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Shortest Coverings of Graphs with Cycles

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It is shown that the edges of a bridgeless graph G can be covered with cycles such that the sum of the lengths of the cycles is at most $|E(G)| + \min\{\frac{2}{3}|E(G)|, \frac{7}{3}(|V(G)| - 1)\}$.

1. INTRODUCTION

1.1. Definitions

All graphs considered are finite, and may contain loops and multiple edges. Let G be a graph. For $S \subseteq V(G)$, we denote by $\omega(S)$ the set of edges of G with exactly one end in S . A k -cut of G is a set of the form $\omega(S)$ ($S \subseteq V(G)$) with $|\omega(S)| = k$. A *bridge* is a 1-cut. A *cycle* in a graph is a connected, 2-regular subgraph. The *length* of a cycle is the number of edges it contains. A *digon* is a cycle of length two. Given the graph G , a *cycle cover* of G is a set \mathcal{C} of cycles of G such that each edge of G belongs to at least one cycle of \mathcal{C} . The *length* of \mathcal{C} is the sum of the lengths of the cycles in \mathcal{C} and is denoted by $l(\mathcal{C})$. It is clear that a graph admits a cycle cover if and only if it contains no bridges. Other definitions for graphs can be found in [2, 3].

1.2. The Main Results

Itai and Rodeh [12] have shown that every connected bridgeless graph G has a cycle cover of length at most $|E(G)| + 2|V(G)| \log |V(G)|$. This upper bound was improved to $\min\{3|E(G)| - 6, |E(G)| + 6|V(G)| - 7\}$ by Itai, Lipton, Papadimitriou, and Rodeh in [11]. The main result of this paper is

THEOREM 1. *Let G be a bridgeless graph. Then G has a cycle cover \mathcal{C} such that $l(\mathcal{C}) \leq \frac{5}{3}|E(G)|$.*

Although Theorem 1 appears to be stronger than the previous results only if G has relatively few edges, we shall use Theorem 1 to improve these results for all graphs.

THEOREM 2. *Every bridgeless graph G has a cycle cover of length at most $|E(G)| + \frac{7}{3}(|V(G)| - 1)$.*

1.3. Relationship with the Chinese Postman Problem

Itai and Rodeh point out in [12] that one may obtain a lower bound for the length of a shortest cycle cover by considering the Chinese postman problem. That is, given a connected graph G , find a closed walk which traverses each edge of G at least once and is as short as possible. An algorithm for finding such a "postman tour" appears in [7]. The problem is equivalent to constructing a graph H_0 such that:

- (i) H_0 is obtained by replacing each edge of G by one or more parallel edges,
- (ii) H_0 is Eulerian, and
- (iii) $|E(H_0)|$ is as small as possible.

A cycle cover \mathcal{C} for a connected bridgeless graph G easily gives rise to a graph H satisfying (i) and (ii), and such that $l(\mathcal{C}) = |E(H)|$, by replacing each edge e of G by a number of parallel edges equal to the number of cycles of \mathcal{C} which contain e . Thus $l(\mathcal{C}) \geq |E(H_0)|$. Indeed, it would seem at first sight that the two problems were equivalent, since a cycle decomposition of H_0 should give rise to a cycle cover of G . This is not necessarily the case, however, since it is possible that every cycle decomposition of H_0 contains digons which correspond to single edges in G . (We shall henceforth refer to such digons of H_0 as *forbidden digons*.) For example, if G is a cubic graph, then every H_0 satisfying (i)–(iii) is obtained by replacing each edge of some 1-factor of G by 2 parallel edges. Thus if \mathcal{C} is a cycle cover of G , then $l(\mathcal{C}) \geq |E(H_0)| = \frac{4}{3}|E(G)|$. If G is the Petersen graph, however, then a shortest cycle cover of G has length 21 (see [12]), and $\frac{4}{3}|E(G)| = 20$. On the

other hand one can prove that for planar graphs such a situation cannot occur.

