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# Optimal Control Theory and the Efficiency of the Swimming Mechanism of the Copepod Zooplankton

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# Optimal Control Theory and the Efficiency of the Swimming Mechanism of the Copepod Zooplankton

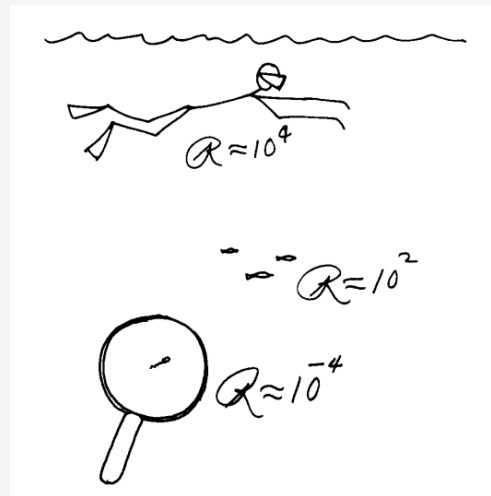
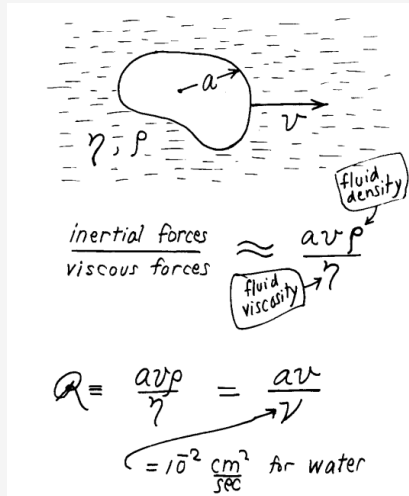
Jérémy Rouot, LAAS-CNRS, Toulouse

with Piernicola Bettiol (LBMA, Brest) and Bernard Bonnard (INRIA, Sophia Antipolis).

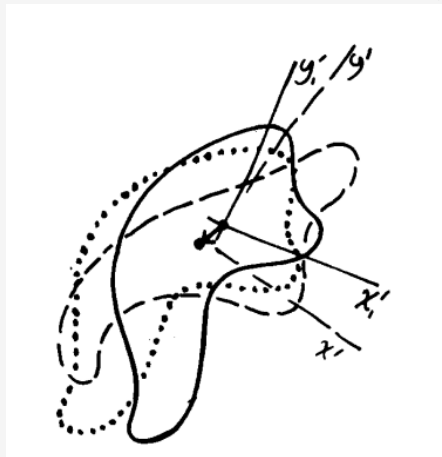
2017, 10<sup>th</sup> July

**IFAC 2017 World Congress, Toulouse, France**

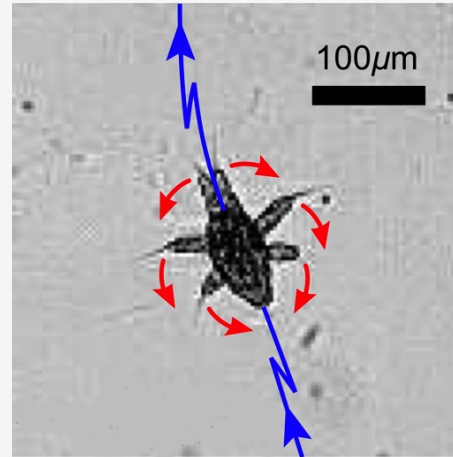
# Life at low Reynolds number - Purcell, 1977



