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1 Hierarchy Builder: algebraic hierarchies made easy 2 in Coq with Elpi

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12 — Abstract —

13 It is nowadays customary to organize libraries of machine checked proofs around hierarchies of
14 algebraic structures [2, 6, 8, 16, 18, 23, 27]. One influential example is the Mathematical Components
15 library on top of which the long and intricate proof of the Odd Order Theorem could be fully
16 formalized [14].

17 Still, building algebraic hierarchies in a proof assistant such as Coq [9] requires a lot of manual
18 labor and often a deep expertise in the internals of the prover [13, 17]. Moreover, according to our
19 experience [26], making a hierarchy evolve without causing breakage in client code is equally tricky:
20 even a simple refactoring such as splitting a structure into two simpler ones is hard to get right.

21 In this paper we describe *HB*, a high level language to *build* hierarchies of algebraic structures
22 and to make these hierarchies *evolve* without breaking user code. The key concepts are the ones of
23 *factory*, *builder* and *abbreviation* that let the hierarchy developer describe an actual interface for
24 their library. Behind that interface the developer can provide appropriate code to ensure backward
25 compatibility. We implement the *HB* language in the `hierarchy-builder` addon for the Coq system
26 using the Elpi [11, 28] extension language.

27 **2012 ACM Subject Classification** General and reference → General literature; General and reference

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30 **1** Introduction

31 Modern libraries of machine checked proofs are organized around hierarchies of algebraic
32 structures [2, 6, 8, 16, 18, 23, 27]. For example the Mathematical Components library for
33 the Coq system [9] provides a very rich, ever growing, hierarchy of structures such as group,
34 ring, module, algebra, field, partial order, order, lattice. . . The hierarchy does not only serve
35 the purpose of organizing knowledge, but also to make it easy to exploit it. Indeed the
36 interactive prover can take advantage of the structure of the library and the relation between
37 its concepts to infer part of information usually left implicit by the user, a capability that
38 turned out to be key to tame the complexity and size of the formal proof of the Odd Order
39 Theorem [14].

40 The hierarchy of the Mathematical Components library is implemented following the
41 discipline of Packed Classes initially introduced in [13] and later also adopted in [2] and
42 [6]. We call Packed Classes a discipline, and not a language, because, in spite of its many
43 virtues, it is unwieldy to use. In particular it leaks to the user many of the technical details
44 of the Coq system. As a result, one needs to be a Coq expert in order to build or modify a
45 hierarchy, and even experts make mistakes as shown in [26]. Another inconvenience of the



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46 Packed Classes discipline is that even simple changes to the hierarchy, such as splitting a
47 structure into two simpler ones, break user code.

48 In this paper we describe \mathcal{HB} , a high level language to *build* hierarchies of algebraic
49 structures and to make these hierarchies *evolve* without breaking user code. The key concepts
50 are the ones of *factory* and *abbreviation* that let the hierarchy developer describe an actual
51 interface for their library. Behind that interface the developer can provide appropriate code
52 to ensure backward compatibility. We implement the \mathcal{HB} language by compiling it to a
53 variant of the Packed Classes discipline, that we call *flat*, in the `hierarchy-builder` addon
54 for Coq. We write this addon using the Elpi [11, 28] extension language.

55

56 To sum up, the main contributions of the paper are:

- 57 ■ the design of the \mathcal{HB} language,
- 58 ■ the compilation of \mathcal{HB} to the (flat) Packed Classes discipline, and
- 59 ■ the implementation of \mathcal{HB} in the `hierarchy-builder` addon for Coq.

60 The paper is organized as follows. Via an example we introduce \mathcal{HB} and its key ideas. We
61 then describe the discipline of Packed Classes and we show how \mathcal{HB} can be compiled to it.
62 We then discuss the implementation of the Coq addon via the Elpi extension language and
63 we position \mathcal{HB} in the literature.

64 **2** \mathcal{HB} by examples: building and evolving a hierarchy

65 The first version of our hierarchy (that we name V1) features only two structures: `Monoid`
66 and `Ring`.

```
1  HB.mixin Record Monoid_of_Type A := {
2    zero : A;
3    add : A -> A -> A;
4    addrA : associative add;           (* `add` is associative.           *)
5    addOr : left_id zero add;         (* `zero` is the left and right neutral *)
6    addr0 : right_id zero add;       (* element with respect to `add`.     *)
7  }.
8  HB.structure Definition Monoid := { A of Monoid_of_Type.axioms A }.
9
10 HB.mixin Record Ring_of_Monoid A of Monoid.axioms A := {
11   one : A;
12   opp : A -> A;
13   mul : A -> A -> A;
14   addNr : left_inverse zero opp add; (* `opp x` is the left and right additive *)
15   addrN : right_inverse zero opp add; (* inverse of `x`.                       *)
16   mulrA : associative mul;           (* `mul` is associative.                 *)
17   mul1r : left_id one mul;           (* `one` is the left and right neutral  *)
18   mulr1 : right_id one mul;         (* element with respect to `mul`.       *)
19   mulrDl : left_distributive mul add; (* `mul` is left and right distributive *)
20   mulrDr : right_distributive mul add; (* over `add`.                          *)
21 }.
22 HB.structure Definition Ring := { A of Monoid.axioms A & Ring_of_Monoid.axioms A }.
```

67 In order to build a structure we need to declare some factories and later assemble them.
68 One kind of factory supported by \mathcal{HB} , the simplest one, is called *mixin* and is embodied by
69 a record, by convention named `axioms`, that gathers operations and properties.

70 Mixins are declared via the `HB.mixin` command that takes a record declaration with a
71 type variable name and a possibly empty list of factories for that type. The code between
72 lines 1 and 7 declares a mixin that can turn a naked type `A` into a monoid, hence we chose
73 the name to be `Monoid_of_Type`.

The `HB.structure` command takes in input the definition `S` for a type `A` together with a list of factories for `A`. It registers a structure `S` in the hierarchy placing any definition specific to that structure inside a Coq module named `S`. Line 8 hence forges the `Monoid` structure.

Line 10 declares a second mixin collecting the operations and properties that are needed in order to enrich a monoid to a ring, hence the name `Ring_of_Monoid`. Indeed this time the type variable `A` is followed by `Monoid.axioms` that enriches `A` with the operations and properties of monoids. As a consequence `add` and `zero` can be used to express the new properties.