PROPOSITION 1. *For every connected bridgeless planar graph G , a shortest cycle cover has length equal to the length of a shortest postman tour.*

To prove this (see also [10]) we need some further definitions. Let v be a vertex of a loopless Eulerian graph H . A *transition at v* is a pair of edges incident to v . A *set of transitions for v* is a partition $T(v)$ of $\omega(\{v\})$ into transitions. If $T(v)$ is defined for every vertex v of H of degree greater than 2, the resulting family \mathcal{E} of transitions is a *transition system* for H . The system is *non-separating* if the graph obtained from H by deleting any one transition of \mathcal{E} is connected. Note that if H has no cut-vertices, every transition system for H is non-separating.

A cycle decomposition \mathcal{C} of H is *compatible with \mathcal{E}* if no cycle of \mathcal{C} contains a transition of \mathcal{E} . It is clear that given \mathcal{E} , a necessary condition for the existence of a cycle decomposition which is compatible with \mathcal{E} is that \mathcal{E} be non-separating. Fleischner has shown

THEOREM 3 [9]. *Let H be a planar loopless Eulerian graph and \mathcal{E} be a non-separating system of transitions for H . Then H has a cycle decomposition which is compatible with \mathcal{E} .*

Proof of Proposition 1. It is easy to see that we may assume that G has no cut-vertices. Let H_0 be a graph satisfying (i)–(iii). It follows from (iii) that H_0 is obtained by replacing each edge of G by at most 2 parallel edges. Whenever e_1 and e_2 are two parallel edges of H_0 which correspond to a single edge of G , let $\{e_1, e_2\}$ be a transition at v for each vertex v incident with both e_1 and e_2 . This family of transitions can be extended to a system of transitions \mathcal{E} for H_0 . Since H_0 has no cut-vertices, \mathcal{E} is non-separating. By Fleischner's theorem, H_0 has a cycle decomposition \mathcal{C} which is compatible with \mathcal{E} , and hence does not contain any forbidden digons. Thus \mathcal{C} gives rise to a cycle cover of G of length $|E(H_0)|$.

2. \mathbb{Z}_2 -FLOWS AND \mathbb{Z}_2 -CYCLES

2.1. Definition

Let $k \geq 1$ and consider the additive group $(\mathbb{Z}_2)^k$. A $(\mathbb{Z}_2)^k$ -flow of the graph G is a mapping ϕ from $E(G)$ to $(\mathbb{Z}_2)^k$, such that: $\forall v \in V(G)$, $\sum_{e \in \omega(\{v\})} \phi(e) = 0$. (The summation and zero symbols refer to the structure of the group $(\mathbb{Z}_2)^k$.)

2.2. Elementary properties

(1) If ϕ is a $(\mathbb{Z}_2)^k$ -flow of G , for any $S \subseteq V(G)$ we have $\sum_{e \in \omega(S)} \phi(e) = 0$.

(2) The *support* of the $(\mathbb{Z}_2)^k$ -flow ϕ , denoted by $\sigma(\phi)$, is the set of edges $e \in E(G)$ such that $\phi(e) \neq 0$.

Then for $F \subseteq E(G)$ the following properties are equivalent:

- (i) F is the support of some \mathbb{Z}_2 -flow of G .
- (ii) Each vertex of G is incident to an even number of edges of F .
- (iii) F can be partitioned into cycles of G .

A subset F of $E(G)$ satisfying (i)–(iii) will be called a \mathbb{Z}_2 -cycle of G .

(3) It easily follows from (1) (or (2)) that if e is a bridge of G , $\phi(e) = 0$ for any $(\mathbb{Z}_2)^k$ -flow ϕ . A $(\mathbb{Z}_2)^k$ -flow ϕ is said to be *nowhere-zero* if $\sigma(\phi) = E(G)$. Thus if a graph has a nowhere-zero $(\mathbb{Z}_2)^k$ -flow, it has no bridges.

