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Graphs and Interconnection networks: diameter and vulnerability,

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► **To cite this version:**

Jean-Claude Bermond, Johny Bond, Madeleine Paoli, Claudine Peyrat. Graphs and Interconnection networks: diameter and vulnerability,. Surveys in Combinatorics, London Math Society Lecture Notes Series 82, 1, Cambridge University Press, pp.1-30, 1983, 10.1017/CBO9781107325548 . hal-02447135

HAL Id: hal-02447135

<https://inria.hal.science/hal-02447135>

Submitted on 21 Jan 2020

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1 INTRODUCTION

It is well known that telecommunication networks or interconnection networks can be modelled by graphs. Recent advances in technology, especially the advent of very large scale integrated (VLSI) circuit technology have enabled very complex interconnection networks to be constructed. Thus it is of great interest to study the topologies of interconnection networks, and, in particular, their associated graphical properties. If there are point-to-point connections, the computer network is modelled by a graph in which the nodes or vertices correspond to the computer centres in the network and the edges correspond to the communication links. When the computers share a communication medium such as a bus, the network is modelled by a hypergraph, where the nodes correspond to the computer centres and the (hyper)edges to the buses. Note that there exists a second important class of networks, the "multistage networks", but we will not consider them. For a survey of interconnection networks, we refer the reader to Feng (1981).

In the design of these networks, several parameters are very important, for example message delay, message traffic density, reliability or fault tolerance, existence of efficient algorithms for routing messages, cost of the networks, ...

One important measure of the power of an interconnection network is the length of the longest path that the messages must travel from one node to another in the network, i.e. the distance between the nodes. It is advantageous to make these distances as small as possible, since this will reduce the message delay and also the message traffic density in the links. Worst case distance corresponds to the diameter of the associated graph or hypergraph. Similarly other network characteristics correspond to parameters of graphs or hypergraphs, e.g. mean distance, symmetry, connectivity, ...

A direct approach to network construction is to consider the graph model of possible links, with lengths and costs. The problem of designing a minimal or even near-minimal diameter subgraph with limited total cost was proved to be NP-complete by Plesník (1981). Also, Yannakakis (1978) proved that determining a connected subgraph with bounded maximal degree (the case of most network applications) of a given graph is NP-complete.

The object of this paper is to survey the results concerning diameter and connectivity in graphs and hypergraphs, in particular those of some importance for interconnection networks. For other results on the diameter we refer the reader to Bermond & Bollobás (1981), and on the connectivity to Mader (1979). Furthermore, we will only consider the deterministic aspect of these problems, though it is worth noting that the probabilistic aspect is considered in many papers and is of great importance for the reliability properties of networks.

Definitions and notation

We use standard terminology (see for example Berge (1973)). We give below some important definitions and notation which might be different from the usual terms.

The graph $G = (X, E)$ has vertex-set X and edge-set E . We denote by $n = |X|$ the number of vertices and by $m = |E|$ the number of edges. The degree $d(x)$ of a vertex x is the number of edges incident with x . We denote by δ the minimum degree and by Δ the maximum degree of the graph. A path with endpoints x and y is called an x - y path. The distance between two different vertices x and y is the length of a shortest x - y path. The diameter is the maximum distance over all the pairs of vertices. It will be denoted by D .

In interconnection networks a practical problem is to interconnect the maximum number of nodes, while minimizing the diameter and knowing that, because of technical constraints, a node must not be incident to more than a fixed number of links. When a graph is associated with the network, this problem, first posed by Elspas (1964) and known as the (Δ, D) -graph problem, can be stated in graph-theoretic terms. How many vertices can a (Δ, D) -graph have, where a (Δ, D) -graph is a graph with maximum degree Δ and diameter at most D ? Let $n(\Delta, D)$ be the maximum number of vertices of a (Δ, D) -graph.

The Moore bound and Moore graphs

A theoretical bound on $n(\Delta, D)$ was given by Moore:

$$n(2, D) \leq 2D+1$$

$$n(\Delta, D) \leq \frac{\Delta(\Delta-1)^D - 2}{\Delta-2} \quad \text{if } \Delta \geq 3.$$

Let the theoretical Moore bound be denoted $n_0(\Delta, D)$. The (Δ, D) -graphs with $n_0(\Delta, D)$ vertices are called Moore graphs. This name was given by Hoffman & Singleton (1960), because E.F. Moore raised the question with Hoffman and thought that eigenvalue techniques could be used to solve the problem (personal communication of Hoffman).

We warn the reader of the fact that the term Moore graphs is also used for what are known as cages. A cage is a Δ -regular of girth g having the minimum possible number of vertices. It can be shown that if g is odd, $g = 2D+1$, the minimum number of vertices is also $n_0(\Delta, D)$ and the cages are of diameter D . Therefore in this case, cages are exactly Moore graphs. If g is even, $g = 2D$, the minimum number $n_1(\Delta, g)$ of vertices is

$$n_1(\Delta, g) = 2D \quad \text{if } \Delta = 2,$$

$$n_1(\Delta, g) = \frac{2(\Delta-1)^D - 2}{\Delta-2} \quad \text{if } \Delta \geq 3.$$

We propose to call these graphs bipartite Moore graphs because they achieve the maximum number of vertices of a bipartite (Δ, D) -graph. For a survey on cages see Wong (1982).

It has been proved by various authors (see Biggs' book (1974)) that the Moore graphs exist only for $\Delta = 2$ ($2D+1$ -cycles), or $D = 1$ ($\Delta+1$ -cliques), or $D = 2$ and $\Delta = 3$ (the Petersen graph), or $D = 2$

and $\Delta = 7$ (the Hoffman-Singleton graph) and possibly for $D = 2$ and $\Delta = 57$. Such a $(57,2)$ -Moore graph has not been exhibited, however Aschbacher (1971) has shown that it cannot be distance transitive. Singleton (1966) proved that bipartite Moore graphs cannot exist unless $\Delta = 2$ (2Δ -cycles) or $D = 2$ (the complete bipartite graph $K_{\Delta,\Delta}$) or $D = 3, 4$ or 6 . In the cases $D = 3, 4$ or 6 , it has been shown, in particular by Benson (1966), that such cages (bipartite Moore graphs) exist if $\Delta = q+1$, q being a prime power. Therefore we know infinite families of bipartite Moore graphs.

Instead of undirected graphs, we can consider directed ones. If D is a digraph with indegree and outdegree at most Δ and diameter exactly D , the number of vertices is no greater than the directed Moore bound: $n_{\text{Dir}}(\Delta, D) = 1 + \Delta + \dots + \Delta^D$. Bridges & Toueg (1980) have proved the impossibility of Moore digraphs (i.e. digraphs achieving the Moore bound) for $D > 1$ and $\Delta > 1$. This was also proved by Plesník & Znáám (1974).