The last line declares the `Ring` structure to hold all the axioms declared so far. We can now inspect the contents of the hierarchy and then proceed to build a theory about abstract rings, register examples (instances) of ring structures and finally use the abstract theory on these examples.

```

1  Print Monoid.type. (* Monoid.type := { sort : Type; ... } *)
2  Check @add.      (* add      : forall M : Monoid.type, M -> M -> M *)
3  Check @addNr.   (* addNr   : forall R : Ring.type, left_inverse zero opp add *)
4
5  Lemma addrC : forall R : Ring.type, commutative (@add R).
6  Proof. (* Proof by Hankel 1867, omitted for brevity *) Qed.
7
8  Definition Z_Monoid_axioms : Monoid_of_Type.axioms Z :=
9    Monoid_of_Type.Axioms Z 0 Z.add Z.add_assoc Z.add_0_l Z.add_0_r.
10
11 HB.instance Z Z_Monoid_axioms.
12
13 Definition Z_Ring_axioms : Ring_of_Monoid.axioms Z _ :=
14   Ring_of_Monoid.Axioms Z 1 Z.opp Z.mul
15   Z.add_opp_diag_l Z.add_opp_diag_r Z.mul_assoc Z.mul_1_l Z.mul_1_r
16   Z.mul_add_distr_r Z.mul_add_distr_l.
17
18 HB.instance Z Z_Ring_axioms.
19
20 Lemma exercise (m n : Z) : (n + m) - n * 1 = m.
21 Proof. by rewrite mulr1 (addrC n) -(addrA m) addrN addr0. Qed.

```

We can print the type for monoids as forged by `HB` (line 1). It packs a carrier, called `sort`, and the collection of operation and properties that we omit for brevity. We can also look at the type of two constants synthesized by `HB` out of the hierarchy declaration. Remark that while the names of the constants come from the names of mixin fields, their types differ. In particular they are quantified over a `Monoid.type` or `Ring.type`, and not a simple type `A` as in the mixins. Moreover we evince that the `sort` projection is declared as an implicit coercion [24] and is automatically inserted in order to make `M -> M -> M` a meaningful type for binary operations on the carrier of `M`. Last, we see that properties are quantified on (hence apply to) the structure they belong to but use, in their statements, operations belonging to simpler structures. For example `addNr` is a property of a ring but its statement mentions `add`, the operation of the underlying monoid.

We then follow the proof of Hankel [5] to show that ring axioms imply the commutativity of the underlying monoid (line 5). This simple example shows we can populate the theory of abstract rings with new results.

We use the `Monoid_of_Type.Axioms abbreviation` (line 8) in order to build an instance of the `Monoid` structure for binary integers `Z`. We then register that monoid instance as the canonical one on `Z` (line 11) via the command `hb.canonical`. We can similarly declare that `Z` forms a ring by using the `Ring_of_Monoid.Axioms abbreviation` (lines 13 and 18). Note that the `Ring_of_Monoid.Axioms abbreviation` is not a plain record constructor for `Ring_of_Monoid.axioms`, since that would require more arguments, namely the monoid ones

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106 (see the `_` at line 13). The abbreviation synthesized by *HB* infers them automatically (as
107 in [17, Section 7]) thanks to the `hb.canonical` declaration given just above.

108 From now on the axioms as well as the abstract theory of rings apply to integers, as shown
109 in lemma `exercise`. The details of the proof do not matter here, what is worth pointing out
110 is that in a single statement we mix monoid (e.g. `+`) and ring (e.g. `-`) operations and in the
111 proof we use monoid axioms (e.g. `addrA`), ring axioms (e.g. `addrN`) and ring lemmas (e.g.
112 `addrC`), all seamlessly.

113 2.1 Evolution of the hierarchy

114 We proceed by accommodating the intermediate structure of Abelian groups.

```
1  HB.mixin Record Monoid_of_Type A := { ... (* unchanged *) ... }.
2  HB.structure Definition Monoid := { A & Monoid_of_Type.axioms A }.
3
4  HB.mixin Record AbelianGroup_of_Monoid A of Monoid.axioms A := {
5    opp : A -> A;
6    addrC : commutative (add : A -> A -> A);
7    addrNr : left_inverse zero opp add;
8  }.
9  HB.structure Definition AbelianGroup :=
10 { A of Monoid.axioms A & AbelianGroup_of_Monoid.axioms A }.
11
12 HB.mixin Record Ring_of_AbelianGroup A of AbelianGroup.axioms A := {
13   one : A;
14   mul : A -> A -> A;
15   mulrA : associative mul;
16   mulr : left_id one mul;           mulr1 : right_id one mul;
17   mulrDl : left_distributive mul add; mulrDr : right_distributive mul add;
18 }.
19 HB.structure Definition Ring :=
20 { A of AbelianGroup.axioms A & Ring_of_AbelianGroup.axioms A }.
21
22 Lemma addrN {R : AbelianGroup.type} : right_inverse zero opp add.
23 Proof. by move=>x; rewrite addrC addrNr. Qed.
```

115 Some operations and properties were moved from the old mixin for rings into a newborn
116 mixin `AbelianGroup_of_Monoid` that gathers the axioms needed to turn a monoid into an
117 Abelian group. Consequently the mixin for rings is now called `Ring_of_AbelianGroup` (instead
118 of `Ring_of_Monoid`) since it expects the type `A` to be already an Abelian group and hence
119 gathers fewer axioms.

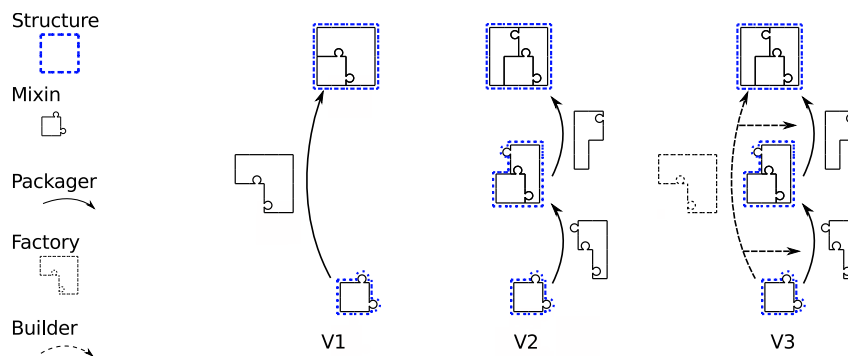
120 While operations moved from one structure to another, some properties undergo a deep
121 change in their status. The lemma `addrC` part of the abstract theory of rings is now an axiom
122 of Abelian groups, while `addrN` is no more an axiom of rings, but rather a theorem of the
123 abstract theory of Abelian groups.

124 With this new version of the hierarchy, that we name `V2`, code written for version `V1`
125 breaks. For example the declaration of the canonical ring over the integers fails, if only
126 because we do not have a `Ring_of_Monoid.Axioms` abbreviation anymore.

127 Our objective is to obtain a version of the hierarchy, that we name `V3`, that does not
128 only feature Abelian groups but that is also backward compatible with `V1`.

