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EXPONENTIAL STABILIZATION OF NONHOLONOMIC SYSTEMS IN POWER FORM*

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Abstract. Systems in canonical power form have recently been used to model the kinematic equations of nonholonomic mechanical systems. The properties of homogeneous systems have been used in (M'Closey and Murray 1993a, M'Closey and Murray 1993b) to derive exponentially stabilizing continuous time-periodic feedbacks for this class of systems. Motivated by this work, the present study extends a control design method previously proposed by Samson to the design of such homogeneous feedbacks. The approach here followed has the advantage of yielding simple and direct stability proofs. Homogeneity-related results needed for time-varying exponential stabilization are also provided.

Keywords. nonholonomic systems, chained and power forms, time-varying stabilization, homogeneous systems.

1. INTRODUCTION

Recent studies on mobile robot control have been using so-called *chained form* and equivalent *power form* systems to model the kinematic equations of non-holonomic wheeled vehicles, see (Murray and Sastry 1991, Tilbury *et al.* 1992, Sørtdalen 1993, M'Closey and Murray 1993a, M'Closey and Murray 1993b, Samson 1993a) for example. It is now well known that nonholonomic systems, although controllable, are not point-stabilizable by means of continuous feedback $u(x)$ where x is the state (Bloch and McClamroch 1989, Samson and Ait-Abderrahim 1991a, d'Andréa Novel *et al.* 1991). This has motivated the study of alternative solutions, such as piecewise continuous feedback, as first proposed by (Bloch and McClamroch 1989) for a knife edge, and smooth time-varying feedback, as first proposed by (Samson 1991) in the case of a three-dimensional unicycle-type vehicle. The idea of smooth time-varying feedback $u(x, t)$, although little exploited before (Samson 1991), may in fact be traced back, for one-dimensional systems, to (Sontag and Sussmann 1980). Various developments have followed, like (Canudas and Sørtdalen 1991, Bloch *et al.* 1992, Sørtdalen and Egeland 1993) for piecewise continuous feedback, or (Pomet 1992, Samson 1992, Teel *et al.* 1992, Gurvits and Li 1992, Samson 1993b, M'Closey and Murray 1993a) for time-varying feedback. The potentialities of time-varying feedback, as a general stabilization technique, has also been explored in (Coron 1992, Coron 1994) where it is established that almost all controllable systems are stabi-

lizable by continuous time-periodic feedback.

As noted in (Samson 1991, Samson and Ait-Abderrahim 1991b), and subsequently in several other references, the convergence provided by smooth time-periodic feedback is only polynomial and not exponential, by contrast with some discontinuous feedbacks (Sørtdalen and Egeland 1993). In (M'Closey and Murray 1993a, M'Closey and Murray 1993b), continuous but non-lipschitz time-periodic control laws which render the closed-loop vector field homogeneous have been proposed for nonholonomic systems in power form. In this case, local stabilisation is equivalent to global stabilization and exponential convergence. However, the stability analysis proposed in (M'Closey and Murray 1993a, M'Closey and Murray 1993b) is not entirely conclusive, despite demonstrative simulation results. Also, the construction of adequate control laws for n -dimensional systems does not seem to be systematic. In (Samson 1993a), the stability proof for a class of exponentially stabilizing continuous time-varying feedbacks is simpler, but is only given for three-dimensional systems. The contribution of the present paper is, on one hand, to propose a systematic way of deriving, for n -dimensional systems in power form, a family of homogeneous continuous time-varying feedback laws, the stabilizing properties of which are easily proved, and, on the other hand, to establish properly the homogeneity-related results needed for time-varying exponential stabilization.