(4) Let ϕ be a $(\mathbb{Z}_2)^k$ -flow. Let ϕ_1, \dots, ϕ_k be \mathbb{Z}_2 -flows such that $\forall e \in E(G): \phi(e) = (\phi_1(e), \dots, \phi_k(e))$. We shall write $\phi = (\phi_1, \dots, \phi_k)$. Then ϕ is nowhere-zero if and only if $\bigcup_{i=1}^k \sigma(\phi_i) = E(G)$. In this case, for every $i \in \{1, \dots, k\}$, there exists a partition $P_i = \{C_i^1, \dots, C_i^{r_i}\}$ of $\sigma(\phi_i)$ into cycles of G , and $\bigcup_{i=1}^k P_i$ is a cycle cover of G of length $\sum_{i=1}^k |\sigma(\phi_i)|$. This number will be denoted by $l(\phi)$. Conversely, if $\mathcal{C} = \{C_1, \dots, C_k\}$ is a cycle cover of G , let $\phi_i (i = 1, \dots, k)$ be the unique \mathbb{Z}_2 -flow such that $\sigma(\phi_i) = C_i$. Then $\phi = (\phi_1, \dots, \phi_k)$ is a nowhere-zero $(\mathbb{Z}_2)^k$ -flow and

$$l(\mathcal{C}) = \sum_{i=1}^k |\sigma(\phi_i)| = l(\phi).$$

We conclude that the minimum of $l(\mathcal{C})$ over the set of cycle covers \mathcal{C} of G is equal to the minimum of $l(\phi)$ over the set of nowhere-zero $(\mathbb{Z}_2)^k$ -flows ϕ of G ($k \geq 1$).

Remark. If G is bridgeless, G has a cycle cover and hence G has a nowhere-zero $(\mathbb{Z}_2)^k$ -flow for some $k \geq 1$ (see the above discussion).

(5) Let $z = (z_1, \dots, z_k) \in (\mathbb{Z}_2)^k$.

The (*Hamming*) *weight* $w(z)$ of z is the number of nonzero components of z , that is, $w(z) = |\{i \in \{1, \dots, k\} : z_i = 1\}|$. Let ϕ be a nowhere-zero $(\mathbb{Z}_2)^k$ -flow of G . A straightforward counting argument yields $l(\phi) = \sum_{e \in E(G)} w(\phi(e))$.

2.3. The Double-Cover Conjecture

A *cycle double-cover* of G is a cycle cover \mathcal{C} of G such that each edge appears in exactly two cycles of \mathcal{C} . The double-cover conjecture asserts that

every bridgeless graph has a cycle double cover [17, Conjecture 3.3]. We shall denote by D_k ($k \geq 2$) the subset of $(\mathbb{Z}_2)^k$ consisting of the elements of weight 2. It is easy to show (see the above discussion in 2.2(4)) that G has a cycle double-cover iff it has a $(\mathbb{Z}_2)^k$ -flow with all edge-values in D_k for some $k \geq 2$ (such a flow will be called a D_k -flow).

Remark. If a graph has a D_k -flow it has a $D_{k'}$ -flow for every $k' \geq k$. Hence the double-cover conjecture can be formulated as follows:

For every bridgeless graph G , there exists a $k \geq 2$ such that G has a D_k -flow. (DCC)

This conjecture is clearly related to the shortest cycle cover problem. In fact Itai and Rodeh rediscover an equivalent form of the (DCC) in [12, Problem (ii)].

We have the following result:

PROPOSITION 2. *If G has a D_k -flow ($k \geq 2$), it has a cycle cover \mathcal{C} with $l(\mathcal{C}) \leq (2(k-1)/k)|E(G)|$*

Proof. Let $\phi = (\phi_1, \dots, \phi_k)$ be a D_k -flow of G . Clearly $l(\phi) = \sum_{e \in E(G)} w(\phi(e)) = 2|E(G)|$.

