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# Homogenization of Maxwell's equations and related scalar problems with sign-changing coefficients

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**Abstract.** In this work, we are interested in the homogenization of time-harmonic Maxwell's equations in a composite medium with periodically distributed small inclusions of a negative material. Here a negative material is a material modelled by negative permittivity and permeability. Due to the sign-changing coefficients in the equations, it is not straightforward to obtain uniform energy estimates to apply the usual homogenization techniques. The goal of this article is to explain how to proceed in this context. The analysis of Maxwell's equations is based on a precise study of two associated scalar problems: one involving the sign-changing permittivity with Dirichlet boundary conditions, another involving the sign-changing permeability with Neumann boundary conditions. For both problems, we obtain a criterion on the physical parameters ensuring uniform invertibility of the corresponding operators as the size of the inclusions tends to zero. In the process, we explain the link existing with the so-called Neumann-Poincaré operator, complementing the existing literature on this topic. Then we use the results obtained for the scalar problems to derive uniform energy estimates for Maxwell's system. At this stage, an additional difficulty comes from the fact that Maxwell's equations are also sign-indefinite due to the term involving the frequency. To cope with it, we establish some sort of uniform compactness result.

**Key words.** Homogenization, Maxwell's equations, metamaterials, sign-changing coefficients, Neumann-Poincaré operator.

## 1 Introduction

Negative index materials (also called left-handed materials) are artificially structured composite materials whose dielectric permittivity  $\varepsilon$  and magnetic permeability  $\mu$  are simultaneously negative in some frequency ranges [40]. In the last two decades, these metamaterials have been the subject of a large number of studies in physics and engineering due to their potential use for several existing applications [23] such as sub-wavelength imaging and focusing, cloaking, sensing or data storage.

Besides their practical applications, negative index materials are also interesting from a mathematical point of view. The reason is that they are usually used in applications jointly with classical (positive) materials so that their mathematical modelling leads to consider operators with coefficients whose sign changes in the domain of interest. Establishing well-posedness results for such problems requires to develop a specific theory and this has been investigated by several

authors. In particular, it has been shown in [8] (see also the recent work [33]) that Maxwell’s equations with sign-changing electromagnetic coefficients  $\varepsilon$ ,  $\mu$  are uniquely solvable – except for a discrete set of frequencies – when two associated scalar problems, one involving  $\varepsilon$  with Dirichlet boundary conditions, another involving  $\mu$  with Neumann boundary conditions, are well-posed.

The goal of this article is to study the homogenization process for time harmonic Maxwell’s equations in presence of  $\delta$ –periodically distributed inclusions of negative material embedded into a dielectric material (see Figure 1 for a typical configuration). The main objective is to clarify if the homogenization process is doable in this context and if so, to determine whether the corresponding homogenized material behaves like a positive or negative material as  $\delta$  tends to zero. For scalar problems, the first homogenization results have been obtained in [18] using the  $T$ -coercivity approach of [12]. More precisely, it is proved therein that for negative contrasts close to 0 (the contrast being defined here as the ratio between the interior and exterior values, see (1)), the scalar problem with Dirichlet boundary conditions can be homogenized. In other words, it is proved that under this assumption on the contrast, the solution of the problem in the composite material is well-defined for  $\delta$  small enough (this is not obvious due to the loss of coercivity due to the sign-changing coefficient) and that it two-scale converges (see Definition 5.1 below) to the solution of a well-posed problem set in a homogeneous material. These results have been extended in [14], through the analysis of the spectrum of the Neumann-Poincaré operator. In particular, the authors show that the homogenization process is possible provided the contrast between the two media (defined using the same convention as above) belongs to  $(-\infty; -1/\alpha)$  or  $(-\alpha; 0)$ ,  $\alpha > 0$  (see Remark 3.1 below). The proof of this result is based on an elegant continuity argument (see [14, Corollary 5.1]). However, it does not provide a precise value for  $\alpha$ .

