

# On deciding stability of high frequency amplifiers

L. Baratchart, S. Fueyo, G. Lebeau, J.-B. Pomet

September 10, 2019

*15th IFAC Workshop on Time Delay Systems*

The logo for Inria, featuring the word "Inria" in a stylized, red, cursive script.

# Motivation

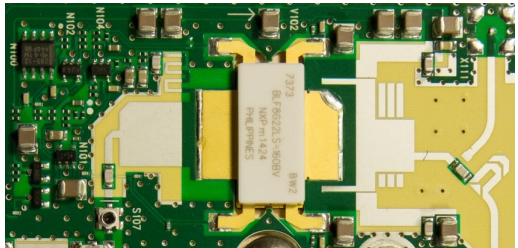
- Amplifiers at high frequency are ubiquitous (Cell phones, relays...). They need to be quick to design and produced in large quantities.
- Computer-assisted design (CAD) and simulation before production.
- Powerful “frequency simulation” tools give a reliable prediction of the response, but that response might be unstable.
- **Need** for a tool to predict stability/unstability in the frequency domain.

# Inside

An amplifier is made of interconnected

- resistors, inductors, capacitors,
- diodes/transistors,
- lossless transmission lines which cannot be neglected at high frequency inducing delays.

Forcing periodic signal ►► periodic solution in the amplifier ►► amplified signal.



# Harmonic Balance

The Harmonic Balance method, through Fourier development, Laplace transform and fixed point methods permits to :

- approximate the periodic solution of the circuit,
- linearize the circuit around the periodic solution,
- give a frequency response to a periodic signal wich disturbs the linearized circuit.

Our focus : Structure of the harmonic transfer function, its singularities, links with stability.

# Summary

- 1 Equations and stability
- 2 Harmonic transfer function
- 3 conclusion

General system,  $T$ -periodic :

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^N B_{1,i}(t)y(t - \tau_i) \\ y(t) = \sum_{i=1}^N B_{2,i}(t)y(t - \tau_i) + A_2(t)x(t), \quad t \geq s, \end{cases}$$

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^N B_{1,i}(t)y(t - \tau_i) \\ y(t) = \sum_{i=1}^N B_{2,i}(t)y(t - \tau_i) + A_2(t)x(t), \quad t \geq s, \end{cases}$$

- $L^2 := \mathbb{R}^n \times L^2([-\tau_N, 0], \mathbb{R}^k)$ .
- Solution operator  $U(t, s) : L^2 \rightarrow L^2$
- Monodromy operator  $U(T, 0)$

•

$L^2$  exponential }  $\Leftrightarrow$  {  $Sp(U(T, 0))$  included in

# Behaviour at high frequency

High frequency system :

$$\begin{cases} x(t) = 0 \\ y(t) = \sum_{i=1}^N B_{2,i}(t)y(t - \tau_i), \quad t \geq s, \end{cases}$$

- $\tilde{L}^2 := \{0_n\} \times L^2([-\tau_N, 0], \mathbb{R}^k)$ .
- Solution operator  $V(t, s) : \tilde{L}^2 \rightarrow \tilde{L}^2$ .
- Monodromy operator  $V(T, 0)$ .
- $\left. \begin{array}{l} L^2 \text{ exponential} \\ \text{stability} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} Sp(V(T, 0)) \text{ included in} \\ \text{disc of radius } r < 1 \end{array} \right.$

# Compact perturbation

## Lemma

*We have :*

$$U(t, s) = V(t, s)P + K(t, s), \quad t \geq s$$

*with  $K(t, s)$  compact operator  $L^2 \rightarrow L^2$  for all  $t, s$  and  $P$  the canonical projection  $L^2 \rightarrow \tilde{L}^2$ .*

## Theorem

*If the high frequency system is exponentially stable then the monodromy operator  $U(T, 0)$  possesses at most a finite number of eigenvalues  $\zeta_1, \dots, \zeta_n$  outside a disk of a radius strictly less than 1.*



## $L^2$ stability equivalent to $C^0$ stability

Proposition (Chitour and al 2016, BFLP 2019)

*A periodic delay system is  $L^2$  exponentially stable if and only if it is  $C^0$  exponentially stable.*

Theorem

*Assuming the system at high frequency is  $L^2$  exponentially stable.  
Then the general system is  $L^2$  exponentially stable if and only if it is  $C^0$  exponentially stable.*

