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To cite this version:

Jérémy Rouot, Piernicola Bettiol, Bernard Bonnard. Optimal Control Theory and the Efficiency of the Swimming Mechanism of the Copepod Zooplankton. 2017. hal-02483344

HAL Id: hal-02483344

https://inria.hal.science/hal-02483344v1

Submitted on 18 Feb 2020

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Optimal Control Theory and the Efficiency of the Swimming Mechanism of the Copepod Zooplankton

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2017, 10th July

IFAC 2017 World Congress, Toulouse, France

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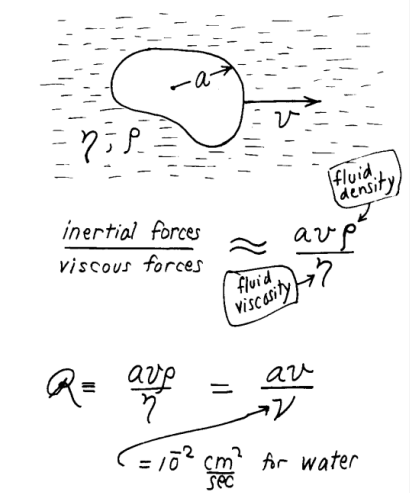
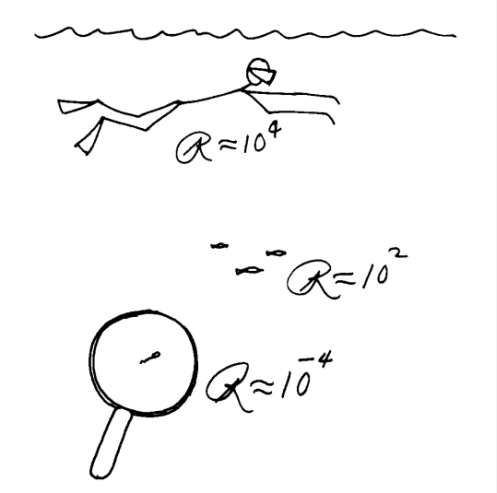
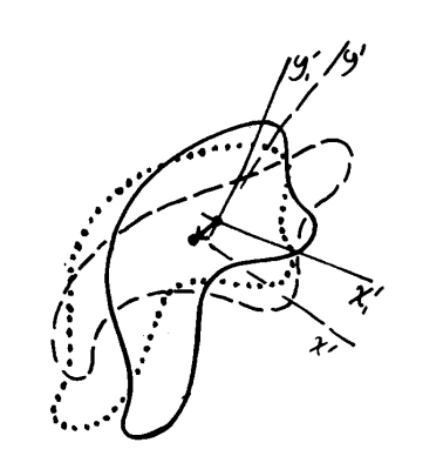
Life at low Reynolds number - Purcell, 1977

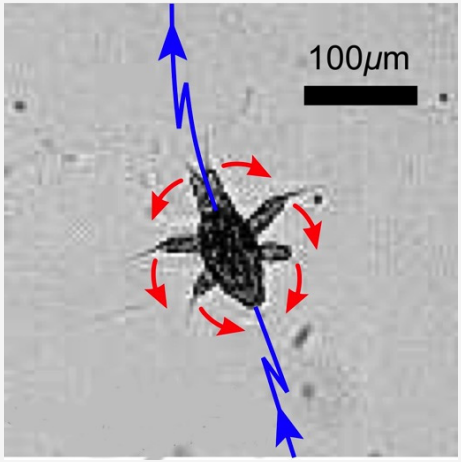




Reynolds number

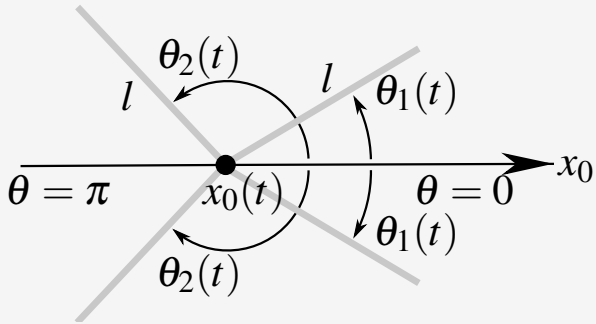


Shape deformations of a microswimmer



Zooplankton

2-link symmetric swimmer.