The paper is organized as follows. Section 2 provides the mathematical setting of the problem and necessary notation. Before studying Maxwell’s system, we collect in Section 3 some useful results concerning two associated scalar problems, a Dirichlet and a Neumann one. In particular, we prove the uniform invertibility of these operators as  $\delta$  tends to zero, for small or large values of the contrast, i.e. for contrasts in  $(-\infty; -1/m) \cup (-1/M; 0)$ , with  $0 < m < M$  (see subsections 3.1 to 3.3). A variational characterization of the bounds  $m$  and  $M$  is also obtained (see (39)). Next, inspired by [14], we discuss in §3.4 the connection with the Neumann-Poincaré operator and the optimality of the obtained conditions. In Section 4, we study the cell problems appearing in the homogenization of Maxwell’s equations. We prove that they are well-posed under the same assumptions as the scalar problems investigated in Section 3. This allows us to define homogenized tensors and we show that they are positive definite under the same assumption on the contrasts, that is for contrasts in  $(-\infty; -1/m) \cup (-1/M; 0)$ . This is also an improvement of the results obtained in [18] and [14]. In Section 5, we finally tackle the homogenization process for Maxwell’s equations with sign-changing coefficients. Combining results from [43] and [21] obtained for classical (positive) electromagnetic materials, we first derive in §5.1 a homogenization result under a uniform energy estimate condition. At this stage, the sign-changing of the physical parameters does not play any role. Related to this part of the work, let us mention the seminal book of [6] as well as [46, 5] for the study of the time-dependent Maxwell equations. For the time harmonic case, we refer to [6, 48, 39, 24, 19, 17, 47, 3, 43, 29]. Then, in §5.2, we establish the needed uniform energy estimates for Maxwell’s equations. This is done by using the results obtained for the scalar problems as well as the  $T$ -coercivity approach presented in [8] and a uniform compactness property. The final homogenization result for Maxwell’s system with sign-changing coefficients is stated in Theorem 5.1.

## 2 Setting of the problem

Let  $\Omega$  be an open, connected and bounded subset of  $\mathbb{R}^3$  with a Lipschitz-continuous boundary  $\partial\Omega$ . Once for all, we make the following assumption:

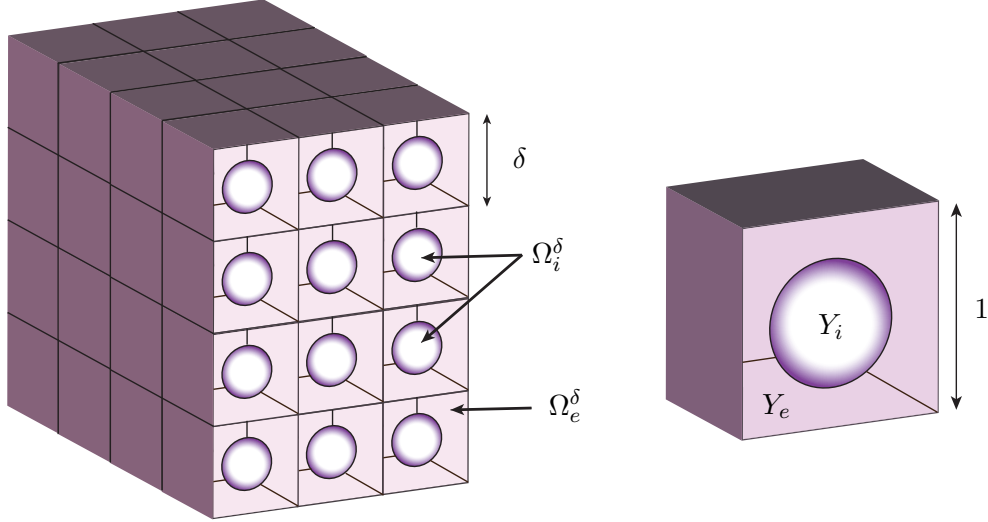


Figure 1: Example of a periodic material and the corresponding reference cell  $Y$ .

**Assumption.** *The domain  $\Omega$  is simply connected and  $\partial\Omega$  is connected.*

When this assumption is not satisfied, the analysis below must be adapted (see some preliminary ideas in [8, §8.2]). We consider a situation where  $\Omega$  is filled with a composite electromagnetic material constituted of periodically distributed inhomogeneous cells of small size  $\delta > 0$ . More precisely, let  $Y = (0; 1)^3$  denote the reference cell and assume that  $Y$  contains two materials:

- a metamaterial with negative dielectric permittivity  $\varepsilon_i < 0$  and magnetic permeability  $\mu_i < 0$  located inside a connected domain  $Y_i \subset Y$  with Lipschitz boundary  $\partial Y_i$  such that  $\overline{Y_i} \subset Y$ ;
- a dielectric material with positive dielectric permittivity  $\varepsilon_e > 0$  and magnetic permeability  $\mu_e > 0$  filling the region  $Y_e := Y \setminus \overline{Y_i}$ .