Large (Δ, D) -graphs

When Moore graphs do not exist, the determination of the exact value of $n(\Delta, D)$ appears to be a very difficult problem. So an interesting problem, especially for practical applications, is to find large lower bounds for $n(\Delta, D)$. For example, Erdős, Fajtlowicz & Hoffman (1980) ask the following question: given a non-negative number σ , is there a graph G with diameter 2, degree Δ and $\Delta^2 + 1 - \sigma$ vertices? They show that it is false for $\sigma = 1$ except if $G = C_4$, and therefore $n(\Delta, 2) \leq n_0(\Delta, 2) - 2$ if $\Delta \geq 4$ and $\Delta \neq 7$ or 57 . In fact, this problem seems very difficult. A simpler one is to ask whether, for a fixed diameter D , there exist infinite families of (Δ, D) -graphs with about $C(D)\Delta^D$ vertices. That would mean $\liminf_{\Delta \rightarrow \infty} n(\Delta, D)\Delta^{-D} \geq C(D)$. We know by the Moore bound that $C(D) \leq 1$, and a question is: can we have $C(D) = 1$? Partial results on this problem will be given later on.

In order to give a lower bound for $n(\Delta, D)$, we have to show the existence of (Δ, D) -graphs with many vertices. For this purpose, there are basically two methods: use random graphs or design such graphs. For applications in interconnection networks, the second method is certainly more interesting than the first one since it really gives good graphs, whereas the first one, though perhaps more powerful, guarantees only their existence. However, we will note later on a few results on diameter in random graphs; for more details we refer the reader to

Klee & Larman (1981), Bollobás (1981) and Bollobás & de la Vega (1983).

Construction of large (Δ, D) -graphs

We will now survey the different types of method which have been used for the construction of large (Δ, D) -graphs.

Geometric methods: for instance, the existence of projective planes of order q , when q is a prime power, permits us to construct graphs with degree $\Delta = q+1$, diameter $D = 2$ and $q^2 + q + 1$ vertices. For these methods, see Delorme (1983a).

Direct methods: a simple example is given by the generalized de Bruijn graphs (de Bruijn (1946)). Let $q > 0$, $m \geq 2$ and $Q = \{0, 1, \dots, q-1\}$. The de Bruijn graph $G = (Q^m, E)$ is the graph with vertex-set Q^m and edge-set E where $(\alpha, \beta) \in E$, if and only if $\alpha = a\gamma$ and $\beta = \gamma b$ for some $a, b \in Q$ and $\gamma \in Q^{m-1}$. It has maximum degree $\Delta = 2q$, diameter $D = m$ and q^m vertices. Since $q^m = \left(\frac{\Delta}{2}\right)^D$, we have: $\lim_{\Delta \rightarrow \infty} n(\Delta, D)\Delta^{-D} \geq 2^{-D}$. These graphs have been generalized by Lam & van Lint (1978), Memmi & Raillard (1982), Delorme & Farhi (1983) and Quisquater (1983a).

Recently, we received a copy of Fiol, Alegre & Yebra (1983). They construct regular digraphs of out- and indegree Δ , diameter D , having $\Delta^D + \Delta^{D-1}$ vertices. In the case $D = 2$, these graphs have the maximum possible number of vertices $(\Delta^2 + \Delta)$ because Moore digraphs (with $\Delta^2 + \Delta + 1$ vertices) do not exist. If we consider the associated undirected graphs, we have, for Δ even, $n(\Delta, D) \geq \left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1}$. The authors also show that their digraphs can be viewed as iterated line digraphs of complete symmetric digraphs. Still more recently, Y. Raillard informed us that Kautz (1969) used these same graphs, which he referred to as state diagrams of shift registers, for the same purpose.

Addition of vertices: in certain graphs, vertices can be added without changing the degree or the diameter of the initial graph. See for example Delorme (1983a, 1983b). Recently, one of us (Bond) has shown that vertices can be added to the de Bruijn graphs. For there is a loop in these graphs for each vertex (a, a, \dots, a) and a double-edge between two vertices of the form (a, b, a, b, \dots) and (b, a, b, a, \dots) . Eliminating these loops and double-edges, there remain q points of degree $2q-2$ and $q(q-1)$ of degree $2q-1$. Then we add q points x_i , $i = 0, 1, \dots, q-1$, with q edges between x_i and vertices $(\alpha, i, \alpha, i, \dots)$ for any $\alpha \in Q$, so that x_i has degree q . We add a vertex x_q (of degree q) connected to $(\alpha, \alpha, \dots, \alpha)$ for any $\alpha \in Q$. So we can add q other vertices y_i ($i \in Q$)

such that they form a complete bipartite graph $K_{q+1,q}$ with the $(q+1)$ vertices $x_i, i = 0,1,\dots,q$. Now, we have a graph with q^{m+2q+1} vertices, all of degree $2q$ except y_i . The resulting completed de Bruijn graph still has diameter m .

Products of graphs: in general, the classic products do not give good results, but Bermond, Delorme & Farhi (1982, 1983) have defined a new product which produces a lot of interesting (Δ, D) -graphs for small values of D and Δ .

Delorme (1983b) considers a product which, from two bipartite graphs, gives a bipartite graph. It consists of taking one of the connected components of the Kronecker product of two graphs; in what follows we will call it also a Kronecker product.

Let us consider two bipartite graphs $(X \cup X', E)$ and $(Y \cup Y', F)$ with diameters D and D' respectively, and degrees $\Delta_X, \Delta_{X'}, \Delta_Y$ and $\Delta_{Y'}$, for the vertices of X, X', Y and Y' respectively. The Kronecker product of these two graphs has vertex set $(X \times Y) \cup (X' \times Y')$, and (x, y) is adjacent to (x', y') if and only if $(x, x') \in E$ and $(y, y') \in F$. The new graph is bipartite with maximum degree $\max(\Delta_X \Delta_Y, \Delta_{X'} \Delta_{Y'})$ and diameter $\max(D, D')$. Thus, if we take the Kronecker product of a bipartite graph $G = (X \cup X', E)$ and its opposite $(X' \cup X, E)$, the resulting graph has degree $\Delta_X \Delta_{X'}$ and diameter $D(G)$.