Dynamic.

$$\dot{x}_0 = \frac{\sum_{i=1}^{2} l^2\, \dot{\theta}_i \sin(\theta_i)}{2 + \sum_{i=1}^{2} \sin^2(\theta_i)}, \quad \dot{\theta}_i = u_i,\ i = 1,2 \quad (\text{state constraint} : 0 \leq \theta_1 \leq \theta_2 \leq \pi).$$

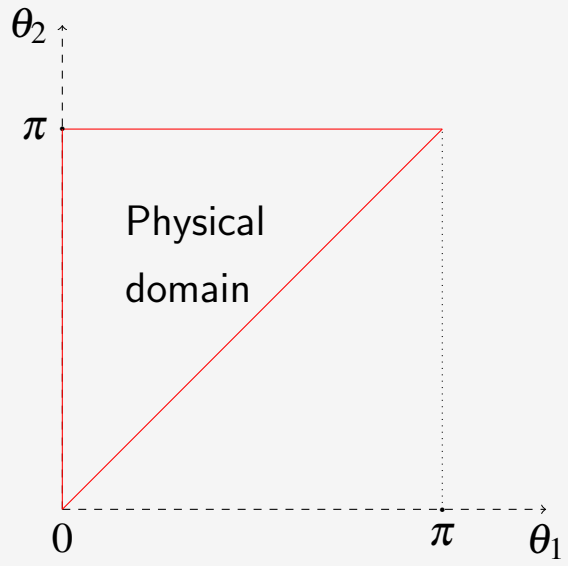
Criterion. minimize drag forces : $\dot{q} S(q) \dot{q}^{\mathsf{T}}$, $\boldsymbol{q} = (\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \boldsymbol{x_0})$ and S is positive definite $\Longrightarrow$ quadratic form in (u_1, u_2).

Aim

- Compute **optimal strokes** minimizing an efficiency.

- Stroke : T-**periodic motion of the shape variables** θ s.t. $x_0(T) - x_0(0) > 0$.

- Admissible stroke : find closed curves in the θ-plane **contained in the triangle** :



$$\mathscr{E}(q(\cdot)) = \textbf{Geometric Efficiency} = x_0(T)/l_{SR}(q)$$

SR Length l_{SR}.

$$l_{SR}(q) = \int_0^T \sqrt{L(q,u)}\,\mathrm{d}t, \qquad L(q,u) = a(q)\,u_1^2 + 2b(q)\,u_1u_2 + c(q)\,u_2^2$$

$$\min_{u(\cdot)} l_{SR}(q) \Leftrightarrow \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t$$

Two steps.

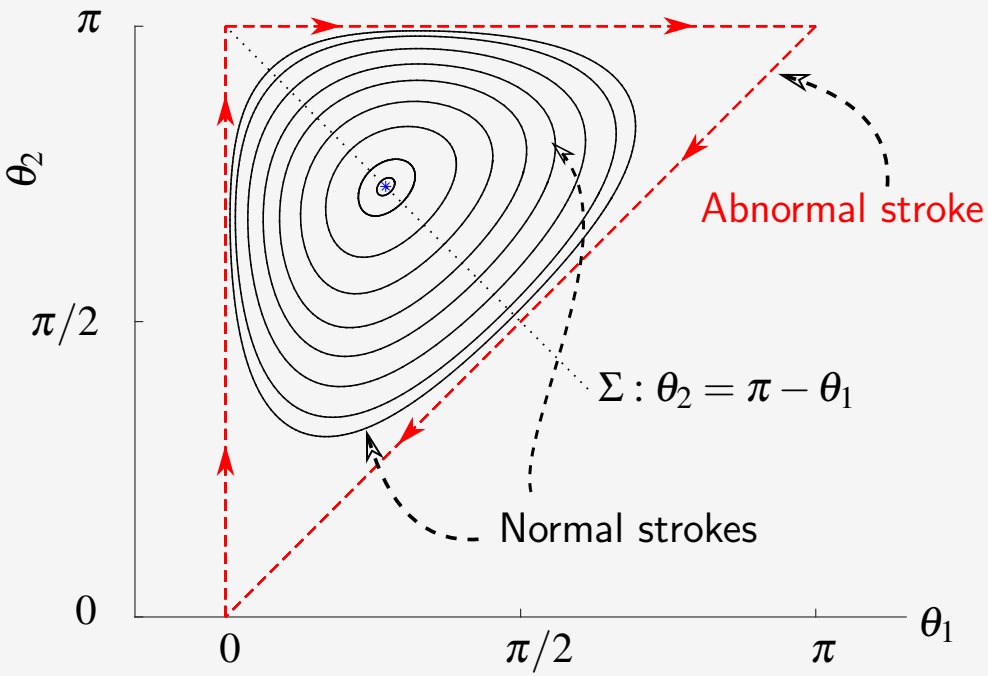
$$\dot{q} = \sum_{i=1}^{2} u_i F_i \leftarrow \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t,$$

$$\max_{u(\cdot),\,x_0(T)} \mathscr{E} \Leftrightarrow \left(\min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t \quad \text{with } \boldsymbol{x_0(T)} \textbf{ fixed}, \text{ then } \underbrace{\text{select } \max_{x_0(T)} \mathscr{E}}_{\text{Transversality cond.}} \right)$$

