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REGULAR FACTORS IN NEARLY REGULAR GRAPHS

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We strengthen classical theorems of Petersen, Ore, Bäbler, Gallai on the existence of regular factors in regular graphs to nearly regular graphs.

1. Introduction

An h-factor of a graph G, h an integer, is a spanning subgraph which is regular of degree h. More generally, given an integer-valued function f defined on $V(G)$, an *f-factor* is a subgraph H of G such that $d_H(x)=f(x)$ for all $x \in V(G)$. Several important theorems of the theory of graph factors give simple sufficient conditions for the existence of h-factors in regular graphs [1, 2, 5].

Theorem (Petersen [5]). *A regular graph of even degree k can be decomposed into $\frac{1}{2}k$ 2-factors.*

Theorem (Ore [4, Theorem 3.2.5]). *Let G be a 2-edge-connected regular graph of odd degree k and h be an even integer $2 \leqslant h \leqslant 2k/3$. Then G contains an h-factor.*

Theorem (k odd: Bäbler [1]; k even: Gallai [2]). *Let G be a connected regular graph of degree k, with an even number of vertices. Let h be an odd integer, $h \leqslant \frac{1}{2}k$. If for all $X \subseteq V$, $|X|$ odd we have $e(X, V-X) \geqslant k/h$, then G has an h-factor.*

In the present note we give short proofs of these theorems by deriving them from Tutte's f-Factor Theorem [7]. We obtain slight generalizations: Due to integrality conditions these existence theorems remain valid for graphs which are almost regular (in a sense to be made precise below).

For completeness we recall Tutte's f-Factor Theorem [7].

* C.N.R.S.

Theorem. *A graph has an f-factor if and only if*

$$f(X)+d_G(Y)-f(Y)\geqslant e_G(X,Y)+\begin{pmatrix}\textit{number of connected components}\\ C \textit{ of } G[V\backslash(X\cup Y)] \textit{ such that}\\ e_G(C,Y)\not\equiv f(C) \bmod 2\end{pmatrix}$$

for all $X, Y\subseteq V(G)$ *with* $X\cap Y=\emptyset$.

The reader is referred to [3] for a short proof of the 1-Factor theorem [6], and to [8] for a short proof of the f-Factor Theorem from the 1-Factor theorem. Another necessary and sufficient condition for the existence of f-factors is given by Ore [4].

2. Notations

Graphs considered in the present note are finite, undirected, and may have loops and multiple edges (a loop incident to a vertex x contributes 2 to the degree of x).

Throughout the paper the following notations are used: Given a graph G the vertex-set of G is denoted by $V(G)$. The degree in G of a vertex x is $d_G(x)$. Given $X\subseteq V(G)$, we set $d_G(X)=\sum_{x\in X} d_G(x)$. the subgraph of G induced by X is denoted by $G[X]$, the number of edges of $G[X]$ by $e_G(X)$. Given $X, Y\subseteq V(G)$ with $X\cap Y=\emptyset$ we denote by $e_G(X,Y)$ the number of edges of G joining X and Y.

If no confusion results subscripts G will usually be omitted.

3. The theorems

Theorem 1 (even factors). *Let G be a graph and h, k be two integers such that h is even and $2\leqslant h\leqslant\frac{2}{3}k$. Suppose that $\sum_{x\in V(G)}|d_G(x)-k|<k/h$ and G is 2-edge connected. Then G contains an h-factor.*

Proof. Tutte's condition for the existence of an h-factor, h even, is:

$$h\,|X|+d(Y)-f\,|Y|\geqslant e(X,Y)+\begin{pmatrix}\text{number of connected components } C\\ \text{of } G[V\backslash(X\cup Y)] \text{ such that}\\ e(C,Y) \text{ is odd}\end{pmatrix}$$

for all $X, Y\subseteq V$ with $X\cap Y=\emptyset$.

The condition is clearly satisfied for $X=Y=\emptyset$. Let $X, Y\subseteq V$ with $X\cap Y=\emptyset$ and $X\cup Y\neq\emptyset$. We distinguish two types of connected components C of

$G[V\setminus(X\cup Y)]$ such that $e(C, Y)$ is odd:

(1) $e(C, X)=0$. Since G is 2-edge-connected $e(C, Y)=e(C, V\setminus C)\geqslant 2$. Hence, $e(C, Y)$ being odd, we have $e(C, Y)\geqslant 3$.

(2) $e(C, X)\geqslant 1$. Since $e(C, Y)$ is odd, we have $e(C, Y)\geqslant 1$.

Let p_1, p_2 be the numbers of components of each type. We have clearly

$$p_2+e(X, Y)\leqslant e(X, V\setminus X)\leqslant d(X),$$

$$3p_1+p_2+e(X, Y)\leqslant e(Y, V\setminus Y)\leqslant d(Y).$$

Hence the inequalities

$$p_2+e(X, Y)\leqslant d(X),$$

$$3p_1+p_2+e(X, Y)\leqslant d(Y).$$

We multiply the first inequality by h/k, the second one by $(k-h)/k$ and add. We obtain

$$3\frac{k-h}{k}p_1+p_2+e(X, Y)\leqslant\frac{h}{k}d(X)+\frac{k-h}{k}d(Y).$$

Since $3(k-h)/k\geqslant 1$ if $h\leqslant\frac{2}{3}k$ it follows, with $p=p_1+p_2$, that

$$p+e(X, Y)\leqslant\frac{h}{k}d(X)+\frac{k-h}{k}d(Y).$$

Suppose

$$h\,|X|+d(Y)-h\,|Y|<p+e(X, Y).$$

Since both sides are integers we have

$$h\,|X|+d(Y)-h\,|Y|+1\leqslant p+e(X, Y)\leqslant\frac{h}{k}d(X)+\frac{k-h}{k}d(Y).$$

