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Averaging for minimum time control problems and applications

18th French - German - Italian Conference on Optimization

Paderborn, September 25 - 28, 2017

Jérémy Rouot

*Joint work with J.-B. Caillau and J.-B. Pomet
from INRIA Sophia Antipolis, France.*

Coordinates (I, φ)

- $I \in U(\subset \mathbb{R}^5)$: geometric elements of the ellipse,
- $\varphi \in S^1$: position on the orbit.

Equations of motion In first approximation,

$$\dot{I} = 0, \quad \dot{\varphi} = \omega(I) > 0$$

I is a slow variable compared to φ .

$$\min_{t_f, |u| \leq 1} t_f$$

$$\text{s.t.} \quad \dot{I} = \varepsilon F_0(I, \varphi, \varepsilon) + \varepsilon \sum_{i=1}^3 u_i F_i(I, \varphi, \varepsilon), \quad I(0) = I_0$$

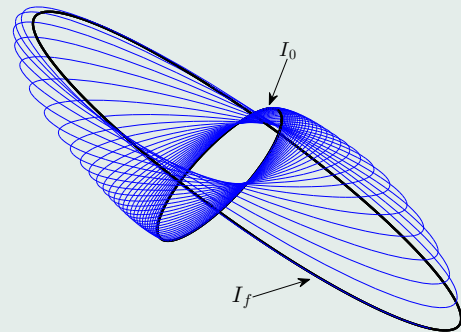
$$\dot{\varphi} = \omega(I) + \varepsilon G_0(I, \varphi, \varepsilon) + \varepsilon \sum_{i=1}^3 u_i G_i(I, \varphi, \varepsilon), \quad I(t_f) = I_f.$$

$0 < \varepsilon \ll 1$, and I_0, I_f are given. $\varphi(0)$ and $\varphi(t_f)$ are free.

$F_i(\cdot, \varphi, \cdot), G_i(\cdot, \varphi, \cdot)$ $i = 0 \dots 3$ are **2π -periodic** w.r.t. φ , $u \in \mathbb{R}^3$.

F_0, G_0 modeled perturbations

- high order terms in the gravitational potential of the Earth,
- luni-solar perturbation, ...



Let $f(X, \psi, \varepsilon), g(X, \psi, \varepsilon)$ be two vector fields **2π -periodic** w.r.t. $\psi \in S^1$.

Unperturbed system $\dot{X} = 0, \quad \dot{\psi} = \omega(X)$

Perturbed system

$$\dot{X} = \varepsilon f(X, \psi, \varepsilon), \quad \dot{\psi} = \omega(X) + \varepsilon g(X, \psi, \varepsilon)$$

where $X \in C \subset \mathbb{R}^n$. There are $C' \subset C$ and $\tau > 0$ s.t.

$$Y' = \bar{f}(Y), \quad \bar{f}(X) = \frac{1}{2\pi} \int_0^{2\pi} f(X, \psi, \varepsilon = 0) d\psi,$$

have a well-defined solution defined on $[0, \tau]$ for every initial condition $Y(0) \in C'$.

Theorem 1. $\exists \varepsilon_0 > 0, 0 < \varepsilon < \varepsilon_0$ and for every initial condition in $C' \times S^1$, the perturbed system has a solution on $[0, \tau/\varepsilon]$ s.t.

$$|X(t) - Y(\varepsilon t)| = O(\varepsilon)$$

PMP: a minimizing trajectory is projection of an extremal $(I(\cdot), \varphi(\cdot), p_I(\cdot), p_\varphi(\cdot))$ solution of the Hamiltonian system

$$H(I, \varphi, p_I, p_\varphi, \varepsilon) = p_\varphi \omega(I) + \varepsilon K_0(I, \varphi, p_I, p_\varphi, \varepsilon),$$

$$K_0 := H_0 + \sqrt{\sum_{i=1}^3 H_i^2}, \quad H_i(I, \varphi, p_I, p_\varphi, \varepsilon) = p_I F_i(I, \varphi, \varepsilon) + p_\varphi G_i(I, \varphi, \varepsilon), \quad i = 0, \dots, 3.$$