Compute points on the Sub-Riemannian sphere $\rightarrow$
provide candidate solutions for the maximum of efficiency problem.

$$\int_0^T L(q,u) \leftarrow \min_{u(\cdot)}$$

Proposition 1. *There is a one parameter family of normal simple loop strokes parameterized by the displacement $x_0(T)$.*

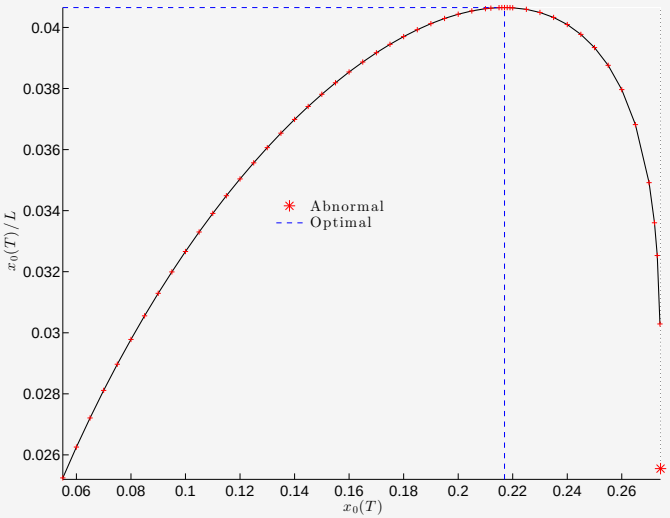


Select on the point on the SR sphere with maximum of efficiency

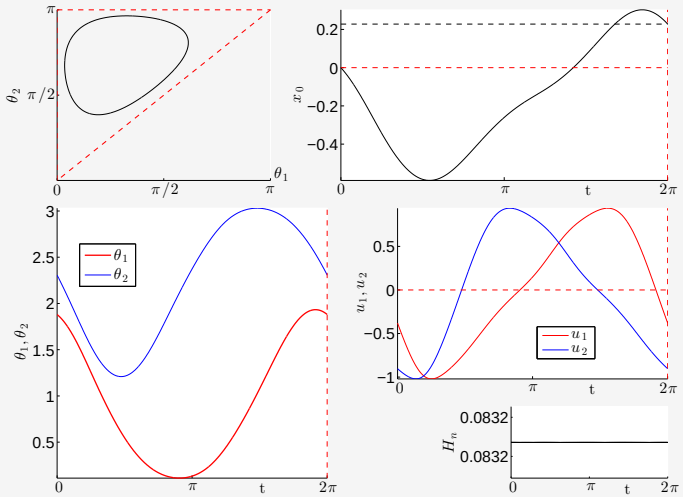
Geometric efficiency : $\mathscr{E} = x_0(T)/l_{SR}(q)$, $\quad$ **($x_0(T)$ free)**

Transversality condition of the maximum Principle

$$p_{x_0}(T) = q^0(T)/x_0(T), \qquad p^0(T) = -1/2$$



Efficiency curve with continuation on $x_0(T)$.



Optimal normal stroke

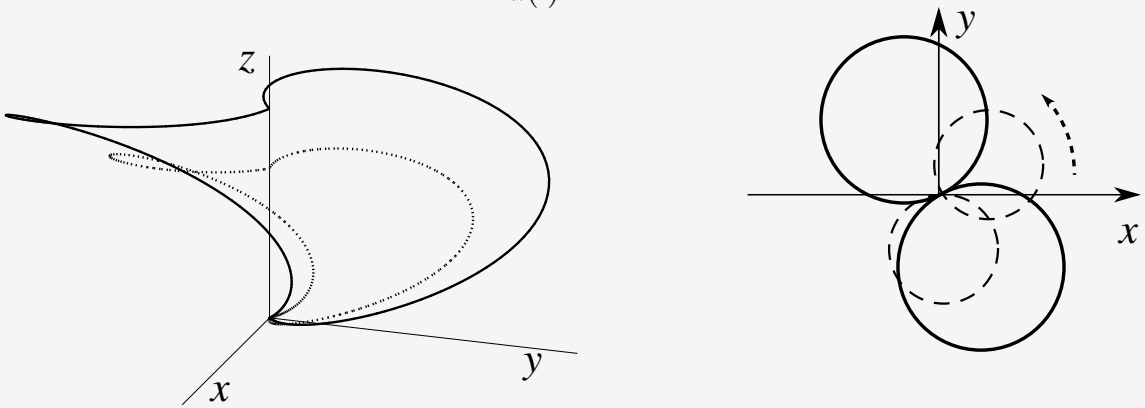
Theorem 2. *The abnormal stroke is not minimizing.*

