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# REGULAR FACTORS IN NEARLY REGULAR GRAPHS

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We strengthen classical theorems of Petersen, Ore, Båbler, Gallai on the existence of regular factors in regular graphs to nearly regular graphs.

## 1. Introduction

An  $h$ -factor of a graph  $G$ ,  $h$  an integer, is a spanning subgraph which is regular of degree  $h$ . More generally, given an integer-valued function  $f$  defined on  $V(G)$ , an  $f$ -factor is a subgraph  $H$  of  $G$  such that  $d_H(x) = f(x)$  for all  $x \in V(G)$ . Several important theorems of the theory of graph factors give simple sufficient conditions for the existence of  $h$ -factors in regular graphs [1, 2, 5].

**Theorem** (Petersen [5]). *A regular graph of even degree  $k$  can be decomposed into  $\frac{1}{2}k$  2-factors.*

**Theorem** (Ore [4, Theorem 3.2.5]). *Let  $G$  be a 2-edge-connected regular graph of odd degree  $k$  and  $h$  be an even integer  $2 \leq h \leq 2k/3$ . Then  $G$  contains an  $h$ -factor.*

**Theorem** ( $k$  odd: Båbler [1];  $k$  even: Gallai [2]). *Let  $G$  be a connected regular graph of degree  $k$ , with an even number of vertices. Let  $h$  be an odd integer,  $h \leq \frac{1}{2}k$ . If for all  $X \subseteq V$ ,  $|X|$  odd we have  $e(X, V - X) \geq k/h$ , then  $G$  has an  $h$ -factor.*

In the present note we give short proofs of these theorems by deriving them from Tutte's  $f$ -Factor Theorem [7]. We obtain slight generalizations: Due to integrality conditions these existence theorems remain valid for graphs which are almost regular (in a sense to be made precise below).

For completeness we recall Tutte's  $f$ -Factor Theorem [7].

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**Theorem.** A graph has an  $f$ -factor if and only if

$$f(X) + d_G(Y) - f(Y) \geq e_G(X, Y) + \left( \begin{array}{l} \text{number of connected components} \\ C \text{ of } G[V \setminus (X \cup Y)] \text{ such that} \\ e_G(C, Y) \not\equiv f(C) \pmod{2} \end{array} \right)$$

for all  $X, Y \subseteq V(G)$  with  $X \cap Y = \emptyset$ .

The reader is referred to [3] for a short proof of the 1-Factor theorem [6], and to [8] for a short proof of the  $f$ -Factor Theorem from the 1-Factor theorem. Another necessary and sufficient condition for the existence of  $f$ -factors is given by Ore [4].

## 2. Notations

Graphs considered in the present note are finite, undirected, and may have loops and multiple edges (a loop incident to a vertex  $x$  contributes 2 to the degree of  $x$ ).

Throughout the paper the following notations are used: Given a graph  $G$  the vertex-set of  $G$  is denoted by  $V(G)$ . The degree in  $G$  of a vertex  $x$  is  $d_G(x)$ . Given  $X \subseteq V(G)$ , we set  $d_G(X) = \sum_{x \in X} d_G(x)$ . the subgraph of  $G$  induced by  $X$  is denoted by  $G[X]$ , the number of edges of  $G[X]$  by  $e_G(X)$ . Given  $X, Y \subseteq V(G)$  with  $X \cap Y = \emptyset$  we denote by  $e_G(X, Y)$  the number of edges of  $G$  joining  $X$  and  $Y$ .

If no confusion results subscripts  $G$  will usually be omitted.