Extremal system

$$\begin{aligned} \dot{I} &= \varepsilon \frac{\partial K_0}{\partial p_I}, & \dot{p}_I &= -p_\varphi \omega' - \varepsilon \frac{\partial K_0}{\partial I} \\ \dot{\varphi} &= \omega + \varepsilon \frac{\partial K_0}{\partial p_\varphi}, & \dot{p}_\varphi &= -\varepsilon \frac{\partial K_0}{\partial \varphi} \end{aligned}$$

Theorem 2. Normalizing on $\mathbf{H} = \varepsilon$ (normal extremals), we have

$$p_\varphi = -\varepsilon h(I, \varphi, p_I, \varepsilon)$$

$$0 = \frac{\partial}{\partial p_I} \left(H(I, \varphi, p_I, -\varepsilon h(I, \varphi, p_I, \varepsilon), \varepsilon) - \varepsilon \right) = \varepsilon \frac{\partial K_0}{\partial p_I} - \varepsilon \frac{\partial h}{\partial p_I} \frac{\partial H}{\partial p_\varphi}$$

$$\Rightarrow \dot{I} = \varepsilon \frac{\partial h}{\partial p_I} \dot{\varphi}, \text{ and similarly, we have: } \dot{p}_I = -\varepsilon \frac{\partial h}{\partial I} \dot{\varphi}.$$

$$\Rightarrow \text{Reparametrization w.r.t.: } s = \varepsilon(\varphi - \varphi_0).$$

Definition 3. *The average boundary value problem is*

$$\begin{aligned} \frac{d\bar{I}}{ds} &= \frac{\partial \bar{h}}{\partial p_I}, & \frac{d\bar{p}_I}{ds} &= -\frac{\partial \bar{h}}{\partial I} \\ \bar{I}(0) &= I_0, & \bar{I}(\bar{s}_f) &= I_f, & \bar{h} &= 0 \end{aligned}$$

where

$$\bar{h}(I, p_I) := \frac{1}{2\pi} \int_0^{2\pi} h(I, \varphi, p_I, \varepsilon=0) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \frac{K_0 - 1}{\omega}(I, \varphi, p_I, p_\varphi=0, \varepsilon=0) d\varphi.$$

Let

$$\bar{S}(s_f, p_0) = (\bar{I}(s_f, p_0) - I_f, \bar{h}(I_0, p_0))$$

be the shooting function associated with the average dynamic

$$\frac{d\bar{z}}{ds} = \overrightarrow{h}(\bar{z}(s)), \quad \bar{z} = (\bar{I}, \bar{p}_I)$$

Remark 4. \bar{h} is smooth outside $\bar{\Sigma}$ where

$$\Sigma = \{(I, p_I, \varphi) \in T^*M \times S^1 \mid p_I \cdot F_i(I, \varphi, p_I, \varepsilon) = 0, \quad i = 1, 2, 3\}$$

in

$$\bar{\Sigma} = \Pi(\Sigma), \quad \Pi : T^*M \times S^1 \rightarrow T^*M.$$

For $\varepsilon > 0$, consider

$$S_\varepsilon(s_f, p_0) = (I(s_f, p_0) - I_f, \bar{h}(I_0, p_0))$$

be the shooting function associated with the non-average dynamic

$$\frac{dz}{ds} = \vec{h}(s/\varepsilon, z(s), \varepsilon), \quad z = (I, p_I).$$

Remark 5. *Note that solutions of $S_\varepsilon(s_f, p_0) = 0$ doesn't define strict extremals in the sense that the true conditions*

$$h(\varphi_0, z_0, \varepsilon) = h(\varphi_f, z_f, \varepsilon) = 0$$

are replaced by

$$\bar{h}(z_0) = 0.$$

They are called "quasi-extremals".

Theorem 6. Let $(I^\varepsilon(\cdot))_\varepsilon$ be a family of *non average* trajectories (with "quasi-extremals" lifts) converging as $\varepsilon \rightarrow 0$ with fixed extremities I_0, I_f . Then the limit is solution of the average boundary value problem.

Theorem 7. Let $\bar{I}(\cdot)$ be a trajectory of the average system which doesn't have conjugate points.

