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# A GRAPH THEORETICAL CHARACTERIZATION OF MINIMAL DEADLOCKS IN PETRI NETS

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## ABSTRACT

*Petri Nets are now a well known formal model of parallel computation. They can be modeled by weighted bipartite digraphs, with two set of vertices: places and transitions. Here we study the properties of minimal deadlocks, which are important for liveness properties of the net. A deadlock  $D$  is a set of places such that  $\Gamma^-(D) \subseteq \Gamma(D)$ . We give a characterization of minimal deadlocks in terms of path properties and show their application to special classes of Petri Nets.*

## 0 INTRODUCTION

Petri Nets were introduced in 1962 to model the dynamic evolution of discrete systems. Since then they have been widely used and they constitute now one of the most complete and advanced formal models of parallel computation. Curiously, although a Petri Net is a graph, very few combinatorial stu-

dies have been carried out. In this paper we give an answer to such a graph problem.

We first recall some definitions and properties of Petri Nets useful to understand the motivation of the problem; the interested reader will find more details in the recent book of G.W. Brams [2]. In the second part, we study in details the properties of deadlocks. In the third part, we solve a combinatorial problem associated to them. (This part can be read independently of the others).

## 1 PETRI NETS

In a Petri Net there are two kinds of elements: *places* and *transitions*. Roughly the set of places is defined in order to represent the set of states of the system, the set of transitions represent the set of rules to go from one state to another. We will associate to each state of the system a marking which consists in putting in each place marks or tokens. (One can think that these tokens correspond to available resources). A transition can be fired (i.e. a rule can be executed) if there are enough resources in the places concerned by this transition and the firing of the transition will result in the transfer of some resources of these places to other ones.

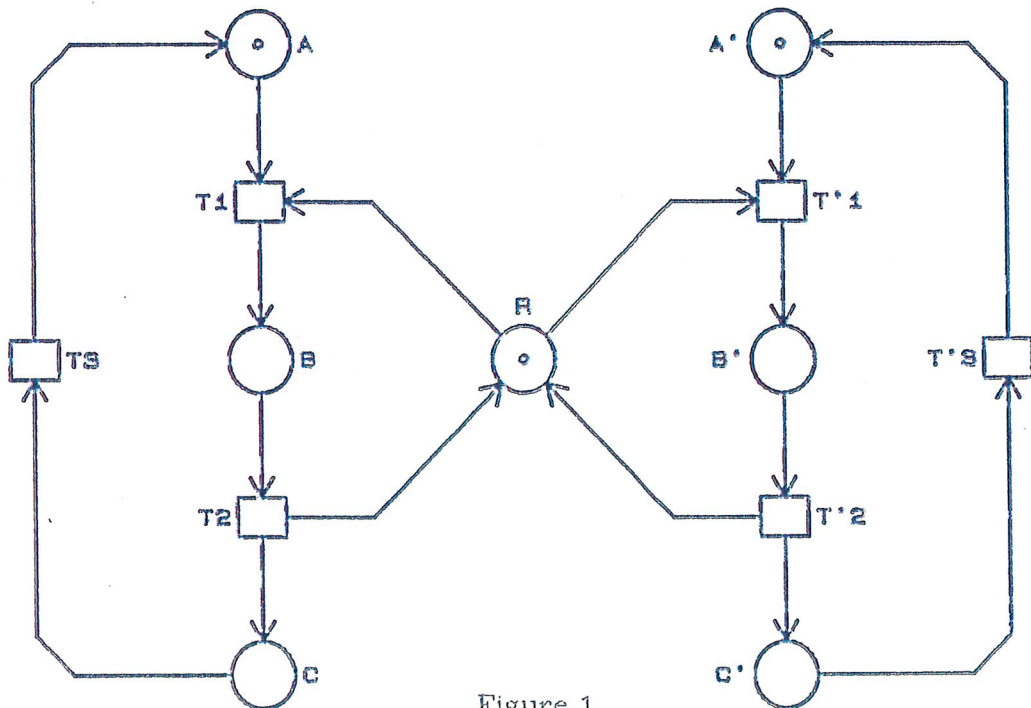


Figure 1

**Example 1.1 :** The Petri Net of *Figure 1* models the mutual exclusion of two processes sharing a common resource. The place R corresponds to this resource. In place A (resp A') process 1 (resp 2) is waiting for the resource. In place B (resp B') the process 1 (resp 2) uses the resource. In place C (resp C') process 1 (resp 2) is working. Transition  $T_1$  can be fired if there is a token in place A and one in place R, that is if process 1 is waiting and the resource is available. When this transition is fired, we go to place B; we delete the tokens in A and R and add one in B. The functioning of  $T_1'$  is the same. When  $T_1$  is fired, it is not possible to fire  $T_1'$  until we have fired  $T_2$ , which corresponds to put the first process in state C and to give back the resource in R; again we model this by deleting the token in B and adding one in C and one in R.