Reynolds number

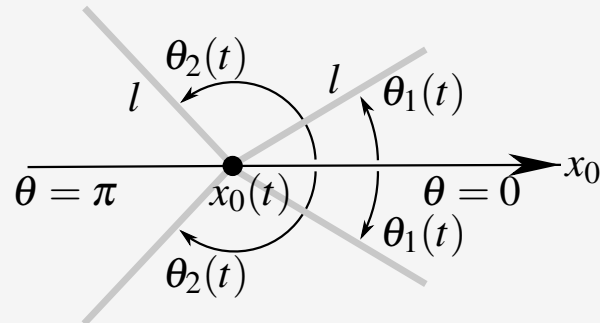


Shape deformations of a microswimmer



Zooplankton

## 2-link symmetric swimmer.

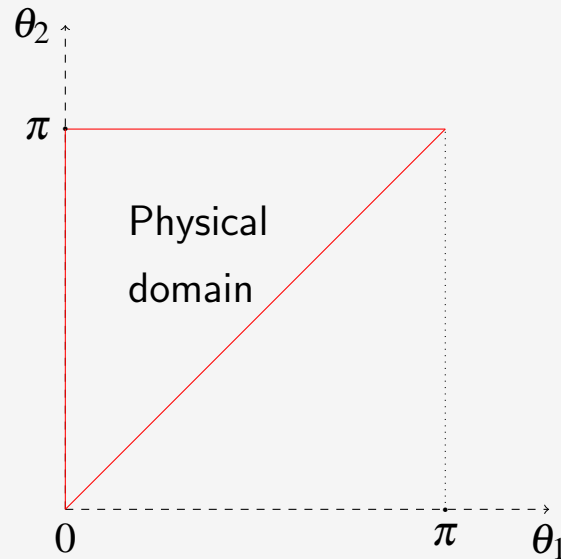


## Dynamic.

$$\dot{x}_0 = \frac{\sum_{i=1}^2 l^2 \dot{\theta}_i \sin(\theta_i)}{2 + \sum_{i=1}^2 \sin^2(\theta_i)}, \quad \dot{\theta}_i = u_i, \quad i = 1, 2 \quad (\text{state constraint : } 0 \leq \theta_1 \leq \theta_2 \leq \pi).$$

**Criterion.** minimize drag forces :  $\dot{q}S(q)\dot{q}^\top$ ,  $\mathbf{q} = (\theta_1, \theta_2, x_0)$  and  $S$  is positive definite  $\implies$  quadratic form in  $(u_1, u_2)$ .

- Compute **optimal strokes** minimizing an efficiency.
- Stroke :  **$T$ -periodic motion of the shape variables  $\theta$**  s.t.  $x_0(T) - x_0(0) > 0$ .
- Admissible stroke : find closed curves in the  $\theta$ -plane **contained in the triangle** :



$$\mathcal{E}(q(\cdot)) = \text{Geometric Efficiency} = x_0(T)/l_{SR}(q)$$

**SR Length**  $l_{SR}$ .

$$l_{SR}(q) = \int_0^T \sqrt{L(q, u)} dt, \quad L(q, u) = a(q) u_1^2 + 2b(q) u_1 u_2 + c(q) u_2^2$$

$$\min_{u(\cdot)} l_{SR}(q) \Leftrightarrow \min_{u(\cdot)} \int_0^T L(q, u) dt$$

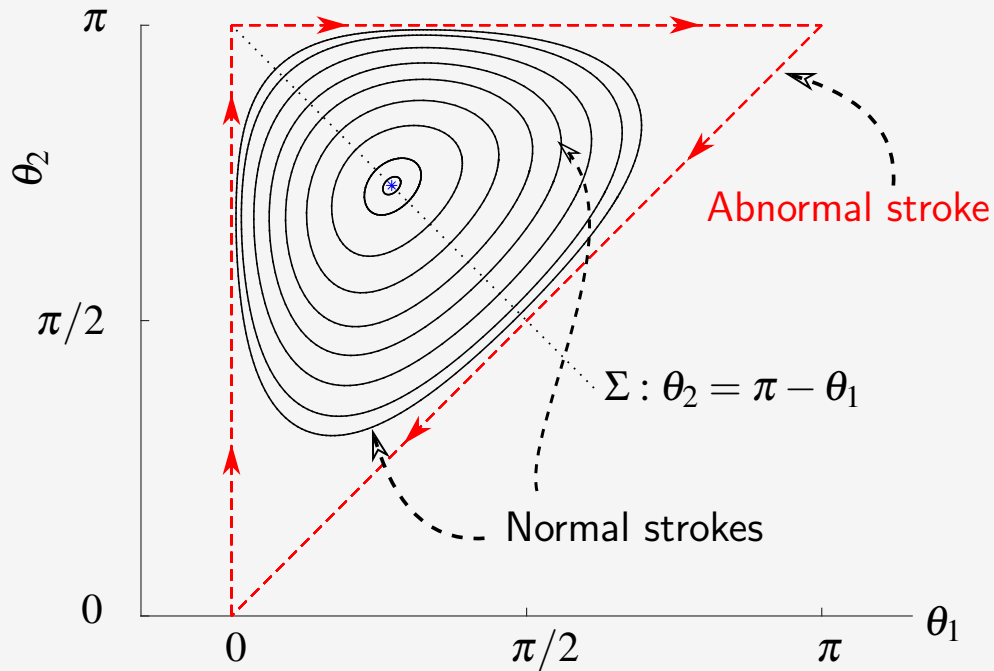
**Two steps.**

$$\max_{u(\cdot), x_0(T)} \mathcal{E} \Leftrightarrow \left( \begin{array}{l} \dot{q} = \sum_{i=1}^2 u_i F_i \leftarrow \min_{u(\cdot)} \int_0^T L(q, u) dt, \\ \min_{u(\cdot)} \int_0^T L(q, u) dt \text{ with } x_0(T) \text{ fixed, then } \underbrace{\text{select max } \mathcal{E}}_{x_0(T)} \\ \text{Transversality cond.} \end{array} \right)$$