2. CHAINED/POWER FORM

It is now well known (Teel *et al.* 1992, Sørtdalen 1993, Samson 1993a) that, by an adequate (local)

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change of coordinates, the kinematic equations of various nonholonomic wheeled mobile robots, such as unicycle-type and car-like vehicles with or without trailers, can be written in the normal (or canonical) form of a *chained system*. A two-input system with a single chain has the form:

$$\left. \begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_j &= x_{j+1} u_1 \quad j = 2, \dots, n-1 \\ \dot{x}_n &= u_2 \end{aligned} \right\} \quad (1)$$

Under the following global change of coordinates:

$$\left. \begin{aligned} y_1 &= x_1 \\ y_j &= \sum_{i=j-2}^{i=n-j} (-1)^i x_1^i x_{n-j+2+i} \quad j = 2, \dots, n \end{aligned} \right\} \quad (2)$$

it is simple to verify that the previous chained system is itself equivalent to the following form, called *power form*, introduced in (Teel *et al.* 1992):

$$\left. \begin{aligned} \dot{y}_1 &= u_1 \\ \dot{y}_j &= \frac{1}{(j-2)!} y_1^{j-2} u_2 \quad j = 2, \dots, n \end{aligned} \right\} \quad (3)$$

i.e.

$$\dot{y} = u_1 f_1(y) + u_2 f_2(y) \quad (4)$$

with:

$$f_1 = \frac{\partial}{\partial y_1} ; \quad f_2 = \sum_{j=2}^n \frac{y_1^{j-2}}{(j-2)!} \frac{\partial}{\partial y_j} . \quad (5)$$

In (Pomet 1992), a systematic method, based on Lyapunov and Lasalle techniques, is given to design smooth time-periodic stabilizing control laws for controllable systems without drift meeting a certain Lie Bracket condition (Pomet 1992; Assumption 1). This condition is satisfied by system 1, written as $\dot{x} = u_1 f_1(x) + u_2 f_2(x)$, because $\text{Rank}\{f_1, f_2, [f_1, f_2], \text{ad}_{f_1}^2 f_2, \dots, \text{ad}_{f_1}^{n-2} f_2\} = n$. Moreover, in the case where one can find system coordinates for which f_1 is a coordinate vector field, the method yields explicit expressions of the control laws. Hence, it is natural to look for coordinates (y_1, \dots, y_n) for which $f_1 = \frac{\partial}{\partial y_1}$. This leads to the change of coordinates (2), which transforms the chain form into the power form.

In (Teel *et al.* 1992), another set of stabilizing smooth time-periodic control laws for systems in power form, with a stability analysis relying upon *Center Manifold* techniques, is considered. This already illustrates the fact that there are multiple ways of designing and analysing stabilizing time-varying feedbacks.

In the present paper, we propose a systematic way of designing stabilizing continuous time-varying feedbacks, with the properties of homogeneity and exponential convergence being obtained as a particular case. Based on Lyapunov analysis complemented by Lasalle's invariance principle, the method takes up the techniques used by Samson in several of his articles and yields simple control expressions.

3. HOMOGENEITY AND EXPONENTIAL STABILIZATION

In this section, after recalling briefly the main definitions concerning homogeneity, we extend a result from (Kawski 1990) to time-varying vector fields, this extension being simplified by arguments taken from (Rosier 1992).

Define, for $\lambda > 0$, the usual dilation operator δ_λ , adapted here to functions which also depend on time:

$$\delta_\lambda(t, y_1, \dots, y_n) = (t, \lambda^{r_1} y_1, \dots, \lambda^{r_n} y_n) \quad (6)$$

with $r_j > 0$, the weight upon the coordinate y_j .

A function h from $\mathbb{R} \times \mathbb{R}^n$ to \mathbb{R} is said to be *homogeneous of degree τ* with respect to the family of dilations δ_λ if

$$h(\delta_\lambda(t, y_1, \dots, y_n)) = \lambda^\tau h(t, y_1, \dots, y_n), \quad (7)$$

and a time-varying vector field f is said to be homogeneous of degree σ with respect to the family of dilations δ_λ if and only if its i th coordinate is a homogeneous function of degree $r_i + \sigma$, i.e.

$$f^i(\delta_\lambda(t, y_1, \dots, y_n)) = \lambda^{r_i + \sigma} f^i(t, y_1, \dots, y_n). \quad (8)$$

Note that since the dilation does not act on t the homogeneity of a time-varying function (resp. vector field) is equivalent to the homogeneity of the corresponding time-invariant functions (resp. vector fields) obtained by fixing time, for any value of time.