Then, there exists a family of non average trajectories $(I^\varepsilon(\cdot))_\varepsilon$ converging to $\bar{I}(\cdot)$ as $\varepsilon \rightarrow 0$ satisfying the boundary values: $I^\varepsilon(0) = I_0$ and $I^\varepsilon(t_f) = I_f + O(\varepsilon)$ and under further assumptions, there exists (φ_0, p_0) s.t.

$$h(\varphi_0, z(0), \varepsilon) = 0 \quad \text{and} \quad h(s_f/\varepsilon + \varphi_0, z(s_f), \varepsilon) = 0.$$

Remark 8. Numerical use: The solution of the *average* system can be used to initialize the shooting function of the *non average* system.

Lemma 9. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuously differentiable function with a regular zero at $x = 0$, and let $f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\varepsilon > 0$, be continuous functions converging uniformly towards f on a neighborhood of the origin when $\varepsilon \rightarrow 0$.

There exists $\varepsilon_0 > 0$ and $x : [0, \varepsilon_0] \rightarrow \mathbb{R}^n$ continuous at $\varepsilon = 0$, s.t. $x(0) = 0$ and

$$f_\varepsilon(x(\varepsilon)) = 0, \quad \varepsilon \in (0, \varepsilon_0].$$

Proof of the convergence Theorem 7. Let (\bar{s}_f, \bar{p}_0) the solution of the average system $(\bar{S}(\bar{s}_f, \bar{p}_0) = 0)$.

1. Apply Lemma 9 with $f = \bar{S}$ and $f_\varepsilon = S_\varepsilon$. Then there exists $p_0(\varepsilon) \rightarrow \bar{p}_0$ s.t. $\bar{h}(I_0, p_0(\varepsilon)) = 0$.

2. Assume

$$\exists \bar{\varphi}_0 \in S^1, \quad h(\bar{\varphi}_0, \bar{z}_0, \varepsilon = 0) = 0, \quad \frac{\partial h}{\partial \varphi}(\bar{\varphi}_0, \bar{z}_0, \varepsilon = 0) \neq 0, \quad (\bar{z}_0 = (I_0, \bar{p}_0)),$$

we construct $\varphi_0(p, \varepsilon)$ and we apply again Lemma 9 with $S_\varepsilon(\varphi_0, s_f, p_0)$ associated to

$$\frac{dz}{ds} = \vec{h}(s/\varepsilon + \varphi_0, z(s), \varepsilon)$$

to satisfy the condition $\mathbf{h}(\varphi_0, \mathbf{z}(0), \varepsilon) = \mathbf{0}$ for the non average extremals.

3. Assume

$$\exists \bar{\varphi}_f \in S^1, \quad h(\bar{\varphi}_f, \bar{z}_f, \varepsilon = 0) = 0, \quad \frac{\partial h}{\partial \varphi}(\bar{\varphi}_f, \bar{z}_f, \varepsilon = 0) \neq 0, \quad (\bar{z}_f = \bar{z}(\bar{s}_f, p_0)),$$

