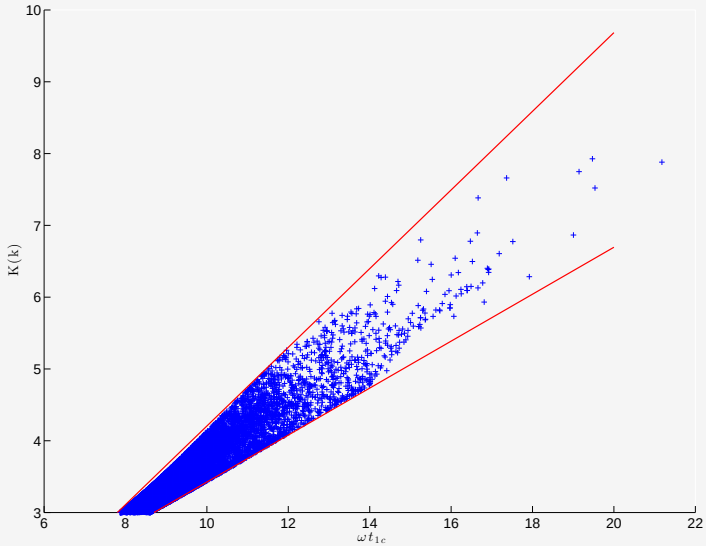


(left) Oscillating case, (right) Rotating case

Consider a family of normal extremals parameterized by $\hat{p}(0)$, we compute numerically the first conjugate time.



