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## Large hypergraphs of diameter one

Jean-Claude Bermond, Johny Bond, Jean-François Saclé

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L A R G E   H Y P E R G R A P H S   O F   D I A M E T E R   O N E

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J.C. Bermond, J. Bond & J.F. Sacle

Universite de Paris-Sud, Laboratoire de Recherche en Informatique,  
E.R.A. 452 du C.N.R.S., F-91405 Orsay

## 1. INTRODUCTION

The problem considered here is motivated by the study of combinatorial properties of interconnection networks (which can be telecommunication networks, microprocessor networks, etc.). In most of the classical networks the connections are point to point and therefore the network is modelled by a graph in which the edges represent the communication links. Now, there is some interest to consider networks in which the computers share a communication medium such as a bus (our interest in that problem came from work with researchers at the French National Telecommunication Research Center (CNET) on a project called REBUS ). In this case the network is well modelled by a hypergraph, where the vertices correspond to the computer centers, and the edges to the buses. In the design and study of these networks, several parameters are of importance, for example message delay, message traffic density, reliability or fault tolerance, cost, etc. A technical constraint is that a node must not belong to too many buses. In other words, every vertex of the hypergraph has a bounded degree (in practice 2, 3 or 4). Furthermore the traffic load on each bus is limited. Since for general networks without symmetry properties, the traffic per bus is not easily calculated, we will restrict our study to networks with some uniformity and, in a first approximation, replace the constraint on traffic load by the limitation of the number of processors connected to each bus. This means that the size  $r$  of the edges of the associated hypergraph is bounded.

We will restrict our attention to the problem of minimizing the message delay, since it is important that the length of the longest path that the message must travel from a node to another in the network, that is the distance between the nodes, is as small as possible. In other words the diameter of the hypergraph must be small (in practice no more than 4). The problem is to construct or design an interconnection network satisfying the two constraints above, with a small diameter  $D$  and having the maximum number of vertices.

The first case to study is the case of diameter 1 and we will see that it gives rise to interesting problems of hypergraphs or set systems. For more details on the problem and the results obtained on it, we refer to the recent survey of Bermond, Bond, Paoli & Peyrat [1983]. The study of diameter  $D > 1$  will be done in a forthcoming article of the authors of this paper.

## 2. NOTATION AND DEFINITIONS

We shall follow the terminology of Berge's book [1973].

- $H=(X,E)$  will denote a hypergraph with vertex set  $X$  and edge set  $E$ .
- the degree of a vertex is the number of edges containing it
- $\Delta$  will denote the maximum degree of all vertices of  $H$ .
- $r$  will denote the maximal size of an edge ; if all the edges have size  $r$ , the hypergraph is  $r$ -uniform (that is also sometimes called in the literature  $r$ -graph).

-a path in a hypergraph connecting  $x$  and  $y$  will be a sequence  $x=x_1, E_1, x_2, \dots, x_i, E_i, x_{i+1}, \dots, E_p, x_{p+1}=y$  with  $\{x_i, x_{i+1}\} \subset E_i$ . Its length will be  $p$ , the number of the edges of the path. The distance between  $x$  and  $y$  is the length of a shortest path between them. The diameter is the maximum of the distances over all pair of vertices.

In other words a hypergraph is of diameter 1 if any pair of vertices belongs to at least one edge.

### 3. $(\Delta, 1, r)$ -HYPERGRAPH PROBLEM

Let us call a  $(\Delta, D, r)$ -hypergraph, a hypergraph of diameter  $D$ , maximum degree  $\Delta$  and maximum size of the edges  $r$ . The problem considered in the introduction is to determine the maximum number of vertices  $n(\Delta, D, r)$  of a  $(\Delta, D, r)$ -hypergraph. Here we restrict our attention to  $D=1$ , therefore we have:

Main Problem: Determine the maximum number of vertices  $n(\Delta, 1, r)$  of a hypergraph with maximum degree  $\Delta$ , maximal size of the edges  $r$  and such that every pair of vertices belongs to at least one edge.

Proposition 1:  $n(\Delta, 1, r) \leq 1 + \Delta(r-1)$

Proof: Choose a vertex  $x$  and consider the edges containing it; they must contain all the other vertices. There are at most  $\Delta$  such edges and each contains  $x$  and at most  $(r-1)$  other vertices.