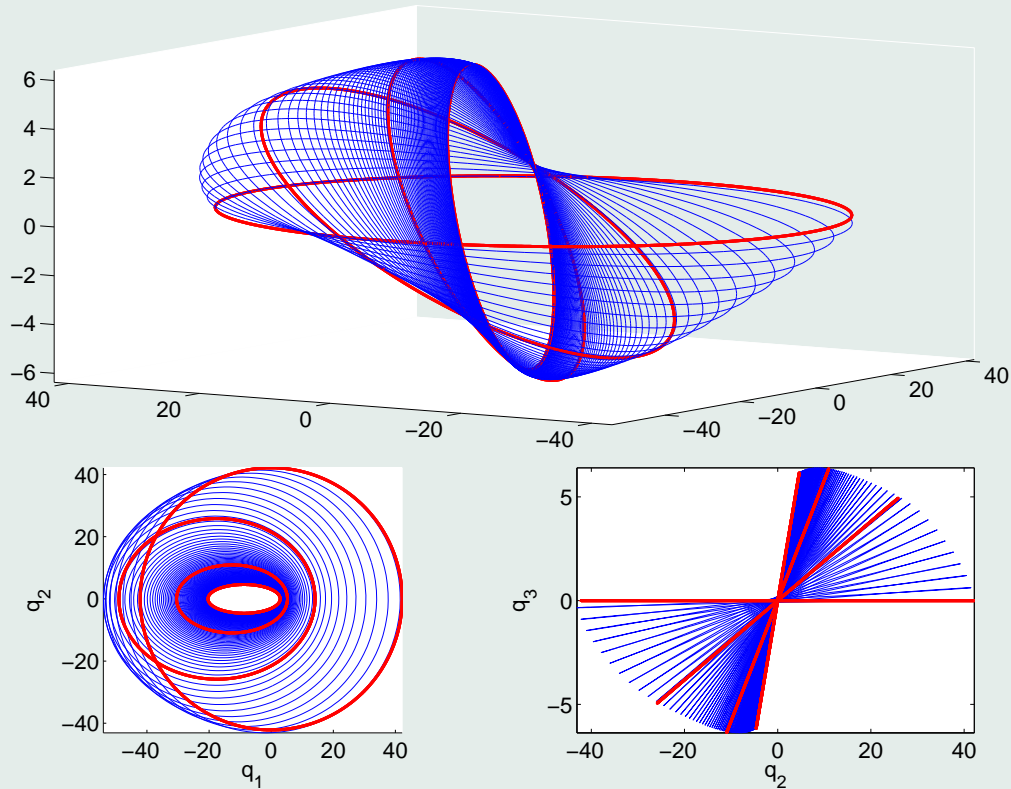
we construct $\varphi_f(\varepsilon)$ (Brouwer fix point theorem) and consider

$$\tilde{s}_f(\varepsilon) = s_f(\varepsilon) + \varepsilon \varphi_f(\varepsilon),$$

so that

$$h(\tilde{s}_f/\varepsilon + \varphi_0, z(\tilde{s}_f), \varepsilon) = 0 \text{ and } I(\tilde{s}_f) = I_f + O(\varepsilon).$$

TRAJECTORIES IN CARTESIAN COORDINATES



(red) Some average orbits of the average system.
(blue) Non average trajectory in Cartesian coordinates.

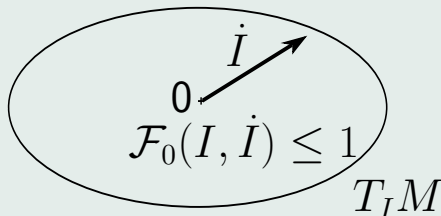
Finsler norm $\mathcal{F} : TM \rightarrow \mathbf{R}$ smooth on $TM \setminus 0$ s.t.

- $\mathcal{F}(x, \lambda v) = \lambda \mathcal{F}(x, v)$, $\lambda > 0$
- $\partial^2 \mathcal{F}^2(x, v) / \partial v^2 > 0$

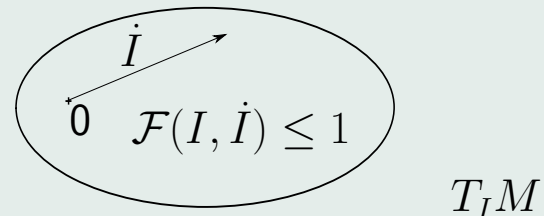
Finsler co-norm $F^* : T^*M \rightarrow \mathbf{R}$ smooth on $T^*M \setminus 0$ s.t.

- $F^*(x, \lambda p) = \lambda F^*(x, p)$, $\lambda > 0$,
- $\partial^2 (F^*)^2(x, p) / \partial p^2 > 0$

Theorem 10. If $\text{rang} \{ \partial^j F_i(I, \varphi, \varepsilon = 0) / \partial \varphi^j, i = 1, \dots, 3, j \geq 0 \} = 6$, then \bar{K}_0 defined a symmetric Finsler co-norm.