Compute points on the Sub-Riemannian sphere  $\rightarrow$   
provide candidate solutions for the maximum of efficiency problem.

$$\int_0^T L(q, u) \leftarrow \min_{u(\cdot)}$$

**Proposition 1.** *There is a one parameter family of normal simple loop strokes parameterized by the displacement  $x_0(T)$ .*

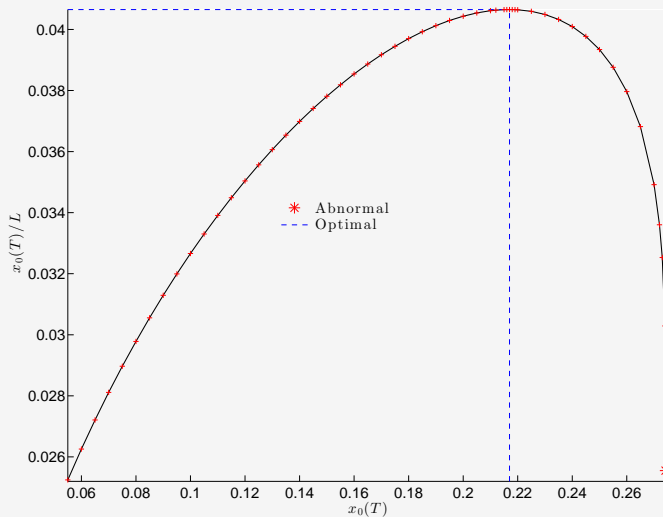


# Select on the point on the SR sphere with maximum of efficiency

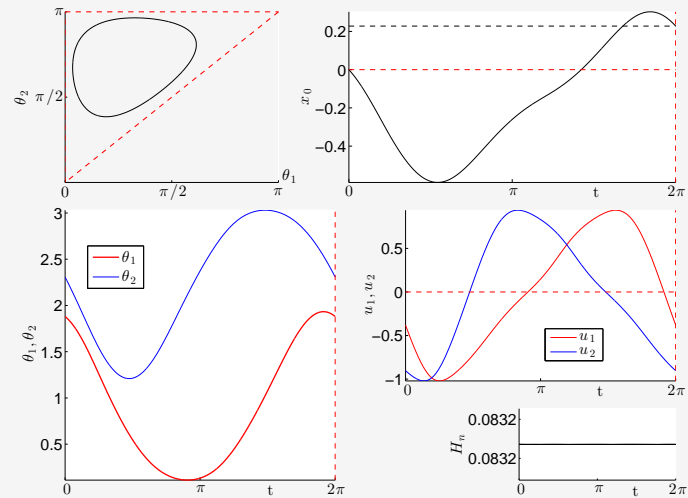
Geometric efficiency :  $\mathcal{E} = x_0(T)/l_{SR}(q)$ ,  $(\mathbf{x}_0(T) \text{ free})$

**Transversality condition of the maximum Principle**

$$p_{x_0}(T) = q^0(T)/x_0(T), \quad p^0(T) = -1/2$$



Efficiency curve with continuation on  $x_0(T)$ .



Optimal normal stroke

**Theorem 2.** *The abnormal stroke is not minimizing.*

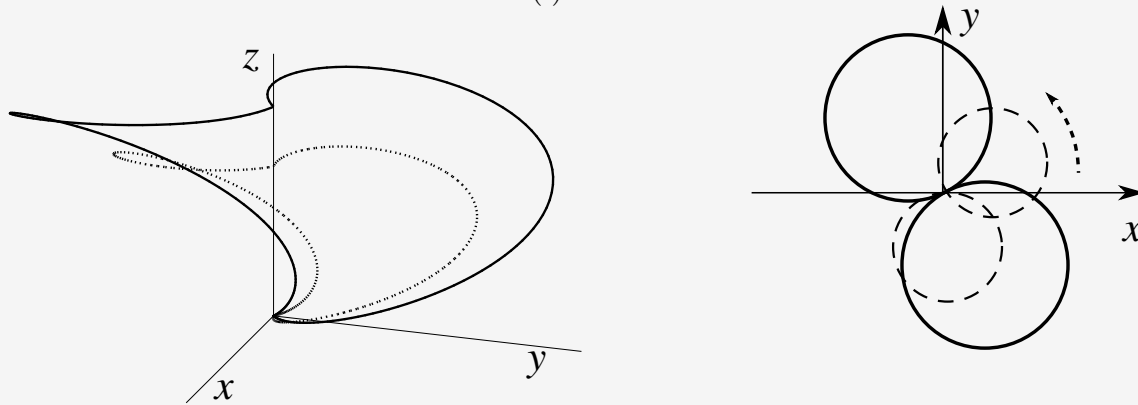


**Normal form of order  $-1$**  Nilpotent model of order  $-1$ , the Brockett-Heisenberg model :

$$\hat{F} = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad \hat{G} = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}$$

where (weight of  $x, y$ )=1 and (weight of  $z$ )=2 (privileged coordinates).