Suppose that f is a homogeneous vector field with degree σ and h is a homogeneous function of degree τ , with respect to the same dilation, then:

$$\left. \begin{aligned} hf &\text{ is a homog. vector field of degree } \tau + \sigma, \\ \frac{\partial h}{\partial t} &\text{ is a homog. function of degree } \tau, \\ L_f h &\text{ is a homog. function of degree } \tau + \sigma. \end{aligned} \right\} \quad (9)$$

with respect to the considered dilation. (Recall that $L_f h(t, y) = \frac{\partial h}{\partial y}(t, y) f(t, y)$ is the Lie derivative of h along f).

Consider now the vector fields f_1 and f_2 associated with the system in power form (3) and given by (5). It is easy to verify that f_1 is homogeneous of degree $-r_1$ with respect to any dilation, and that the vector field f_2 is homogeneous with respect to dilations such that:

$$r_j = r_2 + (j-2)r_1 \quad \text{for } j \geq 2. \quad (10)$$

The degree of homogeneity of f_2 is then $-r_2$. Therefore:

$$\left. \begin{aligned} f_1 &\text{ is a homog. vector field of degree } -r_1, \\ f_2 &\text{ is a homog. vector field of degree } -r_2. \end{aligned} \right\} \quad (11)$$

with respect to the family of dilations satisfying (10). In the sequel, only such dilations will be considered, with r_1 and r_2 kept as degrees of freedom which do not need to be specified at this stage. Note however that only the ratio $\frac{r_2}{r_1}$ really matters since, by a change of variable on λ , it is clear that r_1 can be normalized and set equal to one without any loss of generality.

The following proposition (the second part of it actually) is an extension to time-varying systems of a result stated in (Kawski 1990) (lemma 2.1).

Proposition 1 Suppose that the input functions $u_1(t, y)$ and $u_2(t, y)$ are continuous with respect to y and infinitely differentiable with respect to t , periodic with period $T > 0$, and that

$$\left. \begin{aligned} u_1(t, y) &\text{ is homogeneous of degree } r_1 \\ u_2(t, y) &\text{ is homogeneous of degree } r_2, \end{aligned} \right\} \quad (12)$$

then the closed-loop time-varying vector field $u_1 f_1 + u_2 f_2$ is homogeneous of degree zero. Furthermore, if the origin of the closed-loop system

$$\dot{y} = u_1(t, y) f_1(y) + u_2(t, y) f_2(t, y) \quad (13)$$

is locally asymptotically stable for (13), then it is globally exponentially stable with respect to the dilation

in the sense that there exists $K > 0$ and $a > 0$ such that, along any solution $y(t)$ of the closed-loop system:

$$\rho(y(t)) \leq K e^{-at} \rho(y(0)) \quad (14)$$

where ρ is the homogeneous norm given by:

$$\rho(y) = \left(\sum_{j=1}^n |y_j|^{\frac{p}{r_j}} \right)^{\frac{1}{p}} \quad \text{with } p > 0. \quad (15)$$

◇

Proof : The fact that the closed-loop vector field is homogeneous of degree zero is an obvious consequence of (12), (11) and (9). Then, according to Proposition 4 given in the Appendix, local stabilization of the origin implies the existence of a positive Lyapunov function $\bar{V}(t, y)$, homogeneous of degree one, of class C^∞ except at $y = 0$, continuous everywhere (from Proposition 4, the function is only of class C^0 everywhere due to its degree of homogeneity chosen equal to one), and such that ($\forall y \neq 0$):

$$\dot{\bar{V}} = \frac{\partial \bar{V}}{\partial t} + \sum_{i=1}^n \frac{\partial \bar{V}}{\partial x_i} (u_1 f_1^i + u_2 f_2^i) < 0. \quad (16)$$

Now, since $\dot{\bar{V}}$ may be viewed as $\frac{\partial \bar{V}}{\partial t} + L_{u_1 f_1 + u_2 f_2} \bar{V}$ and since $L_{u_1 f_1 + u_2 f_2}$ is a homogeneous vector field of degree zero, one deduces from (9) that $\dot{\bar{V}}$ is a homogeneous time-varying periodic function with the same degree as \bar{V} . It is thus of degree one. From there, one deduces the existence of two positive numbers K_1 and K_2 such that:

$$K_1 \bar{V}(t, y) \leq -\dot{\bar{V}}(t, y) \leq K_2 \bar{V}(t, y) \quad \forall y \neq 0. \quad (17)$$