Normal form of order -1 Nilpotent model of order -1, the Brockett-Heisenberg model :

$$\hat{F} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad \hat{G} = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$$

where (weight of x, y)=1 and (weight of z)=2 (privileged coordinates).

$$u_1^2 + u_2^2 \leftarrow \min_{u(\cdot)}, \quad \dot{q} = u_1 \hat{F}(q) + u_2 \hat{G}(q)$$



Geodesics $q(\cdot)$ for the model of order -1.

These families of circles are **not generic**

Theorem 3 (Brockett, Chakir et al)**.** *The model of order* -1 *is equivalent to the model of order* 0*.*

Application to the Copepod z cannot be identified to the displacement x_0.
$x = \theta_1 - \theta_{10}$, $y = \theta_2 - \theta_{20}$, $\theta_{20} = \pi - \theta_{10}$. (contact point)

Perturbation of the model of order -1 by terms of order 0 :

$$F_1(x,y,z) = \frac{\partial}{\partial x} + (a_{11}(\theta_{10})xy + a_{02}(\theta_{10})y^2 + y)\frac{\partial}{\partial z}$$
$$F_2(x,y,z) = \frac{\partial}{\partial y} + (a_{11}(\theta_{10})xy + a_{02}(\theta_{10})x^2 - x)\frac{\partial}{\partial z}$$

$$(F_1, F_2) \quad \underset{\varphi}{\Longleftrightarrow} \quad (\hat{F}, \hat{G})$$

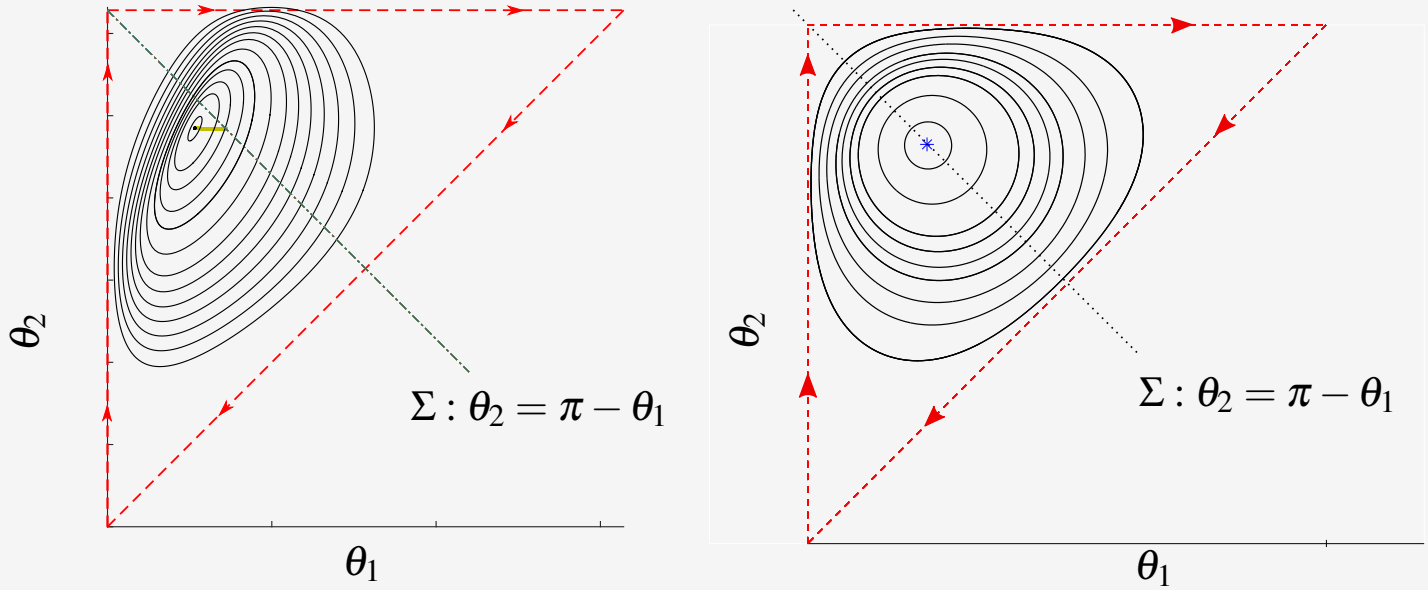
Lemma 4. *The only value of θ_{10} such that the transformation φ doesn't mix up the* **shape variables (x,y) and the displacement z** *corresponds to the center of swimming of the family.*