We emphasize that the assumption  $\overline{Y_i} \subset Y$  is important. When the inclusion  $Y_i$  meets the boundary of the cell  $\partial Y$ , phenomena different from the ones described below can appear. We refer the reader to [14, Appendix A] for more details concerning the scalar problem in this case. To simplify the presentation, we assume that  $\varepsilon_i$ ,  $\varepsilon_e$ ,  $\mu_i$  and  $\mu_e$  are constant. However, we could also consider physical parameters which are elements of  $L^\infty(\Omega, \mathbb{R}^{3 \times 3})$ , the variational techniques we use below would work in a similar way. In our analysis, the following dielectric and magnetic contrasts

$$\kappa_\varepsilon := \frac{\varepsilon_i}{\varepsilon_e} < 0, \quad \kappa_\mu := \frac{\mu_i}{\mu_e} < 0 \quad (1)$$

will play a key role. We define on the reference cell the two real-valued functions  $\varepsilon, \mu \in L^\infty(Y)$  such that

$$\varepsilon(y) = \varepsilon_e \mathbb{1}_{Y_e}(y) + \varepsilon_i \mathbb{1}_{Y_i}(y), \quad \mu(y) = \mu_e \mathbb{1}_{Y_e}(y) + \mu_i \mathbb{1}_{Y_i}(y), \quad (2)$$

where for a set  $S$ ,  $\mathbb{1}_S(\cdot)$  stands for the indicator function of  $S$ . For any  $\delta > 0$  and any integer vector  $k \in \mathbb{Z}^3$ , we define the shifted and scaled sets  $\Omega_{ik}^\delta, \Omega_{ek}^\delta, \Omega_k^\delta$  such that

$$\begin{aligned} \Omega_{ik}^\delta &:= \{x \in \mathbb{R}^3 \mid (x - k)/\delta \in Y_i\} \\ \Omega_{ek}^\delta &:= \{x \in \mathbb{R}^3 \mid (x - k)/\delta \in Y_e\} \\ \Omega_k^\delta &:= \{x \in \mathbb{R}^3 \mid (x - k)/\delta \in Y\}. \end{aligned} \quad (3)$$

We denote by  $K^\delta$  the set of  $k \in \mathbb{Z}^3$  such that  $\Omega_k^\delta \subset \Omega$ . We assume that the metamaterial fills the region

$$\Omega_i^\delta := \bigcup_{k \in K^\delta} \Omega_{ik}^\delta,$$

while the complementary set in  $\Omega$

$$\Omega_e^\delta = \Omega \setminus \overline{\Omega_i^\delta}$$

is occupied by the dielectric. We denote by  $\Omega^\delta$  the interior of

$$\bigcup_{k \in K^\delta} \overline{\Omega_k^\delta}.$$

And we set  $\mathcal{U}^\delta := \Omega \setminus \overline{\Omega^\delta}$ . We define the macroscopic dielectric permittivity  $\varepsilon^\delta$  and the magnetic permeability  $\mu^\delta$  on  $\Omega$  such that

$$\varepsilon^\delta(x) = \varepsilon_e \mathbb{1}_{\Omega_e^\delta}(x) + \varepsilon_i \mathbb{1}_{\Omega_i^\delta}(x), \quad \mu^\delta(x) = \mu_e \mathbb{1}_{\Omega_e^\delta}(x) + \mu_i \mathbb{1}_{\Omega_i^\delta}(x). \quad (4)$$

For a given frequency  $\omega \neq 0$  ( $\omega \in \mathbb{R}$ ), we study time harmonic Maxwell's equations

$$\mathbf{curl} \mathbf{E}^\delta - i\omega \mu^\delta \mathbf{H}^\delta = 0 \quad \text{and} \quad \mathbf{curl} \mathbf{H}^\delta + i\omega \varepsilon^\delta \mathbf{E}^\delta = \mathbf{J} \quad \text{in } \Omega. \quad (5)$$