The reader may have seen the similarity of our problem with problems in design theory. Let us recall that a  $(v,r,\lambda)$ -BIBD (balanced incomplete block design) is a collection of subsets, called blocks, of a given set on  $v$  vertices such that every block contains exactly  $r$  vertices and every pair of vertices belongs to exactly  $\lambda$  blocks. Furthermore in such a BIBD, every vertex belongs to the same number of blocks

$$\Delta = \lambda(v-1)/(r-1)$$

Proposition 2:  $n(\Delta,1,r)=1+\Delta(r-1)$  if and only if there exists a  $(v,r,1)$ -design with  $v=1+\Delta(r-1)$ .

Proof: One can consider the blocks of the design as the edges of a hypergraph and take  $\lambda=1$ . Then a  $(v,r,1)$ -design with  $v=1+\Delta(r-1)$  is a  $(\Delta,1,r)$ -hypergraph. Conversely, in view of the proof of Prop. 1, if  $n(\Delta,1,r)=1+\Delta(r-1)$ , then every pair of vertices belongs to exactly one edge, and the size of each edge is exactly  $r$ .

Remark: Therefore we can use all the results in design theory to obtain examples of  $(\Delta,1,r)$ -hypergraphs which have the maximum number of vertices. For these results we refer the reader to Hanani [1975] or Doyen & Rosa [1980].

Note also that by Fisher's inequality we always have  $\Delta \geq r$ . However in practice  $\Delta$  is small ( $\Delta \leq 4$ ) and  $r$  can be of an order of 15-20, and as a consequence, results concerning designs will not be readily applicable.

Before we proceed further, we pause to give a dual formulation of the problem, in the spirit of Erdős' topics.

Dual formulation: Consider the dual hypergraph  $H^*=(E^*,X^*)$  of  $H$  where the vertices of  $H^*$  represent the edges of  $H$  and the edges of  $H^*$  represent the vertices of  $H$  and where  $e_j^* \in X_i^*$  if and only if  $x_i \in E_j$  in  $H$ . Then the maximum degree of  $H^*$  is equal to  $r$ , and the maximum size of an edge of  $H^*$  is  $\Delta$ . Furthermore the property that  $H$  is of diameter 1 corresponds to the fact that in  $H^*$  any pair of edges intersects in at least one point; a family of edges having this property is usually called an intersecting family. Therefore our problem can be expressed as follows:

Find the maximum number of edges of an intersecting family of edges, where each edge has size at most  $\Delta$  and each vertex maximum degree  $r$ .

Note that without the restriction on the maximum degree, but with a fixed number of vertices, this problem has been extensively studied.

#### 4. THE CASE $r$ SMALL

Let us recall that in the case  $\Delta=r$  the problem of existence of a  $(v,r,1)$ -design with  $v=1+\Delta(r-1)=(r-1)^2+(r-1)+1$ , is equivalent with the existence of a projective plane of  $p^2+p+1$  points ( $p=r-1$ ). It is well known that such a plane exist when  $p$  is a power of prime. Furthermore, it is known that such plane does not exist when  $p$  is congruent to 1 or 2 (mod 4) and cannot be written as the sum of two squares of integers - for example  $p=6$  or  $p=14$ .

In the case  $r=2$  we have in fact a graph of diameter 1, namely the complete graph, and the bound is always attained.

The case  $r=3$  can be solved, using results on coverings, and the case  $r=4$  also, almost entirely.

Proposition 3:  $n(\Delta, 1, 3) = 2\Delta + 1$  if  $\Delta \equiv 0$  or  $1 \pmod{3}$

$n(\Delta, 1, 3) = 2\Delta + 1$  if  $\Delta \equiv 2 \pmod{3}$

Proof: By Proposition 2 we have seen that the value  $1+2\Delta$  can be reached if and only if a  $(v, 3, 1)$ -design exists with  $v=1+2\Delta$ . Such a design is known as a Steiner triple system  $(n, 3, 2)$  with  $n=1+2\Delta$ . A Steiner triple system exists if and only if  $n \equiv 1, 3 \pmod{6}$ , which means  $\Delta \equiv 0, 1 \pmod{3}$  (see any textbook in Combinatorics). It remains to consider the case  $\Delta \equiv 2 \pmod{3}$ , where the value  $1+2\Delta$  can not be achieved, so we have  $n(\Delta, 1, 3) \leq 2\Delta$  for  $\Delta \equiv 2 \pmod{3}$ . This value can be easily reached from an  $n(\Delta-1, 1, 3)$  system, having  $2(\Delta-1)+1=2\Delta-1$  points, by partitioning the vertices in  $(\Delta-1)$  sets containing two vertices and a singleton set. Then add edges containing each, one of these sets and an additional new vertex.