Oscillating case

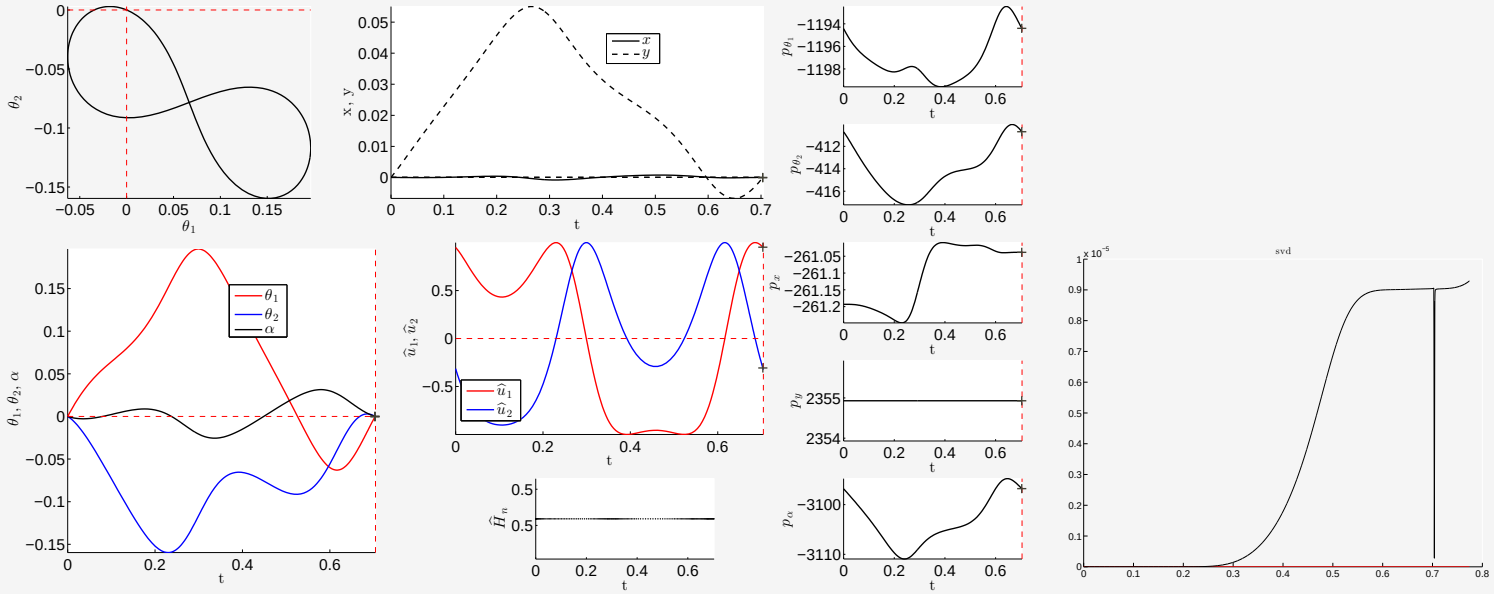


Rotating case

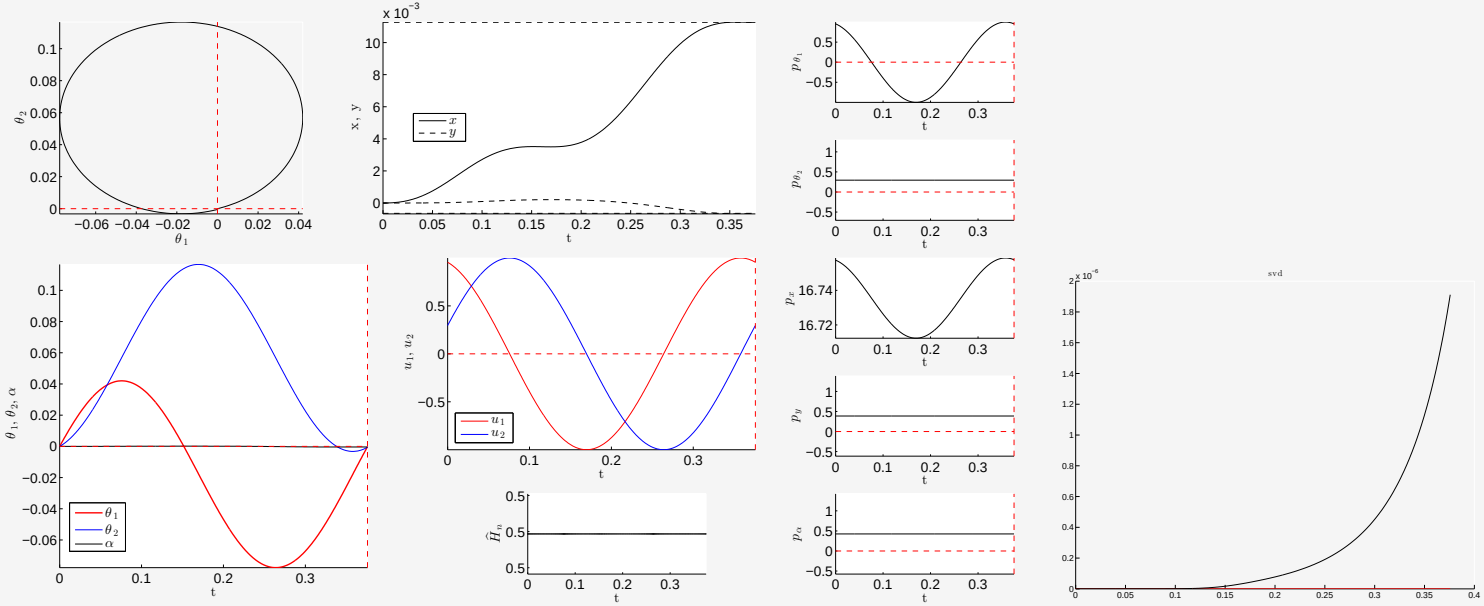
Numerical fact : Conjugate times are located in a sector.

$$0.34\,\omega t_{1c} - 0.4 < K(k) < 0.53\,\omega t_{1c} - 0.8 \quad : \text{Oscillating case,}$$
$$0.33\,\omega t_{1c} + 0.16 < K(k) < 0.55\,\omega t_{1c} - 1.27 \quad : \text{Rotating case.}$$

Initialization of algorithms from the microlocal analytical solutions



(left) Control, state and adjoint of the physical variables (θ, x, y, α) for the **oscillating case** and for the nilpotent system : stroke with **eight shape**,
(right) SVD of the rank condition to compute conjugate points.



(left) Control, state and adjoint of the physical variables (θ, x, y, α) for the **rotating case** and for the nilpotent system : **simple loop stroke**,
(right) SVD of the rank condition to compute conjugate points.

Proposition 15 (Numerical)**.** *For the Cartan problem,* **simple loops** *(rotating case) are the only strokes without conjugate points.*

— nilpotent model $\rightarrow$ give an initial adjoint vector $\hat{p}(0)$ to initialize strokes with small amplitudes associated with the true dynamic and to the cost $\int_0^T (u_1^2 + u_2^2)\,\mathrm{d}t$.