On the other hand, $l(\phi) = \sum_{i=1}^k |\sigma(\phi_i)|$. We may assume without loss of generality that $\forall i \in \{1, \dots, k-1\}, |\sigma(\phi_k)| \geq |\sigma(\phi_i)|$. Consider now the $(\mathbb{Z}_2)^{k-1}$ -flow $\phi' = (\phi_1, \dots, \phi_{k-1})$. It is clearly nowhere-zero. Moreover

$$l(\phi') = l(\phi) - |\sigma(\phi_k)| \leq \frac{k-1}{k} l(\phi) = \frac{2(k-1)}{k} |E(G)|.$$

This completes the proof.

2.4. Some Consequences of Proposition 2

(1) For $k = 2$ the situation is quite simple. If G has a D_2 -flow, it has a cycle cover \mathcal{C} with $l(\mathcal{C}) = |E(G)|$, i.e., $E(G)$ can be partitioned into cycles, and conversely.

(2) For $k = 3$, we obtain that if G has a D_3 -flow it has a cycle cover \mathcal{C} with $l(\mathcal{C}) \leq \frac{4}{3}|E(G)|$.

We may now use the following easy result:

PROPOSITION 3. *A graph has a D_3 -flow iff it has a nowhere-zero $(\mathbb{Z}_2)^2$ -flow.*

Proof. If $\phi = (\phi_1, \phi_2, \phi_3)$ is a D_3 -flow, then $\phi' = (\phi_1, \phi_2)$ is a nowhere-

zero $(\mathbb{Z}_2)^2$ -flow. Conversely, if $\phi' = (\phi_1, \phi_2)$ is a nowhere-zero $(\mathbb{Z}_2)^2$ -flow, $\phi = (\phi_1, \phi_2, \phi_1 + \phi_2)$ is a D_3 -flow.

Now, applying Propositions 2 and 3 together with some known results on the existence of nowhere-zero $(\mathbb{Z}_2)^2$ -flows, we obtain

COROLLARY 1. *Every bridgeless planar graph G has a cycle cover \mathcal{C} with $l(\mathcal{C}) \leq \frac{4}{3} |E(G)|$.*

Proof. Use the four color theorem [1; 13, Proposition 3].

COROLLARY 2. *Let G be a cubic 3-edge-colorable graph. The length of a shortest cycle cover of G is equal to $\frac{4}{3} |E(G)|$.*

Proof. As already seen in Subsection 1.3, the length of a shortest cycle cover of G is at least $\frac{4}{3} |E(G)|$. The equality follows from Propositions 2, 3 and [13, Proposition 2].

COROLLARY 3. *Every bridgeless graph G without 3-cuts has a cycle cover \mathcal{C} with $l(\mathcal{C}) \leq \frac{4}{3} |E(G)|$.*

Proof. Use [13, Proposition 10].

(3) For $k = 4$, we shall use the following observation:

PROPOSITION 4. *If a graph has a D_4 -flow, it has a D_3 -flow.*

Proof. Let $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ be a D_4 -flow. Then it is easy to check that $\phi' = (\phi_1 + \phi_2, \phi_1 + \phi_3, \phi_1 + \phi_4)$ is a D_3 -flow. Hence this case reduces to the previous one.

(4) For $k = 5$, we have nothing but a conjecture which has been proposed by several authors [4, 16], and which is stronger than the double-cover conjecture.

Conjecture. Every bridgeless graph has a D_5 -flow.

By Proposition 1, this implies the following:

Conjecture. Every bridgeless graph G has a cycle cover \mathcal{C} with $l(\mathcal{C}) \leq \frac{8}{5} |E(G)|$.

(5) For $k = 6$, the existence of a D_6 -flow in G implies that G has a cycle cover \mathcal{C} with $l(\mathcal{C}) \leq \frac{5}{3} |E(G)|$.

We shall show (by different methods) that this last property holds for every bridgeless graph G .

3. THE MAIN RESULT

3.1. Introduction

The following result is proved in [13].