129 2.2 The missing puzzle piece

130 The key to make a hierarchy evolve without breaking user code is the full fledged notion
131 of *factory* (the mixins seen so far are degenerate, trivial, factories). Factories, like mixins,
132 are packages for operations and properties but are not directly used in the definition of



■ **Figure 1** The evolution of the hierarchy. V3 is backward compatible with V1, while V2 is not.

133 structures. Instead a factory is equipped with *builders*: user provided pieces of code that
 134 extract from the factory the contents of mixins, so that existing abbreviations can be used.

135 As depicted in figure 1 we change again the hierarchy by declaring a `Ring_of_Monoid` factory,
 136 that, from the user point of view, will look indistinguishable from the old `Ring_of_Monoid`
 137 mixin and hence grant backward compatibility between version V3 and version V1.

```

1  HB.factory Record Ring_of_Monoid A of Monoid.axioms A := {
2  ... (* unchanged, see page 2 line 13 *) ...
3  }.
4  HB.builders Context A (f : Ring_of_Monoid.axioms A).
5
6  Lemma addrC : commutative add. Proof. (* The same proof as before *) Qed.
7
8  Definition to_AbelianGroup_of_Monoid :=
9  AbelianGroup_of_Monoid.Axioms A (opp f) addrC (addNr f). (* addrN unused *)
10 HB.instance A to_AbelianGroup_of_Monoid.
11
12 Definition to_Ring_of_AbelianGroup :=
13 Ring_of_AbelianGroup.Axioms A (one f) (mul f)
14 (mulrA f) (mulr f) (mulr1 f) (mulrDl f) (mulrDr f).
15 HB.instance A to_Ring_of_AbelianGroup.
16
17 HB.end.
```

138 The record `Ring_of_Monoid.axioms` is the same we declared as a mixin in version V1.
 139 In order to make a factory out of it we equip it with two definitions that embody the
 140 *builders*. The first is `to_AbelianGroup_of_Monoid` and explains how to build an `AbelianGroup`
 141 structure out of the factory axioms (named `f`, line 4). This construction is also registered
 142 as canonical for `A`, so that the next construction `to_Ring_of_AbelianGroup` can call the
 143 `Ring_of_AbelianGroup.Axioms` abbreviation that requires `A` to be an `AbelianGroup`. It is worth
 144 pointing out that the proof of `addrC` we had in V1 is now required in order to write the
 145 builder for Abelian groups, while the `addrN` field is not used (the same statement is already
 146 part of the theory of Abelian groups, see line 22 of the previous code snippet).

147 Thanks to this factory we can now declare `Z` to be an instance of a ring using the
 148 `Ring_of_Monoid.Axioms` abbreviation. The associated builders generate, behind the scenes,
 149 instances of the `Ring_of_AbelianGroup.axioms` and `AbelianGroup_of_Monoid.axioms` mixins
 150 that in turn are used to build instances of the `AbelianGroup` and `Ring` structures. Indeed, when
 151 used in the context of the hierarchy version V3, the command `HB.instance Z Z_Ring_axioms`
 152 makes `Z` an instance of *both* structures, and not just the `Ring` one as in version V1. Thanks
 153 to that the proof of `example` can use the theory of both structures, for example `addrC` holds

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154 on Abelian groups, `addrA` holds on monoids, while `addr1` holds on rings. As a result the very
155 same proof works on both version V1 and V3.

156 Last, it is worth pointing out that the new factory makes the following two lines equivalent
157 (the former declares rings in V1) since they both describe the same set of mixins.

```
1 HB.structure Definition Ring :=  
2   { A of Monoid.axioms A & Ring_of_Monoid.axioms A }.  
3 HB.structure Definition Ring :=  
4   { A of AbelianGroup.axioms A & Ring_of_AbelianGroup.axioms A }.
```

158 This is another example of client code that would not break: the client of the hierarchy
159 is allowed to declare new structures on top of existing ones.

160 2.3 \mathcal{HB} in a nutshell

161 By using \mathcal{HB} the hierarchy designer has the following freedoms and advantages.

- 162 ■ Operations and properties (axioms) are made available to the user as soon as they are
163 used in a structure. The hierarchy designer is free to move them from one to another and
164 replace an axiom by a lemma and vice versa.
- 165 ■ Structures cannot disappear but the way they are built may change. The hierarchy
166 designer is free to split structures into smaller ones in order to better factor and reuse
167 parts of the hierarchy and the library that follows it.
- 168 ■ Mixins cannot disappear but can change considerably in their implementation. A mixin
169 can become a full fledged factory equipped with builders to ensure backward compatibility.
- 170 ■ \mathcal{HB} high level commands compile to the discipline of Packed Classes (Coq modules, records,
171 coercions, implicit arguments, canonical structure instances, notations, abbreviations).
172 This process lifts a considerable burden from the shoulders of the hierarchy designer and
173 the final user who are no longer required to master the details of Packed Classes.

174 3 \mathcal{HB} language

175 The Coq terms handled by \mathcal{HB} are subdivided in five categories, the mixins \mathcal{M} , the factories \mathcal{F} ,
176 builders \mathcal{B} , the classes \mathcal{C} and the structures \mathcal{S} . Mixins, factories and instances are tagged
177 by the user, through the commands `HB.mixin`, `HB.factory` and `HB.instance`, and the user
178 may rely on their implementation. However structures and classes are generated with the
179 command `HB.structure` and builders are generated when using `HB.instance` while declaring
180 a factory, and the user may only refer to structure types, but should never rely on their
181 implementation, neither should they rely on explicit builders.

182 3.1 \mathcal{HB} mixins, factories and instances

183 ► **Definition 1** (\mathcal{M} , mixins). A mixin $m \in \mathcal{M}$ is a Coq record with one or more parameters.
184 The first parameter must be a $(T : \text{Type})$, while the other parameters are mixins applied to T
185 and possibly n other distinct mixins. I.e.

```
1 Record m (T : Type)  $\overline{(p : m T p_\sigma)^n} : \text{Type} := \{ .. \}.$ 
```

186 where $\overline{(p : m T p_\sigma)^n} = \overline{(p_i : m_i T p_{\sigma(i,1)} \dots p_{\sigma(i,n_i)})_{i \in \{1, \dots, n\}}}$. For all $i \in \{1, \dots, n\}$ we
187 have $m_i \in \mathcal{M}$ and its arguments consist in n_i of the previously quantified mixin parameters
188 p_k , i.e. for all $k \in \{1, \dots, n_i\}$, we have $\sigma(i, k) \in \{1, \dots, i - 1\}$.

189 ► **Definition 2** (*dep*, mixin dependencies). Given $m \in \mathcal{M}$, we define $\text{dep}(m) \in \wp(\mathcal{M})$ as the
 190 set of all mixins that occur as parameters of m , i.e. $\text{dep}(m) = \{m_1, \dots, m_n\}$.