Indeed, the function $-\frac{\dot{\bar{V}}(t, y)}{\bar{V}(t, y)}$ is well defined and strictly positive away from $y = 0$, and it is homogeneous of degree zero. Therefore, it takes the same value at (t, y) and at $\delta_\lambda(t, y)$ for all $\lambda > 0$ so that one may restrict his attention to $y \in S_\rho^{n-1}$, where S_ρ^{n-1} is the unit sphere associated with the "homogeneous norm" ρ defined by (15). Moreover, time-periodicity allows one to consider that time lives on the compact set $S^1 = \mathbb{R}/T\mathbb{Z}$, instead of \mathbb{R} . On the compact set $S^1 \times S_\rho^{n-1}$, the smooth and strictly positive function $-\frac{\dot{\bar{V}}}{\bar{V}}$ reaches its minimum, denoted as K_1 , and its maximum, denoted as K_2 ; both numbers being strictly positive.

By using the same arguments, one shows that there are positive numbers K_3 and K_4 such that

$$K_3 \rho(y) \leq \bar{V}(t, y) \leq K_4 \rho(y) \quad \forall y \neq 0. \quad (18)$$

Now, from (17): $\dot{\bar{V}} \leq -K_1 \bar{V}$, and thus: $\bar{V}(t, y(t)) \leq \bar{V}(0, y(0)) e^{-K_1 t}$, along any system's solution. From (18), one then obtains: $\rho(y(t)) \leq \frac{K_4}{K_3} e^{-K_1 t} \rho(y(0))$, which is the relation (14) with $K = K_4/K_3$ and $a = -K_1$. ■

4. A CLASS OF STABILIZING CONTROL LAWS

We describe in this section a way of designing time-varying stabilizing feedbacks. Depending on the choice of some design parameters, one obtains control laws which are either smooth, or homogeneous and only continuous, with exponential convergence of

the closed-loop system's solutions to zero in the latter case.

The proposed two-step approach described below is basically the one used in (Samson 1991, Samson 1992, Samson 1993a).

Let us consider any function $W_1(y_2, \dots, y_n)$ such that:

$$\left. \begin{array}{l} W_1 \text{ is continuously differentiable on } \mathbb{R}^{n-1} \\ \text{and of class } C^2 \text{ on } \mathbb{R}^{n-1} - \{0\}, \\ W_1(0) = 0 \text{ and } W_1(x) > 0 \text{ for } x \neq 0, \\ \frac{\partial W_1}{\partial x_i}(x) \neq 0 \text{ for } x \neq 0 \\ W_1 \text{ is proper.} \end{array} \right\} \quad (19)$$

(by *proper* it is meant that the preimage of a compact set is compact, i.e. the set $\{x / W_1(x) \geq K\}$ is bounded for all $K > 0$).

It is clear that W_1 is non-increasing along the solutions if u_2 is chosen such that $u_2 L_{f_2} W_1 \leq 0$, no matter how u_1 is chosen. This may be achieved, for example, by taking:

$$u_2 = -g_2 \text{sign}(L_{f_2} W_1) |L_{f_2} W_1|^\alpha \quad (20)$$

with $\alpha > 0$ and $g_2 > 0$.

Irrespective of the behavior of y_1 (considered as a function of time, governed by u_1), this control guarantees boundedness of (y_2, \dots, y_n) and eventually convergence to zero when $y_1(t)$ is suitably chosen. To this purpose, one may try to determine a continuous feedback u_1 (which has to be time dependent since continuous pure-state feedbacks cannot stabilize the complete system) which keeps y_1 bounded, makes it converge to zero, and produces a $y_1(t)$ which causes (y_2, \dots, y_n) to go to zero. A simple choice, among other possibilities, is:

$$u_1 = -g_1 y_1 + h(W_1(y_2, \dots, y_n)) \sin \frac{2\pi t}{T} \quad (21)$$

with $g_1 > 0$.

Proposition 2 For any function W_1 satisfying (19) and any continuous real function h such that

$$\left. \begin{array}{l} h \text{ is of class } C^1 \text{ on }]0, +\infty), \\ h(s) = 0 \Leftrightarrow s = 0, \end{array} \right\} \quad (22)$$

the control law (21)-(20) globally asymptotically stabilizes the origin for the system (3).