Above  $\mathbf{E}^\delta$  and  $\mathbf{H}^\delta$  are respectively the electric and magnetic components of the electromagnetic field. The source term  $\mathbf{J}$  is the current density. We suppose that the medium  $\Omega$  is surrounded by a perfect conductor and we impose the boundary conditions

$$\mathbf{E}^\delta \times n = 0 \quad \text{and} \quad \mu^\delta \mathbf{H}^\delta \cdot n = 0 \quad \text{on } \partial\Omega, \quad (6)$$

where  $n$  denotes the unit outward normal vector field to  $\partial\Omega$ . Let us introduce some usual spaces in the study of Maxwell's equations, namely

$$\begin{aligned} \mathbf{L}^2(\Omega) &:= (\mathbf{L}^2(\Omega))^3 \\ \mathbf{H}(\mathbf{curl}) &:= \{\mathbf{H} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{H} \in \mathbf{L}^2(\Omega)\} \\ \mathbf{H}_N(\mathbf{curl}) &:= \{\mathbf{E} \in \mathbf{H}(\mathbf{curl}) \mid \mathbf{E} \times n = 0 \text{ on } \partial\Omega\} \\ \mathbf{V}_T(\xi) &:= \{\mathbf{H} \in \mathbf{H}(\mathbf{curl}) \mid \operatorname{div}(\xi \mathbf{H}) = 0, \xi \mathbf{H} \cdot n = 0 \text{ on } \partial\Omega\}, \quad \text{for } \xi \in \mathbf{L}^\infty(\Omega) \\ \mathbf{V}_N(\xi) &:= \{\mathbf{E} \in \mathbf{H}(\mathbf{curl}) \mid \operatorname{div}(\xi \mathbf{E}) = 0, \mathbf{E} \times n = 0 \text{ on } \partial\Omega\}, \quad \text{for } \xi \in \mathbf{L}^\infty(\Omega). \end{aligned}$$

For an open set  $\mathcal{O} \subset \mathbb{R}^3$ , the inner products in  $\mathbf{L}^2(\mathcal{O})$  and  $\mathbf{L}^2(\mathcal{O})$  are denoted indistinctly by  $(\cdot, \cdot)_\mathcal{O}$  and the corresponding norm by  $\|\cdot\|_\mathcal{O}$ . To simplify, in  $\mathbf{L}^2(\Omega)$  and  $\mathbf{L}^2(\Omega)$ , we just denote  $(\cdot, \cdot)$  and  $\|\cdot\|$ . The space  $\mathbf{H}(\mathbf{curl})$  and its subspaces  $\mathbf{H}_N(\mathbf{curl})$ ,  $\mathbf{V}_N(\xi)$ ,  $\mathbf{V}_T(\xi)$  are endowed with the inner product

$$(\cdot, \cdot)_{\mathbf{curl}} := (\cdot, \cdot) + (\mathbf{curl} \cdot, \mathbf{curl} \cdot).$$

The corresponding norm is denoted  $\|\cdot\|_{\mathbf{curl}}$ . Let us recall a well-known property for the particular spaces  $\mathbf{V}_T(1)$  and  $\mathbf{V}_N(1)$  (cf. [44, 4]).

**Proposition 2.1.** *The embeddings of  $\mathbf{V}_T(1)$  in  $\mathbf{L}^2(\Omega)$  and of  $\mathbf{V}_N(1)$  in  $\mathbf{L}^2(\Omega)$  are compact. Moreover, there is a constant  $C > 0$  such that*

$$\|\mathbf{u}\| \leq C \|\mathbf{curl} \mathbf{u}\|, \quad \forall \mathbf{u} \in \mathbf{V}_T(1) \cup \mathbf{V}_N(1).$$

Therefore, in  $\mathbf{V}_T(1)$  and in  $\mathbf{V}_N(1)$ ,  $\|\mathbf{curl} \cdot\|_\Omega$  is a norm which is equivalent to  $\|\cdot\|_{\mathbf{curl}}$ .