Proposition 4:

(i)  $n(\Delta, 1, 4) = 3\Delta + 1$  if  $\Delta \equiv 0, 1 \pmod{4}$

(ii)  $n(\Delta, 1, 4) = 3\Delta$  if  $\Delta \equiv 2 \pmod{4}$

(iii)  $n(\Delta, 1, 4) = 3\Delta$  or  $3\Delta - 1$  if  $\Delta \equiv 3 \pmod{4}$ .

Proof: By Prop. 2, we have  $n(\Delta, 1, 4) = 3\Delta + 1$  if and only if there exists an  $(n, 4, 2)$ -BIBD; it is known, since Hanani, that such systems exist if and only if  $n \equiv 1$  or  $4 \pmod{12}$  in which case  $\Delta \equiv 0$  or  $1 \pmod{4}$ . Therefore we have part (i) of the proposition and that if  $\Delta \equiv 2$  or  $3 \pmod{4}$ ,  $n(\Delta, 1, 4) \leq 3\Delta$ .



The existence of a  $(\Delta, 1, 4)$ -hypergraph with  $3\Delta$  vertices will follow from a more general result on coverings. Recall that a covering of the pairs by  $r$ -tuples consists of a family of  $r$ -tuples such that every pair of vertices belongs to at least one of them. A theoretical bound for the number of  $r$ -tuples is  $m \geq \left\lceil \frac{n}{r} \left\lceil \frac{n-1}{r-1} \right\rceil \right\rceil$ . This bound is obtained by considering that each vertex belongs to at least  $\left\lceil \frac{n-1}{r-1} \right\rceil$   $r$ -tuples. If  $\frac{n-1}{r-1}$  and  $\frac{n(n-1)}{r(r-1)}$  are integers and if there exists a covering, with this number of  $r$ -tuples then necessarily each vertex belongs to exactly  $\frac{n-1}{r-1}$   $r$ -tuples. Therefore in this case we have a  $(\Delta, 1, 4)$ -hypergraph on  $n$  vertices with  $\Delta = \frac{n-1}{r-1}$ . In the case  $r=4$ , we know by the work of different authors (see Mills [1979]) that there exists a covering of pairs with quadruples in number  $\left\lceil \frac{n}{4} \left\lceil \frac{n-1}{3} \right\rceil \right\rceil$ .

If we take  $n=12t+6$  then  $\frac{12t+6}{4} \frac{12t+5}{3}$  is an integer and there exists a  $(4t+2, 1, 4)$ -hypergraph with  $12t+6=3\Delta$  vertices (therefore proving (ii)).

If we take  $n=12t+8$  then  $\frac{12t+8}{4} \frac{12t+7}{3}$  is an integer and there exists a  $(4t+3, 1, 4)$ -hypergraph with  $12t+8=3\Delta-1$  vertices, therefore proving (iii).

Remark: We shall prove that  $n(3, 1, 4)=8$  which is  $3\Delta-1$  and proved  $n(\Delta, 1, 4)=3\Delta$  for  $\Delta=7 \pmod{12}$ .

## 5. THE CASE $\Delta$ SMALL

That is this case which is the most interesting in practice. First as we have seen in Proposition 2 if  $\Delta=r$  then  $n(\Delta,1,\Delta)=\Delta^2-\Delta+1$  if and only if there exists a  $(\Delta^2-\Delta+1,1,\Delta)$  design, which is in fact a projective plane of order  $q$ , where  $\Delta=q+1$ . It is known that such a projective plane exists if  $q$  is a prime power. From this projective plane one can easily construct  $(\Delta,1,r)$ -hypergraphs for  $r \geq \Delta$ , which will turn at least for  $\Delta \leq 4$  to have the maximum number of vertices.