◇

Proof : The boundedness of the system's solutions, which implies that these solutions exist over $[0, +\infty)$, comes from that $W_1(y_2(t), \dots, y_n(t))$ is nonnegative and nonincreasing. It is thus bounded. This implies, from (19), that $(y_2(t), \dots, y_n(t))$ is uniformly bounded, and since

$$\dot{y}_1 = -g_1 y_1 + h(W_1(y_2(t), \dots, y_n(t))) \sin \frac{2\pi t}{T} \quad (23)$$

is the equation of a Hurwitz (stable) linear system perturbed by the additive uniformly bounded term $h(W_1(y_2(t), \dots, y_n(t))) \sin \frac{2\pi t}{T}$, $y_1(t)$ is also uniformly bounded. Moreover $W_1(y_2(t), \dots, y_n(t))$ converges to some limit value, and $W_1(y_1(t), \dots, y_n(t))$, being uniformly continuous with respect to t , tends to zero.

We then apply LaSalle's invariance principle for time-invariant systems (LaSalle 1968; Theorem 2). This can be done by incorporating time in the system state vector ($\dot{i} = 1$). In doing so, at least one of the state components (the time coordinate) is unbounded so that it may seem that LaSalle's principle does not

apply in this case. This difficulty is presently overcome in the following way: since the time-dependence is periodic, it is possible to replace all occurrences of $\frac{2\pi t}{T}$ in the control laws, and thus in the closed-loop vector field, by the “angle” θ which lives in $\mathbb{R}/2\pi\mathbb{Z}$ and is thus naturally bounded. This is in fact equivalent to incorporating $\sin\frac{2\pi t}{T}$ and $\cos\frac{2\pi t}{T}$ in the state vector, instead of t . Now, by Lasalle’s principle, since all the trajectories $(\theta(t), y_1(t), \dots, y_n(t))$ are bounded, and since the function $W_1(y_2, \dots, y_n)$ decreases along these trajectories, all solutions converge to the set consisting of the reunion of the trajectories for which $\dot{W}_1(y_1, \dots, y_n)$ is identically zero.

Consider such a trajectory $(\theta(t), y_1(t), \dots, y_n(t))$. Since \dot{W}_1 is $u_2 L_{f_2} W_1$, and u_2 is given by (20), u_2 is identically zero on this trajectory. Therefore, in view of the system’s equations (3), $y_2(t), \dots, y_n(t)$ are constant. W_1 is also constant on the considered trajectory, since $\dot{W}_1 = 0$. Let c denote its value. By integration of (23), $y_1(t)$ is given by

$$y_1(t) = A \cos\left(\frac{2\pi t}{T} + \varphi\right) + B e^{-g_1 t} \quad (24)$$

where the constant A , B , and φ depend on $h(c)$ and $y_1(0)$, and A is equal to zero if and only if $h(c) = 0$, i.e. if and only if W_1 is itself equal to zero. Now, $\dot{W}_1 = 0$ is equivalent to $L_{f_2} W_1 = 0$, i.e.

$$\sum_{j=2}^n \frac{y_1^{j-2}}{(j-2)!} \frac{\partial W_1}{\partial y_j} = 0, \quad (25)$$

which, from (24), implies

$$\sum_{j=2}^n \frac{(A \cos(\frac{2\pi t}{T} + \varphi) + B e^{-g_1 t})^{j-2}}{(j-2)!} \frac{\partial W_1}{\partial y_j} = 0 \quad (26)$$

where all the coefficients $\frac{\partial W_1}{\partial y_j}$ are constant, since they depend only on y_2, \dots, y_n .

If $A = B = 0$, the relations (23)-(24) imply that y_1 and $h(W_1)$ are equal to zero and, from (22), that $W_1 = 0$. Hence, from (19), $y_j = 0$, ($j = 2, \dots, n$).

If either A or B is nonzero, the functions $g_j(t) : t \mapsto (A \cos(\frac{2\pi t}{T} + \varphi) + B e^{-g_1 t})^{j-2}$ are linearly independent so that the constant coefficients $\frac{\partial W_1}{\partial y_j}$ in (26) must all be equal to zero. From (19), this again implies $y_j = 0$, ($j = 2, \dots, n$) and $W_1 = 0$.