## 3. The theorems

**Theorem 1** (even factors). *Let  $G$  be a graph and  $h, k$  be two integers such that  $h$  is even and  $2 \leq h \leq \frac{2}{3}k$ . Suppose that  $\sum_{x \in V(G)} |d_G(x) - k| < k/h$  and  $G$  is 2-edge connected. Then  $G$  contains an  $h$ -factor.*

**Proof.** Tutte's condition for the existence of an  $h$ -factor,  $h$  even, is:

$$h |X| + d(Y) - f |Y| \geq e(X, Y) + \left( \begin{array}{l} \text{number of connected components } C \\ \text{of } G[V \setminus (X \cup Y)] \text{ such that} \\ e(C, Y) \text{ is odd} \end{array} \right)$$

for all  $X, Y \subseteq V$  with  $X \cap Y = \emptyset$ .

The condition is clearly satisfied for  $X = Y = \emptyset$ . Let  $X, Y \subseteq V$  with  $X \cap Y = \emptyset$  and  $X \cup Y \neq \emptyset$ . We distinguish two types of connected components  $C$  of



$G[V \setminus (X \cup Y)]$  such that  $e(C, Y)$  is odd:

(1)  $e(C, X) = 0$ . Since  $G$  is 2-edge-connected  $e(C, Y) = e(C, V \setminus C) \geq 2$ . Hence,  $e(C, Y)$  being odd, we have  $e(C, Y) \geq 3$ .

(2)  $e(C, X) \geq 1$ . Since  $e(C, Y)$  is odd, we have  $e(C, Y) \geq 1$ .

Let  $p_1, p_2$  be the numbers of components of each type. We have clearly

$$p_2 + e(X, Y) \leq e(X, V \setminus X) \leq d(X),$$

$$3p_1 + p_2 + e(X, Y) \leq e(Y, V \setminus Y) \leq d(Y).$$

Hence the inequalities

$$p_2 + e(X, Y) \leq d(X),$$

$$3p_1 + p_2 + e(X, Y) \leq d(Y).$$

We multiply the first inequality by  $h/k$ , the second one by  $(k-h)/k$  and add. We obtain

$$3 \frac{k-h}{k} p_1 + p_2 + e(X, Y) \leq \frac{h}{k} d(X) + \frac{k-h}{k} d(Y).$$

Since  $3(k-h)/k \geq 1$  if  $h \leq \frac{2}{3}k$  it follows, with  $p = p_1 + p_2$ , that

$$p + e(X, Y) \leq \frac{h}{k} d(X) + \frac{k-h}{k} d(Y).$$

Suppose

$$h |X| + d(Y) - h |Y| < p + e(X, Y).$$

Since both sides are integers we have

$$h |X| + d(Y) - h |Y| + 1 \leq p + e(X, Y) \leq \frac{h}{k} d(X) + \frac{k-h}{k} d(Y).$$

It follows that

$$d(X) - k |X| - (d(Y) - k |Y|) \geq \frac{k}{h}.$$

Since  $X \cap Y = \emptyset$  we have

$$d(X) - k |X| - (d(Y) - k |Y|) \leq \sum_{x \in X \cup Y} |d(x) - k| < \frac{k}{h},$$

a contradiction. Hence Tutte's condition is satisfied. Theorem 1 follows.  $\square$

The theorems of Petersen and Ore quoted in the introduction correspond to the particular cases of Theorem 1 when  $G$  is regular. To obtain Petersen's theorem it suffices to observe that the connected components of a regular graph of even degree are necessarily 2-edge-connected. (Proof: for  $X \subseteq V$  we have  $k |X| = d(X) = 2e(X) + e(X, V \setminus X)$ , hence if  $k$  is even  $e(X, V \setminus X)$  is also even.)

**Theorem 2** (odd factors). Let  $G$  be a connected graph with an even number of vertices, and  $h, k$  be two integers such that  $h$  is odd and  $1 \leq h \leq \frac{1}{2}k$ . Suppose that  $\sum_{x \in V(G)} |d(x) - k| < k/h$  and for all  $X \subseteq V$  with  $e(X, V - X)$  odd or  $|X|$  odd we have  $e(X, V - X) \geq k/h$ . Then  $G$  contains an  $h$ -factor.