Delorme uses for $(X \cup X', E)$ the bipartite Moore graph with diameter 4 and degree $(q+1)$, where q is a prime power, and with $|X'| = |X| = (q+1)(q^2+1)$. He obtains thus a bipartite graph $P = (A \cup B, \mathcal{E})$ of diameter 4, degree $(q+1)^2$, and $2(q+1)^2(q^2+1)^2$ vertices. This graph P has a polarity, i.e. an automorphism f such that $f(A) = B$ and $f(B) = A$ and $f^2 = id$; here $f((x, x')) = (x', x)$. The quotient of this graph P by the polarity is the graph with vertex set A where two distinct vertices a and a' are adjacent if and only if a and $f(a')$ are adjacent in P . The quotient Q has degree $(q+1)^2$, $(q+1)^2(q^2+1)^2$ vertices and diameter D equal to 3. Thus, we obtain a family of graphs of degree Δ , diameter 3 and with about Δ^3 vertices. Therefore: $\liminf_{\Delta \rightarrow \infty} n(\Delta, D) \Delta^{-3} = 1$. (We recall that because of the Moore bound $\liminf_{\Delta \rightarrow \infty} N(\Delta, D) \Delta^{-D} \leq 1$). In the same way, using the bipartite Moore graphs of degree $q+1$ and diameter 6, Delorme obtained by the same methods a family of graphs with degree Δ ,

diameter 5 and about Δ^5 vertices. For $D = 1, 2, \dots, 10$ he gave improved lower bounds u_D for the asymptotic values $\liminf_{\Delta \rightarrow \infty} N(\Delta, D) \Delta^{-D}$.

D	1	2	3	4	5	6	7	8	9	10
u_D	1	1	1	$3^3 \cdot 2^{-7}$	1	$2.5^6 \cdot 6^{-6}$	$6^6 \cdot 7^{-7}$	$3 \cdot 2^{-8}$	$14 \cdot 3^{-8}$	$5 \cdot 2^{-10}$

Compounding methods: they have been developed by Bermond, Delorme & Quisquater (1982b), and also by Uhr (1983). In general, they consist of replacing the vertices in a large graph by copies of the same small graph or by different graphs. Some of these constructions were also considered by Wegner (1976); he proved, for example, that $n(\Delta, D) \geq (\Delta+1)n(\Delta-1, D-2)$.

Jerrum & Skyum (1983) have compounded de Bruijn graphs with a graph G' , which have good properties of average path length, to obtain for Δ fixed, graphs with a diameter equal to $\lambda_{\Delta} \log_2 n + O(1)$. Their values of λ_{Δ} , which depend on Δ only, improved those of the de Bruijn graphs and are now the best obtained by constructive means. For example, they obtained cubic graphs with diameter $1.472169 \log_2 n + O(1)$; a previous construction giving a diameter of $1.5 \log_2 n + O(1)$ was given by Leland & Solomon (1982). However, the results on random graphs of Bollobás and de la Vega (1983) give better values. One of their main results is that, for $\Delta \geq 3$ and $\epsilon > 0$ fixed, if $D = D(n)$ is the least integer satisfying $(\Delta-1)^{D-1} \geq (2+\epsilon)\Delta n \log n$, then almost every Δ -regular graph of order n has diameter at most D . As a corollary of this result, for a fixed degree Δ , the minimum diameter of a graph of order n is bounded above by $\frac{\log n}{\log(\Delta-1)} + O(\log \log n)$. This is the best possible result because of the Moore bound: $D \geq \frac{\log n}{\log(\Delta-1)}$. It would be interesting to construct families of graphs which attain this bound.

Computer searches: these have been done by Quisquater (1983a) using Y -graphs and recently by Doty (1982) using chordal ring. Other algorithmic methods which need the help of a computer have been developed before by Toueg & Steiglitz (1979), Imase & Itoh (1981) and Arden & Lee (1982). Different tables of the largest known (Δ, D) -graphs, for $\Delta \leq 15$, $D \leq 10$, have been given by Storwick (1970), Leland, Finkel, Qiao, Solomon & Uhr (1981), Memmi & Raillard (1982) and Bermond, Delorme & Quisquater (1982a).

Minimum number of edges of a (Δ, D) -graph

Another problem close to the (Δ, D) -graph problem was proposed by Erdős & Rényi (1962). Suppose that there is a (Δ, D) -graph of order n . Let $e_D(n, \Delta)$ be the minimum number of edges of such a graph. An interconnection network whose associated graph achieves this bound is interesting because it has a minimum cost. Bollobás (1978, ch IV) presented, in the case $D \leq 3$, results of Erdős & Rényi (1962), Erdős, Rényi & Sós (1966) and Bollobás (1971). More recently, the case $D = 2$ has been settled almost completely by Pach & Surányi (1981). They defined a function $g(c)$ on $[0, 1]$ and proved that $\lim_{n \rightarrow \infty} e_2(n, \lfloor cn \rfloor) / n = g(c)$ for every c , $0 < c < 1$, except for $c = c_1, c_2, \dots$ where (c_k) is a sequence tending to 0.

Mean distance

We may also consider that the important parameter in an interconnection network is not the maximal time spent to travel from one node to another, but the average time. Therefore, we must study instead of the diameter of the associated graph, its average path length or mean distance. It has been defined by Doyle & Graver (1977):

$$\mu(G) = \frac{1}{\binom{n}{2}} \sum_{v, w \in X} d(v, w).$$

Few things have been done on this parameter except for certain particular cases. The mean distance is easily calculated if the graph has great properties of symmetry, for example if it is distance degree regular, which means that each vertex has exactly D_j vertices at distance j from it. As this sort of graph is far from our topic, for other properties of them, we refer the reader to a survey of Bloom, Kennedy & Quintas (1981). Buckley & Superville (1981) have determined the mean distance for various classes of graphs, and Buckley (1981) for their line-graphs.

Cerf, Cowan, Mullin & Stanton (1975) introduced a problem similar to the (Δ, D) -graph problem: given the number n of vertices and the degree Δ of a graph, what is the lower bound of the mean distance? The graphs which achieve this bound are called generalized Moore graphs. The same authors (1974a, 1974b, 1976) described these graphs or established their non-existence up to $n = 34$, for $\Delta = 3$.

Other problems of distance have been investigated in the literature, for example problems of radius, eccentricity or of "locations in networks" (see the survey of Tansel, Francis & Lowe (1983) which contains many references).

3 CONNECTIVITY

In the design and analysis of interconnection networks, one of the fundamental considerations is the reliability, in particular that the centres can communicate in case of link or node failure. Reliability has been defined in a number of different ways, either with deterministic or probabilistic criteria. Maybe the probabilistic criteria fit better with the real world, indeed vertices or links can fail randomly: one of the main problems is to compute the probability that there is an operating path between two nodes knowing the probability of vertex or link failure. We will not discuss this non-combinatorial aspect here; we refer the reader to the surveys of Frank & Frisch (1970, 1971), Wilkov (1972) and Hwang (1978) in the case of multistage networks. Note that Ball (1980) has recently shown that virtually all network reliability analysis problems of practical interest are NP-hard.

In the case of deterministic networks, the criteria indicate how difficult it is to disrupt the operation of the whole network (or some part of it). Therefore one aim is to maximize the number of nodes or links that must fail in order to disrupt the operation of the network, while taking into account that there are cost constraints. The main criteria are the classical connectivity or edge-connectivity and the local connectivity.