Proposition 5: If there exists a projective plane of order  $q$ , then for  $\Delta=q+1$  there exists a  $(\Delta,1,r)$ -hypergraph for  $r \geq \Delta$ , with  $\rho_{\Delta} = \Delta r - (\Delta-1) \left[ \frac{r}{\Delta} \right]$  vertices, that is  $n(\Delta,1,r) \geq \Delta r - (\Delta-1) \left[ \frac{r}{\Delta} \right]$ .

Proof: For  $r = \Delta$  let us simply consider the projective plane with  $\Delta^2 - \Delta + 1$  vertices. Otherwise let  $r = t\Delta - \alpha$ , with  $t \geq 2$  and  $\alpha \leq \Delta - 1$ . Let  $x_0$  be any vertex of the projective plane and let  $L_1, L_2, \dots, L_{\alpha}$  be any  $\alpha$  lines containing  $x_0$ . Then replace each vertex  $x$  by a set  $S_x$  of cardinality  $t - \alpha$  if  $x = x_0$ ;  $t - 1$  if  $x$  is a vertex of  $L_i$  ( $1 \leq i \leq \alpha$ ) different from  $x_0$  and  $t$  otherwise. Replace a line  $L$  containing the vertices  $x_1, x_2, \dots, x_{\Delta}$  by an edge containing the sets  $S_{x_1}, S_{x_2}, \dots, S_{x_{\Delta}}$ . Either the line  $L$  contains  $x_0$  and the edge obtained is of size at most  $t\Delta - \alpha$ . Otherwise the line  $L$  intersects each of the lines  $L_i$  ( $1 \leq i \leq \alpha$ ) and again its size is at most  $t\Delta - \alpha$ .

The degree of any vertex of  $S_x$  is that of  $x$  that is  $\Delta$ . Furthermore if we consider two vertices either they belong to the same  $S_x$  and therefore to many edges or they belong to two different  $S_x$  and  $S_y$  and therefore to the edge associated to the line containing  $x$  and  $y$ . The total number of vertices is:

$$t(\Delta^2 - \Delta + 1) - \alpha - \alpha(\Delta - 1) = \Delta(t\Delta - \alpha) - (\Delta - 1)t = \Delta r - (\Delta - 1) \frac{r}{\Delta}.$$

Thus we have constructed a hypergraph with edge size at most  $r$  having  $\rho_{\Delta}$  vertices (this hypergraph has always  $\Delta$  edges).

Proposition 6: If  $\Delta < 4$ , then  $n(\Delta, 1, r) \leq \Delta r - (\Delta - 1)r / \Delta = \rho_{\Delta}$ .

Before embarking on the proof which will be somewhat long, in particular for  $\Delta = 4$ , let us remark that this bound is different from the bound of Proposition 1 and that with Proposition 5 we have (1, 2, 3 being prime):

Theorem:

- (i)  $n(2, 1, r) = 2r - \lceil r/2 \rceil$ ;
- (ii)  $n(3, 1, r) = 3r - 2 \lceil r/3 \rceil$ ;
- (iii)  $n(4, 1, r) = 4r - 3 \lceil r/4 \rceil$ .

Proof of Proposition 6: The proof is somewhat similar for the cases  $\Delta = 2, 3, 4$  and therefore let us give some lemmas concerning the cardinality of the intersection of  $k$  edges.

Let  $M_k$  denote the maximum cardinality of the intersection of any  $k$  edges of a  $(\Delta, 1, r)$ -hypergraph. Then  $r \geq M_1 \geq M_2 \geq \dots \geq M_{\Delta} \geq 1$ . Let in what follows  $\rho_{\Delta} = \Delta r - (\Delta - 1) \lceil r / \Delta \rceil$ . Let  $n$  be the number of vertices of the hypergraph.

Lemma 1:  $M_{k+1} \geq \lceil M_k / \Delta \rceil$ , in particular  $M_2 \geq \lceil r / \Delta \rceil$ .

Proof: Let  $E_1, E_2, \dots, E_k$  be  $k$  edges whose intersection  $A$  is of cardinality  $M_k$  and let  $x$  be a vertex not belonging to any  $E_i$ , then  $x$  can be related to the points of  $A$  by at most  $\Delta$  edges and one of these edges intersects  $A$  in at least  $|A| / \Delta$  vertices therefore  $M_{k+1} \geq \lceil |A| / \Delta \rceil = \lceil M_k / \Delta \rceil$ .