8-FLOW THEOREM. *Every bridgeless graph has a nowhere-zero $(\mathbb{Z}_2)^3$ -flow.*

Using this result only, one can prove that every bridgeless graph G has a cycle cover \mathcal{C} with $l(\mathcal{C}) \leq \frac{12}{7}|E(G)|$. We now present the proof of Theorem 1. It uses a refinement of an alternative proof for the 8-flow Theorem (the idea is indicated in [13, Sect. V]).

3.2. LEMMA. *Every bridgeless graph G has a \mathbb{Z}_2 -cycle C such that $|C| \geq \frac{2}{3}|E(G)|$ and C intersects every 3-cut of G .*

Proof. It is clear that to prove the lemma, it is enough to prove it for loopless 2-edge-connected graphs. Let G be such a graph, and let $v \in V(G)$. A *splitting of G at v* is the graph G' obtained by replacing v by two distinct vertices v' and v'' , each edge of G with end vertices v, x ($x \in V(G) - \{v\}$) being replaced by an edge with end vertices v', x or v'', x in such a way that v' has degree 2 in G' . A *splitting of G* is any graph obtained from G by a succession of vertex-splittings. It follows from a result of Fleischner [8] (see also Mader [14]) that G has a 2-edge-connected splitting G' which has no vertices of degree greater than three. Note that identifying edges of G' with edges of G in the obvious way, every \mathbb{Z}_2 -cycle of G' is a \mathbb{Z}_2 -cycle of G , and every 3-cut of G is a 3-cut of G' . We conclude that:

(1) *To prove the lemma it is enough to prove it for loopless 2-edge-connected graphs with no vertex of degree greater than three.*

Let G be such a graph. If G has no vertices of degree 3, the result is clear. Otherwise there exists a cubic 2-edge-connected graph H such that G can be obtained from H by replacing each edge e of H by a simple path P_e of length $f(e) \geq 1$. For $F \subseteq E(H)$ we shall denote by $f(F)$ the sum $\sum_{e \in F} f(e)$. It follows from a result of Edmonds [6] that there exists an integer $k \geq 1$ and a family (M_1, \dots, M_{3k}) of $3k$ perfect matchings of H (not necessarily distinct) such that every edge of H appears in exactly k of the M_i 's.

Let K be a 3-cut of H . For every perfect matching M of H , $E(H) - M$ is a 2-factor of H and hence a \mathbb{Z}_2 -cycle. Hence $|K \cap (E(H) - M)|$ is even, so that $|K \cap M|$ equals 1 or 3. Now each one of the 3 edges of K appears in exactly k of the M_i 's ($i = 1, \dots, 3k$), so that $\sum_{i=1}^{3k} |K \cap M_i| = 3k$. It follows that $\forall i \in \{1, \dots, 3k\}, |K \cap M_i| = 1$.

Finally we note that $\sum_{i=1}^{3k} f(M_i) = kf(E(H))$. Hence there exists

$i \in \{1, \dots, 3k\}$ with $f(M_i) \leq \frac{1}{3}f(E(H))$. Then $F = E(H) - M_i$ is a \mathbb{Z}_2 -cycle of H which intersects every 3-cut of H and such that $f(F) \geq \frac{2}{3}f(E(H))$. Let C be the subset of edges of G equal to $\bigcup_{e \in F} P_e$. Clearly C is a \mathbb{Z}_2 -cycle of G and $|C| = f(F) \geq \frac{2}{3}f(E(H)) = \frac{2}{3}|E(G)|$. Moreover, no 3-cut of G contains two edges of a single path P_e , $e \in E(H)$ (the remaining edge of the 3-cut would be a bridge). Hence every 3-cut of G is obtained by considering some 3-cut $\{e_1, e_2, e_3\}$ of H and choosing exactly one edge from each of $P_{e_1}, P_{e_2}, P_{e_3}$. It follows that C intersects every 3-cut of G . This completes the proof.

3.3. A Consequence of the Lemma

PROPOSITION 5. *Every connected bridgeless graph G has a postman tour of length at most $\frac{4}{3}|E(G)|$.*

Proof. Let C be a \mathbb{Z}_2 -cycle of G with $|C| \geq \frac{2}{3}|E(G)|$. Replace every edge of $E(G) - C$ by two parallel edges. This yields an Eulerian graph H with $|E(H)| \leq \frac{4}{3}|E(G)|$.