$$u_1^2 + u_2^2 \leftarrow \min_{u(\cdot)}, \quad \dot{q} = u_1 \hat{F}(q) + u_2 \hat{G}(q)$$



Geodesics  $q(\cdot)$  for the model of order  $-1$ .

These families of circles are **not generic**

**Theorem 3** (Brockett, Chakir et al). *The model of order  $-1$  is equivalent to the model of order  $0$ .*

**Application to the Copepod**  $z$  cannot be identified to the displacement  $x_0$ .  
 $x = \theta_1 - \theta_{10}$ ,  $y = \theta_2 - \theta_{20}$ ,  $\theta_{20} = \pi - \theta_{10}$ . (contact point)

**Perturbation of the model of order  $-1$  by terms of order  $0$  :**

$$F_1(x, y, z) = \frac{\partial}{\partial x} + (a_{11}(\theta_{10})xy + a_{02}(\theta_{10})y^2 + y) \frac{\partial}{\partial z}$$

$$F_2(x, y, z) = \frac{\partial}{\partial y} + (a_{11}(\theta_{10})xy + a_{02}(\theta_{10})x^2 - x) \frac{\partial}{\partial z}$$

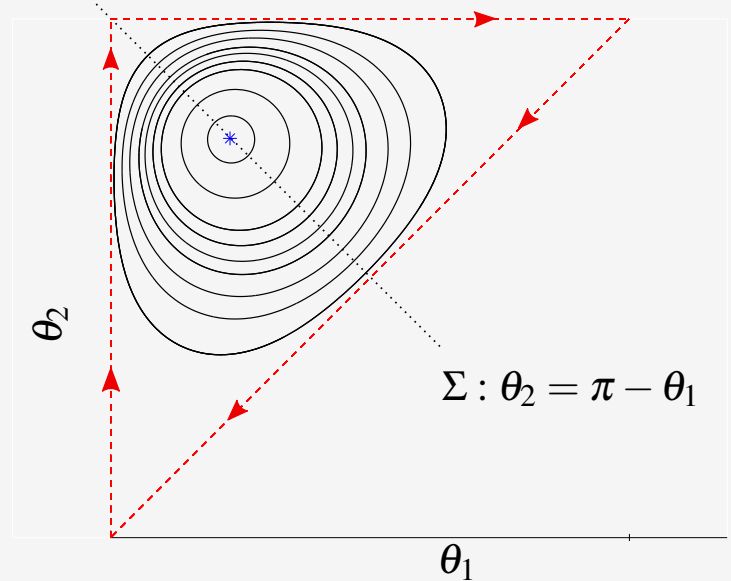
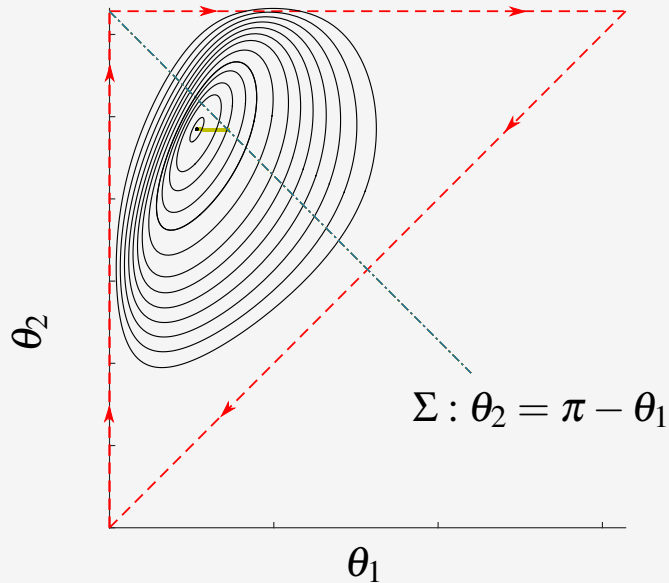
$$(F_1, F_2) \underset{\varphi}{\iff} (\hat{F}, \hat{G})$$