**Definition 1.2 :** A marked Petri Net is a pair  $\langle R, M_0 \rangle$  where:

$R = \langle P, T, \Gamma, V \rangle$  is a finite weighted bipartite directed graph with vertex set  $P \cup T$  where P is the set of places and T the set of transitions and  $P \cap T = \emptyset$ ;  $\Gamma$  is the function successor which associates to a vertex of P (resp. T) a subset of T (resp. P);  $(x, y)$  is an arc of R if and only if y belongs to  $\Gamma(x)$ .  
 $V: P \times T \cup T \times P \rightarrow \mathbb{N}$ , where  $\mathbb{N}$  is the set of natural numbers is a mapping such that  $V(x, y) = 0$  if and only if  $y \notin \Gamma(x)$ . V is called the valuation of the Petri Net and consists to associate to each arc of the net a positive integer. (we shall write  $V(x, y)$  for the valuation of the arc  $(x, y)$ );

$M_0$  is a mapping from P to  $\mathbb{N}$  which is called *the initial marking* and if  $M_0(p) = k$ , we will say that the place p contains k tokens (or marks).

**Remarks:** Usually places are pictured with circles and transitions with bars, squares or rectangles. In graph theory,  $\Gamma(x)$  it is also noted  $\Gamma^+(x)$ , and we denote by  $\Gamma^-(x)$  the set  $\{y / x \in \Gamma(x)\}$ .

Petri Nets can also be defined in a purely algebraic way [8].

The dynamic behavior of a Petri Net is taken into account by considering at any time the marking of the net, where the marking of a place is the number of tokens in this place. The way of passing from one marking to another one is the firing of a transition defined as follows.

**Definition 1.3 :** A transition t is *fireable* at a marking M if and only if:

$$\text{for every } p \text{ of } P, M(p) \geq V(p, t).$$

If t is fireable at M, we can *fire* t and *reach* M' with:

for every  $p$  of  $P$ ,  $M'(p) = M(p) - V(p,t) + V(t,p)$ .

In the example 1.1.  $P = \{A,B,C,R,A',B',C'\}$ ;  $T = \{T_1, T_2, T_3, T_1', T_2', T_3'\}$  and the valuation of each arc is 1.  $M_0(A) = 1, M_0(A') = 1, M_0(R) = 1, M_0(p) = 0$  otherwise. When  $T_1$  is fired, we reach  $M'$  with  $M'(B) = 1, M'(A') = 1$ , and  $M'(p) = 0$  otherwise.

**Example 1.4:** In the example of Figure 2,  $P = \{A,B\}$ ;  $T = \{t,r\}$ ;  $V(A,r) = 2$ ;  $V(t,A) = 3$ ; and  $V(x,y) = 1$  for the other arcs.  $M_0(A) = 2$ ;  $M_0(B) = 0$ .

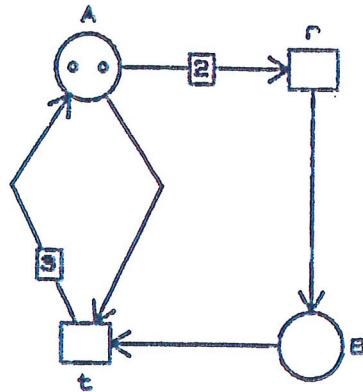


Figure 2

In what follows, we will denote that  $t$  is fireable at  $M$  by  $M(t>$ , and that  $M'$  is reached by  $M(t>M'$ . We recall the following definitions for a Petri Net  $\langle R, M_0 \rangle$ .

**Definition 1.5:** Let  $s = t_1, t_2, \dots, t_i$  be a sequence of transitions;  $s$  is a *firing sequence* fireable at  $M_0$  if and only if there exists a sequence of markings  $M_1, \dots, M_i$  such that for every  $j$ ,  $1 \leq j \leq i$ ,  $M_{j-1}(t_j > M_j$  and we will write  $M_0(s > M_i$

**Notation 1.6:** The set of markings reachable from a marking  $M$  is denoted by

$$\text{Acc}(R, M_0) = \{m \mid \exists s \in T^*, M_0(s > m\}$$

where  $T^*$  is the set of all finite sequences on  $T$ .