191 ► **Remark 3.** For all $i \in \{1, \dots, n\}$, we have $\text{dep}(m_i) = \{m_{\sigma(i,1)}, \dots, m_{\sigma(i,m_i)}\}$.

192 ► **Definition 4.** If $f : \mathcal{X} \rightarrow \wp(\mathcal{Y})$, then $f^* : \wp(\mathcal{X}) \rightarrow \wp(\mathcal{Y})$ is defined as $f^*(X) = \bigcup_{x \in X} f(x)$.

193 ► **Proposition 5.** *dep* is transitively closed since records are well typed in the empty context
 194 and describes a DAG since Coq does not admit circular definitions: $\text{dep}^*(\text{dep}^*(M)) \subseteq$
 195 $\text{dep}^*(M)$

196 ► **Definition 6** (factories \mathcal{F} , builders \mathcal{B} and from). A builder $\mu \in \mathcal{B}$ is any function which para-
 197 meters are the carrier type (T : **Type**), an arbitrary number of distinct mixins $\{m_1, \dots, m_n\}$
 198 and an additional argument f , which we call a factory, and which depends on the mixins. Its
 199 return type is a mixin $m_{n+1} \in \mathcal{M}$ i.e.

1 **Definition** $f (T : \mathbf{Type}) \overline{(p : m \ T \ p_\sigma)^n} : \mathbf{Type} := \dots$

2 **Definition** $\mu (T : \mathbf{Type}) (p : m \ T \ p_\sigma)^n : f \ T \ p_1 \dots p_n \rightarrow m_{n+1} \ T \ p_{\sigma(n+1,1)} \dots p_{\sigma(n+1,n_i)} := \dots$

200 We define $\text{from}(f, m_{n+1}) = \mu$ to be the (unique) builder for m_{n+1} from the factory f .

201 ► **Definition 7** (*provides*, *requires*). We define the following functions

202 ■ $\text{requires}(f) = \{m_1, \dots, m_n\} \in \wp(\mathcal{M})$ is the set of mixins a factory $f \in \mathcal{F}$ depends on. As
 203 a consequence all the builders of a given factory have the same set of dependencies.

204 ■ $\text{provides}(f) = \{m \mid \text{from}(f, m) \text{ is defined}\} \in \wp(\mathcal{M})$ the set of mixins that a factory $f \in \mathcal{F}$
 205 provides, through its builders.

206 Moreover the following properties hold for all $f \in \mathcal{F}$:

207 1. $\text{dep}^*(\text{requires}(f)) \subseteq \text{requires}(f)$

208 2. $\text{requires}(f) \cap \text{provides}(f) = \emptyset$

209 3. $\forall m \in \text{provides}(f)$, $\text{from}(f, m)$ is defined

210 Mixins are declared by the user as the fundamental building blocks of a hierarchy. As
 211 the next proposition shows they shall not be distinguished from regular factories, since they
 212 are a special cases.

213 ► **Proposition 8** ($\mathcal{M} \subseteq \mathcal{F}$). For all $m \in \mathcal{M}$ we have $\text{requires}(m) = \text{dep}(m)$, $\text{provides}(m) =$
 214 $\{m\}$ and $\text{from}(m, m) = \left(\mathbf{fun} \ T \ \overline{(p : m \ T \ p_\sigma)^n} \ (x : m \ T \ p_1 \dots p_n) \Rightarrow x \right) \in \mathcal{B}$.

215 ► **Coq command** to declare a new mixin:

1 **HB.mixin Record** $M \ T$ of $f_1 \dots f_n := \{ \dots \}$.

216 Creates a module M with a record $M.\text{axioms}$, which depends on the mixins $\text{requires}(M.\text{axioms}) =$
 217 $\text{provides}^*(\{f_1, \dots, f_n\})$, and registers this new record both as a mixin and a factory. It also
 218 creates a abbreviation $M.\text{Axioms}$.

219 ► **Coq command** to declare an instance: **HB.instance** $X \ b_1 \dots b_k$, synthesizes terms
 220 corresponding to all the mixins that can be built from the b_i . Indeed if $b_i : f_i \ T \dots$, then this
 221 command creates elements of type $\text{provides}^*(\{f_1, \dots, f_k\})$. This command also generates
 222 unification hints as described in Section 4.

223 ► **Coq command** to declare a new factory and generate new builders:


```

1  HB.factory Record M T of f1 ... fn := { .. }.
2  HB.builders Context T (a : M.axioms A).
3
4  Definition bn+1 : fn+1 T .. := ...
5  HB.instance T bn+1.
6  ..
7  Definition bn+k : fn+k T .. := ...
8  HB.instance T bn+k.
9  HB.end.

```

224 Creates a module M with a record $M.axioms$, which depends on the mixins $requires(M.axioms) =$
225 $provides^*(\{f_1, \dots, f_n\})$ and registers this new record as a factory. We then use the fact-
226 ory instances b_{n+1}, \dots, b_{n+k} provided by the user and derive builders from them, so that
227 $provides(M.axioms) = provides^*(\{f_{n+1}, \dots, f_{n+k}\})$.

228 It is thus necessary that $requires^*(\{f_{n+1}, \dots, f_{n+k}\}) \subseteq provides^*(\{f_1, \dots, f_n\})$.

229 Note that the b_i are not builders since their return types are not necessarily mixins, but
230 could be a factory. However, since all factories provide mixin through builders, we can obtain
231 builders out of each b_i by function composition.

232 3.2 \mathcal{HB} classes and structures

233 ► **Definition 9** (\mathcal{C} , class). A class $c \in \mathcal{C}$ is a Coq record with one parameter ($T : \mathbf{Type}$). The
234 type of each field is a mixin in \mathcal{M} applied to T and, if needed, any number of other fields:

```

1  Record c (T : Type) := { p1 : m1 T; .. ; pi : mi T pσ(i,1) ... pσ(i,ni); .. }.

```

235 ► **Definition 10** (def , class definition). We call $def(c) \in \wp(\mathcal{M})$ the set of mixins mentioned
236 in the fields of the class, i.e. $\{m_1, \dots, m_n\}$. Given that class records are well typed in the
237 empty context the set of mixin records is closed transitively. The implementation enforces
238 that no two class records contains the same set of mixins (disregarding the order of the fields),
239 i.e. def is injective.

240 ► **Programming invariant for def** For all $f \in \mathcal{F}$:

- 241 1. $\exists c \in \mathcal{C}$, $def(c) = requires^*(f) \cup provides^*(f)$,
- 242 2. $\exists C \subseteq \mathcal{C}$, $def^*(C) = requires^*(f)$.

243 Classes could also be seen as factories.

244 ► **Proposition 11** ($\mathcal{C} \subseteq \mathcal{F}$). for all $m \in def(c)$ we have $from(c, m_i) = p_i$, and it follows that
245 $requires(c) = \emptyset$, $provides(c) = def(c)$.