Lemma 2: If  $M_{\Delta} \geq \lceil r/\Delta \rceil$ , then  $n \leq \rho_{\Delta}$ .

Proof: Let us consider  $\Delta$  edges intersecting in  $M_{\Delta}$  vertices, any vertex of the intersection having a degree  $\Delta$  these edges cover all the vertices of the hypergraph. Therefore

$$n \leq \Delta r - (\Delta-1) M_{\Delta} \leq \Delta r - (\Delta-1) \lceil r/\Delta \rceil = \rho_{\Delta}.$$

Corollary: For  $\Delta = 2n(2,1,r) \leq \rho_2 = 2r - \lceil r/2 \rceil$ .

Indeed by Lemma 1  $M_2 \geq \lceil r/2 \rceil$  and by Lemma 2 then  $n \leq \rho_2$ .

Lemma 3: If  $M_2 \geq (\Delta-1)r/\Delta$  then  $n \leq \rho_{\Delta}$ .

Proof: Consider two edges  $E_1$  and  $E_2$ , intersecting in  $M_2$  vertices, and consider a given vertex  $x$  in the intersection. Then the edges  $E_1, E_2$  and the  $\Delta-2$  other edges containing  $x$  cover all the vertices. Therefore  $n \leq \Delta r - M_2 - (\Delta-2) \leq \Delta r - (\Delta-1)r/\Delta - (\Delta-2) \leq \rho_{\Delta}$ .

Lemma 4: Let  $E_1, E_2, \dots, E_k$  be  $k$  edges intersecting in  $M_k$  vertices and let  $C_k$  be the set of vertices not belonging to any of the  $E_i$ . If  $M_{k+1} < M_k$ , then  $|C_k| \leq (\Delta-2)^2 M_2$ .

Proof: Let  $B_k$  the intersection of  $E_i$  ( $i=1, \dots, k$ ), and let  $x$  be a vertex in  $B_k$ ,  $x$  belongs to at most  $(\Delta-k)$  other edges which can intersect  $C_k$ . Therefore one of them, say  $F$  intersects  $C_k$  in at least  $|C_k|/(\Delta-k)$  vertices. If  $M_k > M_{k+1}$ ,  $B_k \not\subseteq F$  therefore there exists an other vertex  $y$  in  $B_k - F$  and another edge  $F'$  containing  $y$  which intersects also  $C_k$  into at least  $|C_k|/(\Delta-k)$  vertices. But then  $|F \cap F'| \geq |C_k|/(\Delta-k)^2$  and  $M_2(\Delta-k)^2 \geq |C_k|$ .

Lemma 5: If  $M_{\Delta-1} \geq \lceil r/\Delta \rceil$  then  $n \leq \rho_{\Delta}$ .

Proof: By lemma 2 we can suppose  $M_{\Delta} \leq \lceil r/\Delta \rceil$ . Therefore  $M_{\Delta} < M_{\Delta-1}$  and by lemma 4,  $|C_{\Delta-1}| \leq M_2$ . By lemma 3,  $M_2 \leq (\Delta-1)r/\Delta - 1$ . Let  $E_1, E_2, \dots, E_{\Delta-1}$  be  $\Delta-1$  edges intersecting in  $M_{\Delta-1}$  vertices.

Then  $n \leq |E_1 \cup E_2 \cup \dots \cup E_{\Delta-1}|$ , and we have

$$\begin{aligned} n &\leq (\Delta-1)r - (\Delta-2)M_{\Delta-1} + |C_{\Delta-1}|. \text{ So we have} \\ n &\leq (\Delta-1)r - (\Delta-2)r/\Delta + (\Delta-1)r/\Delta - 1, \text{ and} \\ n &\leq \rho_{\Delta} - r + r/\Delta + (\Delta-1)r/\Delta - 1 = \rho_{\Delta}. \end{aligned}$$

Corollary: For  $\Delta=3$ ,  $n(3,1,r) \leq \rho_3 = 3r - 2\lceil r/3 \rceil$ .

Proof: By lemma 1  $M_2 \geq \lceil r/\Delta \rceil$  and by lemma 5  $n \leq \rho_3$ .

Proof of the case  $\Delta=4$ .