It follows that

$$d(X)-k\,|X|-(d(Y)-k\,|Y|)\geqslant\frac{k}{h}.$$

Since $X\cap Y=\emptyset$ we have

$$d(X)-k\,|X|-(d(Y)-k\,|Y|)\leqslant\sum_{x\in X\cup Y}|d(x)-k|<\frac{k}{h},$$

a contradiction. Hence Tutte's condition is satisfied. Theorem 1 follows. □

The theorems of Petersen and Ore quoted in the introduction correspond to the particular cases of Theorem 1 when G is regular. To obtain Petersen's theorem it suffices to observe that the connected components of a regular graph of even degree are necessarily 2-edge-connected. (Proof: for $X\subseteq V$ we have $k\,|X|=d(X)=2e(X)+e(X, V\setminus X)$, hence if k is even $e(X, V\setminus X)$ is also even.)

Theorem 2 (odd factors). *Let G be a connected graph with an even number of vertices, and h, k be two integers such that h is odd and $1 \leqslant h \leqslant \frac{1}{2}k$. Suppose that $\sum_{x \in V(G)} |d(x)-k| < k/h$ and for all $X \subseteq V$ with $e(X, V-X)$ odd or $|X|$ odd we have $e(X, V-X) \geqslant k/h$. Then G contains an h-factor.*

Note. The condition on $e(X, V-X)$ is obviously satisfied if the edge connectivity of G is at least k/h.

Proof. Tutte's condition for the existence of an h-factor, h odd, is:

$$h\,|X|+d(Y)-h\,|Y| \geqslant e(X, Y)+\begin{pmatrix}\text{number of connected components } C \\ \text{of } G[V\setminus(X\cup Y)] \text{ such that} \\ e(C, Y) \not\equiv |C| \bmod 2\end{pmatrix}$$

for all $X, Y \subseteq V$ with $X \cap Y = \emptyset$.

The condition is clearly satisfied for $X = Y = \emptyset$. Let $X, Y \subseteq V$ with $X \cap Y = \emptyset$ and $X \cup Y \neq \emptyset$. We distinguish four types of connected components C of $G[V\setminus(X\cup Y)]$ such that $e(C, Y) \not\equiv |C| \bmod 2$.

(1) $|C|$ even, $e(C, X) = 0$. Since $e(C, Y)$ is odd we have $e(C, Y) = e(C, V\setminus C) \geqslant k/h$.

(2) $|C|$ even, $e(C, X) \geqslant 1$. Since $e(C, Y)$ is odd we have $e(C, Y) \geqslant 1$.

(3) $|C|$ odd, $e(C, Y) = 0$. We have $e(C, X) = e(C, V\setminus C) \geqslant k/h$ by hypothesis.

(4) $|C|$ odd, $e(C, Y) \geqslant 1$. Since $e(C, Y)$ is even we have $e(C, Y) \geqslant 2$.

Let p_1, p_2, p_3, p_4 be the number of components of each type. We have clearly

$$p_2+\frac{k}{h}p_3+e(X, Y) \leqslant e(X, V\setminus X) \leqslant d(X),$$

$$\frac{k}{h}p_1+p_2+2p_4+e(X, Y) \leqslant e(Y, V\setminus Y) \leqslant d(Y).$$

Hence the inequalities

$$p_2+\frac{k}{h}p_3+e(X, Y) \leqslant d(X),$$

$$\frac{k}{h}p_1+p_2+2p_4+e(X, Y) \leqslant d(Y).$$

We multiply the first inequality by h/k, the second one by $(k-h)/k$, and add. We obtain

$$\frac{k-h}{h}p_1+p_2+p_3+2\frac{k-h}{k}p_4+e(X, Y) \leqslant \frac{h}{k}d(X)+\frac{k-h}{k}d(Y).$$

Since $(k-h)/h \geqslant 1$ and $2(k-h)/k \geqslant 1$ if $h \leqslant \frac{1}{2}k$, it follows with $p = p_1+p_2+p_3+p_4$, that

$$p+e(X, Y) \leqslant \frac{h}{k}d(X)+\frac{k-h}{k}d(Y).$$

As in the proof of Theorem 1 this inequality implies that Tutte's condition is satisfied. □

The theorem of Bäbler and Gallai corresponds to the particular cases of Theorem 2 when G is regular. If G is regular of degree k we have $k\,|X| = d(X) = 2e(X) + e(X, V-X)$. Hence if k is odd, $e(X, V-X)$ is odd if and only if $|X|$ is odd and if k is even, $e(X, V-X)$ is always even. Therefore the hypothesis of Theorem 2 reduces to: for all $X \subseteq V$, $|X|$ odd implies $e(X, V-X) \geqslant k/h$.

Finally, we note that the bounds on h and $\sum_{x \in V} |d(x)-k|$ in Theorems 1 and 2 are probably tight. (We leave as an exercise to the reader the construction of effective examples).

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