Classically, one proves that if  $(\mathbf{E}^\delta, \mathbf{H}^\delta)$  satisfies (5)-(6), then  $\mathbf{E}^\delta$  and  $\mathbf{H}^\delta$  are respectively solutions of the problems

$$\left| \begin{array}{l} \text{Find } \mathbf{E}^\delta \in \mathbf{H}(\mathbf{curl}) \text{ such that:} \\ \mathbf{curl}((\mu^\delta)^{-1} \mathbf{curl} \mathbf{E}^\delta) - \omega^2 \varepsilon^\delta \mathbf{E}^\delta = i\omega \mathbf{J} \quad \text{in } \Omega \\ \mathbf{E}^\delta \times n = 0 \quad \text{on } \partial\Omega, \end{array} \right. \quad (7)$$

$$\left| \begin{array}{l} \text{Find } \mathbf{H}^\delta \in \mathbf{H}(\mathbf{curl}) \text{ such that:} \\ \mathbf{curl}((\varepsilon^\delta)^{-1} \mathbf{curl} \mathbf{H}^\delta) - \omega^2 \mu^\delta \mathbf{H}^\delta = \mathbf{curl}((\varepsilon^\delta)^{-1} \mathbf{J}) \quad \text{in } \Omega \\ \mu^\delta \mathbf{H}^\delta \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \\ (\varepsilon^\delta)^{-1}(\mathbf{curl} \mathbf{H} - \mathbf{J}) \times \mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (8)$$

In the following, we will focus our attention on the problem (7) for the electric field. The analysis for the magnetic field is quite similar. The variational formulation of (7) writes

$$(\mathcal{P}^\delta) \quad \left| \begin{array}{l} \text{Find } \mathbf{E}^\delta \in \mathbf{H}_N(\mathbf{curl}) \text{ such that for all } \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}): \\ ((\mu^\delta)^{-1} \mathbf{curl} \mathbf{E}^\delta, \mathbf{curl} \mathbf{E}') - \omega^2(\varepsilon^\delta \mathbf{E}^\delta, \mathbf{E}') = i\omega(\mathbf{J}, \mathbf{E}'). \end{array} \right. \quad (9)$$

Before studying the behaviour of some solutions of  $(\mathcal{P}^\delta)$  as  $\delta$  tends to zero, we must clarify the properties of this problem for a fixed  $\delta > 0$ . With the Riesz representation theorem, define the linear and continuous operator  $\mathcal{A}_N^\delta(\omega) : \mathbf{H}_N(\mathbf{curl}) \rightarrow \mathbf{H}_N(\mathbf{curl})$  such that for all  $\omega \in \mathbb{C}$ ,

$$(\mathcal{A}_N^\delta(\omega) \mathbf{E}, \mathbf{E}')_{\mathbf{curl}} = ((\mu^\delta)^{-1} \mathbf{curl} \mathbf{E}, \mathbf{curl} \mathbf{E}') - \omega^2(\varepsilon^\delta \mathbf{E}, \mathbf{E}'), \quad \forall \mathbf{E}, \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}). \quad (10)$$

The features of  $\mathcal{A}_N^\delta(\omega)$  are strongly related to the ones of two scalar operators that we define now. Set

$$\begin{aligned} \mathbf{H}_0^1(\Omega) &:= \{\varphi \in \mathbf{H}^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega\} \\ \mathbf{H}_\#^1(\Omega) &:= \{\varphi \in \mathbf{H}^1(\Omega) \mid \int_\Omega \varphi \, dx = 0\}. \end{aligned}$$

In  $\mathbf{H}_0^1(\Omega)$  and in  $\mathbf{H}_\#^1(\Omega)$  (since  $\Omega$  is connected),  $\|\nabla \cdot\|$  is a norm which is equivalent to the usual norm of  $\mathbf{H}^1(\Omega)$ . We define the two linear and continuous operators  $A_\varepsilon^\delta : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  and  $B_\mu^\delta : \mathbf{H}_\#^1(\Omega) \rightarrow \mathbf{H}_\#^1(\Omega)$  such that

$$\begin{aligned} (\nabla(A_\varepsilon^\delta \varphi), \nabla \varphi') &= (\varepsilon^\delta \nabla \varphi, \nabla \varphi'), & \forall \varphi, \varphi' \in \mathbf{H}_0^1(\Omega) \\ (\nabla(B_\mu^\delta \varphi), \nabla \varphi') &= (\mu^\delta \nabla \varphi, \nabla \varphi'), & \forall \varphi, \varphi' \in \mathbf{H}_\#^1(\Omega). \end{aligned}$$