**Definition 1.7:** A transition  $t$  is *live* for  $M_0$  if and only if for every  $M$  of  $\text{Acc}(R, M_0)$ , there exists  $s$  of  $T^*$  such that  $M(st) \in \langle R; M_0 \rangle$ .  $\langle R; M_0 \rangle$  is said to be live if and only if every transition of  $T$  is live.

The net of example 1.1 is live for the initial marking  $M_0$ . In example 1.4, it can be seen that if  $M(A) + 2M(B) = 2k+1$ , with  $k$  a positive integer, then the net is live; otherwise it is not.

The problem of deciding if a Petri Net is live is exp-space hard [9]. Therefore it is interesting to find classes of Petri Nets for which the problem is easier to solve.

## 2 DEADLOCKS, TRAPS and COMMONER'S PROPERTY.

We can imagine the firing of transitions as a move of tokens along paths in the Petri Net. A subset of places, such that a token cannot enter in it if it does not contain already a certain amount of tokens, will be called a deadlock and similarly a subset of places such that some tokens can never quit it will be called a trap. These two notions have been introduced in [3] and [6]. Let us give precise definitions.

**Definition 2.1** A place  $p$  is *deficient in tokens* (shortly deficient) for a marking  $M$  if and only if:

$$\Gamma(p) \neq \emptyset \text{ and } M(p) < \min_{t \in \Gamma(p)} \{V(p,t)\}.$$

A subset of places  $Q$  is deficient for  $M$  if and only if each place of  $Q$  is deficient for  $M$ .

When a subset of places  $Q$  is deficient for a marking  $M$ , then no transition of  $\Gamma(Q)$  is fireable at  $M$ .

**Definition 2.2** Let  $D$  be a subset of places,  $D$  is a *deadlock* if and only if  $\Gamma(D) \subseteq D$ .

**Definition 2.3** Let  $S$  be a subset of places,  $S$  is a *trap* if and only if for every  $t$  in  $\Gamma(S)$ , there exists  $p$  in  $\Gamma(t) \cap S$  such that either  $\Gamma(p) = \phi$ , or  $V(t,p) \geq \text{Min}_{t' \in \Gamma(p)} (V(p,t'))$ .

**Remark 2.4:** We also have that  $S$  is not deficient after the firing of a transition of  $\Gamma(S)$ . If  $S$  is a trap, we have  $\Gamma(S) \subseteq \Gamma^-(S)$ . The union of two deadlocks (or two traps) is a deadlock (resp. a trap). So, we can speak about the maximal deadlock or trap of a given subset. But the intersection of two deadlocks (or two traps) is not necessarily a deadlock (resp. a trap).

**Example 2.5 :** In the net of Figure 3,  $\{A,B\}$  and  $\{A,C\}$  are both deadlocks and traps.  $\{A,B\} \cap \{A,C\} = \{A\}$  is neither a deadlock ( $t \in \Gamma^-(\{A\})$  and  $t \notin \Gamma(\{A\})$ ) nor a trap ( $r \in \Gamma(\{A\})$  and  $V(r,A) \leq \text{Min} ( V(A,r) , V(A,s) )$ ).

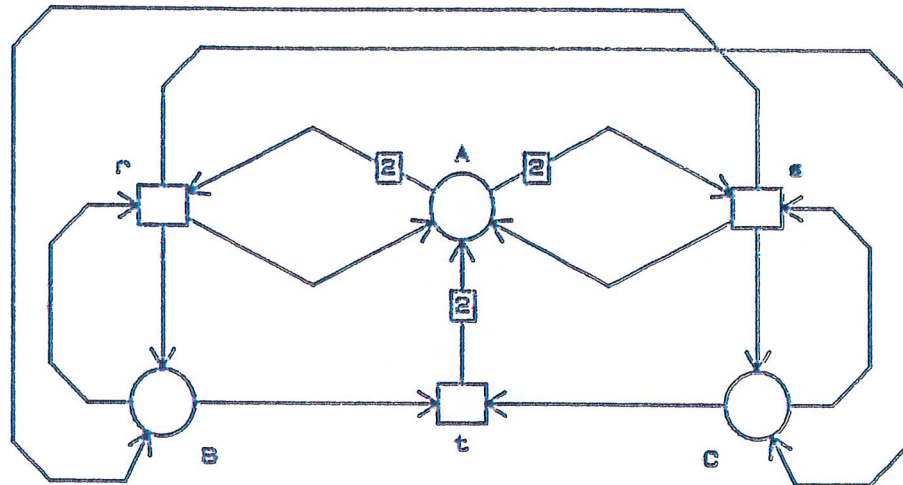


Figure 3

The fundamental property of a deadlock  $D$  is the following : if  $D$  is deficient for  $M$ , it remains deficient for any marking reachable from  $M$ .