$$\varphi(x,y,z) = (x + c_{011}(\theta_{10})\boldsymbol{yz} + c_{001}(\theta_{10})\boldsymbol{z},\ y - c_{011}(\theta_{10})\boldsymbol{xz} + c_{001}(\theta_{10})\boldsymbol{z},\ z + P(x,y,z,\theta_{10}))$$

$$\underset{\text{Lem. 5}}{\Longrightarrow} c_{011}(\theta_{10}) = c_{001}(\theta_{10}) = 0 \quad \Leftrightarrow \quad \theta_{10} = \text{center of the family.}$$

Theorem 5. *The center of swimming is a SR-invariant.*

Different costs to minimize



One parameter family of simple loops with different metrics.

SR-invariant and inverse problem : given two swimmers with their centers of swimming, we can recover the metric.

Second order sufficient optimality conditions

for a Copepod stroke

$$\begin{cases} \min & J(q(.),u(.)) := c(q(0),q(T)) \\ \text{s.t.} & \dot{q}(t) \;=\; F(q(t),u(t)) \\ & m(q(0),q(T)) = 0\,, \end{cases}$$

$H(q,u,p) := p \cdot F(q,u)$ and $h(q_0,q_T) := c(q_0,q_T) + \nu \cdot m(q_0,q_T)$.
Consider a **normal extremal** $(\overline{q},\overline{p})$ associated with $\overline{u}$.

Second variation

$$\delta^2 J(\delta q(.),\delta u(.)) := 1/2\,[\delta q(0)^\intercal \quad \delta q(T)^\intercal]\,\boldsymbol{C}\,[\delta q(0) \quad \delta q(T)]^\intercal$$
$$+\,1/2\int_0^T (\delta q(t)^\intercal \boldsymbol{\partial}_{qq}\boldsymbol{H}(t)\delta q(t) + 2\,\delta q(t)^\intercal \boldsymbol{\partial}_{qu}\boldsymbol{H}(t)\delta u(t) + \delta u(t)^\intercal \boldsymbol{\partial}_{uu}\boldsymbol{H}(t)\delta u(t))\,\mathrm{d}t$$

where $\quad \boldsymbol{C} := \mathrm{Hessian}_{q_0,q_T}(h)$.

$$\begin{cases} \min & \delta^2 J(\delta q(.), \delta u(.)) \\ \text{s.t.} & \dot{\delta q} := \boldsymbol{\partial}_q \boldsymbol{F}(t)\, \delta q(t) + \boldsymbol{\partial}_u \boldsymbol{F}(t)\, \delta u(t) \\ & \nabla_{q_0} m(\overline{q}(0), \overline{q}(T))\, \delta q(0) + \nabla_{q_T} m(\overline{q}(0), \overline{q}(T))\, \delta q(T) = 0 \end{cases}$$

Classical optimality conditions

- *2nd order necessary conditions :* $\delta^2 J(\delta q(.), \delta u(.)) \geq 0$,
- *2nd order sufficient conditions :* $\delta^2 J(\delta q(.), \delta u(.))$ coercive.

Monodromy matrix. $\Phi(.,.)$ associated with the linearized Hamiltonian system :

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,s) = \boldsymbol{Z}\Phi(t,s) \\ \Phi(s,s) = \mathrm{Id}, \end{cases}$$

where

$$Z := \begin{bmatrix} \partial_q F - \partial_u F [\partial_{uu} H]^{-1} \partial_{qu} H^T & -\partial_u F [\partial_{uu} H]^{-1} \partial_u F^T \\ -\partial_{qq} H + \partial_{qu} H [\partial_{uu} H]^{-1} \partial_{qu} H^T & -\partial_q F [\partial_{uu} H]^{-1} \partial_u F^T \end{bmatrix}.$$

Define

$$\mathscr{W} := \begin{bmatrix} \phi_{22}\phi_{12}^{-1} & \phi_{21} - \phi_{22}\phi_{12}^{-1}\phi_{11} \\ -\phi_{12} & \phi_{12}^{-1}\phi_{11} \end{bmatrix}, \quad \Phi(0,T) =: \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

Theorem 6 (**Standard conditions**). *Assume*

(i) : $\boldsymbol{\partial}_{uu}\boldsymbol{H}(t) \leq -\varepsilon\, Id$ *on* $[0,T]$, $\overline{u}(.)$ *bounded and* $(\boldsymbol{\partial}_{\boldsymbol{q}}\boldsymbol{F}(.), \boldsymbol{\partial}_{\boldsymbol{u}}\boldsymbol{F}(.))$ *is controllable on* $[0,T]$,