246 However since classes are generated, their implementation may change, thus users should
247 not rely on constructors of classes. Hence the only way users may refer to a class is as an
248 argument of **HB.mixin**, **HB.factory** or **HB.structure**.

249 ► **Definition 12** (\mathcal{S} Structure). A structure $s \in \mathcal{S}$ is a dependent pair: a Coq record where
250 the value of the first field occurs in the type of the second. The first field is ($sort : \mathbf{Type}$)
251 and the second field is ($class : c sort$) for some $c \in \mathcal{C}$. As a consequence structures are in
252 bijection with classes.

253 ► **Coq command** to declare a class and structure: **HB.structure Definition** $M := \{ A \text{ of}$
254 $f_1 \dots f_n \}$. crafts a class $M.class_of \in \mathcal{C}$ where $def(c) = provides^*(\{f_1, \dots, f_n\})$ and the
255 corresponding structure $M.type \in \mathcal{S}$, together with unification hints as described in Section 4.

256 ► **Definition 13** ($\leq \in \mathcal{C} \times \mathcal{C}$, subclass). $c_1 \leq c_2$ iff $def(c_2) \subseteq def(c_1)$

257 3.3 Automatic inference of mixins

258 Since mixins may change but factories stay the same, \mathcal{HB} commands must never rely on a
 259 particular set of mixins as arguments, and factory arguments must never be given explicitly
 260 by the user. As described in Sections 3.1 and 3.2, \mathcal{HB} commands take a list of factories as
 261 arguments, which they expand into lists of mixins behind the scene. However factory types
 262 and constructors have mixin arguments that must be inferred automatically when used. To
 263 this end, the commands `HB.mixin` and `HB.factory` generate abbreviations for the user to
 264 replace uses of constructors of factories. These commands first create a record f_{aux} with a
 265 constructor F_{aux} and then create abbreviations f and F that automatically fill the mixin
 266 arguments of f_{aux} and F_{aux} respectively. See [17, Section 7] for a detailed description of how
 267 to implement these abbreviations in Coq.

```

1 Record  $f_{aux}$  T  $\overline{(p : m T p\sigma)^n} := F_{aux} \{ \dots \}$ .
2 Notation  $f$  T := ( $f_{aux}$  T ... (*  $p_1 \dots p_n$  inferred when T is known *)).
3 Notation F T  $x_1 \dots x_k := (F_{aux}$  T ... (*  $p_1 \dots p_n$  inferred when T is known *)  $x_1 \dots x_k$ ).
4 Definition b :  $f$  T := F T  $x_1 \dots x_n$ 

```

268 4 The target language: Coq with Packed, flat, Classes

269 The language of packed classes were introduced in [13] and used directly to describe the
 270 algebraic hierarchy of the Mathematical Components library. It is based on a disciplined use
 271 of records and projections and on the possibility of extending the elaborator of Coq via the
 272 declaration of Canonical Structures instances. In this section we describe the *flat* variant of
 273 Packed Classes, the target language of \mathcal{HB} .

274 4.1 Describing structures with records and projections

275 We describe mathematical structures with three kinds of dependent records: mixins, classes,
 276 and structures. As shown in section 2, a mixin gathers operators and axioms newly introduced
 277 by a structure. As in [13, section 2.4][26, section 2], a class record is parametrized by the
 278 carrier type (T : **Type**) and gathers all the operators and axioms of a structure by assembling
 279 mixins, and a **Structure** type (a record) bundles a carrier and its class instance, as follows.

```

1 Module Monoid.
2 Record axioms (T : Type) : Type :=
3   Class { Monoid_of_Type_mixin : Monoid_of_Type.axioms T; }.
4 Structure type : Type := Pack { sort : Type; class : axioms sort; }.
5 End Monoid.

```

280 The `Monoid` module plays the role of a name space and forces us to write qualified names such
 281 as `Monoid.type`; as a consequence we can reuse the same unqualified names for other structures,
 282 i.e., class and structure record can always be named as `axioms` and `type` respectively.

283 Mixins and classes are internal definitions to structures; in contrast, `Monoid.type` is part
 284 of the interface of the monoid structure. As seen in section 2, we declare `Monoid.sort` as
 285 an implicit coercion and lift monoid operators and axioms from projections for the monoid
 286 mixin to definitions and lemmas for `Monoid.type` as follows.

```

1 Coercion Monoid.sort : Monoid.type >-> Sortclass.
2
3 Definition zero {T : Monoid.type} : T :=
4   Monoid_of_Type.zero T (Monoid.Monoid_of_Type_mixin T (Monoid.class T)).
5 Definition add {T : Monoid.type} : T -> T -> T :=
6   Monoid_of_Type.add T (Monoid.Monoid_of_Type_mixin T (Monoid.class T)).
7 Lemma addrA {T : Monoid.type} : associative (@add T).
8 (* Two monoid axioms `addOr` and `addr0` are omitted. *)

```

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Next we define Abelian groups. Since the monoid structure is the bottom of this hierarchy its class record `Monoid.axioms` consists just one mixin. In contrast the class record of Abelian groups consists of two mixins where the second one depends on the former (since Abelian groups inherit from monoids).

```

1 Module AbelianGroup.
2 Record axioms (T : Type) : Type := Class {
3   Monoid_of_Type_mixin : Monoid_of_Type.axioms T;
4   AbelianGroup_of_Monoid_mixin : AbelianGroup_of_Monoid.axioms T Monoid_of_Type_mixin; }.
5 Structure type : Type := Pack { sort : Type; class : axioms sort }.
6 End AbelianGroup.

```

As in the monoid structure, we declare `AbelianGroup.sort` as an implicit coercion and lift additive inverse `opp` and Abelian group axioms. Since Abelian groups inherit from monoids, we also declare an implicit coercion from Abelian groups to monoids. A coercion between structures can be defined in two steps: first a coercion between the class record of the superclass to the class record of the subclass (line 2); and a coercion between structure records (line 4) relying on the first one.

```

1 Coercion AbelianGroup.sort : AbelianGroup.type >-> Sortclass.
2 Coercion AbelianGroup_class_to_Monoid_class (T : Type) (c : AbelianGroup.axioms T) :
3   Monoid.axioms T := Monoid.Class T (AbelianGroup.Monoid_of_Type_mixin T c).
4 Coercion AbelianGroup_to_Monoid (T : AbelianGroup.type) : Monoid.type :=
5   Monoid.Pack (AbelianGroup.sort T) (AbelianGroup.class T).
6
7 Definition opp {T : AbelianGroup.type} : T -> T := ... .
8 (* Two Abelian group axioms `addrC` and `addNr` are omitted. *)

```

Generally, a coercion from a structure A to another structure B can (and should) have the following form, thanks to the corresponding coercion between classes `A_class_to_B_class`.

```

1 Coercion A_to_B (T : A.type) : B.type :=
2   B.Pack (A.sort T) ((* A_class_to_B_class _ *) (A.class T)).

