



**HAL**  
open science

## On locally hamiltonian graphs

Jean-Claude Bermond, Maryvonne Maheo, Henri Thuillier

► **To cite this version:**

Jean-Claude Bermond, Maryvonne Maheo, Henri Thuillier. On locally hamiltonian graphs. 1980.  
hal-02426796

**HAL Id: hal-02426796**

**<https://inria.hal.science/hal-02426796>**

Preprint submitted on 2 Jan 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON LOCALLY HAMILTONIAN GRAPHS

J.-C. BERMOND, M. MAHEO et H. THUILLIER

Laboratoire de Recherche en Informatique

E.R.A. 452, Bâtiment n° 490

Université de Paris-Sud - 91405 ORSAY (France)

## Résumé :

Un graphe est dit localement hamiltonien si pour tout sommet le sous graphe engendré par les voisins de ce sommet est hamiltonien. Dans cet article nous montrons que la théorie de "la fermeture" due à Bondy et Chvátal ne peut s'appliquer aux graphes localement hamiltoniens, bien qu'il existe un théorème analogue au théorème d'Ore. Puis nous construisons en réponse à un problème d'Oberly et Sumner des graphes connexes localement hamiltoniens mais non hamiltoniens.

## Abstract :

*A graph is locally hamiltonian if for every vertex, the subgraph generated by the neighbours of this vertex is hamiltonian. In this article we show that the closure method of Bondy and Chvátal does not work, although there exists a sufficient condition of Ore's type. Then, we answer by the negative a problem of Oberly and Sumner by exhibiting connected locally hamiltonian but non-hamiltonian graphs.*

## INTRODUCTION

In what follows  $G=(X,E)$  will be a simple graph, with vertex set  $X$  and edge set  $E$ . We denote by  $N(x)$  the neighbour set of the vertex  $x$ ,  $N(x) = \{y : \{x,y\} \in E\}$ , and  $d(x) = |N(x)|$ .

A graph  $G$  is locally hamiltonian if for each vertex  $x$  the subgraph induced by  $N(x)$  is hamiltonian. This concept has been introduced by Skupien [see 6,7] who gave in particular some sufficient conditions for a graph to be locally hamiltonian.

In this article we show first that the closure method of Bondy and Chvátal [2] does not work although there exists a sufficient condition of Ore's type. Secondly we answer by the negative a problem of Oberly and Sumner [5] by exhibiting connected locally hamiltonian graphs which are non hamiltonian.

### 1. Sufficient conditions for a graph to be locally hamiltonian

We are interested here in sufficient conditions concerning the degrees analogous to that for hamiltonian graphs (see [1]).

In [6] Skupien proved

Theorem (Skupien [6]) : Let  $G$  be a simple graph of order  $n$  such that

- (i) the degree of any vertex satisfies  $d(x) \geq 3$
- (ii) for every triple of points  $\{x,y,z\}$  such that the subgraph generated by  $\{x,y,z\}$  contains two edges,  $d(x)+d(y)+d(z) \geq 2n$ .

Then  $G$  is locally hamiltonian.

Corollary : Let  $G$  be a simple graph of order  $n$  such that for every vertex  $x$   $d(x) \geq \frac{2n}{3}$ . Then  $G$  is hamiltonian.

In [2] Bondy and Chvátal developed a powerful method. They said that a property  $P$  defined on all graphs of order  $n$  is  $k$ -stable if, for every pair of non adjacent vertices  $x,y$  such that  $d(x)+d(y) \geq k$ ,  $G$  has property  $P$  if and only if  $G+\{x,y\}$  has property  $P$ .

In particular they show that the property of being hamiltonian is  $n$ -stable. From that, follow the classical theorems of Ore and Dirac.

Theorem (Ore) : Let  $G$  be a simple graph of order  $n$  such that, for every pair of non adjacent vertices  $x$  and  $y$ ,  $d(x)+d(y) \geq n$ ; then  $G$  is hamiltonian.