**Property 2.6 :** Let  $D$  be a subset of places such that there does not exist a place  $p$  with  $\Gamma(p) = \phi$ . Then  $D$  is a deadlock if and only if every marking  $M$  for which  $D$  is deficient is such that  $D$  remains deficient for any marking of

$\text{Acc}(R, M)$ .

**Proof :** Let  $D$  be a deadlock and let  $M_1$  be a marking for which  $D$  is deficient. Suppose that there exists a marking  $M_n$  of  $\text{Acc}(R, M_1)$  for which  $D$  is not deficient, and let  $s = t_2, \dots, t_n$  be such that  $M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} M_3, \dots, (t_n \rightarrow M_n$ . Let  $M_1$  be the first marking in the sequence for which  $D$  is not deficient, necessarily  $t_1 \in \Gamma(D)$ ; but  $D$  is deficient for  $M_{i-1}$  then  $t_i \notin \Gamma(D)$ . This is in contradiction with  $\Gamma(D) \subseteq \Gamma(D)$ .

Conversely, suppose there exists  $t \in \Gamma(D) - \Gamma(D)$ . We can find a marking  $M$  such that  $D$  is deficient for  $M$  (for example by putting no tokens in  $D$ ; as  $\Gamma(p) \neq \emptyset$  every place is deficient). We can also choose  $M$  such that for a given  $n$   $M \xrightarrow{t^n}$  (where  $t^n$  is the sequence where  $t$  is fired  $n$  times) belongs to  $\text{Acc}(R, M)$  (it suffices to put enough tokens in the places of  $\Gamma(t)$ ). But if  $n$  is large enough after  $n$  firings of the transition  $t$ , a place of  $D$  is no more deficient (choose a place  $p$  of  $\Gamma(t) \cap D$  and  $n > V(p, r)$  for one transition  $r$  of  $\Gamma(p)$ ). Therefore we have a contradiction and  $\Gamma(D) \subseteq \Gamma(D)$ .  $\square$

**Property 2.7 :** Let  $A$  be a subset of places such that there does not exist a place  $p$  with  $\Gamma(p) = \emptyset$ . Then  $A$  is a trap if and only if every marking  $M$  for which  $A$  is non deficient remains non deficient for any marking of  $\text{Acc}(R, M)$ .

**Proof:** It is similar to that of property 2.6.

We are now ready to explain the basic idea of Commoner. If a deadlock  $D$  is deficient for a marking  $M$ , it remains deficient and that means that any transition of  $\Gamma(D)$  is not live. On the other hand if a trap is not deficient, it will remain not deficient. Therefore if we want a net to be live, it is necessary that no deadlock becomes deficient. A step to achieve that is that every deadlock contains a non deficient trap. This is the so-called *deadlock-trap property* and it depends of the structure of the Petri Net if this property is a necessary, a sufficient or a necessary and sufficient condition of liveness for a given marked Petri Net (see [2],[7],[10]).

A first restriction on the structure of a Petri Net consists in giving constraints on the valuation  $V$  of the net.

**Definition 2.8 :** A valuation  $V$  of a Petri Net is said to be *homogeneous* if and only if for every place  $p$  and for every pair of transitions  $t$  and  $t'$  of  $\Gamma(p)$ ,  $V(p, t) = V(p, t')$



**Definition 2.9 :** A valuation  $V$  of a net is said to be *non blocking* if and only if for every transition  $t$ , for every  $p \in \Gamma^-(t)$  either  $\Gamma^+(p) = \emptyset$  or  $V(t,p) \geq \min_{t' \in \Gamma^+(p)} (V(p,t'))$ .

If  $V$  is non blocking, a subset of places  $S$  is a trap if and only if:  $\Gamma^-(S) \subseteq \Gamma^+(S)$ .

We slightly generalize a result of [7] for Petri Net with an homogeneous valuation.

**Property 2.10 :** Let  $\langle R; M_0 \rangle$  be a marked Petri Net with an homogeneous valuation. If every deadlock contains a non deficient trap for  $M_0$ , then there does not exist a marking  $M$  of  $\text{Acc}(R, M_0)$  such that no transition is fireable at  $M$ .