Boundary value problem

$$\begin{cases} \dot{q} = \frac{\partial \widetilde{H}_n}{\partial p}, \quad \dot{p} = -\frac{\partial \widetilde{H}_n}{\partial q}, \\ \theta_j(T) = \theta_j(0) \quad j = 1,2, \\ x(0) = y(0) = \alpha(0) = 0, \\ x(T)^2 + y(T)^2 = r_1,\ \alpha(T) = 0, \\ p_{\theta_j}(T) = p_{\theta_j}(0),\ j = 1,2, \quad p_\alpha(0) = p_\alpha(T) \end{cases}$$

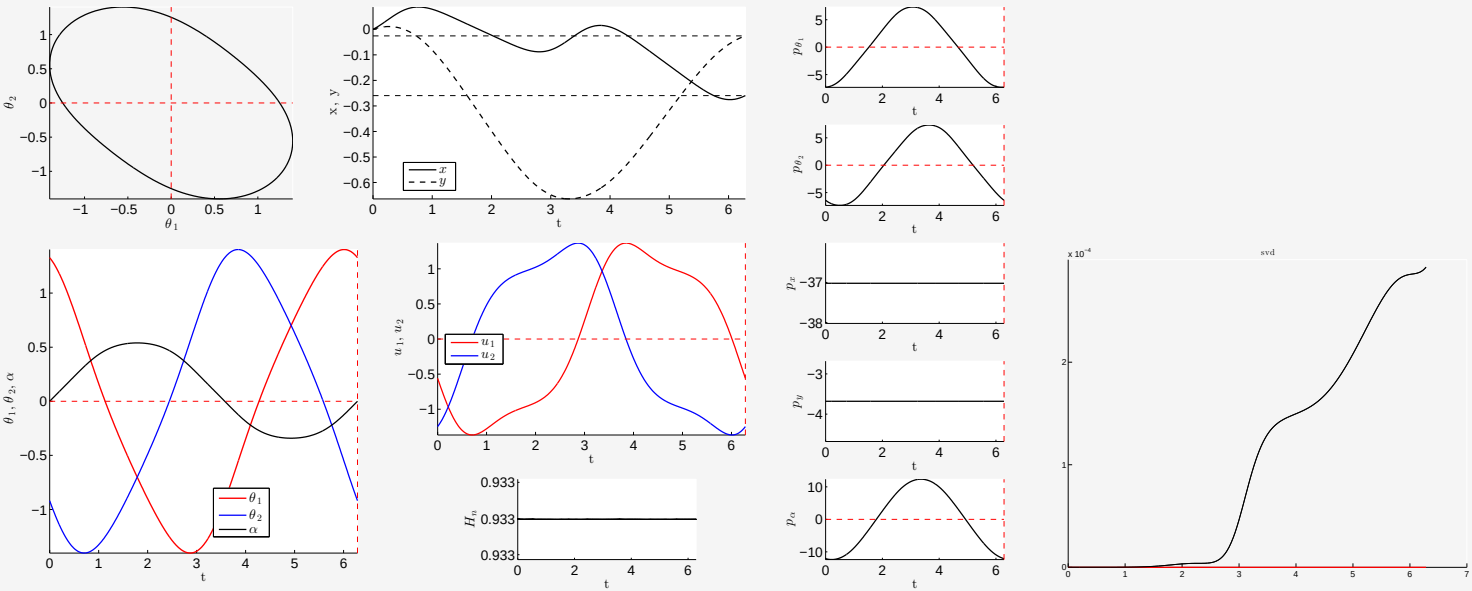
$\widetilde{H}_n$: Hamiltonian associated with the cost $\int_0^T (u_1^2 + u_2^2)\,\mathrm{d}t$.

— **2 discrete continuations** :