Remark. Propositions 5 and 1 together give another proof of Corollary 1 which does not rely on the four color theorem.

3.4. Proof of Theorem 1

Let G be a bridgeless graph with $|E(G)| = m$. By the lemma, there exists a \mathbb{Z}_2 -flow ϕ_1 of G such that $|\sigma(\phi_1)| \geq 2m/3$ and $\sigma(\phi_1)$ intersects every 3-cut of G . For each edge e of $\sigma(\phi_1)$, add to G an edge e' parallel to e (i.e., with the same pair of ends). We obtain a new bridgeless graph G' which contains G as a subgraph. Moreover it is clear that G' has no 3-cuts. By Proposition 10 of [13], G' has a nowhere-zero $(\mathbb{Z}_2)^2$ -flow $\phi' = (\phi'_2, \phi'_3)$.

For $e \in E(G)$ and $i \in \{2, 3\}$ let $\phi_i(e) = \phi'_i(e)$ if $e \notin \sigma(\phi_1)$ and $\phi_i(e) = \phi'_i(e) + \phi_1(e)$ if $e \in \sigma(\phi_1)$. This defines two \mathbb{Z}_2 -flows ϕ_2, ϕ_3 of G . Since $\phi' = (\phi'_2, \phi'_3)$ is nowhere-zero, the $(\mathbb{Z}_2)^2$ -flow (ϕ_2, ϕ_3) of G takes nonzero values on $E(G) - \sigma(\phi_1)$. It follows that (ϕ_1, ϕ_2, ϕ_3) is a nowhere-zero $(\mathbb{Z}_2)^3$ -flow of G .

Consider the vector space $[GF(2)]^3$ (over $GF(2)$) of the 3-tuples $x = (\alpha_1, \alpha_2, \alpha_3)$ ($\alpha_i \in GF(2)$, $i = 1, 2, 3$). To every element $x = (\alpha_1, \alpha_2, \alpha_3)$ of this space we associate the flow $\phi_x = \sum_{\alpha_i=1} \phi_i$. In particular,

$$\phi_{(1,0,0)} = \phi_1, \quad \phi_{(0,1,0)} = \phi_2, \quad \text{and} \quad \phi_{(0,0,1)} = \phi_3.$$

It is easy to show that for every basis $\{x_1, x_2, x_3\}$ of $[GF(2)]^3$, $(\phi_{x_1}, \phi_{x_2}, \phi_{x_3})$ is a nowhere-zero $(\mathbb{Z}_2)^3$ -flow of G . Denote by X the set $[GF(2)]^3 - \{(0, 0, 0)\}$ and by X' the set $X - \{(1, 0, 0)\}$. One can easily check that each edge appears in exactly 4 of the $\sigma(\phi_x)$ ($x \in X$), and hence $\sum_{x \in X} |\sigma(\phi_x)| = 4m$. Then $\sum_{x \in X'} |\sigma(\phi_x)| = 4m - |\sigma(\phi_1)| \leq 4m - \frac{2}{3}m = \frac{10}{3}m$. Let \mathcal{B} be the set of bases of

$[GF(2)]^3$ which do not contain the vector $(1, 0, 0)$. Every vector of X' appears in exactly 8 elements of \mathcal{B} . Hence $\sum_{B \in \mathcal{B}} (\sum_{x \in B} |\sigma(\phi_x)|) = 8 \sum_{x \in X'} |\sigma(\phi_x)| \leq 80m/3$. Since $|\mathcal{B}| = 16$, there exists $B \in \mathcal{B}$ with

$$\sum_{x \in B} |\sigma(\phi_x)| \leq \frac{1}{16} \frac{80m}{3} = \frac{5m}{3}.$$

Then the supports of the \mathbb{Z}_2 -flows ϕ_x for $x \in B$ will give a cycle cover \mathcal{C} with $l(\mathcal{C}) \leq 5m/3$. This completes the proof.