**Lemma 4.** *The only value of  $\theta_{10}$  such that the transformation  $\varphi$  doesn't mix up the shape variables  $(x,y)$  and the displacement  $z$  corresponds to the center of swimming of the family.*

$$\varphi(x, y, z) = (x + c_{011}(\theta_{10}) \mathbf{yz} + c_{001}(\theta_{10}) \mathbf{z}, y - c_{011}(\theta_{10}) \mathbf{xz} + c_{001}(\theta_{10}) \mathbf{z}, z + P(x, y, z, \theta_{10}))$$

$$\xRightarrow{\text{Lem.5}} c_{011}(\theta_{10}) = c_{001}(\theta_{10}) = 0 \quad \Leftrightarrow \quad \theta_{10} = \text{center of the family.}$$

**Theorem 5.** *The center of swimming is a SR-invariant.*



One parameter family of simple loops with different metrics.

**SR-invariant and inverse problem :** given two swimmers with their centers of swimming, we can recover the metric.

# Second order sufficient optimality conditions

for a Copepod stroke

$$\begin{cases} \min & J(q(\cdot), u(\cdot)) := c(q(0), q(T)) \\ \text{s.t.} & \dot{q}(t) = F(q(t), u(t)) \\ & m(q(0), q(T)) = 0, \end{cases}$$

$H(q, u, p) := p \cdot F(q, u)$  and  $h(q_0, q_T) := c(q_0, q_T) + v \cdot m(q_0, q_T)$ .

Consider a **normal extremal**  $(\bar{q}, \bar{p})$  associated with  $\bar{u}$ .

## Second variation

$$\begin{aligned} \delta^2 J(\delta q(\cdot), \delta u(\cdot)) &:= 1/2 [\delta q(0)^\top \quad \delta q(T)^\top] \mathbf{C} [\delta q(0) \quad \delta q(T)]^\top \\ &+ 1/2 \int_0^T (\delta q(t)^\top \partial_{qq} \mathbf{H}(t) \delta q(t) + 2 \delta q(t)^\top \partial_{qu} \mathbf{H}(t) \delta u(t) + \delta u(t)^\top \partial_{uu} \mathbf{H}(t) \delta u(t)) dt \end{aligned}$$

where  $\mathbf{C} := \text{Hessian}_{q_0, q_T}(h)$ .

$$\begin{cases} \min & \delta^2 J(\delta q(\cdot), \delta u(\cdot)) \\ \text{s.t.} & \dot{\delta} q := \partial_q \mathbf{F}(t) \delta q(t) + \partial_u \mathbf{F}(t) \delta u(t) \\ & \nabla_{q_0} m(\bar{q}(0), \bar{q}(T)) \delta q(0) + \nabla_{q_T} m(\bar{q}(0), \bar{q}(T)) \delta q(T) = 0 \end{cases}$$

### Classical optimality conditions

- 2nd order necessary conditions :  $\delta^2 J(\delta q(\cdot), \delta u(\cdot)) \geq 0$ ,
- 2nd order sufficient conditions :  $\delta^2 J(\delta q(\cdot), \delta u(\cdot))$  coercive.

**Monodromy matrix.**  $\Phi(.,.)$  associated with the linearized Hamiltonian system :

$$\begin{cases} \frac{d}{dt} \Phi(t, s) = \mathbf{Z} \Phi(t, s) \\ \Phi(s, s) = \text{Id}, \end{cases}$$

where

$$\mathbf{Z} := \begin{bmatrix} \partial_q F - \partial_u F [\partial_{uu} H]^{-1} \partial_{qu} H^T & -\partial_u F [\partial_{uu} H]^{-1} \partial_u F^T \\ -\partial_{qq} H + \partial_{qu} H [\partial_{uu} H]^{-1} \partial_{qu} H^T & -\partial_q F [\partial_{uu} H]^{-1} \partial_u F^T \end{bmatrix}.$$

Define

$$\mathcal{W} := \begin{bmatrix} \phi_{22} \phi_{12}^{-1} & \phi_{21} - \phi_{22} \phi_{12}^{-1} \phi_{11} \\ -\phi_{12} & \phi_{12}^{-1} \phi_{11} \end{bmatrix}, \quad \Phi(0, T) =: \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

**Theorem 6 ( Standard conditions ).** Assume

(i) :  $\partial_{uu}\mathbf{H}(t) \leq -\varepsilon \text{Id}$  on  $[0, T]$ ,  $\bar{u}(\cdot)$  bounded and  $(\partial_q \mathbf{F}(\cdot), \partial_u \mathbf{F}(\cdot))$  is controllable on  $[0, T]$ ,

(ii) : the extremal  $(\bar{q}(\cdot), \bar{u}(\cdot), \bar{p}(\cdot))$  **doesn't have conjugate points** on  $[0, T]$ ,