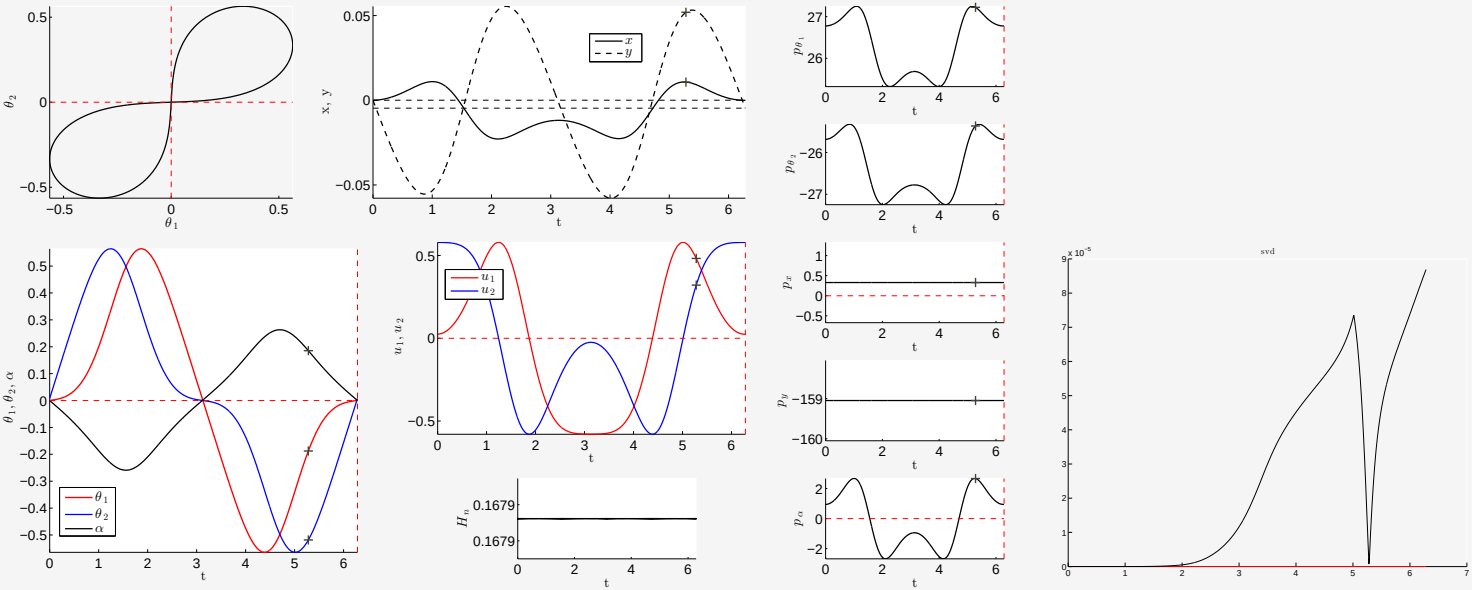
— on the radius r_1

— from the cost $\int_0^T (u_1^2 + u_2^2)\,\mathrm{d}t$ to the mechanical cost.

Discrete continuation on the amplitudes of the strokes



(left) Normal stroke with simple loop for the true dynamic, minimizing the cost $\int_0^T (u_1^2 + u_2^2)\mathrm{d}t$ $(T = 2\pi,\ r_1 = 0.068)$ *(right)* Test for the conjugate points : no conjugate points on $[0, 2\pi]$).

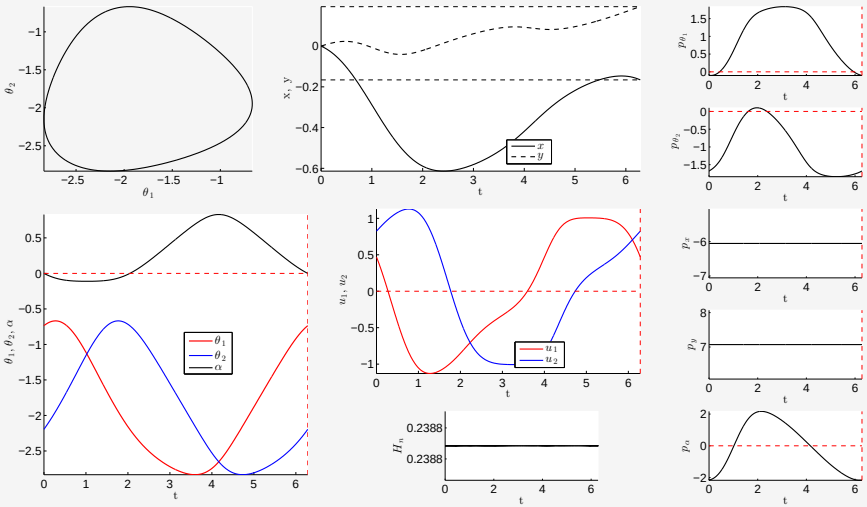


(left) Normal stroke with eight shape loop for the true dynamic, minimizing the cost $\int_0^T (u_1^2 + u_2^2)\mathrm{d}t$ $(T = 2\pi,\ r_1 = 4.6e{-}4)$ *(right)* Test for the conjugate points.