Hence, all solutions for which $L_{f_2} W_1$ is identically zero are contained in the set defined by $y_2 = 0, \dots, y_n = 0$. From LaSalle’s invariance principle, this implies that all trajectories converge to this set. Therefore, for any trajectory, $y_j(t)$ ($j = 2, \dots, n$) and $W_1(t)$ tend to zero, and so does $h(W_1(t))$. In view of (23), this in turn implies that $y_1(t)$ tends to zero. ■

This stabilization result is true for any W_1 and h meeting the assumptions (19) and (22) respectively. In particular, if h is differentiable everywhere and α is chosen larger than one (or $\alpha = 1$ with W_1 being twice differentiable everywhere), then the obtained time-varying controls are differentiable everywhere.

Another possibility, motivated by Proposition 1, consists in choosing the functions W_1 and h so that:

$$\left. \begin{aligned} W_1 \text{ is homog. of degree } q > r_2 + (n-2)r_1, \\ h(s) = g_0 |s|^{r_1/q} \text{ with } g_0 \neq 0, \\ \alpha = \frac{r_2}{q - r_2}. \end{aligned} \right\} (27)$$

In this case, one easily verifies that the control laws u_1 and u_2 considered previously are homogeneous of degree r_1 and r_2 respectively, so that Proposition 1 applies. Note that h , and thus u_1 , cannot be differentiable everywhere due to the ratio r_1/q which is smaller than one in this case. The result is summarized in the following proposition:

Proposition 3 If h and W_1 satisfy (19) and (27), then the control law given by (21)-(20) makes the origin globally exponentially stable with respect to the considered family of dilations δ_λ , in the sense of the inequality (14).

◇

To illustrate this result, one may, for example, choose:

$$W_1(y_2, \dots, y_n) = \sum_{j=2}^n a_j |y_j|^{\frac{q}{r_2 + (j-2)r_1}} \quad (28)$$

with $a_j > 0$ ($j = 2, \dots, n$) and $q > r_2 + (n-2)r_1$. W_1 is then homogeneous of degree q and continuously differentiable. One also has in this case:

$$L_{f_2} W_1 = \sum_{j=2}^n \frac{q}{r_2 + (j-2)r_1} \frac{a_j}{(j-2)!} y_1^{j-2} \text{sign}(y_j) |y_j|^{\frac{q}{r_2 + (j-2)r_1} - 1}. \quad (29)$$

Explicit expressions for u_1 and u_2 are then obtained from (21) and (20).

Choosing $r_1 = r_2 = 1$ and $q = 6$, we have simulated this solution for a four-dimensional ($n = 4$) system in power form representing the kinematics of a car-like vehicle.

Recall that the state components x_1 and x_2 of the equivalent four-dimensional chained system may physically be interpreted as the cartesian coordinates x and y of the vehicle’s point located at mid-distance of the rear wheels (see (Sordalen 1993), (Samson 1993a), for example). In view of the change of coordinates (2), these cartesian coordinates are given by y_1 and $y_4 - y_1 y_3 + \frac{1}{2} y_1^2 y_2$ respectively. The vehicle’s orientation angle θ is given by $\arctan(x_3)$ ($= \arctan(-y_3 + y_1 y_2)$), u_2 is the velocity of the front steering wheel angle, and the vehicle’s translational velocity v is equal to $\frac{u_1}{\cos(\theta)}$.

The simulation results are presented in **Fig.1-Fig.4**.

APPENDIX

The purpose of this appendix is to extend to homogeneous periodic time-varying systems a theorem by (Rosier 1992; theorem 2), itself a generalization of a theorem by (Hahn 1967), stating that a continuous homogeneous time-invariant vector field with a unique globally asymptotically stable equilibrium admits a homogeneous Lyapunov function. This result is the core of the exponential convergence property associated with asymptotically stable homogeneous vector fields of degree zero (see the proof of Proposition 1).

Proposition 4 Suppose that $x = 0$ is an asymptotically stable equilibrium of

$$\dot{x} = F(t, x), \quad (30)$$

with $F \in C^0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ being T -periodic ($F(t + T, x) = F(t, x)$) and homogeneous of degree zero with

respect to the dilation δ_λ defined in (6).