(ii) : the extremal $(\overline{q}(.), \overline{u}(.), \overline{p}(.))$ **doesn't have conjugate points** *on* $[0,T]$,

(iii) : there exists $\gamma > 0$ *t.q.*

$$\left[\xi_0^T \;\; \xi_1^T\right] \mathscr{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right|^2 ,$$

for all vectors $\xi_0, \xi_1 \in \mathbb{R}^n \setminus \{0\}$ *s.t.*

$$\nabla_{q_0} m((\overline{q}(0), \overline{q}(T)))\xi_0 + \nabla_{q_T} m((\overline{q}(0), \overline{q}(T)))\xi_1 = 0.$$

Then $(\overline{q}(.), \overline{u}(.))$ *is a* $W^{1,\infty}$*-minimizer and* **locally unique**.

Boundary values

$$\theta_j(0) = \theta_j(T) \quad j = 1,2,$$
$$x_0(0) = 0,\ x(T) = x_T, \quad x_T \text{ is fixed}$$

Proposition 7. *Take $I = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ and let $(\overline{q}(\cdot), \overline{u}(\cdot), \overline{p}(\cdot))$ be a normal extremal. For all $a \in I$ and $t \in [0,T]$, we define $q^a(\cdot) = (\theta_1^a(\cdot), \theta_2^a(\cdot), x^a(\cdot))$, $u_1^a(\cdot), u_2^a(\cdot)$ and $p^a(\cdot)$ by*

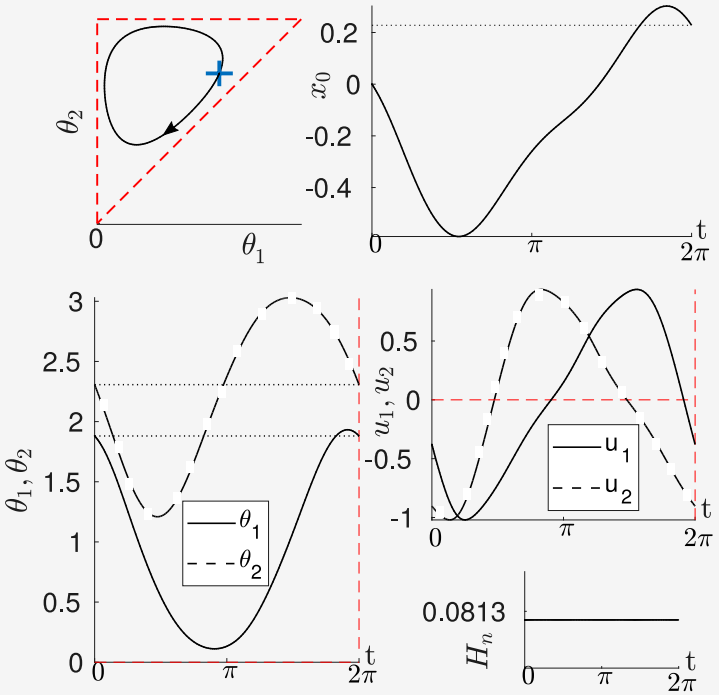
$$\theta_j^a(t) = \overline{\theta}_j(t+a), \quad u_j^a(t) = \overline{u}_j(t+a) \quad \text{for } j = 1,2,$$
$$x^a(t) = \overline{x}(t+a) - \overline{x}(a),\ p^a(t) = (\overline{p}_1(t), \overline{p}_2(t+a), \overline{p}_3(t+a)).$$

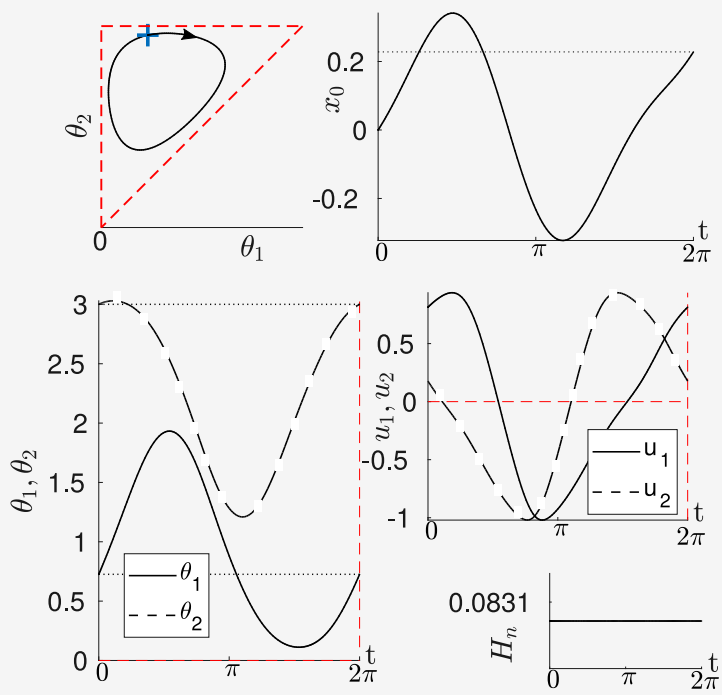
Then, for $\varepsilon > 0$ small enough, the normal extremal $(\overline{q}(.), \overline{p}(.), \overline{u}(.))$ is continuously embedded in the family of extremals $(q^a(.), p^a(.), u^a(.))_{a \in I}$.