3.5. 4-Covers

We observe that using the seven \mathbb{Z}_2 -cycles $\sigma(\phi_x)$ ($x \in [GF(2)]^3 - \{(0, 0, 0)\}$) defined in the above proof it is possible to obtain a cycle cover \mathcal{C} such that every edge appears in exactly 4 cycles of \mathcal{C} . Calling such a cycle cover a *cycle 4-cover*, we have

PROPOSITION 6. *Every bridgeless graph has a cycle 4-cover.*

4. PROOF OF THEOREM 2

Let G be a bridgeless graph. We may assume G is connected. Let H be a subset of $E(G)$ such that the graph $(V(G), H)$ is 2-edge-connected and minimal with this property. It is easy to show, using [5, 15], that $|H| \leq 2|V(G)| - 2$. By Theorem 1, $(V(G), H)$ has a cycle cover \mathcal{C}_1 with $l(\mathcal{C}_1) \leq \frac{5}{3}|H|$. Let $F = E(G) - H$, and consider a spanning tree T contained in H . For every e in F , there is a unique \mathbb{Z}_2 -flow ϕ_e such that $e \in \sigma(\phi_e) \subseteq T \cup \{e\}$. Let $\phi = \sum_{e \in F} \phi_e$. Then clearly $F \subseteq \sigma(\phi) \subseteq T \cup F$. Let \mathcal{C}_2 be a cycle decomposition of $\sigma(\phi)$. Now $\mathcal{C}_1 \cup \mathcal{C}_2$ is a cycle cover \mathcal{C} of G , with

$$l(\mathcal{C}) = l(\mathcal{C}_1) + l(\mathcal{C}_2) = l(\mathcal{C}_1) + |\sigma(\phi)| \leq \frac{5}{3}|H| + |T \cup F|.$$

Since $|T \cup F| = |T| + |F| = |V(G)| - 1 + |E(G)| - |H|$ we have $l(\mathcal{C}) \leq |E(G)| + |V(G)| - 1 + \frac{2}{3}|H| \leq |E(G)| + \frac{7}{3}(|V(G)| - 1)$. This completes the proof.

5. VERTEX CYCLE COVERS

Given a graph G , a *vertex cycle cover* of G is a set of cycles \mathcal{C} of G such that each vertex of G belongs to at least one cycle of \mathcal{C} .

PROPOSITION 7. *Let G be a graph such that each vertex of G lies in a cycle. Then G has a vertex cycle cover \mathcal{C} such that $l(\mathcal{C}) \leq \frac{10}{3}(|V(G)| - 1)$.*

Proof. We may assume that G is 2-edge-connected. Let H be a critically 2-edge-connected spanning subgraph of G , so that $|E(H)| \leq 2|V(G)| - 2$. By Theorem 1, H has a cycle cover \mathcal{C} such that $l(\mathcal{C}) \leq \frac{5}{3}|E(H)|$. Clearly \mathcal{C} is a vertex cycle cover of G and $l(\mathcal{C}) \leq \frac{10}{3}(|V(G)| - 1)$.

6. COVERING OF THE VERTICES OF A STRONG DIGRAPH WITH CIRCUITS

In this section, circuit means "directed circuit." Let $f(2p) = p^2 + p$ and $f(2p + 1) = (p + 1)^2$.

PROPOSITION 8. *For any strong digraph D with n vertices, there exists a vertex circuit cover \mathcal{C} such that $l(\mathcal{C}) \leq f(n)$.*

Proof. Let k be the length of the longest circuit of D and let C_0 be such a longest circuit. We can cover the vertices of D with C_0 and for each vertex not in C_0 with a circuit of length at most k . Therefore we can cover with $(n - k + 1)$ circuits of length at most k . This yields a vertex circuit cover \mathcal{C} with $l(\mathcal{C}) \leq k(n - k + 1)$. But it is known that $\max_k k(n - k + 1) = f(n)$.