Then, for any $\alpha > 0$, and any integer p such that $p < \frac{\alpha}{\max\{r_j\}}$, there exists a function $\bar{V}(t, x)$ which satisfies:

1. $\bar{V} \in C^p(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R} \times \mathbb{R}^n - \{0\}, \mathbb{R})$
2. \bar{V} is T -periodic : $\bar{V}(t + T, x) = \bar{V}(t, x)$
3. \bar{V} is homogeneous of degree α with respect to the dilation δ_λ :

$$\bar{V}(\delta_\lambda(t, x)) = \lambda^\alpha \bar{V}(t, x) \quad (31)$$
4. $\bar{V}(t, 0) = 0$ for all t , $\bar{V}(t, x) > 0$ for $x \neq 0$ and \bar{V} is proper with respect to x (for all K , the set of x 's such that $\bar{V}(t, x) \leq K$ for at least one t is bounded).
5. $\dot{\bar{V}}(t, x) = \frac{\partial \bar{V}}{\partial t}(t, x) + \sum_{i=1}^n \frac{\partial \bar{V}}{\partial x_i} F^i(t, x) < 0 \forall x \neq 0$

◇

Sketch of proof : The only difference with (Rosier 1992; theorem 2) is the periodic time-dependence of both F and \bar{V} .

Using homogeneity and periodicity, local stability of $\dot{x} = F(t, x)$ implies "strong stability" on \mathbb{R}^n (similar to (Rosier 1992; proposition 1)).

The next step consists in using a theorem from (Kurzweil 1956) in order to exhibit a smooth non homogeneous Lyapunov function. Actually, the case of periodic time-dependence is considered in (Kurzweil 1956; theorem 7). The function $V(t, x)$ so obtained is of class C^∞ everywhere and has all the properties of the function $\bar{V}(t, x)$ described in Proposition 4, except for homogeneity.

One may then proceed with the construction of a homogeneous function \bar{V} from the non-homogeneous function V , as done in (Rosier 1992; proposition 2). This construction is the following. Let $a \in C^\infty(\mathbb{R}, \mathbb{R})$ be a nondecreasing function on \mathbb{R} , constant and equal to 0 on $(-\infty, 1]$, and constant and equal to 1 on $[2, +\infty)$. \bar{V} is then defined by:

$$\bar{V}(t, x) = \begin{cases} \int_0^{+\infty} \frac{1}{\mu^{\alpha+1}} a(V(\delta_\mu(t, x))) d\mu & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

One easily verifies that \bar{V} is time-periodic with the same period as V and that it is homogeneous with degree α (by a change of integration variable). The subsequent arguments given by Rosier to prove his theorem remain valid when \bar{V} is time-periodic. The fact that $\bar{V}(t, x)$ is usually not differentiable more than p times at $x = 0$, with $p < \frac{\alpha}{\max\{r_j\}}$, basically comes from that the function $s \mapsto s^{\alpha/r}$ ($\alpha > 0$, $r > 0$, α/r not in \mathbb{N}), which is homogeneous of degree α with respect to the dilation $\delta_\lambda(s) = \lambda^r s$, is only of class C^q with $q < \alpha/r$. ■