Classical results

Recall that an x - y separating set S is a non-empty subset of vertices of G whose deletion destroys all the paths between x and y . The local connectivity between x and y is defined as the minimum cardinality of an x - y separating set. The connectivity of a non-complete graph is the minimum value of the local connectivity over all pairs of vertices. The connectivity will be denoted by $\kappa(G)$. By convention we set $\kappa(K_n) = n-1$. Analogous definitions can be given for local edge-connectivity and edge-connectivity; the latter will be denoted by $\lambda(G)$.

For an excellent survey of connectivity see Mader (1979). We will mention here only a very small number of results, which are of

interest for our purpose. Recall first that by Menger's theorem the local connectivity between x and y is equal to the maximum number of vertex-disjoint paths between x and y . A similar result holds with edges instead of vertices. Using this or flow techniques there exist good (i.e. polynomial) algorithms to determine the local or the global connectivity (or edge-connectivity) of a graph (see for example Even's book (1979) or Chvátal's book (1983)). An algorithm in $O(n^3)$ for finding maximum flows is given in Malhotra, Kumar & Maheshwari (1978).

Symmetric graphs

Another well-known result says that, in any graph G , $\kappa \leq \lambda \leq \delta$. When the graph has some symmetries the following results are interesting; in particular they ensure that the graph has the best connectivity possible without any computation. A graph is said to be vertex-transitive if for every pair of vertices x and y there exists an automorphism f of the graph such that $f(x) = y$. Edge-transitive graphs are defined analogously.

Watkins (1970) and Mader (1970) have shown that if G is a connected edge-transitive graph then $\kappa = \lambda = \delta$. In the case of connected vertex-transitive graphs they have proved that $\kappa \geq 2 \lfloor \frac{\delta}{3} \rfloor + 2$. Furthermore Mader (1971) proved that in any connected vertex-transitive graph $\lambda = \delta$. Mader (1970) also showed that if G is a connected vertex-transitive graph without a complete graph on 4 vertices then $\kappa = \delta$.

This result of Mader (1970) can be used to show that each of the graphs defined by Memmi & Raillard (1982) (see §2) has its connectivity equal to its degree. This result has been proved directly by exhibiting the paths by Amar (1983).

Maximum connectivity

Note that as $\kappa \leq \lambda \leq \delta$ the maximum connectivity or edge-connectivity of a graph with n vertices and m edges is $\lfloor \frac{2m}{n} \rfloor$. Harary (1962) exhibited constructions of graphs having this maximum connectivity. In fact, that follows from the construction of graphs on n vertices, of connectivity κ , having the minimum number of edges (and corresponds to the problem of designing a minimum cost network, in which all the edges have the same cost with a uniform reliability). If the connectivity k is even, the best known of these graphs is $C_n^{\frac{1}{2}k}$ the $\frac{1}{2}k$ power of a cycle C_n (that is the graph in which two vertices i and j are joined if $|j-i| \leq \frac{1}{2}k \pmod{n}$). If the connectivity k is odd and the

number of vertices is even, we take $C_n^{\frac{1}{2}(k-1)}$ plus the diameters (edges between i and $i + \frac{1}{2}n \pmod{n}$). The same construction works in the case k odd, n odd, but one vertex is of degree $k+1$.

Recently it has been proved by Paoli, Wong & Wong (1983) that these graphs (or some variant of them) are also optimal for designing a minimum cost network with the property that even after k links (or nodes) have failed it still can function as a basic ring network. In graph theoretical terms this means that the above graphs are k -edge (vertex) hamiltonian.

Optimization problems

For constructions and properties of the graphs with maximum connectivity see either Wilkov's survey (1972) or Boesch's survey (1983). Note that the construction of minimum graphs on n vertices with connectivity κ is a particular case of this generalized synthesis problem: given a weighted graph on n vertices in which every edge e has a cost $c(e)$ and given an integer $n \times n$ matrix of required vertex (edge) connectivities U_{ij} , find a partial graph of G of minimum total cost such that the local connectivity between vertices x_i and x_j is at least U_{ij} ($1 \leq i < j \leq n$).

The case considered above deals with a uniform cost and $U_{ij} = \kappa$. There is a wide literature on this general problem, but we will not deal with it as it has been surveyed very recently by Christofides and Whitlock (1981).

Connectivity and diameter

Unfortunately most of these graphs, in particular those constructed as powers of cycles have a great diameter. For example $C_n^{\frac{1}{2}k}$ has diameter $\lfloor \frac{n}{k} \rfloor$. Therefore it would be of interest to construct maximally connected graphs with the smallest diameter possible. For example, an important problem is to build regular graphs with connectivity equal to the degree Δ and minimizing the diameter D , which, because of the Moore bound, is at least $\frac{\log n}{\log(\Delta-1)}$. Since this lower bound is very sporadically achieved, only by the Moore graphs, Wilkov (1972) asked for an algorithm doing this construction for every n ; but this seems to be a very difficult problem. However, Schumacher (1983) has given a polynomial algorithm (complexity $O(n^2)$) to construct Δ -regular, Δ -connected graphs on n vertices with diameter at most twice the Moore bound.

Generalizations

Finally let us note that here we have considered essentially as the criterion of reliability the classical connectivity or edge-connectivity.

One objection to using only these parameters is that they fail to differentiate between the different types of disconnected graphs which result from removing κ vertices or λ edges. It is not the same in practice to isolate a single vertex or to divide the graph into two equal components. Several generalized measures of connectedness have been proposed in the literature. A recent complete survey of these parameters has been done by Boesch (1983). We will see in the next paragraphs some measures involving connectivity and diameter.

4 DISJOINT PATHS

Mengerian-type problems

In the reliability analysis of a communication network it is often required to find the maximum number of vertex-disjoint paths between two given vertices of the network with the requirement that the length of each path does not exceed a given value. With this additional constraint the problem becomes more complicated than the k -connectivity.

Let $I_k(x,y)$ be the maximum number of vertex-disjoint paths between x and y of length not greater than k . Itai, Perl & Shiloach (1982) have shown that the determination of $I_k(x,y)$ is an NP-complete problem if $k \geq 5$. This fact is not surprising because, for $k \geq 5$, there does not exist a Mengerian-type theorem.

Let $S_k(x,y)$ be the minimum number of vertices whose deletion destroys all x - y paths of length not greater than k . We always have $I_k(x,y) \leq S_k(x,y)$, but in general equality does not hold.

For $k \geq 5$ Lovász, Neumann-Lara & Plummer (1978) have exhibited graphs with $I_k(x,y) < S_k(x,y)$. In fact they have shown that

$$\sqrt{\frac{k}{2}} \leq \sup_G \frac{S_k(x,y)}{I_k(x,y)} \leq \lfloor \frac{k}{2} \rfloor.$$

The graph of figure 1 has $I_5(x,y) = 1$ and $S_5(x,y) = 2$.