**Proof :** Suppose in contrary that such a marking  $M$  exists, and consider  $D = \{p \in P \mid p \text{ is deficient for } M\}$ . Let  $t$  be a transition of  $\Gamma^-(D)$ ,  $t$  is not fireable; then there exists  $p$  in  $\Gamma^-(t)$  such that  $M(p) < V(p,t)$ . As  $V$  is homogeneous,  $p$  is deficient for  $M$  and  $t \in \Gamma^-(D)$ . Then  $D$  is a deadlock with no non deficient trap for  $M$ . □

**Remark 2.11 :** This property does not hold in general.

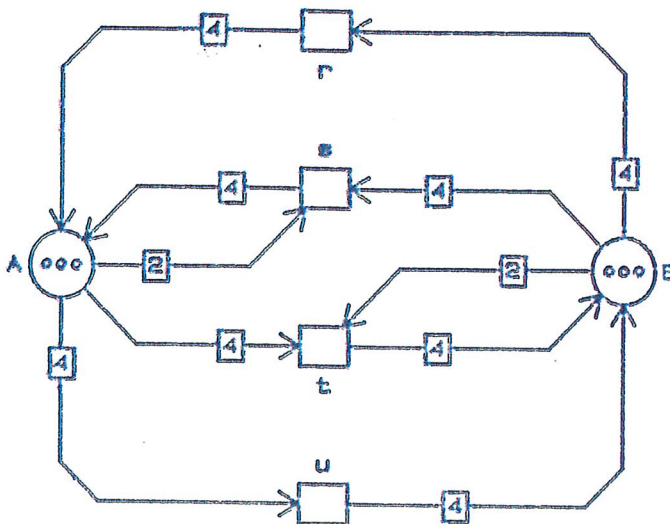


Figure 4

In the example of Figure 4, there is only one deadlock {A,B} which is also a non deficient trap for  $M_0$  and there is no transition fireable at  $M_0$ .

Now let us consider Petri Nets with non blocking valuations. The following proposition generalizes a result of Commoner in [3] given without proof. Furthermore, that proof gives a straightforward algorithm to get the maximal trap of a given deadlock.

**Property 2.12 :** Let  $R = \langle P, T, \Gamma, V \rangle$  be a net such that  $V$  is a non blocking valuation. Let  $D$  be a deadlock and  $S_D$  its maximal trap. If  $S_D$  is deficient for a marking  $M$ , then there exists a marking  $M'$  such that for every  $p$  in  $P$ ,  $M'(p) \geq M(p)$  and such that no transition of  $\Gamma(D)$  is live in  $\langle R; M' \rangle$

**Proof :** First, let us construct a trap of  $D$  maximal for the inclusion. Let  $D_0 = \phi$  We successively construct :

$$T_i = \Gamma \left( D - \bigcup_{j=0}^{i-1} D_j \right) - \Gamma^{-1} \left( D - \bigcup_{j=0}^{i-1} D_j \right)$$

$D_i = \Gamma^{-1}(T_i) \cap \left( D - \bigcup_{j=0}^{i-1} D_j \right)$  for  $i > 0$  until  $T_i = \phi$ . We claim that when  $T_i = \phi$ , then

$$S_D = D - \bigcup_{j=0}^{i-1} D_j \text{ is a maximal trap of } D.$$

Note that such an  $i$  exists because  $D$  and  $T$  are finite. We have :

$\Gamma(S_D) - \Gamma^{-1}(S_D) = \phi$ ; so  $\Gamma(S_D) \subseteq \Gamma^{-1}(S_D)$ . As the valuation is non blocking,  $S_D$  is a trap. Suppose that  $S_D$  is not maximal and let  $S \supset S_D$  be a maximal trap. Let  $k$  be the first index such that there exists  $p$  in  $D_k \cap (S - S_D)$ . By the construction there exists  $t$  in  $T_k \cap \Gamma(p)$  (in particular  $t \in \Gamma(S)$ ), such that  $\Gamma(t) \cap D \subseteq \bigcup_{j=0}^{k-1} D_j$ ; therefore  $t \notin \Gamma(S)$  contradicting the definition of a trap.