\mathcal{F}_0 Symmetric Finsler norm



\mathcal{F} Non-symmetric Finsler norm

Earth oblateness perturbation

$$\begin{aligned} H &= p_\varphi \omega(I) + \varepsilon_{J_2} H_0(I, p_I, \varphi) + \varepsilon_{th} K_0(I, p_I, \varphi, p_\varphi, \varepsilon) \\ &= p_\varphi \omega(I) + \varepsilon \left[\underbrace{\lambda H_0(I, p_I, \varphi) + (1 - \lambda) K_0(I, p_I, \varphi, p_\varphi, \varepsilon)}_{K^\lambda(I, p_I, \varphi, p_\varphi, \varepsilon)} \right] \\ &= p_\varphi \omega(I) + \varepsilon K^\lambda(I, p_I, \varphi, p_\varphi, \varepsilon) \end{aligned}$$

where $\varepsilon = \varepsilon_{J_2} + \varepsilon_{th}$, $\lambda = \varepsilon_{J_2}/\varepsilon$.

We denote by $\overline{K^\lambda}$ the average of K^λ w.r.t. φ .

- $Q : \text{Is } (I, p_I) \mapsto \overline{K}^\lambda(I, p_I) = \lambda \overline{H}_0(I, p_i) + (1 - \lambda) \overline{K}_0(I, p_I)$ a Finsler co-norm ?

Lemma 11. For $I \in M$ fixed, consider the (unique) one-form $\overline{F}_0^{*,\lambda}(I)$ solution of

$$\overline{F}_0^{*,\lambda}(I) = \underset{p \in \mathbb{R}^n}{\text{argmin}} \left[1/2 (1 - \lambda)^2 \overline{K}_0(I, p)^2 - \lambda \langle p, \overline{F}_0(I) \rangle \right].$$

Proposition 12. Let $I \in M$ fixed and λ_c s.t. $\overline{K}_0^{\lambda_c}(I, \overline{F}_0^{\lambda_c^*}(I)) = 1$. For all $\lambda < \lambda_c$,

$$T_I^*M \ni p \mapsto \overline{K}^\lambda(I, p) \in \mathbb{R}^+$$

is a Finsler co-norm (non-symmetric).

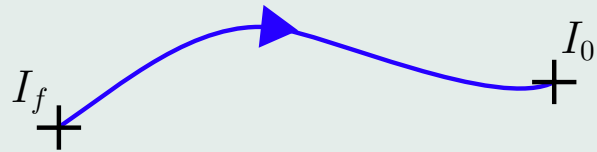
Proposition 13. λ_c can be computed as

$$\lambda_c(I) = \frac{1}{\overline{K}_0(I, p^*) + 1},$$

where $p^* = \langle \pi^*, \overline{F}_0(I) \rangle \pi^*$, $\pi^* = \underset{\pi | \overline{K}_0(I, \pi) = 1}{\text{argmax}} \langle \pi, \overline{F}_0(I) \rangle$

Let I_f be a fixed extremity.
 We construct I_0 by integrating the uncontrolled flow on $[0, \tau_d]$.