By lemma 1  $M_2 \geq \lceil r/4 \rceil$ . By lemma 3 we can suppose (otherwise the proof is finished), that  $M_2 < 3\lceil r/4 \rceil$ ;  $M_3 < \lceil r/4 \rceil$ . Let  $E_1$  and  $E_2$  be two edges intersecting in  $M_2$  points. Let  $A = E_1 \cup E_2$  and  $B = E_1 \cap E_2$  and let  $C = X - A$ . By lemma 4 we have  $|C| \leq 4M_2$  (1). Furthermore  $n \leq |A| + |C| \leq 2r + 3M_2$  that is  $n \leq 2r + 3M_2$  (2) which can be expressed as  $M_2 \geq (n-2r)/3$  (2').

Let  $F_1$  be an edge intersecting  $C$  and  $B$  chosen in such a way that  $F_1 \cap B$  is as large as possible. Let  $B_1 = F_1 \cap B$ ;  $C_1 = F_1 \cap C$ ;  $\bar{C}_1 = C - C_1$ . If  $x \in B_1$ , then the last edge containing  $x$  must contain  $\bar{C}_1$ . If there exist two distinct edges intersecting  $B_1$  and  $\bar{C}_1$ , then  $|\bar{C}_1| \leq M_2$  and  $n \leq |A| + |F_1| - |B_1| + |\bar{C}_1| \leq 2r - M_2 + r - 1 + M_2 = 3r - 1 \leq \rho_4$ .

Therefore we can suppose (otherwise the proof is finished) that there exists one edge  $F'_1$  containing  $B_1$  and  $\bar{C}_1$ . By the maximality of  $B_1$ ,  $F'_1 \cap B = B_1$ . Let  $C'_1 = F'_1 \cap C$ , then  $|C'_1| \geq |\bar{C}_1|$ . As  $B_1 \neq B$ , let  $F_2$  be an edge intersecting  $B - B_1$  and  $C$  and chosen in such a way that  $F_2 \cap B$  is as large as possible. Let  $B_2 = F_2 \cap B$ . As above there exists another edge  $F'_2$  containing  $B_2$ . We can construct a partition of  $B$  into sets  $B_i$ ,  $1 \leq i \leq k$ , where  $k \leq 4$  (because the degree of a vertex is at most 4). Let  $F_i$  and  $F'_i$  be such that  $B_i = F_i \cap B = F'_i \cap B$ . Let  $C_i = F_i \cap C$ ;  $C'_i = F'_i \cap C$ ;  $C_i \cup C'_i = C$ . Therefore  $|C| \leq 2r - 2|B_1| \leq 2r - 2|B_1|$  (3).

To finish the proof we consider the 3 possible values of  $k$ .

Case 1:  $k=4$ .

The 4 edges containing a vertex  $z$  of  $C$  intersect  $B$ . One of them, say  $F$  intersects  $A=E_1 \cup E_2$  in at least  $|A|/4 = (n-|C|)/4$  vertices. Let  $x$  be a vertex of  $B \cap F$ . Then by considering the 4 edges containing  $x$   $n \leq 2r - M_2 + r - (n-|C|)/4 + r - 1 = 4r - r/4 - 1 + (|C|/4 - M_2)$  then by (1)  $n \leq 4r - r/4 - 1$  or  $n \leq (16r-4)/5$ . A calculation shows that  $(16r-4)/5 \leq \rho_4$ , except if  $r=5$  or  $9$ .

If  $r=5$ ,  $M_2=14$  and we cannot have 15 because  $M_2 \leq \lfloor 15/4 \rfloor - 1 = 3$  and therefore we cannot partition  $B$  into 4 disjoint sets.

If  $r=9$ ,  $M_2=27$ . Suppose  $n=28$ . Then by (2')  $M_2 \geq 4$  and  $|C| \geq n - |A| \geq 14$ . But as above  $|C| \leq 2r - (n-|C|)/4 - 1 = 10 + |C|/4$  therefore  $|C| \leq 40/3$  and  $|C| \leq 13$  is a contradiction.

Case 2:  $k=3$ .

In that case  $|B_i| \geq M_2/3$ . Any edge intersecting  $B$  and  $C$  intersects  $C$  in at most  $2M_2$  vertices, otherwise there will be one of the two other edges intersecting  $B$  intersecting it in  $M_2+1$  vertices. Without loss of generality let us suppose  $|C_1| \geq |C|/2$  (otherwise interchange  $F_1$  and  $F_1'$ ) and  $|C_2| \geq |C|/2$ . Therefore there exists  $z \in C_1 \cap C_2$  (if  $|C_1| = |C|/2$  and  $C_1 \cap C_2 = \emptyset$  choose  $C_1'$  instead of  $C_1$ ). The third edge containing  $z$  and  $B_3$  has at least  $|C| - 2M_2$  vertices in  $C$ .