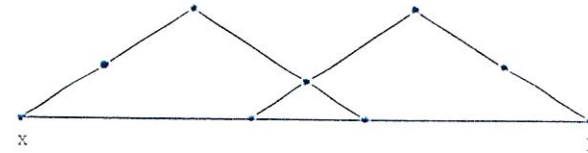


Fig. 1

More recently Boyles & Exoo (1982) found that $\sup \frac{S_k(x,y)}{I_k(x,y)} \geq \lfloor \frac{k+3}{4} \rfloor$,

thereby disproving a conjecture of Lovász and Neumann-Lara mentioned in Bollobás' book (1978) (ex n° 17, p.48) and in Bondy & Murty (1976) (conjecture n° 17). Chung (1983) improved the lower bound to $\frac{1}{3}(k+1)$ and asks whether $\sup \frac{S_k(x,y)}{I_k(x,y)} = \frac{1}{3}k + o(1)$.

However if $k \leq 4$ we have $I_k(x,y) = S_k(x,y)$ and the determination of $I_k(x,y)$ is a polynomially solvable problem.

We remark that if $k \geq n-1$ then $I_k(x,y) = S_k(x,y)$ is nothing else but Menger's theorem. In the particular case where k is the distance between the two vertices we also have:

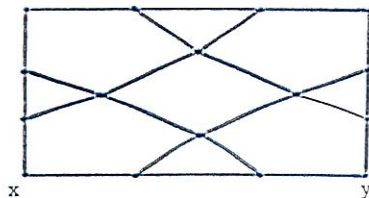
$$I_k(x,y) = S_k(x,y) \quad k = d(x,y).$$

This result is due independently to Entringer, Jackson & Slater (1977) and Lovász, Neumann-Lara & Plummer (1978). Itai, Perl & Shiloach (1982) have shown that the determination of the maximal number of vertex-disjoint paths of shortest length is a polynomially solvable problem. By applying flow techniques the algorithm requires $O(|V|^2|E|)$ time. Itai, Perl & Shiloach (1982) and Hartman & Rubin (1978) have found analogous results for edge-disjoint paths.

The above results concern local properties of the graph. It would be interesting to measure how the deletion of edges or vertices increases the diameter of the graph. A parameter for this purpose has been proposed by Boesch, Harary & Kabell (1981) who defined the persist-ence $\rho_0(G)$ as the minimum number of vertices which must be removed in order to increase the diameter or yield a trivial graph.

Unfortunately, contrary to the claim in their article, the equality $\rho_0(G) = \inf S_{D(G)}(x,y)$ is not true. Indeed, Bondy & Hell (1983) have found an infinite family of counterexamples such that $\rho_0(G)$ is strictly greater than $\inf S_{D(G)}(x,y)$ (see also Exoo (1982)). Figure 2 is a counterexample with $D(G) = 5$, $\rho_0(G) = 2$ and $S_5(x,y) = 1$.

Fig. 2



An interesting problem due to Elspas is to determine the maximum number of vertices of graphs of maximal degree Δ such that each pair of vertices are joined by μ edge-disjoint paths of length at most k (where $I_k(x,y) = \mu$). Some particular cases of this problem have been considered by Quaife (1969).

Geodetic graphs

Instead of searching for graphs with a great number of disjoint paths between vertices, some authors have studied graphs in which any two vertices x and y of V are joined by exactly one shortest path. Such graphs are called geodetic graphs. In some sense these graphs are very vulnerable and therefore less interesting for applications. A survey of the main results and open problems concerning geodetic graphs can be found in Bosák (1978). Geodetic graphs homeomorphic to a complete graph were characterized by Stemple (1979).

A graph G is said to be strongly geodetic if each pair of vertices is connected by at most one path of length less than or equal to the diameter of G . Bosák, Kotzig & Známk (1968) proved that a strongly geodetic graph is either a forest (union of trees) or a Moore graph.

5 DIAMETER VULNERABILITY AND EXTREMAL PROBLEMS

Deletion and augmentation problems

Plesník (1975a) has shown that the deletion of an edge in a graph of diameter D gives either a disconnected graph or a graph of diameter at most $2D$. This bound was greatly generalized by Chung and Garey (1983). They have shown that if s edges are removed from a graph of diameter D , the resulting graph, if it is connected, has diameter at most $(s+1)D + O(s)$. To prove this result they showed that the least possible diameter D' of a graph obtained by adding s edges to the path on n vertices satisfies

$$\frac{n}{s+1} - 1 \leq D' < \frac{n}{s+1} + 3.$$

They also considered the same problem for cycles.

An interesting problem would be to consider the same augmentation problem with the maximum degree of the resulting graph fixed. For example, the following natural question due to Farley & Hedetniemi (1978) is of interest: given a cycle of length $2n$, what is the least possible diameter of a cubic graph obtained by adding a matching of n edges?

These augmentation problems are also of interest when considering adding facilities to a network in order to upgrade its performance. In this spirit, as asked by Chung and Garey, it would be interesting to determine the computational complexity of the following problem: given s , D and G , determine whether there exists a supergraph of G obtained by adding s edges (vertices) that has diameter no more than D and, if so, find such a supergraph. And if this problem is NP complete, do good heuristics exist?

Chung and Garey also studied the vertex-deletion case; however in this case the corresponding maximum diameter is unbounded in terms of s and $D(G)$; nevertheless they proved that the removal of s vertices in a k -connected graph ($k > s$) gives a graph of diameter at most $\frac{n-s-2}{k-s} + 1$. Similarly it would be interesting to know the complexity of the following problem: given s , D and G determine whether there exists a subgraph of G obtained by deleting s edges (vertices) that has diameter no more than D and, if so, find such a subgraph.

(Δ, D, D', s) -graphs

A related problem is to find $N(\Delta, D, D', s)$, the maximum order of a (Δ, D, D', s) -graph, i.e. a (Δ, D) -graph such that the maximum diameter of the graph obtained from G by deleting s vertices is D' . Some values of $N(\Delta, D, D', s)$ were obtained by Bond, Moulinoux & Peyrat (1983). They obtained bounds for $N(\Delta, 2, D', 1)$ and, for "large" graphs of diameter 2, proved that $D' = 3$. They also showed that $N(\Delta, D, D, 1) \geq (\Delta/2)^D + (\Delta/2)^{D-1}$.

In order to find graphs of diameter D such that after the deletion of s vertices the diameter remains small, one can consider graphs of diameter D and girth $2D$, such that every two vertices at distance D are connected by $s+1$ vertex-disjoint paths. The existence of some of these graphs is studied in Gewirtz (1969). For $s=1$ see also Roos & van Zanten (1983c).