(iii) : there exists  $\gamma > 0$  t.q.

$$\begin{bmatrix} \xi^T & \xi^T \\ \xi_0 & \xi_1 \end{bmatrix} \mathcal{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left\| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right\|^2,$$

for all vectors  $\xi_0, \xi_1 \in \mathbb{R}^n \setminus \{0\}$  s.t.

$$\nabla_{q_0} m((\bar{q}(0), \bar{q}(T))) \xi_0 + \nabla_{q_T} m((\bar{q}(0), \bar{q}(T))) \xi_1 = 0.$$

Then  $(\bar{q}(\cdot), \bar{u}(\cdot))$  is a  $W^{1, \infty}$ -minimizer and **locally unique**.



## Boundary values

$$\begin{aligned}\theta_j(0) &= \theta_j(T) \quad j = 1, 2, \\ x_0(0) &= 0, \quad x(T) = x_T, \quad x_T \text{ is fixed}\end{aligned}$$

**Proposition 7.** *Take  $I = (-\varepsilon, \varepsilon)$ ,  $\varepsilon > 0$  and let  $(\bar{q}(\cdot), \bar{u}(\cdot), \bar{p}(\cdot))$  be a normal extremal. For all  $a \in I$  and  $t \in [0, T]$ , we define  $q^a(\cdot) = (\theta_1^a(\cdot), \theta_2^a(\cdot), x^a(\cdot))$ ,  $u_1^a(\cdot), u_2^a(\cdot)$  and  $p^a(\cdot)$  by*

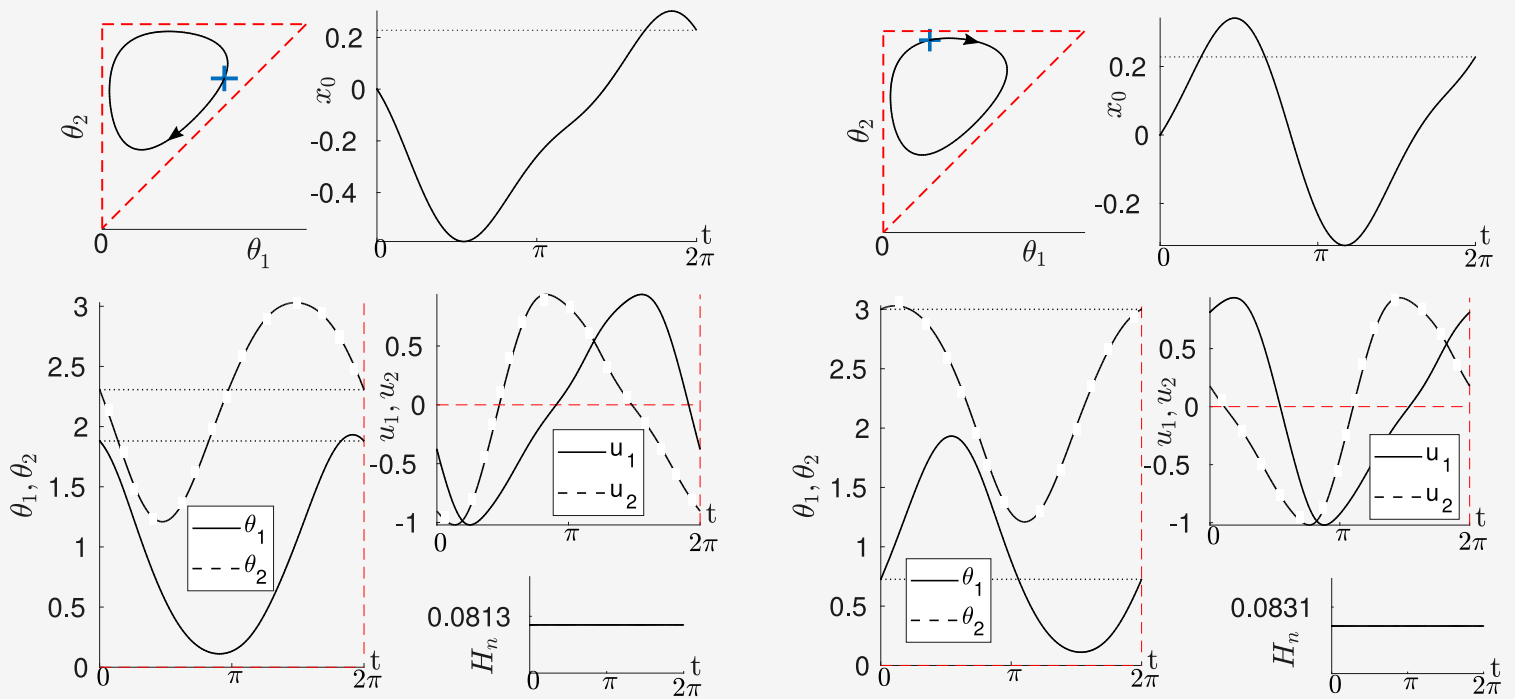
$$\begin{aligned}\theta_j^a(t) &= \bar{\theta}_j(t+a), \quad u_j^a(t) = \bar{u}_j(t+a) \quad \text{for } j = 1, 2, \\ x^a(t) &= \bar{x}(t+a) - \bar{x}(a), \quad p^a(t) = (\bar{p}_1(t), \bar{p}_2(t+a), \bar{p}_3(t+a)).\end{aligned}$$