Theorem (Dirac) : Let  $G$  be a simple graph of order  $n$  such that  $d(x) \geq \frac{n}{2}$  for every vertex  $x$ , then  $G$  is hamiltonian.

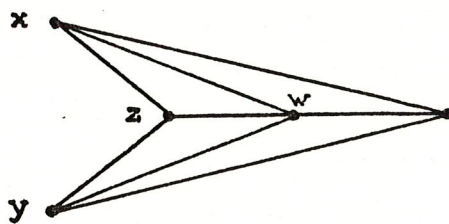
One can think that the property of being locally hamiltonian is  $k$ -stable for some  $k$ . If true, then it would have been possible to apply Bondy and Chvátal's method and deduce powerful theorems on the degree sequence like Chvátal's theorem on hamiltonian graphs. However we have :

Proposition : The property for a graph of order  $n$  of being locally hamiltonian is not  $k$ -stable for any  $k$ .

Proof : Indeed, if such a  $k$  exists, we must have  $k \leq 2n-4$  as, if  $x$  and  $y$  are non adjacent vertices,  $d(x)+d(y) \leq 2n-4$ .

Consider the following graph of order  $n$ ,  $n \geq 4$ .  $G$  consists of a complete graph of order  $n-1$ , minus an edge  $\{x,y\}$ , plus a vertex  $z$  joined to  $x,y$  and an other vertex  $w$ . (see figure 1 for  $n=5$ ).

Fig. 1



We have  $d(x)+d(y)=2n-4$ .  $G+\{x,y\}$  is locally hamiltonian, but  $G$  is not locally hamiltonian as the subgraph generated by  $N(z)$  is non hamiltonian.

However we have the following theorem of Ore's type

□



Theorem : Let  $G$  be a simple graph of order  $n$  such that, for every pair of non adjacent vertices  $x, y, d(x)+d(y) \geq \frac{3n-2}{2}$  ; then  $G$  is locally hamiltonian. Furthermore the result is the best possible.

Proof : If  $G$  is complete the theorem is immediate. Otherwise for every pair of non adjacent vertices  $x, y, d(x)+d(y) \leq 2n-4$  and by hypothesis  $d(x)+d(y) \geq \frac{3n-2}{2}$  thus  $n \geq 6$ .

First we show that for every vertex  $x : d(x) \geq \frac{n+2}{2}$  (1)

Indeed, suppose  $d(x) < \frac{n+2}{2}$  ; as  $n \geq 6, d(x) < n-1$  and therefore there exists a vertex  $y$  non adjacent to  $x$  and  $d(y) \leq n-2$ ; thus  $d(x)+d(y) < \frac{n+2}{2} + n-2 = \frac{3n-2}{2}$  , contradicting the hypothesis.

Now, if  $y \in N(x)$  then :

$$d(y) \leq 1 + d_{N(x)}(y) + |X - (\{x\} \cup N(x))| \quad \text{and thus}$$

$$d_{N(x)}(y) \geq d(x)+d(y)-n \quad (2)$$

If the subgraph generated by  $N(x)$  is complete, then it is hamiltonian. Otherwise let  $y$  and  $z$  be two non adjacent vertices of  $N(x)$ . By (2) we have

$$d_{N(x)}(y) + d_{N(x)}(z) \geq 2d(x) + d(y) + d(z) - 2n$$

By the hypothesis  $d(y)+d(z) \geq \frac{3n-2}{2}$  and

$$d_{N(x)}(y) + d_{N(x)}(z) \geq d(x) + d(x) - \frac{n+2}{2}$$

and thus by (1)  $d_{N(x)}(y) + d_{N(x)}(z) \geq d(x) = |N(x)|$

Therefore the subgraph generated by  $N(x)$  satisfies the hypothesis of Ore's theorem and is hamiltonian.