An extremal problem

Murty & Vijayan (1964) proposed the problem of minimizing the number $f_v(n, D, D', s)$ (resp. $f_e(n, D, D', s)$) of edges in a graph of order n and diameter D and having the property that if any s vertices (resp. edges) are deleted, the remaining graph has diameter at most D' . These functions have been studied by many authors, but their determination seems a difficult problem in general: essentially the cases solved are $s = 1$ or $D \leq 4$; see the survey of Caccetta (1979). Recently Usami (private communication of Enomoto) has shown that $f_v(n, D, 6, 1) = \max \left\{ n, \left\lfloor \frac{4n-8}{3} \right\rfloor \right\}$ for $4 \leq D \leq 6$. The asymptotic behaviour of $f_v(n, D, D', 1)$ for large D' is known; it has been proved that if $D \geq 2$ and $D' \geq 2D - 2$ then $\lim_{n \rightarrow \infty} \frac{f_v(n, D, D', 1)}{n} = \frac{D}{D-1}$ (see Bollobás' book (1978 ch. IV 2)). Recently, Enomoto & Usami (1983) have determined the exact value by showing that $f_v(n, D, D', 1) = \left\lfloor \frac{Dn-2D-1}{D-1} \right\rfloor$ if $n > D' \geq 2D - 1$. This settles a conjecture of Caccetta (1979), and as an immediate corollary, determines the minimum number of edges of a 2-connected graph of diameter D (in the case $D' = n-1$). The same result seems to have been obtained by Suvikov (1978) according to Mathematical Reviews (80C 05095). Finally, this last result settles a conjecture of Glivjak (1976) by proving that the minimum number of edges of an equi-essentric graph of diameter D is $\left\lfloor \frac{Dn-2D-1}{D-1} \right\rfloor$ (an equi-essentric graph is a graph in which $\max_{y \in G} d(x, y) = D$, for every x), for such a graph is 2-connected.

Enomoto (private communication, November 1982) has also shown that the minimum number of edges of a graph of order n , minimum degree 2 and diameter D is also equal to $\left\lfloor \frac{Dn-2D-1}{D-1} \right\rfloor$ if D is odd, $n > 2D$, but to $\min \left\{ \left\lfloor \frac{Dn-2D-1}{D-1} \right\rfloor, \left\lfloor \frac{(D+1)(n-1)}{D} \right\rfloor \right\}$ if D is even and $n > 2D$. As a corollary, he determined $f_e(n, D, D', 1)$ for large D' .

Note that the case $D' = n-1$ corresponds to the case where G is s -connected or s -edge-connected. In that sense the result of Chung and Garey mentioned at the beginning of this paragraph shows that, if $D' \geq (s+1)D+s$, then $f_e(n, D, D', s) = f_e(n, D, n-1, s)$. The case where D' is close to D , and in particular $D' = D$, is very far from being settled. Bollobás (1978, ch. IV 3) has conjectured that

$$\lim_{n \rightarrow \infty} \frac{f_e(n, D, D', 1)}{n} = \frac{\lfloor \frac{1}{2} D \rfloor + 1}{\lfloor \frac{1}{2} D \rfloor}. \text{ Bollobás \& Eldridge (1976) have shown that}$$

if $n \geq 2s$, then $f_v(n, 2, 2, s) = (s+1)(n-s-1)$. Entringer & Jackson (1982) have determined $f_v(n, 2, 2, s)$ when $n \leq 2s - 1$. This result is obtained in fact by determining the minimum number of edges of a graph with n vertices and diameter 2, which is s -geodetically connected (i.e. a connected graph such that the removal of at least s vertices is required to increase the distance between any pair of vertices).

Diameter-critical graphs

A completely different point of view consists of considering vertex- (resp. edge-) critical graphs, i.e. graphs such that the deletion of any vertex (resp. edge) increases the diameter.

The characterization of these graphs seems a very difficult problem. Glivjak (1975) has proved the impossibility of a characterisation of edge-critical graphs by finite extension or by forbidden subgraphs. However partial results have been obtained (see for example Plesník (1975b)) and, in particular, Plesník and Murty and Simon have conjectured that any edge-critical graph of diameter 2 on n vertices has at most $\lfloor n^2 \rfloor$ edges. Some progress has been made by Caccetta and Häggkvist (1979) who have obtained the bound $\frac{1}{2}(1 + \sqrt{5})n^2 < 0.27 n^2$. A general conjecture on the maximum number of edges of an edge-critical graph of diameter D has been proposed by Krishnamoorthy & Nandakumar (1981) who disproved an earlier conjecture of Caccetta & Häggkvist (1979).

Other extremal problems

Another extremal problem consists of finding the maximum diameter of a graph with given minimum degree and connectivity. This has been solved by Amar, Fournier & Germa (1983) who determine the minimum order $f(D, \delta, \kappa)$ of a graph with diameter D , minimum degree δ and connectivity κ :

$$f(1, \delta, \kappa) = \delta + 1$$

$$f(2, \delta, \kappa) = \delta + 2$$

$$f(D, \delta, \kappa) = \begin{cases} 2(\delta+1) + (D-3)\kappa & \text{if } D \geq 3 \text{ and } 1 \leq \kappa \leq \delta \leq 3\kappa - 1, \\ (t+1)(\delta+1) + e\kappa & \text{if } D \geq 3 \text{ and } \delta \geq 3\kappa - 1, \end{cases}$$

$$\text{where } D = 3t + e, e \in \{0, 1, 2\}.$$

Analogous results for hypergraphs have been found by Raynal (1983).

Myers, Klee & Quaipe, furthermore, studied the cases where D is δ -regular, in particular $\kappa = 1$ or $\Delta = 3$ (see Myers (1981) for a survey on these results). For digraphs see Hirschberg & Wong (1979).

Cayley graphs

Let $S_N = \{\Pi_1, \Pi_2, \dots, \Pi_{N!}\}$ be the symmetric group of permutations on N elements. Let $\Sigma_r = \{\sigma_1, \dots, \sigma_r\}$ be a system of r generators of S_N . The associated Cayley graph has:

(1) as vertices the $N!$ permutations in S_N ,

(2) two vertices, indexed by the permutations Π_i and Π_j in S_N , are adjacent if and only if $\Pi_i \sigma_k = \Pi_j$, for some $\sigma_k \in \Sigma_r$. By an enumerative argument, the diameter D is greater than $\log_x(N!)$. Quisquater (1983b) has improved results of Golunkov (1973) by showing that for each N and r , there exists a system Σ_r such that the diameter $D = c_r \log_x(N!)$. This result has some applications for dynamic memories. Other properties of Cayley graphs have been studied by Tchuenté (1982).