Now let  $M$  be a marking for which  $S_D$  is deficient. Let us construct  $M'$  from  $M$  by adding tokens in such a manner that starting at  $M'$  we can fire a sequence of transitions which uses all the tokens in  $D - S_D$  and add no tokens in  $S_D$ . To do that consider the places of  $D_{i-1}$  and let  $m$  be the maximum number of tokens of a place of  $D_{i-1}$ ; then add tokens to all the places of  $\Gamma^{-1}(T_{i-1})$  in order to fire each transition of  $\Gamma^{-1}(T_{i-1})$  exactly  $m$  times. The marking  $M_1$  obtained is such that  $M_1(p) \geq M(p)$  for every place  $p$ . If we fire each transition of  $T_{i-1}$   $n$  times, then we obtain a marking where the places of  $D_{i-1}$  have no more marks. In this construction and firing the places of  $S_D$  have not been affected. Repeat the construction for  $D_{i-2}$  and so on until  $D_1$ . We finally obtain a marking  $M'$  and a firing of transition of  $\bigcup_{j=1}^{i-1} T_j$  such that for the marking obtained

each place of  $D - S_D$  has no token and  $S_D$  is deficient. Therefore no transition of  $\Gamma(D)$  is live.

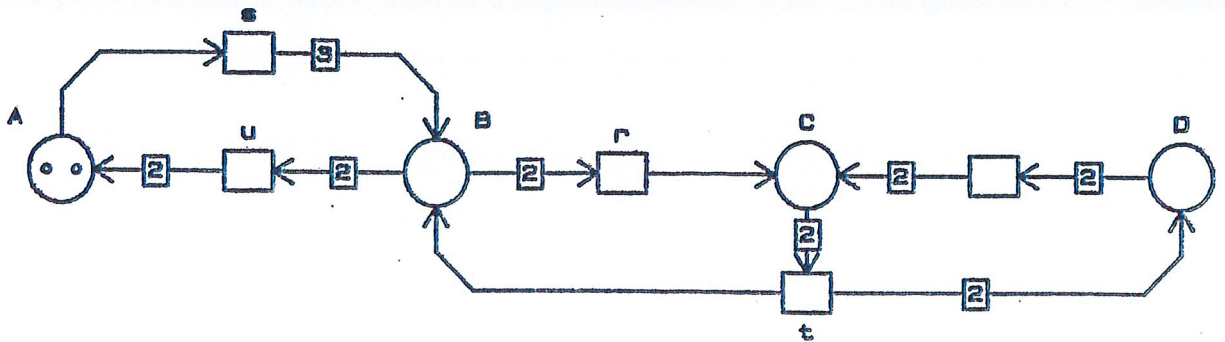


Figure 5

**Remark 2.13 :** This result does not hold in general. In the example of Figure 5,  $\{C, D\}$  is the maximal trap of  $\{A, B, C, D\}$ , and for each marking  $M' \geq M_0$  the net is not bounded.

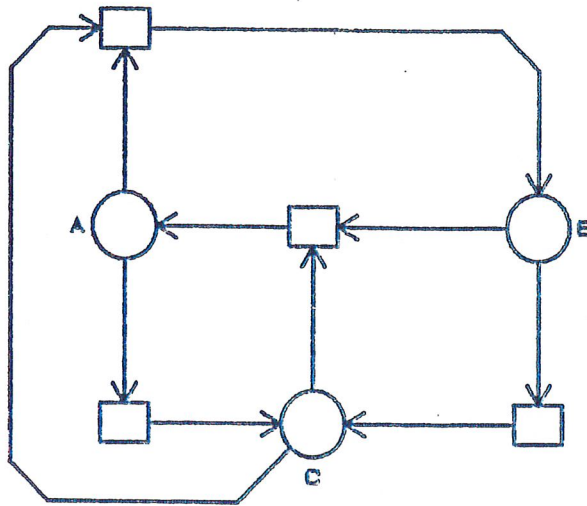
**Problem :** It will be interesting to find for what classes of Petri Nets the deadlock-trap property is either sufficient or necessary for the liveness of the Petri Net.

### 3. A CHARACTERIZATION OF MINIMAL DEADLOCKS

As we have seen in § 2 it is interesting to study deadlocks and traps. Here we restrict our attention to deadlocks; similar results can be obtained in exactly the same manner for traps. In particular, in order to check if every deadlock contains a non deficient trap it suffices to verify it for every minimal deadlock. We present a characterization of minimal deadlocks in terms of path properties. This result was also presented at the conference on Petri Nets held in Toulouse [1].

Recall that a deadlock  $D$  is a subset of places such that  $\Gamma^-(D) \subseteq \Gamma(D)$ .  $D$  is minimal if and only if it does not contain any deadlock except  $D$  and  $\emptyset$ .