These strokes have the SAME COST and satisfy the SAME BOUNDARY CONDITIONS

$\Longrightarrow$ **Standard conditions fail** because of non-unique minimizers.

Families of extremals with same cost and same boundary conditions





Theorem 8 (**Refined Conditions**, Gavriel, Vinter (2014)). *Assume the reference normal extremal* $(\overline{q}(.),\overline{u}(.),\overline{p}(.))$ *is continuously embedded in a family of extremals and*

(i) : $\boldsymbol{\partial}_{uu}\boldsymbol{H} \leq -\varepsilon\, Id$ *on* $[0,T]$, $(\boldsymbol{\partial}_{\boldsymbol{q}}\boldsymbol{F}(.),\boldsymbol{\partial}_{\boldsymbol{u}}\boldsymbol{F}(.))$ *is controllable on* $[0,T]$,

(ii) : the extremal $(\overline{q}(.),\overline{u}(.),\overline{p}(.))$ **doesn't have conjugate points** *on* $[0,T]$,

(iii) : there exists $\gamma > 0$ *s.t.*

$$\left[\xi_0^T \ \xi_1^T\right] \mathscr{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right|^2 ,$$

for all vectors $\xi_0, \xi_1 \neq 0$ *s.t.*

$$\nabla_{q_0} m((\overline{q}(0),\overline{q}(T)))\xi_0 + \nabla_{q_T} m((\overline{q}(0),\overline{q}(T)))\xi_1 = 0 \qquad \text{and} \qquad \boldsymbol{\Gamma}^T \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = 0 \ .$$

where $\Gamma := \left. \begin{bmatrix} \nabla_a q^a(0) \\ \nabla_a q^a(T) \end{bmatrix} \right|_{a=0}$.

Then $(\overline{q}(.),\overline{u}(.))$ *is a* **local** $W^{1,\infty}$**-minimizer**.

Computation. Define the matrix N_s from the subspace $\mathscr{L}_s$ s.t.

$$\mathscr{L}_s = \{\, (\xi_0, \xi_T) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \nabla_{q_0,q_T} m(q_0, q_T)\, (\xi_0 \quad \xi_T)^\intercal = 0 \,\} =: \mathrm{Im}(N_s)$$

Standard conditions. Does the matrix $\mathscr{W}_s := N_s^\intercal\, (\mathscr{W}^\intercal + \mathscr{W})\, N_s \in \mathscr{M}_2$ is positive-definite ?

Consider

$$\Gamma_r = \big(\nabla_a q^a(0) \;\; \nabla_a q^a(T)\big)_{a=0} = \big(\dot{\bar{q}}(0) \;\; \dot{\bar{q}}(T)\big)$$

and the linear subspace $\mathscr{L}_r$ s.t.

$$\mathscr{L}_r := \mathscr{L}_s \cap \{(\xi_0, \xi_T) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \Gamma_r^\intercal\, (\xi_0 \quad \xi_T)^\intercal = 0\} =: \mathrm{Im}(N_r)$$

Refined conditions. Does $\mathscr{W}_r := N_r^\intercal\, (\mathscr{W}^\intercal + \mathscr{W})\, N_r > 0$?

Relative tolerance	(Standard condition) $\mathrm{Spec}(W_s)$	(Refined condition) $\mathrm{Spec}(W_r)$
10^{-5}	**6.89e-4** 3.42	22.5
10^{-8}	**-9.12e-7** 3.42	22.5

— Standard conditions fail : $\mathscr{W}_s$ has a zero eigenvalue.
— **BUT** Refined conditions are satisfied : $\mathscr{W}_r$ is positive-definite.

Theorem 9 (Numerical)**.** *The simple loop normal stroke* $(\overline{q},\overline{u})$ *is* $W^{1,\infty}-optimal$.