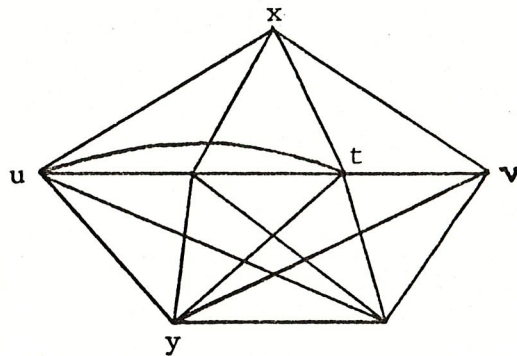
□

To show that the bound is the best possible, consider the following graph of odd order  $n, n=2p+1$ :  $G$  consists of two complete graphs  $K_{p+\alpha}$  and  $K_{p+\beta}$  having a  $K_p$  in common and where  $\alpha > 0, \beta > 0, \alpha + \beta = p$ , plus a vertex  $x$  joined to the vertices of the  $K_{p+\alpha}$  and  $K_{p+\beta}$  not in  $K_p$  and to a particular vertex  $t$  of the

$K_p$ .  $G$  is not locally hamiltonian as  $N(x)$  consists of two complete graphs  $K_{\alpha+1}$  and  $K_{\beta+1}$  having only  $t$  in common and thus  $t$  is cut-vertex. Furthermore, if two vertices are non adjacent, they are either of the form :  $x$  and  $y$  with  $y \in K_p$ ,  $y \neq t$  and then

$$d(x)+d(y) = \alpha + \beta + 1 + n - 2 = n + p - 1 = \frac{3n-3}{2}$$

or of the form  $u$  and  $v$  with  $u$  (resp.  $v$ ) a vertex of the  $K_{p+\alpha}$  (resp.  $K_{p+\beta}$ ) not in the  $K_p$  and then  $d(u)+d(v) = p+\alpha+p+\beta = 3p = \frac{3n-3}{2}$



Case  $n = 7$   
 $p = 3$   
 $\alpha = 2$   
 $\beta = 1$

Fig. 2

Remark 1 : The theorem does not imply Skupien's theorem ; and it is not implied by it.

Remark 2 : Analogous results can be obtained for other "local properties" (see [ 4]). For example it is proved in [ 4] that :

Proposition : The property for a graph of being locally  $k$ -connected is not  $k$ -stable for any  $k$ .

Theorem : Let  $G$  be a simple graph of order  $n \geq k+2$  such that, for every pair of non adjacent vertices  $x, y$ ,  $d(x)+d(y) \geq \frac{3n+k-4}{2}$  ; then  $G$  is locally  $k$ -connected and the result is the best possible.

Other conditions for a graph to be locally  $k$ -connected can be found in [3].

2. Connected locally hamiltonian and non hamiltonian graphs

In [5] Oberly and Sumner asked the following question : if  $G$  is connected and locally hamiltonian, does it imply that  $G$  is hamiltonian. We answer this problem by the negative by exhibiting an infinite family of connected locally hamiltonian, non hamiltonian graphs, denoted  $G(r,j)$  where  $r$  and  $j$  are integers satisfying :  $5 \leq r < n/2$  and  $2 \leq j \leq r-3$ .  $G(r,j)$  is a graph of order  $n$  with vertex set  $X = A \cup B \cup C$  where  $|A| = |B| = r$  and  $|C| = n-2r > 0$ .

Furthermore we have

- (i)  $A$  is an independent set (there is no edge between two vertices of  $A$ )
- (ii)  $B \cup C$  is a clique (the subgraph generated by  $B \cup C$  is complete)
- (iii) there is no edge between  $A$  and  $C$
- (iv) let the vertices of  $A$  (resp.  $B$ ) be denoted  $a_i, (b_i)$

where  $i=1,2,\dots,r$ . Then  $a_i$  is joined exactly to  $b_i, b_{i+1}, \dots, b_{i+j}$  where the subscripts of the  $b$ 's are to be taken modulo  $r$ .

Clearly  $G(r,j)$  is connected, furthermore  $G(r,j)$  is not hamiltonian ; indeed, if we delete the vertices of  $B$ , the subgraph generated by  $A \cup C$  has  $r+1$  components (by (i), (ii) and the fact that  $C \neq \emptyset$  as  $n > 2r$ ). Thus  $G(r,j)$  is non hamiltonian (in fact not 1-tough).