Theorem 6.1 of [8] guarantees that when  $A_\varepsilon^\delta$  and  $B_\mu^\delta$  are isomorphisms, then  $\mathcal{A}_N^\delta(\omega) : \mathbf{H}_N(\mathbf{curl}) \rightarrow \mathbf{H}_N(\mathbf{curl})$  is an isomorphism for all  $\omega \in \mathbb{C} \setminus \mathcal{S}$  where  $\mathcal{S}$  is a discrete set with no accumulation point. In the next section, we give conditions on the contrasts  $\kappa_\varepsilon, \kappa_\mu$  ensuring that  $A_\varepsilon^\delta$  and  $B_\mu^\delta$  are isomorphisms.

### 3 Uniform invertibility of the two scalar problems

We shall say that the operators  $A_\varepsilon^\delta : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  and  $B_\mu^\delta : \mathbf{H}_\#^1(\Omega) \rightarrow \mathbf{H}_\#^1(\Omega)$  are uniformly invertible as  $\delta$  tends to zero if there is  $\delta_0 > 0$  such that  $A_\varepsilon^\delta, B_\mu^\delta$  are invertible for all  $\delta \in (0; \delta_0]$  together with the estimate

$$\|(A_\varepsilon^\delta)^{-1}\| + \|(B_\mu^\delta)^{-1}\| \leq C,$$

where  $C > 0$  is a constant which is independent of  $\delta \in (0; \delta_0]$ . The uniform invertibility of  $A_\varepsilon^\delta$  has been considered in the articles [18, 14]. Below we combine the approaches presented in these two articles and we adapt the analysis in order to obtain a criterion ensuring the uniform invertibility of  $B_\mu^\delta$ .

**Remark 3.1.** *The result of uniform invertibility of [14, Theorem 5.2] is based on the result of Theorem 4.3 of the same article. However, its domain of validity is not completely satisfactory because the constant  $m$  defined in [14, Theorem 4.3] is in fact equal to zero. This has been corrected by the authors and a new proof can be found in the erratum [13].*

#### 3.1 First $\delta$ -dependent criteria

In a pedagogical aim, we first present some  $\delta$ -dependent criteria.

### 3.1.1 Criterion of invertibility for the operator $A_\varepsilon^\delta$

In order to get a criterion on the contrast  $\kappa_\varepsilon$  ensuring that  $A_\varepsilon^\delta : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  is an isomorphism, we first introduce a well-chosen decomposition of the space  $\mathbf{H}_0^1(\Omega)$ .

**Lemma 3.1.** *We have the decomposition  $\mathbf{H}_0^1(\Omega) = \mathcal{H}_D^\delta \oplus^\perp \mathbf{H}_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$  where  $\mathcal{H}_D^\delta := \{\varphi \in \mathbf{H}_0^1(\Omega) \mid \Delta\varphi = 0 \text{ in } \Omega_e^\delta \cup \Omega_i^\delta\}$ .*

**Remark 3.2.** *The index  $D$  in the notation  $\mathcal{H}_D^\delta$  stands for Dirichlet and refers to the homogeneous Dirichlet boundary condition imposed on  $\partial\Omega$  to the elements of  $\mathcal{H}_D^\delta$ . We emphasize that the functions of  $\mathbf{H}_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$  vanish on  $\partial\Omega_i^\delta$ .*