```

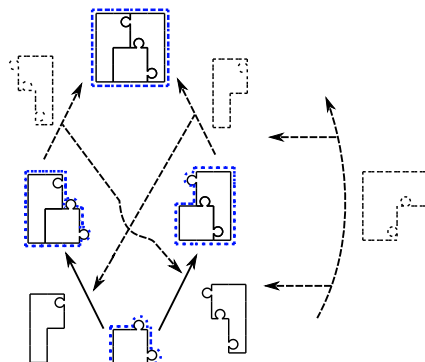
4.2 Multiple inheritance

In order to introduce multiple inheritance, we extend the hierarchy described in section 2.2 with the structure of semirings as depicted in figure 2. Semirings introduce the binary multiplication operator `mul` and its identity element `one`.

```

1 HB.mixin Record SemiRing_of_Monoid A
2   of Monoid.axioms A := {
3     one : A;
4     mul : A -> A -> A;
5     (* 7 axioms are omitted. *)
6   }.
7
8 HB.factory Record Ring_of_AbelianGroup A
9   of AbelianGroup.axioms A := {
10    (* 2 operators and 5 axioms are omitted. *)
11  }.
12 HB.builders Context A (a : Ring_of_AbelianGroup A).
13 ..
14 HB.end.

```



■ **Figure 2** V4 introduces multiple inheritance. For brevity we omit the factories/builders for the upper arrows.

Since semirings and Abelian groups do not inherit from each other, the definition of semirings and Abelian groups are quite similar:

```

1 Module SemiRing.
2 Record axioms (T : Type) : Type := Class {
3   Monoid_of_Type_mixin : Monoid_of_Type.axioms T;
4   SemiRing_of_Monoid_mixin : SemiRing_of_Monoid.axioms T Monoid_of_Type_mixin; }.
5 Structure type : Type := Pack { sort : Type; class : axioms sort; }.
6 End SemiRing.

```

307 We define implicit coercions from the semiring structure to the carrier and the monoid
308 structure, and we lift semiring operators and axioms as follows:

```

1 Coercion SemiRing.sort : SemiRing.type -> Sortclass.
2 Coercion SemiRing_to_Monoid : SemiRing.type -> Monoid.type.
3
4 Definition one {T : SemiRing.type} : T := ... .
5 Definition mul {T : SemiRing.type} : T -> T -> T := ... .
6 (* 7 semiring axioms are omitted. *)

```

309 The class record is defined by gathering the monoid, Abelian group, and semiring mixins.
310 Since the rings inherit from monoids, semirings, and Abelian groups, we define implicit
311 coercions from the ring structure to those three structures.

```

1 Module Ring.
2 Record axioms (T : Type) : Type := Class {
3   Monoid_of_Type_mixin : Monoid_of_Type.axioms T;
4   SemiRing_of_Monoid_mixin : SemiRing_of_Monoid.axioms T Monoid_of_Type_mixin;
5   AbelianGroup_of_Monoid_mixin : AbelianGroup_of_Monoid.axioms T Monoid_of_Type_mixin; }.
6 Structure type : Type := Pack { sort : Type; class : axioms sort; }.
7 End Ring.
8
9 Coercion Ring.sort : Ring.type -> Sortclass.
10 Coercion Ring_to_Monoid : Ring.type -> Monoid.type.
11 Coercion Ring_to_SemiRing : Ring.type -> SemiRing.type.
12 Coercion Ring_to_AbelianGroup : Ring.type -> AbelianGroup.type.

```

312 4.3 Linking structures and instances via Canonical Structures

313 We recall here how to use the Canonical Structures [17, 25] mechanism, which lets the user
314 extend the elaborator of Coq, in order to handle inheritance and inference of structure [1,
315 26, 13]. This software component takes in input a term as written by the user and has to
316 infer all the missing information that is necessary in order to make the term well typed.

317 A first example of elaboration that requires canonical structures is $0 + 1$. After removing
318 all syntactic facilities, the underlying Coq term is `(add _ (zero _)) (one _)` where `_` stands
319 for an implicit piece of information to be inferred and the constants `add` and `zero` come from
320 the monoid structure while `one` comes from semirings. When the term is type checked each `_`
321 is replaced by a unification variable written `?v`. Respectively, the head and the argument of
322 the top application can be typed as follows:

```

1   add ?M (zero ?M) : Monoid.sort ?M -> Monoid.sort ?M
2   one ?SR : SemiRing.sort ?SR

```

323 where `?M : Monoid.type` and `?SR : SemiRing.type`. In order to type check the applica-
324 tion Coq has to unify `Monoid.sort ?M` with `SemiRing.sort ?SR`, which is not trivial: it
325 amounts at finding a structure that is both a monoid and a semiring, possibly the smal-
326 lest one [26, Sect. 4]. This piece of information can be inferred from the hierarchy and
327 its inheritance relation (Definition 13) and we can tell Coq to exploit it by declaring
328 `SemiRing_to_Monoid : SemiRing.type -> Monoid.type` as canonical. With that hint Coq will
329 pick `?M` to be `SemiRing_to_Monoid ?SR` as the canonical solution for this unification problem.
330 In general, all the coercions between structures must be declared as canonical.

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331 Another example of elaboration problem is `-1`, which hides the term `opp _ (one _)`. Here
332 `opp` and `one` are respectively from Abelian groups and semirings, which do not inherit each
333 other but whose smallest common substructure is rings; thus we have to extend the unifier
334 to solve a unification problem `AbelianGroup.sort ?AbG = SemiRing.sort ?SR` by instantiating
335 `?AbG` and `?SR` with `Ring_to_AbelianGroup ?R` and `Ring_to_SemiRing ?R` respectively where `?R` is
336 a fresh unification variable of type `Ring.type`. This hint can be given as follows:

```
1 Canonical AbelianGroup_to_SemiRing (T : Ring.type) :=  
2   SemiRing.Pack (AbelianGroup.sort (Ring_to_AbelianGroup T)) (Ring.class T).
```

337 Similarly, one can apply an algebraic theory to an instance (an example) of that structure,
338 e.g., 2×3 where 2 and 3 have type \mathbb{Z} . The same mechanism of canonical structures let us
339 extend the unifier to solve `SemiRing.type ? = \mathbb{Z}` .

340 5 The implementation in Coq-Elpi

341 The implementation is based on the Elpi extension language for Coq. In this section we
342 introduce the features of the programming language that came in handy in the development
343 of *HB* and comment a few code snippets.