**Assumption : High frequency system exponentially stable**  
 **$\Leftarrow$  always true for "realistic" circuit.**

# Input-Output system

$$\left\{ \begin{array}{l} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^N B_i^1(t)y(t - \tau_i) + C_1(t)u(t) \\ y(t) = \sum_{i=1}^N B_i^2(t)y(t - \tau_i) + A_2(t)x(t) + C_2(t)u(t) \\ z(t) = \sum_{i=1}^N B_i^3(t)y(t - \tau_i) + A_3(t)x(t) + C_3(t)u(t), \quad t \geq 0, \end{array} \right.$$

- $x(t), y(t), z(t) = 0$  for  $t < 0$ ,
- Input  $u \in L_{loc}^2([0, +\infty), \mathbb{R})$  current perturbation, output  $z$  the voltage,
- All coefficients are  $T$  - periodic.

- $X(t, \alpha)$  response at time  $t$  to an impulse at time  $\alpha$   
 $z(t) = \int_0^t X(t, \alpha) u(\alpha) d\alpha$
- $G(s, t) = \int_0^{+\infty} X(t, t - \alpha) e^{-s\alpha} d\alpha$  : Laplace Transform
- $G_k(s) = \frac{1}{T} \int_0^T G(s, t) e^{ik\omega_0 t} dt$  with  $\omega_0 := \frac{2\pi}{T}$

### Definition (Harmonic Transfer Function HTF)

The infinite matrix  $H(s)$  defined by  $H_{m,n}(s) := G_{n-m}(s + \frac{2i\pi m}{T})$  for  $s \in \mathbb{C}$  is called the harmonic transfer function.

$$Z(s) := \int_0^{+\infty} z(t)e^{-st} dt \text{ and } U(s) := \int_0^{+\infty} u(t)e^{-st} dt.$$

$$\begin{pmatrix} \vdots \\ Z(s+i\omega_0) \\ Z(s) \\ Z(s-i\omega_0) \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \cdots & G_2(s-i\omega_0) & G_1(s) & G_0(s+i\omega_0) & \cdots \\ \cdots & G_1(s-i\omega_0) & G_0(s) & G_{-1}(s+i\omega_0) & \cdots \\ \cdots & G_0(s-i\omega_0) & G_{-1}(s) & G_{-2}(s+i\omega_0) & \cdots \\ \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ U(s+i\omega_0) \\ U(s) \\ U(s-i\omega_0) \\ \vdots \end{pmatrix}$$

- HTF is an operator valued analytic map  
(values: continuous ops  $l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$ )
- its entries  $\{G_n\}$  are complex valued analytic maps

# Structure of the Harmonic Transfer Function

Define  $z_{j,k} = \frac{\ln(\zeta_j) + 2ik\pi}{T}$  for  $j$  in  $\{1 \dots n\}$ ,  $k$  in  $\mathbb{Z}$ .

## Theorem

*In  $\{s \in \mathbb{C}, \Re(s) \geq \gamma\}$  for some  $\gamma < 0$ ,*

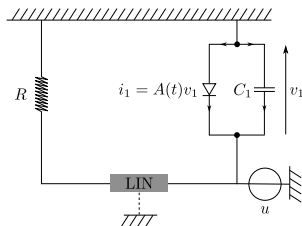
- $H$  is a meromorphic operator  $l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$  with possibly poles at  $\{z_{j,k}, j \in \{1 \dots n\}, k \in \mathbb{Z}\}$ .*

*Under observability/controllability assumptions,*

- for all  $j$ , there is at least a  $k$  such that  $z_{j,k}$  is a pole of  $H$ , and also a pole of one  $G_n$ .*

*If no **pole** in right half plane, exponential  $C^0$  stability.*

## Example : Brayton 1976



### Theorem

*If  $T/\tau_1 \notin \mathbb{Q}$ , all points of a certain vertical line in the left-half plane are essential singularities of the HTF, as an operator valued analytic map.*

But are they singularities for some  $G_n$  ?

## Contribution

- Math. foundation of HF amplifiers stability decision in CAD
- Projection on the unstable part and rational approximation to find the poles
- Advances in stability of periodic delay systems

## Open questions

- For fixed  $j$ , which  $z_{j,k}$  is a pole of which  $G_n$ ?  
(In practice, few  $G_n$  are computed.)
- Bound on the number of unstable poles?
- May the (stable) singularities of the  $G_n$ 's be other than poles ?