**Example 3.1 :**



$\{A, B, C\}$  is a deadlock,  
 $\{A, B\}$  and  $\{A, C\}$  are minimal  
deadlocks

Figure 6

**Theorem 3.2 :** Let  $D$  be a set of places. Then  $D$  is a minimal deadlock if and only if:

a) For every set  $\mathbf{C}$  of paths in  $D \cup \Gamma^-(D)$  beginning by a place, and for every pair  $(p, q)$  of places of  $D$ , there exists an elementary path  $C_p$  from  $p$  to  $q$  in  $D \cup \Gamma^-(D)$  such that for each edge  $(x, t)$  of  $C_p$  such that  $t$  belongs to  $T$  we have : either  $(x, t)$  is an edge of a path of  $\mathbf{C}$ , or  $t$  does not appear in any path of  $\mathbf{C}$ .

b)  $D \cup \Gamma^-(D)$  generates a strongly connected subgraph.

**Proof :** First, let us consider a minimal deadlock  $D$ .

Point a) directly follows from the proposition :

**Proposition 3.3 :** Let  $D$  be a minimal deadlock; let  $\mathbf{C}$  be a set of paths in  $D \cup \Gamma^-(D)$  beginning by a place of  $D$  and let  $p, q$  be two places of  $D$ . Then there exists an elementary path  $C_p$  from  $p$  to  $q$  such that :

- i)  $\{p, q\} \subseteq C_p \subseteq D \cup \Gamma^-(D)$
- ii) No transition  $t$  of  $C_p$  has an input place in the subpath of  $C_p$  from  $t$  to  $q$ .
- iii) For each edge  $(x, t)$  of  $C_p$  such that  $t$  belongs to  $T$  we have :

either  $(x,t)$  is an edge of a path of  $\mathbf{C}$  or  $t$  does not appear in any path of  $\mathbf{C}$ .

**Proof :** Let  $R = \{ x \in D \text{ such that there exists an elementary path } C_x \text{ from } x \text{ to } q \text{ satisfying i), ii), iii) - as in the proposition.} R \neq \emptyset \text{ as } q \in R.$

We are going to show that  $R$  is a deadlock, as  $R \subseteq D$  and  $D$  is a minimal deadlock, then  $R = D$  and  $p$  will belong to  $R$ .

Let  $t \in \Gamma^-(R)$ ,  $t \in \Gamma^-(D)$ ; therefore there exists  $u \in D \cap \Gamma^-(t)$ . If  $u \in R$ , then  $t$  belongs to  $\Gamma(R)$  and we are done.

Otherwise, let  $r \in \Gamma(t) \cap R$ ;  $r \in R$ , thus there exists a path  $C_r$  from  $r$  to  $q$  satisfying i), ii), iii). Let  $C_u$  be the path obtained by the concatenation of  $(u,t,r)$  and  $C_r$ . If  $C_u$  is non elementary, then  $t$  belongs to  $C_r$  and therefore as any place of  $C_r$  is in  $R$ ,  $t$  belongs to  $\Gamma(R)$ .

$C_u$  satisfies by construction i); as  $u$  does not belong to  $R$ , it does not satisfy ii) or iii). If  $C_u$  does not satisfy ii),  $\Gamma^-(t) \cap C_r \neq \emptyset$  and  $t \in \Gamma(R)$ . If  $C_u$  satisfies ii) but not iii), as  $C_r$  satisfies itself iii), there exists  $(\pi,t)$  in  $\mathbf{C}$ , with  $\pi \neq u$  and  $(u,t)$  does not appear in  $\mathbf{C}$ ;  $\pi \in \mathbf{C} \cap \Gamma^-(t)$ , but the path consisting in the concatenation of  $(\pi,t,r)$  and  $C_r$  satisfies i), ii), iii) and therefore  $t$  belongs to  $\Gamma(R)$  a contradiction.

Point b) is proved by using point a) and the following lemma :

**Lemma 3.4:** If  $D$  is any set of places such that for every pair of vertices  $x$  and  $y$ , there exists a path from  $x$  to  $y$  whose places all belong to  $D$ , then the subgraph generated by  $D \cup (\Gamma(D) \cap \Gamma^-(D))$  is strongly connected

**Proof :** To prove the lemma, let  $H$  be the subgraph generated by  $D \cup (\Gamma(D) \cap \Gamma^-(D))$  and let  $x$  and  $y$  be two vertices of  $H$ .