REFERENCES

- Bloch, A. M. and N. H. McClamroch (1989). Control of mechanical systems with classical nonholonomic constraints. In '28th IEEE Conf. on Decision and Control'. Tampa, FL. pp. 201–205.
- Bloch, A. M., N. H. McClamroch and M. Reyhanoglu (1992). 'Control and stabilization of nonholonomic dynamic systems'. *IEEE Transactions on Automatic Control* **37**(11), 1746–1757.
- Canudas, C. and O. J. Sordalen (1991). Exponential stabilization of mobile robots with nonholonomic constraints. In '29th IEEE Conf. on Decision and Control'. Brighton, UK. pp. 201–205.
- Coron, J.-M. (1992). 'Global asymptotic stabilization of for controllable systems without drift'. *Math. of Control, Signals and Systems* **5**, 295–312.
- Coron, J.-M. (1994). 'Linearized control and smooth stabilization'. *SIAM J. on Control and Optim.* **32**(2), 358–386.
- d'Andréa Novel, B., G. Bastin and G. Campion (1991). Modeling and control of nonholonomic wheeled robots. In 'Conf on Robotics and Autom.'. IEEE. Sacramento, CA, USA.
- Gurvits, L. and Z. X. Li (1992). Smooth time-periodic feedback solutions for nonholonomic motion planning. In Z. X. Li and J. Canny (Eds.). 'Progress in Nonholonomic Motion Planning'. Kluwer Academic Publisher.
- Hahn, W. (1967). *Stability of Motion*. Vol. 138 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, New-York.
- Kawski, M. (1990). 'Homogeneous stabilizing feedback laws'. *Control Th. and Adv. Technol.* **6**, 497–516.
- Kurzweil, J. (1956). 'On the inversion of Lyapunov's second theorem on stability of motion'. *A.M.S. Translations, ser. II* **24**, 19–77.
- LaSalle, J.-P. (1968). 'Stability theory for ordinary differential equations'. *J. of Diff. Eq.* **4**, 57–65.
- M'Closey, R. T. and R. M. Murray (1993a). Convergence rates for nonholonomic systems in power form. In 'American Control Conf.'. IEEE. San Francisco, CA, USA.
- M'Closey, R. T. and R. M. Murray (1993b). Nonholonomic systems and exponential convergence: Some analysis tools. In '32th IEEE Conf. on Decision and Control'. San Antonio, TX, USA.
- Murray, R. M. and S. S. Sastry (1991). Steering nonholonomic systems with sinusoids. In 'Conf. on Robotics and Autom.'. IEEE. Sacramento, CA, USA.
- Pomet, J.-B. (1992). 'Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift'. *Systems and Control Letters* **18**, 147–158.
- Rosier, L. (1992). 'Homogeneous Lyapunov function for homogeneous continuous vector field'. *Systems and Control Letters* **19**, 467–473.
- Samson, C. (1991). Velocity and torque feedback control of a nonholonomic cart. In 'Int. Workshop in Adaptive and Nonlinear Control : Issues in Robotics'. Vol. 162. Springer-Verlag. Proceedings of a Conf. held in Grenoble, France, 1990.
- Samson, C. (1992). Path following and time-varying feedback stabilization of a wheeled mobile robot. In 'Int. Conf. ICARCV'92'. Singapore. pp. RO-13.1.
- Samson, C. (1993a). Mobile robot control, part II : Control of chained systems and application to path following and time-varying point-

- stabilization of wheeled vehicles. Report 1994. INRIA. Sophia Antipolis, France.
- Samson, C. (1993b). 'Time-varying feedback stabilization of car-like wheeled mobile robots'. *The International J. of Robotics Research* **12**(1), 55-64.
- Samson, C. and K. Ait-Abderrahim (1991a). Feedback control of a nonholonomic wheeled car in cartesian space. In 'Conf. on Robotics and Autom.'. IEEE. Sacramento, CA, USA.
- Samson, C. and K. Ait-Abderrahim (1991b). Feedback stabilization of a nonholonomic wheeled mobile robot. In 'IEEE/RSJ Int. Workshop on Intelligent Robots and Systems, IROS'91'. Osaka, Japan. pp. 1242-1247.
- Sontag, E. D. and H. J. Sussmann (1980). Remarks on continuous feedback. In '19th IEEE Conf. on Decision and Control'. Albuquerque, NM, USA.
- Sørdalen, O. J. (1993). Conversion of the kinematics of a car with n -trailers into a chained form. In 'Int. Conf. on Robotics and Automation'. IEEE. Atlanta, USA.
- Sørdalen, O. J. and O. Egeland (1993). Exponential stabilization of chained form nonholonomic systems. In '2nd European Control Conference'. Groningen, The Netherlands. pp. 1438-1443.
- Teel, A. R., R. M. Murray and G. Walsh (1992). Nonholonomic control systems : from steering to stabilization with sinusoids. In '31th IEEE Conf. on Decision and Control'. Tucson, AZ, USA.
- Tilbury, D., J.-P. Laumond, R. Murray, S. Sastry and G. Walsh (1992). Steering car-like systems with trailers using sinusoids. In 'Conf. on Robotics and Autom.'. IEEE. Nice, France. pp. 1993-1998.