Let  $A_i = F_i \cap A$  then  $|A_1| + |A_2| + |A_3| \leq 2(r - |C|/2) + r - |C| + 2M_2 = 3r - 2|C| + 2M_2$ . The last edge intersects  $A$  in at most  $2M_2$  vertices. Therefore  $|A| \leq 3r - 2|C| + 4M_2$  and  $n \leq 3r - |C| + 4M_2$  (4). On another way, as  $F_i$  and  $F_i'$  cover  $C$ ,  $|C| \leq 2r - |A_i| + |B_i|$  for every  $i$  and  $3|C| \leq 6r - (|A_1| + |A_2| + |A_3|) - M_2$ . Combining with  $|A| \leq |A_1| + |A_2| + |A_3| + 2M_2$  and  $|A| + |C| = n$  we obtain  $n + 2|C| \leq 6r + M_2$  or  $n \leq 6r - 2|C| + M_2$  (5). Recall that  $n = |A| + |C| \leq 2r - M_2 + |C|$  (6) then (4) and (6) give  $2n \leq 5r + 3M_2$  (5) and (6) give  $3n \leq 10r - M_2$  and therefore  $11n \leq 35r$  or  $n \leq 35r/11$ . In general  $35r/11 \leq \rho_4$  except for  $r=5, 6, 9, 13, 17$ . These cases can be dealt directly by looking more precisely at the structure.

Case 3:  $k=2$ .

Now  $|B_1| \geq M_2/2$  and  $|C| \leq 2r - M_2$  by (3). Therefore  $n = |A| + |C| \leq 4r - 2M_2$ . Combining with (2)  $n \leq 2r + 3M_2$  we obtain  $5n \leq 16r$  or  $n \leq 16r/5$ . In general  $16r/5 \leq \rho_4$  except  $r=5,6,9,10,13,17,21,25$ . There again a careful examination of the structure proved that the values  $n = \rho_4 + 1$  or  $\rho_4 + 2$  are impossible.

## 6. ANOTHER CONSTRUCTION

We can construct  $(\Delta, 1, r)$ -hypergraphs having a large number of vertices by using composition methods like in design theory. Let us give an example. Suppose there exists a  $(t, 1, r)$ -hypergraph on  $p$  vertices, we will construct a  $(\Delta, 1, r)$ -hypergraph on  $pk$  vertices as follows. Let us partition the  $pk$  vertices into  $k$  sets  $X_i$ ,  $i=1, \dots, k$ , each of size  $p$ . Consider a covering of the pairs of vertices  $(x_i, x_j)$  with  $x_i \in X_i$ ,  $x_j \in X_j$  and  $i \neq j$  with sets of size  $r$  in such a way that each vertex belongs to at most  $t$  such sets. The union of these sets and of the edges of  $k$   $(t, 1, r)$ -hypergraphs constructed on each  $X_i$  forms a  $(\Delta, 1, r)$ -hypergraph on  $pk$  vertices where  $\Delta = p + t$ .

As a particular case let  $k=r$ . In this case we know that there exists a partition of the pairs of vertices  $x_i, x_j$  with  $x_i \in X_i$ ,  $x_j \in X_j$ ,  $i \neq j$  with sets of size  $r$  if and only if there exist  $r-2$  orthogonal latin squares of order  $p$ . In such a case every vertex belongs to exactly  $p$  edges and we have the Proposition:

Proposition 7: If there exists a  $(t,1,r)$ -hypergraph on  $p$  vertices and  $(r-2)$  orthogonal latin squares of order  $p$  then there exists a  $(t+p,1,r)$ -hypergraph on  $pr$  vertices.

For example that can be applied with  $r=4$ ,  $p \neq 2,6$  because there exist 2 orthogonal latin squares of any order  $p \neq 2,6$ . In the case  $r=5$  as there exist 3 orthogonal latin squares of order  $p \neq 2,3,6,10,14$  we can apply the proposition with  $r=5$ ,  $p \neq 2,3,6,10,14$ .

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