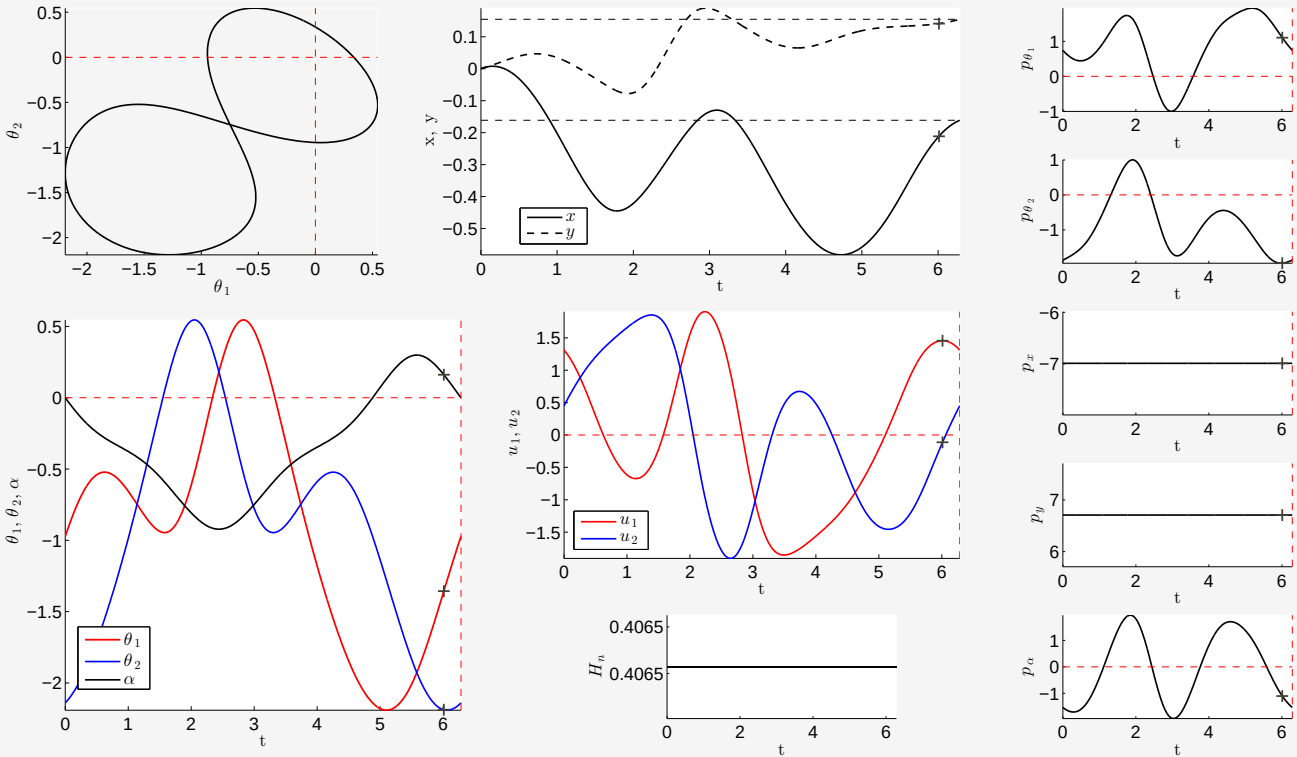
Discrete continuation on the cost

- $\widetilde{H}_n$: normal Hamiltonian for the cost $\int_0^T u_1^2 + u_2^2 \, \mathrm{d}t$ **cost**,
- H_n : normal Hamiltonian for the mechanical cost.