*Then, for  $\varepsilon > 0$  small enough, the normal extremal  $(\bar{q}(\cdot), \bar{p}(\cdot), \bar{u}(\cdot))$  is continuously embedded in the family of extremals  $(q^a(\cdot), p^a(\cdot), u^a(\cdot))_{a \in I}$ .*

These strokes have the SAME COST and satisfy the SAME BOUNDARY CONDITIONS

$\implies$  **Standard conditions fail** because of non-unique minimizers.

# Families of extremals with same cost and same boundary conditions



**Theorem 8 ( Refined Conditions, Gavriel, Vinter (2014) ).** Assume the reference normal extremal  $(\bar{q}(\cdot), \bar{u}(\cdot), \bar{p}(\cdot))$  is continuously embedded in a family of extremals and

- (i) :  $\partial_{uu}\mathbf{H} \leq -\varepsilon \text{Id}$  on  $[0, T]$ ,  $(\partial_q \mathbf{F}(\cdot), \partial_u \mathbf{F}(\cdot))$  is controllable on  $[0, T]$ ,
- (ii) : the extremal  $(\bar{q}(\cdot), \bar{u}(\cdot), \bar{p}(\cdot))$  **doesn't have conjugate points** on  $[0, T]$ ,
- (iii) : there exists  $\gamma > 0$  s.t.

$$\begin{bmatrix} \varepsilon^T & \varepsilon^T \\ \xi_0 & \xi_1 \end{bmatrix} \mathcal{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left\| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right\|^2,$$

for all vectors  $\xi_0, \xi_1 \neq 0$  s.t.

$$\nabla_{q_0} m((\bar{q}(0), \bar{q}(T))) \xi_0 + \nabla_{q_T} m((\bar{q}(0), \bar{q}(T))) \xi_1 = 0 \quad \text{and} \quad \Gamma^T \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = 0.$$

$$\text{where } \Gamma := \left[ \begin{array}{c} \nabla_a q^a(0) \\ \nabla_a q^a(T) \end{array} \right] \Big|_{a=0}.$$

Then  $(\bar{q}(\cdot), \bar{u}(\cdot))$  is a **local**  $W^{1,\infty}$ -**minimizer**.

**Computation.** Define the matrix  $N_s$  from the subspace  $\mathcal{L}_s$  s.t.

$$\mathcal{L}_s = \{ (\xi_0, \xi_T) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \nabla_{q_0, q_T} m(q_0, q_T) (\xi_0 \quad \xi_T)^\top = 0 \} =: \text{Im}(N_s)$$

**Standard conditions.** Does the matrix  $\mathcal{W}_s := N_s^\top (\mathcal{W}^\top + \mathcal{W}) N_s \in \mathcal{M}_2$  is positive-definite?

Consider

$$\Gamma_r = (\nabla_a q^a(0) \quad \nabla_a q^a(T))_{a=0} = (\dot{q}(0) \quad \dot{q}(T))$$

and the linear subspace  $\mathcal{L}_r$  s.t.

$$\mathcal{L}_r := \mathcal{L}_s \cap \{ (\xi_0, \xi_T) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \Gamma_r^\top (\xi_0 \quad \xi_T)^\top = 0 \} =: \text{Im}(N_r)$$

**Refined conditions.** Does  $\mathcal{W}_r := N_r^\top (\mathcal{W}^\top + \mathcal{W}) N_r > 0$ ?

Relative tolerance	(Standard condition) Spec( $W_s$ )	(Refined condition) Spec( $W_r$ )
$10^{-5}$	<b>6.89e-4</b> 3.42	22.5
$10^{-8}$	<b>-9.12e-7</b> 3.42	22.5