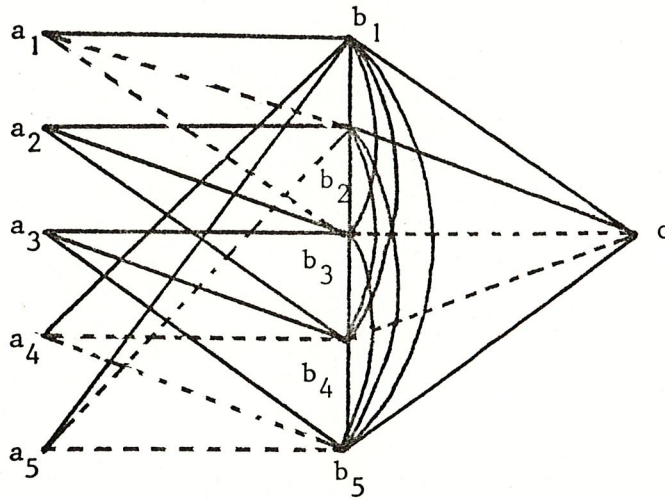
$G(r,j)$  is locally hamiltonian : if  $a_i \in A$ ,  $N(a_i)$  is a complete graph generated by  $b_i, b_{i+1}, \dots, b_{i+j}$  of order  $j+1$  and thus hamiltonian (as  $j \geq 2$ ) ; if  $c \in C$ ,  $N(c)$  is also a complete graph of order  $r+|C|-1 \geq 3$ . Now if  $b_i \in B$ ,  $N(b_i) = C \cup (B - \{b_i\}) \cup \{a_i, a_{i-1}, \dots, a_{i-j}\}$  and the subgraph generated by  $N(b_i)$  contains the hamiltonian cycle :

$$b_{i-j}, a_{i-j}, b_{i-j+1}, a_{i-j+1}, \dots, b_{i-1}, a_{i-1}, b_{i+1}, a_i, b_{i+2}, P$$

where  $P$  is a hamiltonian path between  $b_{i+2}$  and  $b_{i-j}$  in the complete subgraph generated by  $C \cup \{b_{i+2}, b_{i+3}, \dots, b_{i-j}\}$  (note that as  $j \leq r-3$ ,  $i+2 < i-j+r$ ).

In Figure 3 we give the smallest graph possible  $r=5$ ,  $j=2$ , and  $n=11$  (the hamiltonian cycle in  $N(b_1)$  is indicated by dashed lines).

Fig. 3



Remark : Other counterexamples have been found by C. Barefoot, R. Entringer and L. Clark (Communication of D. Sumner).



BIBLIOGRAPHY

- [ 1 ] J.C. BERMOND, Hamiltonian graphs, Chap. 6 of Selected Topics in Graph Theory, Academic Press, 1979, 127-167.
- [ 2 ] J.A. BONDY and V. CHVATAL, A method in graph theory, Discrete Math. 15 (1976), 111-135.
- [ 3 ] G. CHARTRAND and R.E. PIPPERT, Locally connected graphs, Časopis Pěst. Mat. 99 (1974), 158-163.
- [ 4 ] M. MAHEO : Thesis, University of Orsay.
- [ 5 ] D.J. OBERLY and D.P. SUMNER, Every connected, locally connected nontrivial graph with no induced claw is hamiltonian, J. Graph. Theory, 3, 1979, 351-356.
- [ 6 ] Z. SKUPIEN, Problemy i twierdzenia dotyczace grafow lokalnie hamiltonowskich (in Polish) (english summary), Zesk. Nauk. AGH, Nr 208, Mat. Fiz. Chem. 2 (1969), 23-30.
- [ 7 ] Z. SKUPIEN, On generalization of locally hamiltonian graphs, in Graphs, Hypergraphs and Block system, Proc. Coll. Zielona Gora, 1976, 263-271.