The result is best possible in the sense that there exists a strong digraph D of order n such that for any covering family \mathcal{C} , $l(\mathcal{C}) \geq f(n)$. Consider the digraph D consisting of a directed circuit of length $k = \lfloor n/2 \rfloor$ in which we have replaced one vertex by a stable set of $(n - k + 1)$ vertices (see Fig. 1). Each vertex y_i belongs to the unique circuit $C_i = (x_1, y_i, x_2, \dots, x_{k-1})$. Therefore to cover all the y_i we need to use all the circuits C_i . But $\sum l(C_i) = k(n - k + 1) = \lfloor n/2 \rfloor (\lfloor n/2 \rfloor + 1) = f(n)$.

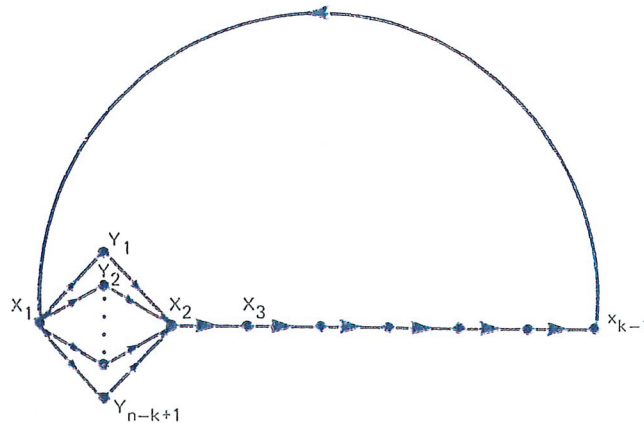


FIGURE 1.

7. OPEN PROBLEMS

7.1.

In view of Theorem 1, the main problem is to find the infimum ρ of the set of numbers r with the property that every bridgeless graph G has a cycle cover \mathcal{C} with $l(\mathcal{C}) \leq r|E(G)|$. All we know is that $\frac{7}{5} \leq \rho \leq \frac{5}{3}$. The lower bound $\frac{7}{5}$ is given by the Petersen graph (see subsection 1.3). In fact, by combining several Petersen graphs together as in Fig. 2, we obtain an infinite family of graphs G whose shortest cycle cover \mathcal{C} satisfies $l(\mathcal{C}) = \frac{7}{5}|E(G)|$. We note further that both the Blanuša snarks on 18 vertices, the flower snark on 20 vertices, and both the Loupekhine snarks on 22 vertices, have cycle covers of length $\frac{4}{3}|E(G)|$.

7.2.

A problem related to Theorem 2 is proposed by Itai and Rodeh [12, Open Problem (i)]. Does every bridgeless graph G have a cycle cover \mathcal{C} with $l(\mathcal{C}) \leq |E(G)| + |V(G)| - 1$? They prove this for graphs with two edge disjoint spanning trees. By Theorem 1, the result is true for graphs G with $|E(G)| \leq \frac{3}{2}(|V(G)| - 1)$ (e. g., subdivisions of cubic graphs containing at least three vertices of degree 2).

By Proposition 1 and using the obvious property that a shortest postman tour of G has length at most $|E(G)| + |V(G)| - 1$ it follows that the result is also true for planar graphs. On the other hand, it can be checked that if G is the complete bipartite graph $K_{n,3}$, the length of a shortest cycle cover is $|E(G)| + |V(G)| - 3$.

7.3.

Finally we propose the following conjecture

Every 2-connected graph G has a vertex cycle cover of length at most $2|V(G)| - 2$.

Note that this conjecture would be best possible because of the complete bipartite graph $K_{n,2}$ (n odd).

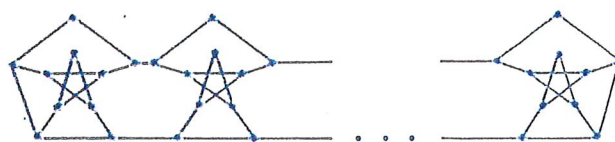


FIGURE 2.

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