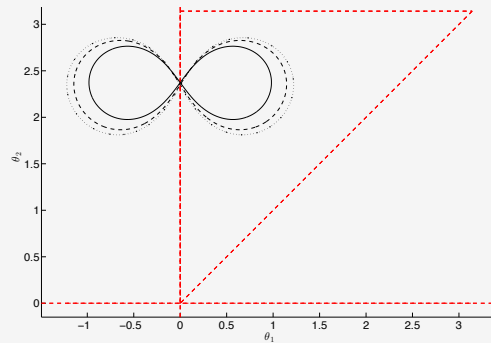
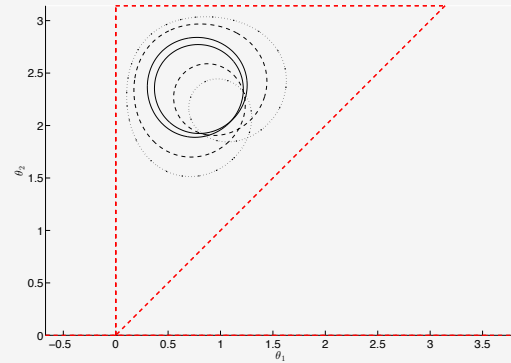
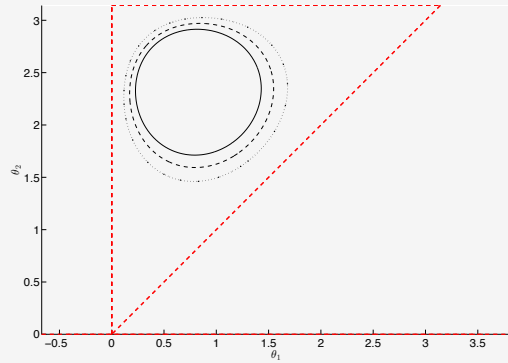
- **Standard conditions fail** :  $\mathcal{W}_s$  has a zero eigenvalue.
- **BUT Refined conditions are satisfied** :  $\mathcal{W}_r$  is positive-definite.

**Theorem 9** (Numerical). *The simple loop normal stroke  $(\bar{q}, \bar{u})$  is  $W^{1,\infty}$  – optimal.*



- *Contact point* : expression of the generic normal form at any point inside the triangle  $\rightarrow$  unique family of simple loops.
- *Martinet point* : compute the normal form for a point on the edges  $\rightarrow$  locate the eight loops.
- estimation of the first conjugate time using normal forms.
- swimmer model with more than 2 pairs of links.

# Generate normal strokes

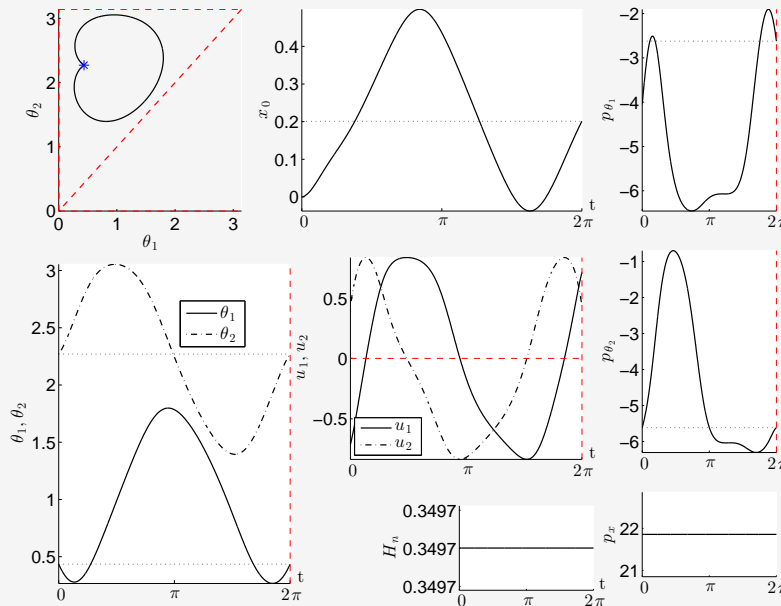




$$\mathcal{E}' = \frac{x_0(T) \mathbf{m}(\boldsymbol{\theta}(0))}{l(q)}, \quad (m \text{ smooth})$$

## Transversality condition of the maximum Principle

$$p_{\theta}(0) - p_{\theta}(2\pi) = \lambda \frac{\partial \mathcal{E}'}{\partial \theta(0)}$$



**Theorem 10** (Chakir, Gauthier, Kupka, 1996). *The generic model is given by the normal form of order 1*

$$F = \hat{F} + yQ(x,y)\frac{\partial}{\partial z}, \quad G = \hat{G} - xQ(x,y)\frac{\partial}{\partial z},$$

$Q$  quadratic in  $(x,y)$ .

**Remark 11.** *This normal form can be used to approximate the one parameter family of simple strokes.*