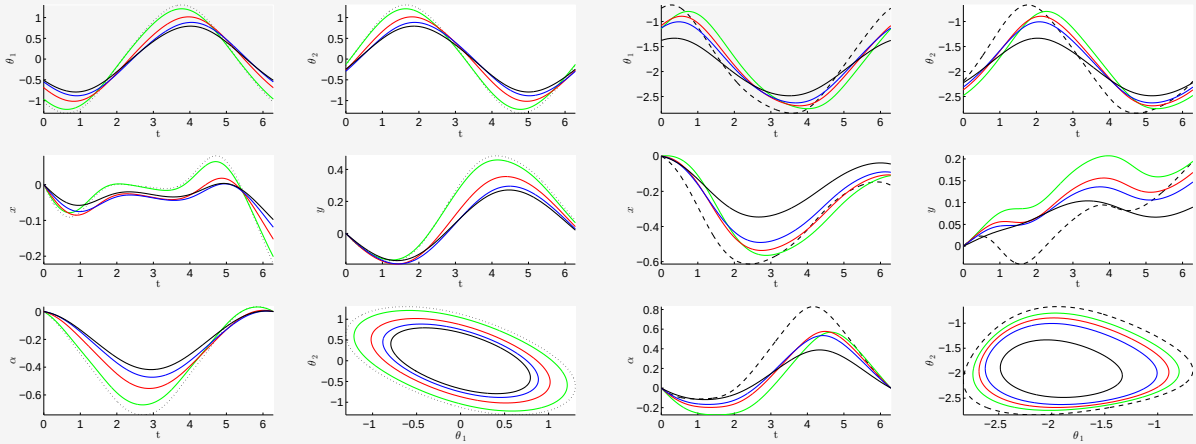
$$H = \lambda H_n + (1-\lambda)\widetilde{H}_n, \ \lambda \in [0,1].$$



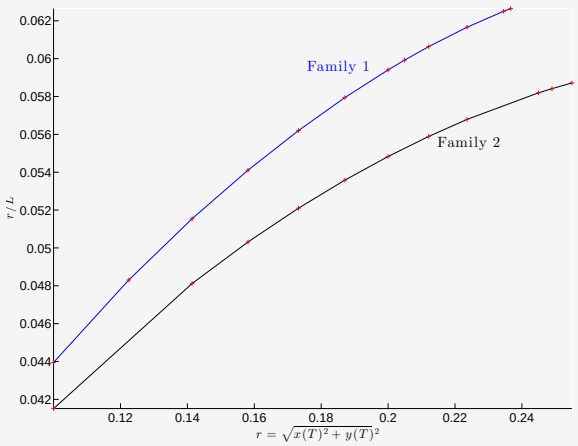
Stroke for the Purcell swimmer minimizing the mechanical energy ($T = 2\pi$, $r_1 = 0.065$).



States and adjoints for the Purcell swimmer (mechanical cost)



Two families of normal strokes for the Purcell swimmer.



Geometric efficiency :

$$\mathscr{E}(\gamma) = \sqrt{x(T)^2 + y(T)^2} / l_{SR}(\gamma)$$

Used to compare family of simple loops.

$$\begin{cases} \min & J(q(.),u(.)) := c(q(0),q(T)) \\ \text{sb. to} & \\ & \dot{q}(t) \;=\; F(q(t),u(t)) \\ & u(t) \;\in\; U \\ & m(q(0),q(T)) = 0\,, \end{cases}$$

$H(q,u,p) := p \cdot F(q,u)$ and $h(q_0,q_T) := c(q_0,q_T) + \nu \cdot m(q_0,q_T)$.
Consider a **normal extremal** $(\overline{q},\overline{p})$ associated to u.

Second variation

$$\delta^2 J(\delta q(.),\delta u(.)) := 1/2\,[\delta q(0)^\intercal \quad \delta q(T)^\intercal]\,\boldsymbol{C}\,[\delta q(0) \quad \delta q(T)]^\intercal$$
$$+\,1/2\int_0^T (\delta q(t)^\intercal \boldsymbol{\partial}_{qq}\boldsymbol{H}(t)\delta q(t) + 2\,\delta q(t)^\intercal \boldsymbol{\partial}_{qu}\boldsymbol{H}(t)\delta u(t) + \delta u(t)^\intercal \boldsymbol{\partial}_{uu}\boldsymbol{H}(t)\delta u(t))\,\mathrm{d}t$$

where $\quad \boldsymbol{C} := \mathrm{Hessian}_{q_0,q_T}(h)$.

$$\begin{cases} \min & \delta^2 J(\delta q(.), \delta u(.)) \\ \text{sous} & \\ & \dot{\delta q} := \boldsymbol{\partial}_q \boldsymbol{F}(t)\, \delta q(t) + \boldsymbol{\partial}_u \boldsymbol{F}(t)\, \delta u(t) \\ & \nabla_{q_0} m(\overline{q}(0), \overline{q}(T))\, \delta q(0) + \nabla_{q_T} m(\overline{q}(0), \overline{q}(T))\, \delta q(T) = 0\,, \end{cases}$$