6 HYPERGRAPHS

In this section, we consider the case where sets of processors share a communication medium such as a bus. As we said in the introduction, these networks are represented by hypergraphs: the vertices represent the processors and the (hyper-)edges the buses. Although these networks are now less used than the classical ones with point-to-point connections, their importance is growing and they might be very important in the near future. Our interest in this subject came from discussions with persons working at the French National Telecommunications Centre (CNET), on a project called REBUS. Examples of such networks are the well-known d -dimensional hypercubes of width r . The vertices can be seen as d -tuples (a_1, \dots, a_d) , whose coordinates can take the values $0, 1, 2, \dots, r-1$; therefore the total number of vertices is r^d . An edge consists of all the vertices having $d-1$ coordinates fixed. So a vertex belongs to exactly d edges and an edge contains exactly r vertices. Particular cases are the binary hypercube ($r=2$) which is a graph, and the "grid" or "mesh" ($d=2$).

The grid structure is essentially that used in the computer interconnection systems Illiac IV or MICRONET (see for example the article of Wittie & Van Tilborg (1980)). Other such networks have been proposed by Finkel & Solomon (1980); Wittie (1981): the "Dual bus hypercubes", and by Finkel & Solomon (1981): the "Lens structured networks".

As we have already said in the introduction, the important parameters in such networks are the message delay, the reliability, the

existence of efficient routing algorithms, etc., which correspond, as in the case of graphs, to parameters of the associated hypergraph. A technical constraint is that a node must not belong to too many buses; i.e. a vertex of the hypergraph has bounded degree. Furthermore, the traffic load on each bus is limited. Since, for general networks without symmetry properties, the traffic per bus is not easily calculated, we will restrict our study to networks with some uniformity and, to a first approximation, replace the constraint of traffic load by the limitation of the number of processors connected to each bus. This means that the size r of the edges of the associated hypergraph is bounded.

A comparison of the different criteria on a few structures has been done by Wittie (1981); however, it is impossible to optimize all the criteria at the same time. From a combinatorial point of view, a study of the extremal problems is worthwhile. We have begun such a study and our aim in what follows is to indicate some further directions of research; indeed as we shall see, most of the area is wide open.

Notation and definitions

Our notation and definitions will be:

- $H = (X, E)$, for a hypergraph with vertex-set X , and (hyper)edge-set E ,
- the degree of a vertex is the number of edges containing it,
- Δ for the maximum degree over all vertices of H ,
- r for the maximal size of an edge,
- a path in a hypergraph connecting x and y will be a sequence $x = x_1, E_1, x_2, \dots, x_i, E_i, x_{i+1}, \dots, E_{p-1}, x_p = y$ with $\{x_i, x_{i+1}\} \subset E_i$. Its length will be $p-1$, the number of edges of the path. Distance and diameter are then defined and denoted as in the case of graphs.

The (Δ, D, r) -hypergraph problem

A (Δ, D, r) -hypergraph will be a hypergraph of diameter D , maximum degree Δ and maximum edge-size r . The problem identical to the (Δ, D) -graph problem is to determine the maximum number of vertices $n(\Delta, D, r)$ of a (Δ, D, r) -hypergraph. As for graphs, $n(\Delta, D, r)$ is bounded above by a "Moore bound":

$$n(\Delta, D, r) \leq 1 + \Delta(r-1) \sum_{i=0}^{D-1} (\Delta-1)^i (r-1)^i.$$

In the case $r=2$, we obtain the classical Moore bound.

The first problem is to determine if this bound can be reached. This problem has been considered by different persons independently (Kuich & Sauer (1969), Bose & Dowling (1971), Fuglister (1977), Damerell & Georgiacodis (1981a, 1981b)). It appears also as a particular case of more general structures. The hypergraphs which attain the bound are often called Moore geometries.

First look at the case $D = 1$, that is when each pair of vertices belongs to at least one edge. The reader will see the similarity with design theory. Let us recall that a (v, k, λ) -BIBD (balanced incomplete block design) is a collection of subsets (called blocks) of a given set on v vertices such that every block contains exactly k vertices and every pair of vertices belongs to exactly λ blocks. In fact, the bound $n(\Delta, 1, r) = 1 + \Delta(r-1)$ is attained if and only if there exists an $(n, r, 1)$ -BIBD. Therefore this case is extensively studied: for values of the parameters for which there exists an $(n, r, 1)$ design, see for example Hanani (1975) or Doyen & Rosa (1980). We will come back to the case $D = 1$ later.

For $D > 1$, successive efforts of different persons using techniques similar to those used for graphs, have proved that the bound can be attained only if $D = 2$. The case $D = 3$ has been excluded by Fuglister (1977), the case $D > 4$ by Damerell & Georgiacodis (1981b) and the case $D = 4$ by Ott (1983). The case $D = 2, r = 2$ corresponds to the Moore graphs; for $D = 2, r > 2$, it has been proved by Bose & Dowling (1971) (see also Kantor (1977)) that, if $r \neq 5$, Moore hypergraphs can exist only for a finite number of values. For $r = 3$, there cannot exist Moore hypergraphs. For $r = 4$, there might exist one Moore hypergraph with $\Delta = 7$, 400 vertices and 700 edges. For $r = 5$, there might exist Moore hypergraphs of degree s^2 with $s = 0, 1$, or $4 \pmod{5}$. The cases $D = 2, r = 3$ and $r = 4$ have also been done by Kuich & Sauer (1969) who studied hypercages, i.e. regular hypergraphs of degree Δ , with uniform size r , of girth g and having the minimum possible number of vertices. In the case g odd, $g = 2D+1$, it can be shown that these hypergraphs have diameter D . They studied the case $g = 5$ (note that there is a small computational mistake in the last corollary for the case $r = 4$). However, it is not known if such Moore hypergraphs exist for $r \geq 3$.

We may also note that these Moore hypergraphs appear as particular cases of more general structures. Very recently, for example, Buekenhout (1983) has defined (g, d_p, d_L) -gons as bipartite graphs with the two disjoint vertex-sets P and L respectively of points and lines, with girth $2g$, point diameter d_p and line diameter d_L where:

$$d_p = \max_{p \in P} \max_{x \in P \cup L} (d(p, x)),$$

$$d_L = \max_{\ell \in L} \max_{x \in P \cup L} (d(\ell, x)).$$

Delorme (1983c) has also defined a class of bipartite graphs, generalizing distance-regular graphs. Particular cases of these graphs are related to hypergraphs in the following sense: to each hypergraph $H = (X, E)$, we associate a bipartite graph with vertex set $X \cup E$, where a vertex x is joined to an edge e if x belongs to e in H . The point diameter of the resulting graph is $2D$ or $2D+1$, and the line diameter between $2D-1$ and $2D+2$.

Another generalization of Moore hypergraphs is due to Roos & van Zanten (1983a, b). They consider Δ -regular, r -uniform hypergraphs such that any pair of vertices at distance strictly less than D are connected by a unique shortest path, and pairs of vertices at distance D by a constant number of paths of length D .