344 Coq-Elpi [28] is a Coq plugin embedding Elpi and providing an extensive, high level,
345 API to access and script the Coq system at the vernacular level. This API lives in the
346 `coq` namespace and lets one easily declare records, coercions, canonical structures, modules,
347 implicit arguments, etc. The most basic Coq data type exposed to Elpi is the one of references
348 to global declarations:

```
1 kind gref type.           % The data type of references to global terms  
2 type indt inductive -> gref. % eg: Coq.Init.Datatypes.nat  
3 type indc constructor -> gref. % eg: Coq.Init.Datatypes.O  
4 type const constant -> gref. % eg: Coq.Init.Peano.plus
```

349 The arguments of the three constructors are opaque to Elpi, that can only use values of
350 these types via dedicated APIs. For example the API for declaring an inductive type will
351 generate a value of type `inductive` that is printed as, for example, `<nat>`.

352 Elpi [11] is a dialect of λ Prolog [19], an higher order logic programming language that
353 makes it easy to manipulate abstract syntax tree with binders. Coq-Elpi takes full advantage
354 of this capability by representing Coq terms in Higher Order Abstract Syntax [20] style,
355 reusing the binder of the programming language in order to represent Coq's ones. Here is an
356 excerpt of the data type of Coq terms:

```
1 kind term type.           % The data type of Coq terms  
2 type global gref -> term. % eg: nat, 0, S, plus, ...  
3 type fun term -> (term -> term) -> term. % eg: fun x : t => b(x)  
4 type app list term -> term. % eg: app [hd|args]  
5 ... % all other term constructors are omitted for brevity
```

357 Note that the `fun` constructor holds a λ Prolog function. In this syntax the Coq term
358 `(fun x : nat => x x)` becomes `(fun (global (indt <nat>)) x\ app[x, x])` where `x\` binds
359 `x` in the body of the function. Substitution of a bound variable for a term can be computed
360 by applying a term (of function type) to an argument.

361 Data types with binders are also used as input to high level APIs that build terms behind
362 the scenes. For example a Coq record is just an inductive type and the API to declare one
363 must allow the type of a field to depend on the fields that comes before it. Note that the
364 `field` constructor takes a coercion flag, the name of the field, its type and binds a term in
365 the remaining record declaration.

```

1 kind indt-decl type.           % The type of an inductive type declaration
2 type record      string -> term -> string -> record-decl -> indt-decl.
3 type field      bool -> string -> term -> (term -> record-decl) -> record-decl.
4 type end-record record-decl.
5 ... % constructors for non-record inductive types are omitted for brevity
6
7 external pred coq.env.add-indt i:indt-decl, o:inductive.
8 external pred coq.CS.canonical-projections i:inductive, o:list (option constant).

```

366 The `pred` keyword documents types and modes (input or output) of the arguments of
 367 a predicate, while `external` signals that the predicate is a builtin (in other words it is
 368 implemented in OCaml rather than λ Prolog).

369 We comment these two builtin predicates and the `indt-decl` type while looking at the
 370 code of `declare-structure` that is in charge of scripting the following Coq code:

```

1 Structure type : Type := Pack { sort : Type; class : axioms sort }.

```

371 The following Elpi code builds the declaration, type checks it, adds it to the Coq
 372 environment and finally returns the projections for the sort and the class fields.

```

1 pred declare-structure i:ref, o:term, o:term, o:term.
2
3 declare-structure ClassName Structure SortProjection ClassProjection :- std.do! [
4   StructureDeclaration =
5     record "type" {{ Type }} "Pack" (
6       field tt "sort" {{ Type }} s\
7       field ff "class" (app [global ClassName, s]) _\
8     end-record),
9   coq.typecheck-indt-decl StructureDeclaration,
10  coq.env.add-indt StructureDeclaration StructureName,
11  coq.CS.canonical-projections StructureName [some SortP, some ClassP],
12  Structure = global (indt StructureName),
13  SortProjection = global (const SortP),
14  ClassProjection = global (const ClassP),
15 ].

```

373 Note that the binder `s\` at line 6 lets one mention the first field in the type of the second.
 374 The syntax `{{ Type }}` is a quotation: it lets one use the syntax of Coq to write an Elpi
 375 expression of type `term`. The API `coq.CS.canonical-projections` lets us find the projections
 376 automatically generated by Coq for a given record. The last detail worth mentioning is that
 377 this program makes no use of backtraking: the `std.do!` combinator signals that.

378 In the simple case of the structure record, the number of fields, and hence the number of
 379 binders, is fixed. On the contrary the class record has one field per mixin and each of them
 380 can depend on the previous ones. In order to synthesize terms with binders in an inductive
 381 fashion (using a recursive predicate) λ Prolog lets one postulate fresh nominal constants using
 382 the `pi` operator and attach to the nominal some knowledge in the form of a clause via the `=>`
 383 operator. This process is called binder mobility: the binder is moved from the data (that we
 384 are building) to the program (the context of the current computation). This feature is key
 385 to the following code that synthesizes the declaration of the fields of the class record.

```

1 pred synthesize-fields.field-for-mixin i:mixinname, o:term.
2 pred synthesize-fields i:list mixinname, i:term, o:record-decl.
3
4 synthesize-fields [] _ Decl :- Decl = end-record.
5 synthesize-fields [M|ML] T Decl :- std.do! [
6   get-mixin-modname M ModName, Name is ModName ^ "_mixin",
7   dep1 M Deps,
8   std.map Deps synthesize-fields.field-for-mixin Args,
9   Type = app[ global M, T | Args ],