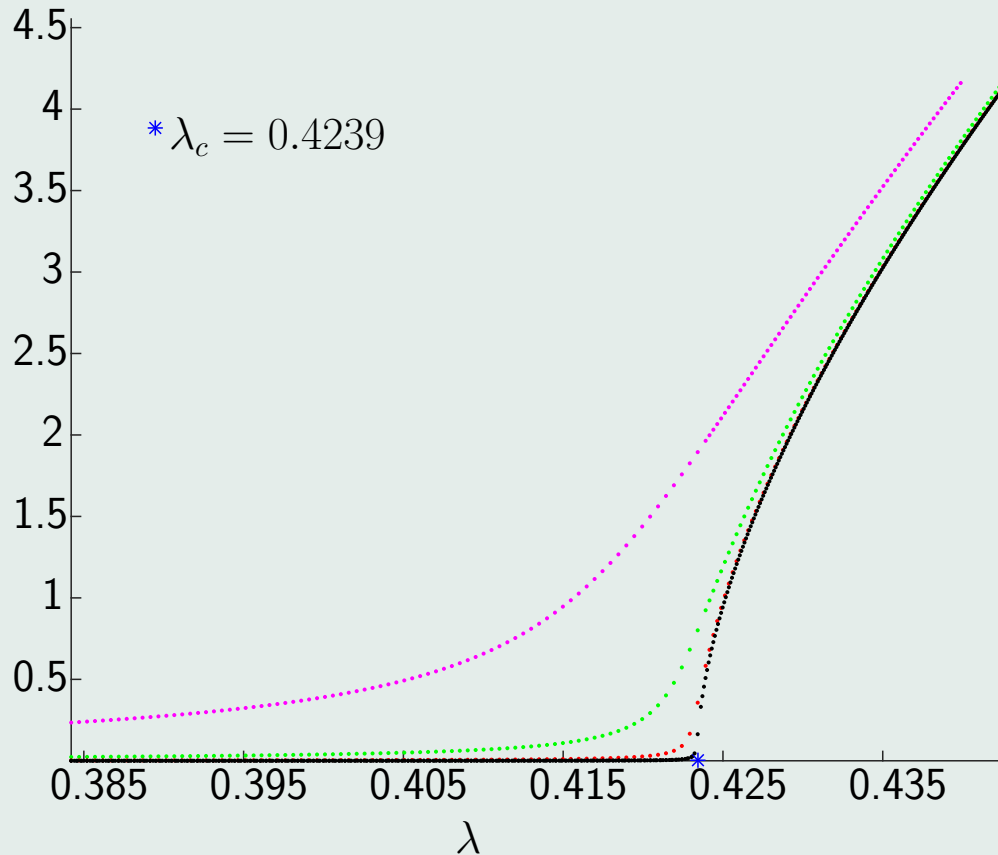
Uncontrolled dynamic during τ_d



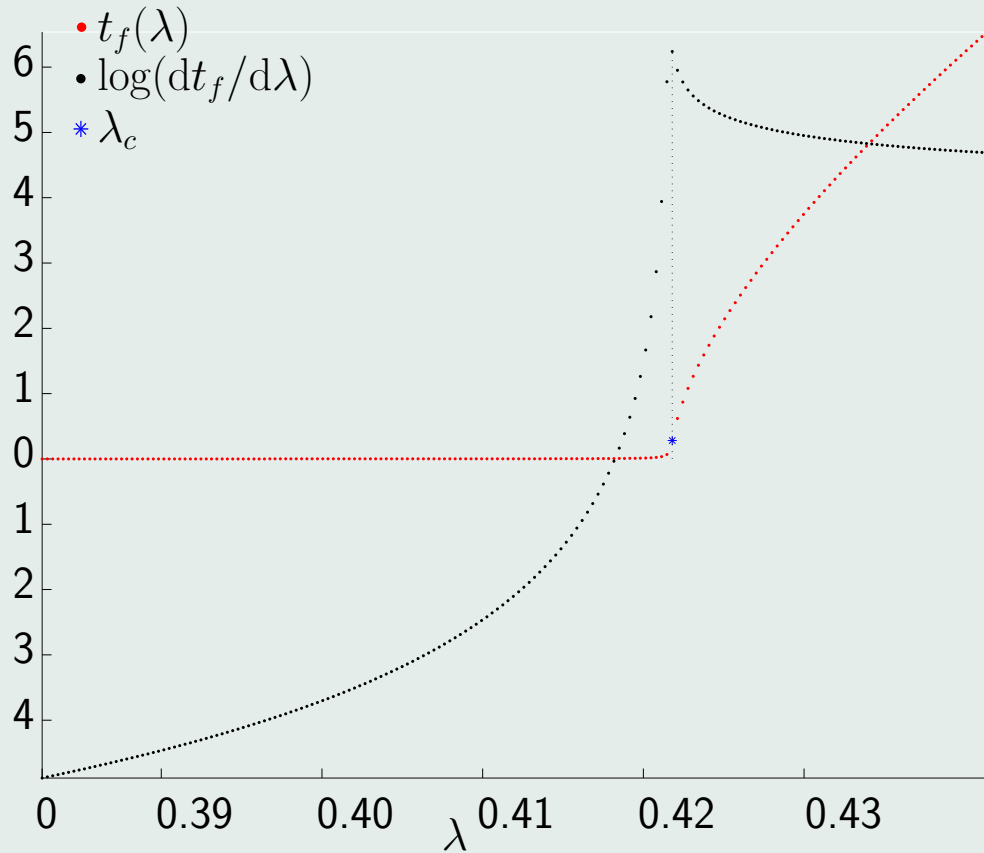
Boundary value problems

	Average BVP	Non average BVP
Extremal system	\overline{K}^λ	$H/\varepsilon = K^\lambda + p_\varphi \omega(I, \varphi)/\varepsilon$
Free parameters	λ, τ_d	$\lambda, \tau_d, \varepsilon$
Boundary values	$I(0) = I_0, I(\tau_f) = I_f$	$I(0) = I_0, I(t_f) = I_f$
	$\overline{K}^\lambda = 1$	$p_\varphi(0) = 0, p_\varphi(t_f) = 0, H = \varepsilon$