Monodromy matrix. $\Phi(.,.)$ associated with the linearized Hamiltonian system :

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,s) = \boldsymbol{Z}\Phi(t,s) \\ \Phi(s,s) = \mathrm{Id}, \end{cases}$$

where

$$\boldsymbol{Z} := \begin{bmatrix} \boldsymbol{\partial_q F} - \boldsymbol{\partial_u F}[\boldsymbol{\partial}_{uu}\boldsymbol{H}]^{-1}\boldsymbol{\partial}_{qu}\boldsymbol{H}^T & -\boldsymbol{\partial_u F}[\boldsymbol{\partial}_{uu}\boldsymbol{H}]^{-1}\boldsymbol{\partial_u F}^T \\ -\boldsymbol{\partial}_{qq}\boldsymbol{H} + \boldsymbol{\partial}_{qu}\boldsymbol{H}[\boldsymbol{\partial}_{uu}\boldsymbol{H}]^{-1}\boldsymbol{\partial}_{qu}\boldsymbol{H}^T & -\boldsymbol{\partial}_q\boldsymbol{F}[\boldsymbol{\partial}_{uu}\boldsymbol{H}]^{-1}\boldsymbol{\partial}_u\boldsymbol{F}^T \end{bmatrix}.$$

Define

$$\mathscr{W} := \begin{bmatrix} \phi_{22}\phi_{12}^{-1} & \phi_{21} - \phi_{22}\phi_{12}^{-1}\phi_{11} \\ -\phi_{12} & \phi_{12}^{-1}\phi_{11} \end{bmatrix}, \quad \Phi(0,T) =: \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

Theorem 16 (Standard conditions). *Assume*

(i) $\boldsymbol{\partial}_{uu}\boldsymbol{H}(t) \leq -\varepsilon Id$ *on* $[0,T]$, $\overline{u}(.)$ *bounded and* $(\boldsymbol{\partial_q F}(.), \boldsymbol{\partial_u F}(.))$ *is controllable on* $[0,T]$,

(ii) : the extremal $(\overline{q}(.), \overline{u}(.), p(.))$ **doesn't have conjugate points** *on* $[0,T]$,

(iii) : there exists $\gamma > 0$ *t.q.*

$$\left[\xi_0^T \ \xi_1^T\right] \mathscr{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right|^2,$$

for all vectors $\xi_0, \xi_1 \in \mathbb{R}^n \setminus \{0\}$ *s.t.*

$$\nabla_{q_0} m((\overline{q}(0), \overline{q}(T)))\xi_0 + \nabla_{q_T} m((\overline{q}(0), \overline{q}(T)))\xi_1 = 0.$$

Then $(\overline{q}(.), \overline{u}(.))$ *is a* $W^{1,\infty}$*-minimizer and locally* **unique**.

Boundary values

$$\theta_j(0) = \theta_j(T) \quad j = 1,2, \quad \alpha(0) = \alpha(T),$$
$$x(0) = 0,\ y(0) = 0,\ x(T)^2 + y(T)^2 = r \text{ (r is fixed) }.$$

Proposition 17. *Take $a = (\phi, \sigma) \in I = (-\varepsilon, \varepsilon)^2$ for $\varepsilon > 0$. $(\overline{q} = (\overline{\theta}_1, \overline{\theta}_2, \overline{x}, \overline{y}, \overline{\alpha})$ and $\overline{u} = (\overline{u}_1, \overline{u}_2)$ a* **normal extremal**.
For all $a \in I$ and $t \in [0,T]$, define $q^a(.) = (\theta_1^a(.), \theta_2^a(.), x^a(.), y^a(.), \alpha^a(.))$ where

$$u_j^a(t) = \overline{u}_j(t+\sigma),\ \theta_j^a(t) = \overline{\theta}_j(t+\sigma) \quad j = 1,2,$$
$$x^a(t) = \cos(\phi)\,(\overline{x}(t+\sigma) - \overline{x}(\sigma)) - \sin(\phi)\,(\overline{y}(t+\sigma) - \overline{y}(\sigma)),$$
$$y^a(t) = \sin(\phi)\,(\overline{x}(t+\sigma) - \overline{x}(\sigma)) + \cos(\phi)\,(\overline{y}(t+\sigma) - \overline{y}(\sigma)),$$
$$\alpha^a(t) = \overline{\alpha}(t+\sigma) + \phi.$$

Then $(\overline{q}(.), p(.), \overline{u}(.))$ is continuously embedded in a family of extremals $(q^a(.), p^a(.), u^a(.))_{a \in I}$ where $p^a(.)$ is the adjoint vector associated with $(q^a(.), u^a(.))$.