```

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```
10   Decl = (field ff Name Type f\ Fields f),
11   pi m\
12     synthesize-fields.field-for-mixin M m =>
13     synthesize-fields ML T (Fields m)
14 ]).
```

386 The first predicate `synthesize-fields.field-for-mixin` is used to link a mixin to a
387 nominal that corresponds to the record field for that mixin. It has no clauses in the base
388 program but some clauses are added dynamically by `synthesize-fields`.

389 The second predicate proceeds by recursion on the (topologically sorted) list of mixins,
390 and terminates when the list is empty. If the list contains a mixin `M` then it crafts a `Name` for
391 it (line 6), fetches its dependencies (line 7) and finds the (previously declared) record fields
392 holding these mixins (line 8). The `(std.map L1 P L2)` predicate relates the two lists `L1` and
393 `L2` point wise using the predicate `P`.

394 Line 9 builds the type of the field: the mixin name applied to the type (sort) and all its
395 dependencies. Note that the `Fields` variable, representing the declaration of the next fields,
396 is under the binder for `f` (the current field). In order to make the recursive call under that
397 binder (line 13) and recursively process `ML` we postulate a nominal `m` (line 11) that is a term
398 satisfying any future dependency on the current mixin (line 12) and we replace `f` by `m` in
399 `Fields` by writing `(Fields m)`.

400 6 Related work

401 The most closely related work is the one about Packed Classes [13] on which we build. The
402 main differences are that `HB` is a higher level language that is compiled down to (flat)
403 Packed Classes. The systematic use of factories makes the user interface of a hierarchy stable
404 under the insertion of structures, a property that Packed Classes lacks. Finally many of the
405 intricacies of Packed Classes are hidden to the user by the compilation step, in particular
406 creating all the necessary coercions and canonical structure declarations, especially in the case
407 of diamonds or when merging several libraries, which used to cause the need for *a posteriori*
408 validation of a hierarchy design [26]. It also opens the way to automatically detect and solve
409 problems tied to non judgmental commutative diagrams when forgetting structure [1].

410 In [7] Carette and O'Connor describe the language of Theory Presentation Combinators
411 that can be used to describe a hierarchy of algebraic structures. They focus on the categorical
412 semantics of the language that is built upon the category of context. They do not describe
413 any actual compilation to the language of a mainstream interactive prover, indeed they claim
414 their language to be mostly type theory independent. We know they considered targeting
415 type theory and the language of unification hints [4] (a super set of the one of Canonical
416 Structures), but we could not find any written trace of that. Language wise they provides
417 keywords such as `combine` and `over` to share (reuse) a property when defining a new structure.
418 For example in order to avoid restating the commutativity property when defining Abelian
419 groups they combine a commutative monoid and a group forcing the subjacent monoid
420 to coincide: `CommutativeGroup := combine CommutativeMonoid , Group over Monoid`.
421 In our language `HB` the same role is played by *mixins*. A mixin lets one write once and for
422 all a property and abstract it over types and operations so that it can be reused in all the
423 structures that need it. One operation `HB` allows for but that does not seem to be possible
424 in the setting of Presentation Combinators is the one of replacing an axioms with a lemma
425 and vice versa. As shown in subsection 2.1 `HB` supports that.

426 The MMT system [22] provides a framework to describe formal languages in a logical
427 framework, providing good support for binders and notations. It also provides an expressive

428 module system to organize theories and express relations among them in the form of functors.
429 At the time of writing it provides limited support for elaborating user input taking systematic
430 advantage of the contents of the theories. The elaborator can be extended by the user writing
431 Scala code, and in principle use the contents of the libraries to make sense of an incomplete
432 expressions, but no higher level language or mechanism is provided.

433 The library of the Mizar system features a hierarchy of algebraic structures [15]. In spite
434 of lacking dependent types, Mizar provides the concept of attributed types and adjectives
435 that can be used to describe the signature of structures as one would do with a dependently
436 typed record and their properties as one would define a conjunctive predicate. The Mizar
437 language also provides the notion of cluster that is used to link structures: by showing that
438 property P implies property Q one can inform automation that structures characterized by
439 P are instances of structures characterized by Q . The foundational theory of Mizar features
440 an extensional notion of equality that makes it easy to share the signature or the properties
441 of structures by just requiring a proof of their equivalence that is in turn used by automation
442 to treat equivalent structures as equal.

443 The concepts of factory and builder presented here is akin to the `AbstractFactory` and
444 `Builder` pattern from "the Gang Of Four" design patterns [12] in the sense that factories are
445 used to build an arbitrary number of objects (here mixin instances).

446 **7 Conclusion**

447 In this paper we design and implement \mathcal{HB} , a high level language to describe hierarchies of
448 algebraic structures. The implementation of \mathcal{HB} is based on the Elpi extension language for
449 the Coq system and is available at <https://github.com/math-comp/hierarchy-builder>.
450 The implementation amounts to approximately a thousand lines of (commented) Elpi code
451 and tree hundred lines of Coq vernacular. It took less than one month to implement \mathcal{HB}
452 while it took several years of attempts and fruitful discussions with the other developers of
453 the Mathematical Components library to design it, as well as Coq users and developers and
454 members of Dagstuhl seminar 18341. In particular, discussions with Georges Gonthier, Assia
455 Mahboubi and Florent Hivert were instrumental in coining the concepts formalized here.

456 The \mathcal{HB} language is only loosely tied to Coq or even Type Theory. We believe it could
457 be adopted with no major change to other tools. Indeed the properties and invariants that
458 link factories, mixins and classes are key to rule out meaningless or ambiguous sentences
459 and are not specific to our logic setting. Moreover most logics feature packing construction
460 similar to records.

461 The compilation scheme we present in section 4 is tied to dependent records, that are
462 available in Coq but also in other provers based on Type Theory such a Matita [3] and
463 Lean [10]. We chose the flat variant of Packed Classes as the target for \mathcal{HB} because the
464 Mathematical Components library uses Packed Classes as well: As of today Packed Classes
465 offer the best compromise between flexibility and performance [17, Section 8] in Coq. Of
466 course one could imagine a future were other approaches such as telescopes [21] or unbundled
467 classes [27] would offer equal or better performances in Coq, or in another system. It is a
468 virtue of our work to provide a user language that is separate from the implementation one.
469 We believe it would take a minor coding effort to retarget \mathcal{HB} to another bundling approach.

470 We leave to future work extending \mathcal{HB} to support structures parametrized by structures
471 such as the one of module over a ring. We also leave to future work the automatic synthesis
472 of the notion of morphism between structures of the hierarchy.

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572 **A** Coq reference

```

1 Section OperationProperties.
2 Variable T : Type.
3 Variable e : T.
4 Variable inv : T -> T.
5 Variable op : T -> T -> T.
6 Variable add : T -> T -> T.
7
8 Definition left_id := forall x, op e x = x.
9 Definition right_id := forall x, op x e = x.
10
11 Definition left_inverse := forall x, op (inv x) x = e.
12
13 Definition commutative := forall x y, op x y = op y x.
14 Definition associative := forall x y z, op x (op y z) = op (op x y) z.
15
16 Definition left_distributive := forall x y z, op (add x y) z = add (op x z) (op y z).
17 Definition right_distributive := forall x y z, op x (add y z) = add (op x y) (op x z).

```

573 **B** Proof of addrC

```

1 Lemma addrC {R : Ring.type} : commutative (@add R).
2 Proof.
3 have innerC (a b : R) : (a + b) + (a + b) = (a + a) + (b + b).
4   by rewrite -[a+b]mul1r -mulrD1 mulrDr !mulrD1 !mul1r.
5 have addKl (a b c : R) : a + b = a + c -> b = c.
6   apply: can_inj (add a) (add (-a)) _ _ _ .
7   by move=> x; rewrite addrA addNr addOr.
8 have addKr (a b c : R) : b + a = c + a -> b = c.
9   apply: can_inj (add ^~ a) (add ^~ (-a)) _ _ _ .
10  by move=> x; rewrite /= -addrA addrN addrO.
11 move=> x y; apply: addKl (x) _ _ _; apply: addKr (y) _ _ _ .
12 by rewrite -!addrA [in RHS]addrA innerC !addrA.
13 Qed.

```