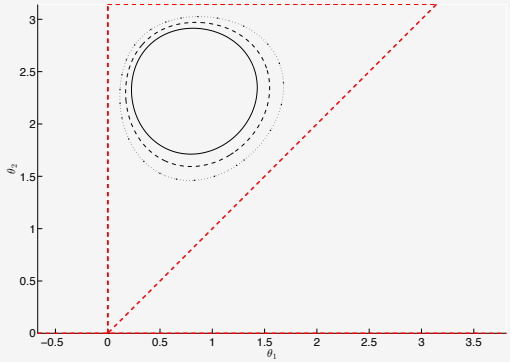
— *Contact point* : expression of the generic normal form at any point inside the triangle $\rightarrow$ unique family of simple loops.

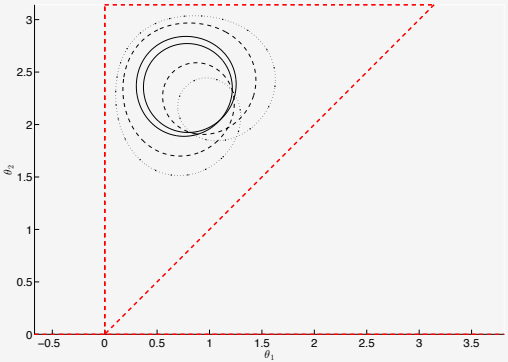
— *Martinet point* : compute the normal form for a point on the edges $\rightarrow$ locate the eight loops.

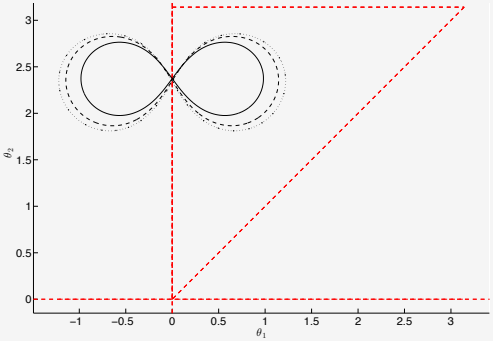
— estimation of the first conjugate time using normal forms.

— swimmer model with more than 2 pairs of links.

Generate normal strokes





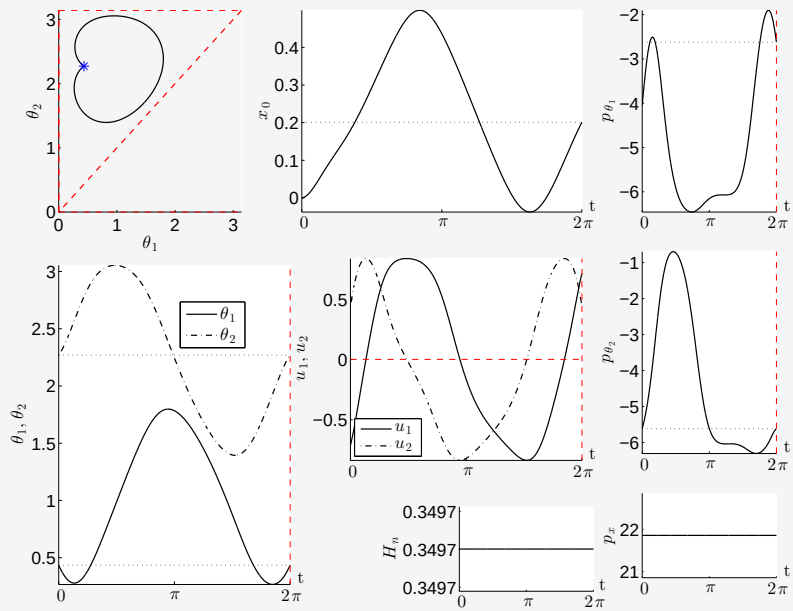


Efficiency depending upon $\theta(0)$

$$\mathscr{E}' = \frac{x_0(T)\,\boldsymbol{m}(\boldsymbol{\theta}(\mathbf{0}))}{l(q)}, \quad (m \text{ smooth})$$

Transversality condition of the maximum Principle

$$p_\theta(0) - p_\theta(2\pi) = \lambda \frac{\partial \mathscr{E}'}{\partial \theta(0)}$$



Theorem 10 (Chakir, Gauthier, Kupka, 1996)**.** *The* **generic model** *is given by the normal form of order* 1

$$F = \hat{F} + yQ(x,y)\frac{\partial}{\partial z},\ G = \hat{G} - xQ(x,y)\frac{\partial}{\partial z},$$

Q *quadratic in* (x,y)*.*

Remark 11. *This normal form can be used to approximate the one parameter family of simple strokes.*