$\Longrightarrow$ **Standard conditions fail**

Theorem 18 (Refined Conditions, Gavriel, Vinter (2014)). *Assume the reference normal extremal* $(\overline{q}(.), \overline{u}(.), p(.))$ *is embedded in a family of extremals and*

(i) $\boldsymbol{\partial}_{uu}\boldsymbol{H} \leq -\varepsilon Id$ *on* $[0,T]$, $\overline{u}(.)$ *is bounded and* $(\boldsymbol{\partial}_{\boldsymbol{q}}\boldsymbol{F}(.), \boldsymbol{\partial}_{\boldsymbol{u}}\boldsymbol{F}(.))$ *is controllable on* $[0,T]$,

(ii) : the extremal $(\overline{q}(.), \overline{u}(.), p(.))$ **doesn't have conjugate points** *on* $[0,T]$,

(iii) : there exists $\gamma > 0$ *s.t.*

$$\left[\xi_0^T \ \xi_1^T\right] \mathscr{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right|^2 ,$$

for all vectors $\xi_0, \xi_1 \in \mathbb{R}^n \setminus \{0\}$ *s.t.*

$$\nabla_{q_0} m((\overline{q}(0), \overline{q}(T)))\xi_0 + \nabla_{q_T} m((\overline{q}(0), \overline{q}(T)))\xi_1 = 0 \qquad \textit{and} \qquad \boldsymbol{\Gamma}^T \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = 0 \ .$$

where $\Gamma := \left. \begin{bmatrix} D_\alpha q^\alpha(0) \\ D_\alpha q^\alpha(T) \end{bmatrix} \right|_{\alpha=0}$.

Then $(\overline{q}(.), \overline{u}(.))$ *is a* **local** $W^{1,\infty}$**-minimizer**.

Define the matrix N_s from the subspace $\mathscr{L}_s$ s.t.

$$\mathscr{L}_s = \{\, (\xi_0, \xi_T) \in \mathbb{R}^5 \times \mathbb{R}^5 \mid \nabla_{q_0,q_T} m(q_0, q_T)\, (\xi_0 \quad \xi_T)^\intercal = 0 \,\} =: \mathrm{Im}(N_s)$$

Standard conditions. Does the matrix $\mathscr{W}_s := \underbrace{N_s^\intercal\, (\mathscr{W}^\intercal + \mathscr{W})\, N_s}_{4\times 4}$ is positive-definite ?

Consider

$$\Gamma_r = \begin{pmatrix} \nabla_\phi q^a(0) & \nabla_\phi q^a(T) \\ \nabla_\sigma q^a(0) & \nabla_\sigma q^a(T) \end{pmatrix}_{\phi=0,\sigma=0}$$

and the linear subspace $\mathscr{L}_r$ s.t.

$$\mathscr{L}_r := \mathscr{L}_s \cap \{(\xi_0, \xi_T) \in \mathbb{R}^5 \times \mathbb{R}^5 \mid \Gamma_r^\intercal\, (\xi_0 \quad \xi_T)^\intercal = 0\} =: \mathrm{Im}(N_r)$$

Refined conditions. Does the matrix $\mathscr{W}_r := \underbrace{N_r^\intercal\, (\mathscr{W}^\intercal + \mathscr{W})\, N_r}_{2\times 2}$ is positive-definite ?

Relative tolerance	(Standard Conditions) Spec($\mathscr{W}_s$)	(Refined Conditions) Spec($\mathscr{W}_r$)
10^{-4}	1319.91 3.44629 -2.61575e-5 -4.17860e-3	36179.7 13.8018
10^{-7}	1320.17 3.44676 9.81190e-6 -5.40128e-6	36186.9 13.8037

— Standard conditions fail : $\mathscr{W}_s$ has zero eigenvalues.
— Refined conditions are satisfied : $\mathscr{W}_r$ is positive-definite.

Theorem 19 (Numerical)**.** *The simple loop normal stroke* $(\overline{q}, \overline{u})$ *is* $W^{1,\infty} - optimal$.