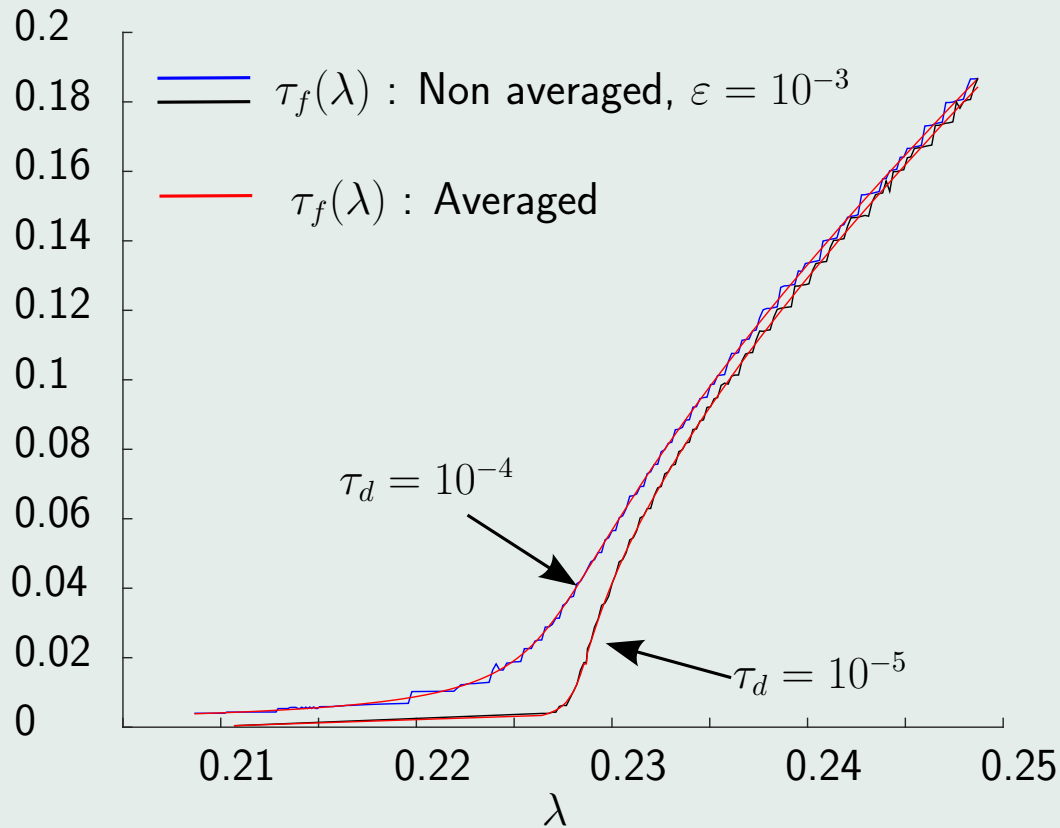
Settings We take ε small and analyze the dependance on λ when $\tau_d \rightarrow 0$ for both systems.



Curves $\lambda \mapsto \tau_f(\lambda)$ where $\tau_d \in \{1e-2, 1e-3, 1e-4, 1e-5\}$
for the average system.



Minimum time w.r.t. to λ for the **average** system
and its derivative w.r.t. λ .



Curves $\lambda \rightarrow \tau_f$ for the **average** and **non average** system
 where $\varepsilon = 10^{-3}$ et $\tau_d \in \{1e-4, 1e-5\}$.

- relate the metric property to controllability property
- continuity of the value function at the critical value λ_c .
- averaging with several angles (and other perturbations): resonances