Case 1 :  $x \in D$  : If  $y$  belongs to  $D$ , by hypothesis there exists a path from  $x$  to  $y$  in  $H$ . If  $y$  belongs to  $(\Gamma(D) \cap \Gamma^-(D))$ , there exists  $z$  in  $\Gamma^-(y) \cap D$  and by hypothesis a path from  $x$  to  $z$  in  $H$  and therefore one from  $x$  to  $y$ .

Case 2 :  $x \in \Gamma(D) \cap \Gamma^-(D)$  : there exists  $z$  in  $\Gamma(x) \cap D$  ; by case 1 there exists a path from  $z$  to  $y$  in  $H$  and therefore a path from  $x$  to  $y$  in  $H$ .

Conversely let us note that if  $D \cup \Gamma^-(D)$  is a strongly connected graph ( condition b) then,  $D$  is a deadlock. Indeed it is clear that  $\Gamma^-(D) \subseteq \Gamma(D)$ .  $D$  does not need to be minimal. Suppose that  $D$  satisfies condition a) and is not minimal. Let  $D_0$  be properly included in  $D$  and let  $\mathbf{C}$  be a set of paths  $(p_0,t,q_0)$

where  $\{p_0, q_0\} \subseteq D_0$ , such that every transition of  $\Gamma^-(D_0)$  is in at least one path. Let  $p \in D - D_0$  and  $q \in D_0$ . Then the path  $C_p$  from  $p$  to  $q$  uses necessarily a transition  $t$  of  $\Gamma^-(D_0)$  and an edge  $(x, t)$  with  $x \notin D_0$ ; therefore there exists a path  $(p_0, t, q_0)$  in  $C$ , contradicting a).  $\square$

We think that this last result gives a powerful tool to construct new classes or to better understand the structure of existing ones. Here we apply our theorem for three classes of Petri Nets. These applications are straightforward and we are looking for more complex cases.

**Definitions 3.5:** A Petri Net  $\langle P, T, \Gamma \rangle$  is said to be *simple* [3] if and only if :  $\forall p, q \in \Gamma(p) \cap \Gamma(q) \neq \emptyset \Rightarrow$  either  $\Gamma(p) \subseteq \Gamma(q)$  or  $\Gamma(q) \subseteq \Gamma(p)$

A Petri Net  $\langle P, T, \Gamma \rangle$  is said to be *N.S.K. (nicht-selbst-Kontrollierend)* [4] if and only if there does not exist a place  $p$  in  $P$  such that :  $\Gamma(p) = \{t_1, \dots, t_m\}$  and there exists  $i$  of  $\{1, \dots, m\}$  with :

- i) there exists an elementary path from  $p$  to a transition  $t_j$ , containing  $t_i$  and  $i \neq j$ .
- ii) there exists a circuit containing  $p$  and  $t_i$ .

In such classes, we can easily deduce from our theorem the following proposition :

**Corollary 3.6 :** Let  $D$  be a minimal deadlock in a simple Petri Net or a N.S.K. Petri Net, then :  $\forall t \in I^-(D) : |\Gamma^-(t) \cap D| = 1$ .

From this last corollary, we easily derive a theorem on liveness and safeness for F.C. nets in [5], N.S.K. in [4] (see [10]).

**Definition 3.7 :**  $G$  is said to be backward-conflict-free (b.c.f.) if and only if every place has one and only one input transition :

$$\forall p \in P, \sum_{t \in T} V(t, p) = 1$$

**Corollary 3.8 :** Let  $G$  be a backward-conflict-free net,  $D$  is a minimal deadlock if and only if  $D \cup \Gamma^-(D)$  generates an elementary circuit.

**Proof :** If  $D \cup \Gamma^-(D)$  generates an elementary circuit, then  $D$  is clearly a minimal deadlock.

Conversely, let  $D$  be a minimal deadlock. Then  $D \cup \Gamma^-(D)$  generates a strongly connected subgraph.  $\Gamma^-(D) \neq \emptyset$  by definition of a b.c.f.; there exists an elementary circuit  $B$  in the subgraph generated by  $D \cup \Gamma^-(D)$ . We apply the theorem with  $C$  consisting of two paths whose union is  $B$ . Suppose that  $D - B \neq \emptyset$  and let  $p$  be in  $D - B$ ,  $q$  in  $D \cap B$ : any path from  $p$  to  $q$  meets the circuit  $B$  for the first time necessarily in a transition  $t$  from a place  $x$  and the edge  $(x,t)$  does not satisfy condition a) for  $C$ .  $\square$

From this property it is easy to deduce that the deadlock-trap property of Commoner is a necessary and sufficient condition of liveness for the b.c.f. nets (see [10]).

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