Case $D=1$

Let us come back to the case $D = 1$: Moore hypergraphs exist if and only if there exist $(n, r, 1)$ -BIBDs, but in this case $\Delta \geq r$. The case $\Delta = r$ ($|X| = |E|$) corresponds to finite projective planes, which exist for $\Delta = q+1$ where q is a prime power: in that case $|X| = |E| = q^2 + q + 1$. However, for practical purposes, the interesting case is when Δ is small and therefore $\Delta < r$. By replacing, in a projective plane of order q (when it exists), each vertex j by k vertices j_1, j_2, \dots, j_k and each line with vertex set L by a bus with vertex set $B = \{j_i \mid i = 1, 2, \dots, k \mid j \in L\}$, we obtain a hypergraph of degree $\Delta = q+1$, uniform size $k(q+1)$ with $k(q^2 + q + 1)$ vertices and diameter 1. We conjecture that we always have: $n(\Delta, D, 1) \leq \lfloor \frac{(\Delta^2 - \Delta + 1)r}{\Delta} \rfloor$. We can prove this for $\Delta = 2$ and $\Delta = 3$, and in this case the bound is attained for an infinite number of values.

Dual hypergraphs

Let us recall that if $H = (X, E)$ is a hypergraph, its dual hypergraph $H^* = (E^*, X^*)$ is the hypergraph whose vertices (resp. edges) are in one-to-one correspondence with the edges (resp. vertices) of H and where the vertex e_j^* belongs to the edge X_i^* if and only if x_i belongs to E_j . If H is a (Δ, D, r) -hypergraph with n vertices and m edges, then H^* is an (r, D^*, Δ) -hypergraph with m vertices and n edges. Though we know that we have the relation $D^* = D$ or $D^* = D+1$ or $D^* = D-1$ between the diameter D^* of H^* and the diameter D of H , the characterization or the construction of graphs satisfying one of those equalities seems to be a difficult problem, even if H is a graph (in this case D^* is equal to the diameter of the line-graph). However, if H is bipartite then $D^* \leq D$. But, here again the characterization of the graphs satisfying $D^* = D$ appears to be difficult. The dual version of the determination of $n(\Delta, 1, r)$ can be reformulated as follows: find the maximum cardinality of a set system (intersecting family), that is a family of sets E_i such that $E_i \cap E_j \neq \emptyset$, $|E_i| \leq \Delta$ and every vertex belongs to at most r sets. Without the last condition, the problem is well known; the solutions are the Δ -systems. Apparently there is no article with the last restriction on the degree.

Case $\Delta = 2$

One important practical case is the determination of $(2, D, r)$ -hypergraphs having a large number of vertices. Since the dual of a $(2, D, r)$ -hypergraph is a graph, the problem becomes an extremal problem in graph theory. Let us define the distance between two edges e and e' of a graph as the minimum number of vertices in a path $e = e_1, x_1, e_2, x_2, \dots, e_i, x_i, \dots, e_p = e'$ joining e and e' . Let the line diameter of a graph be the maximum of the distances over all pairs of edges. The line diameter of a graph is therefore the diameter of its dual hypergraph and also the diameter of its line-graph. The determination of $n(2, D, r)$ is equivalent to that of the maximum number of edges of a graph G whose line diameter is D and whose maximum degree is r . Recall that if a graph is r -regular, finding the maximum number of edges is equivalent to finding the maximum number of vertices ($m = \frac{1}{2}nr$).

As any graph of diameter $D-1$ and any bipartite graph of diameter D have line-diameter at most D , we can use either $(r, D-1)$ -graphs or (r, D) -bipartite graphs to construct large $(2, D, r)$ -hypergraphs.

For example, consider the case $D = 2$. We can use the best-known $(r, 1)$ -graph, which is trivially K_{r+1} and therefore we have a $(2, 2, r)$ -hypergraph with $\frac{1}{2}r(r+1)$ vertices. We can consider the best $(r, 2)$ -bipartite graph, which is $K_{r, r}$, and therefore we have a $(2, 2, r)$ -hypergraph with r^2 vertices (note that it is the grid or mesh considered in the introduction to this section). But in this case, the graph $C_5 \otimes S_t$, obtained from C_5 by replacing each vertex i ($i = 1, 2, 3, 4$ or 5) by t vertices and each edge by a complete bipartite graph $K_{t, t}$, has $5t$ vertices, $5t^2$ edges and degree $2t$. We can check that it has line diameter 2 , and therefore its dual also has diameter 2 , so it is a $(2, 2, r)$ -hypergraph on $\frac{5}{4}r^2$ vertices with $r = 2t$. Therefore $n(2, 2, r) \geq \frac{5}{4}r^2$ if r is even. Trotter (private communication, June 1982) proved that $n(2, r, r) \leq \frac{3}{2}r^2$, and very recently, answering one of our conjectures, it has been shown by Kleitman (1983) that every graph of maximum degree r and line diameter 2 has at most $\frac{5}{4}r^2$ vertices. Thus $n(2, 2, r) \leq \frac{5}{4}r^2$ and if r is even $n(2, 2, r) = \frac{5}{4}r^2$ (compare this bound with the Moore bound $2r^2 - 2r + 1$). In the general case, we do not know the exact answer. Let v_D be $\liminf_{r \rightarrow \infty} n(2, D, r)r^{-D}$; by Benson (1966) and Delorme (1983b)'s results on bipartite graphs, we have: $v_3 \geq 1$, $v_4 \geq 1$, $v_6 \geq 1$ and in general $v_D \geq (\frac{1}{2})^{D-1}$. (Note that this result improved the bound proposed by Finkel & Solomon (1980).)

General constructions

Most of the known constructions for graphs can be extended to hypergraphs. For example, we can use the compounding technique developed in Bermond, Delorme & Quisquater (1982b) by using BIBD instead of complete graphs. If we replace each vertex of an $(n(r-1)+1, r, 1)$ -BIBD by a (Δ, D, r) -hypergraph on n vertices, we obtain a $(\Delta+1, 2D+1, r)$ -hypergraph on $n^2(r-1) + n$ vertices. Therefore, by using $n(2, 1, r) = \frac{3}{2}r$ if r is even (see case $D = 1$), we obtain $n(3, 3, r) \geq \frac{9r^3 - 9n^2 + 6r}{4}$ if r is even. Such constructions are described in Bermond, Bond & Saclé (1983). One can also use the hypergraph structure, for example by taking r copies of a (Δ, D, r) -hypergraph and joining by an edge the vertices having the same label. Other special constructions have also been developed by Quisquater (private communication, November 1982).

Here we have considered only the generalization of the (Δ, D) -graph problem; but it would be worth extending the problems that we have studied in the previous sections for graphs to the case of hypergraphs.

Acknowledgements

We wish to thank all those who helped us with the preparation of this survey, in particular J.A. BONDY, C. DELORME, J.J. QUISQUATER and U. SCHUMACHER.

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