**Note.** The condition on  $e(X, V - X)$  is obviously satisfied if the edge connectivity of  $G$  is at least  $k/h$ .

**Proof.** Tutte's condition for the existence of an  $h$ -factor,  $h$  odd, is:

$$h|X| + d(Y) - h|Y| \geq e(X, Y) + \left( \begin{array}{l} \text{number of connected components } C \\ \text{of } G[V \setminus (X \cup Y)] \text{ such that} \\ e(C, Y) \not\equiv |C| \pmod{2} \end{array} \right)$$

for all  $X, Y \subseteq V$  with  $X \cap Y = \emptyset$ .

The condition is clearly satisfied for  $X = Y = \emptyset$ . Let  $X, Y \subseteq V$  with  $X \cap Y = \emptyset$  and  $X \cup Y \neq \emptyset$ . We distinguish four types of connected components  $C$  of  $G[V \setminus (X \cup Y)]$  such that  $e(C, Y) \not\equiv |C| \pmod{2}$ .

(1)  $|C|$  even,  $e(C, X) = 0$ . Since  $e(C, Y)$  is odd we have  $e(C, Y) = e(C, V \setminus C) \geq k/h$ .

(2)  $|C|$  even,  $e(C, X) \geq 1$ . Since  $e(C, Y)$  is odd we have  $e(C, Y) \geq 1$ .

(3)  $|C|$  odd,  $e(C, Y) = 0$ . We have  $e(C, X) = e(C, V \setminus C) \geq k/h$  by hypothesis.

(4)  $|C|$  odd,  $e(C, Y) \geq 1$ . Since  $e(C, Y)$  is even we have  $e(C, Y) \geq 2$ .

Let  $p_1, p_2, p_3, p_4$  be the number of components of each type. We have clearly

$$p_2 + \frac{k}{h} p_3 + e(X, Y) \leq e(X, V \setminus X) \leq d(X),$$

$$\frac{k}{h} p_1 + p_2 + 2p_4 + e(X, Y) \leq e(Y, V \setminus Y) \leq d(Y).$$

Hence the inequalities

$$p_2 + \frac{k}{h} p_3 + e(X, Y) \leq d(X),$$

$$\frac{k}{h} p_1 + p_2 + 2p_4 + e(X, Y) \leq d(Y).$$

We multiply the first inequality by  $h/k$ , the second one by  $(k - h)/k$ , and add. We obtain

$$\frac{k - h}{h} p_1 + p_2 + p_3 + 2 \frac{k - h}{k} p_4 + e(X, Y) \leq \frac{h}{k} d(X) + \frac{k - h}{k} d(Y).$$

Since  $(k - h)/h \geq 1$  and  $2(k - h)/k \geq 1$  if  $h \leq \frac{1}{2}k$ , it follows with  $p = p_1 + p_2 + p_3 + p_4$ , that

$$p + e(X, Y) \leq \frac{h}{k} d(X) + \frac{k - h}{k} d(Y).$$



As in the proof of Theorem 1 this inequality implies that Tutte's condition is satisfied.  $\square$

The theorem of Babler and Gallai corresponds to the particular cases of Theorem 2 when  $G$  is regular. If  $G$  is regular of degree  $k$  we have  $k|X| = d(X) = 2e(X) + e(X, V - X)$ . Hence if  $k$  is odd,  $e(X, V - X)$  is odd if and only if  $|X|$  is odd and if  $k$  is even,  $e(X, V - X)$  is always even. Therefore the hypothesis of Theorem 2 reduces to: for all  $X \subseteq V$ ,  $|X|$  odd implies  $e(X, V - X) \geq k/2$ .

Finally, we note that the bounds on  $h$  and  $\sum_{x \in V} |d(x) - k|$  in Theorems 1 and 2 are probably tight. (We leave as an exercise to the reader the construction of effective examples).

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