*Proof.* Let  $\varphi$  be a given element of  $\mathbf{H}_0^1(\Omega)$ . Introduce  $\tilde{\varphi} \in \mathbf{H}_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$  the function such that  $\Delta\tilde{\varphi} = \Delta\varphi$  in  $\Omega_e^\delta \cup \Omega_i^\delta$ . Then we have  $\varphi = (\varphi - \tilde{\varphi}) + \tilde{\varphi}$  and clearly  $\varphi - \tilde{\varphi} \in \mathcal{H}_D^\delta$ . Now if  $\varphi_1$  and  $\varphi_2$  are elements of  $\mathcal{H}_D^\delta$  and  $\mathbf{H}_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$ , a direct integration by parts gives

$$(\nabla\varphi_1, \nabla\varphi_2) = \int_{\Omega_e^\delta \cup \Omega_i^\delta} \Delta\varphi_1\varphi_2 \, dx + \int_{\partial\Omega_i^\delta} \frac{\partial\varphi_1}{\partial n_i} \varphi_2 \, d\sigma + \int_{\partial\Omega_e^\delta} \frac{\partial\varphi_1}{\partial n_e} \varphi_2 \, d\sigma = 0.$$

Here and below,  $n_e = -n_i$  stands for the unit normal vector to  $\partial\Omega_i^\delta$  pointing to  $\Omega_i^\delta$ . This gives the desired result.  $\square$

In what follows, some particular elements of  $\mathcal{H}_D^\delta$  will play a key role. For  $k \in K^\delta$ , define the function  $\varphi_D^k \in \mathcal{H}_D^\delta$  such that

$$\varphi_D^k = \begin{cases} 1 & \text{in } \Omega_{ik}^\delta \\ 0 & \text{in } \Omega_{ik'}^\delta \quad \text{for } k' \neq k. \end{cases} \quad (11)$$

Then set

$$\hat{\mathcal{H}}_D^\delta := \{\varphi \in \mathcal{H}_D^\delta \mid (\nabla\varphi, \nabla\varphi_D^k) = 0, \quad \forall k \in K^\delta\} \quad (12)$$

so that we have

$$\mathbf{H}_0^1(\Omega) = \hat{\mathcal{H}}_D^\delta \oplus^\perp \text{span}_{k \in K^\delta} \{\varphi_D^k\} \oplus^\perp \mathbf{H}_0^1(\Omega_e^\delta \cup \Omega_i^\delta). \quad (13)$$

Finally, we define the constants

$$m_D^\delta := \inf_{\varphi \in \hat{\mathcal{H}}_D^\delta \setminus \{0\}} \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}, \quad M_D^\delta := \sup_{\varphi \in \hat{\mathcal{H}}_D^\delta \setminus \{0\}} \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}. \quad (14)$$

Before proceeding, let us discuss a few features of the constants  $m_D^\delta$ ,  $M_D^\delta$ . First, observe that the functions  $\varphi_D^k$  satisfy

$$\|\nabla\varphi_D^k\|_{\Omega_i^\delta}^2 = 0 \quad \text{and} \quad \|\nabla\varphi_D^k\|_{\Omega_e^\delta}^2 \neq 0.$$

As a consequence, the infimum of (14) considered over  $\hat{\mathcal{H}}_D^\delta \setminus \{0\}$  is zero. On the other hand, the next lemma guarantees that the supremum of (14) considered over  $\hat{\mathcal{H}}_D^\delta \setminus \{0\}$  coincides with  $M_D^\delta$ .

**Lemma 3.2.** *The constant  $M_D^\delta$  defined in (14) satisfies*

$$M_D^\delta = \sup_{\varphi \in \hat{\mathcal{H}}_D^\delta \setminus \{0\}} \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}. \quad (15)$$

*Proof.* Since  $\hat{\mathcal{H}}_D^\delta \subset \mathcal{H}_D^\delta$ , clearly we have

$$M_D^\delta \leq \sup_{\varphi \in \mathcal{H}_D^\delta \setminus \{0\}} \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}. \quad (16)$$

Now we establish the other inequality. If  $\varphi \in \mathcal{H}_D^\delta \setminus \{0\}$ , we have the decomposition  $\varphi = \hat{\varphi} + \Phi$  with  $\hat{\varphi} \in \hat{\mathcal{H}}_D^\delta$  and  $\Phi \in \text{span}_{k \in K^\delta} \{\varphi_D^k\}$ . Since  $\Phi$  is constant in each of the  $\Omega_{ik}^\delta$ ,  $k \in K^\delta$ , there holds

$$\|\nabla\varphi\|_{\Omega_i^\delta}^2 = \|\nabla\hat{\varphi}\|_{\Omega_i^\delta}^2. \quad (17)$$

As a consequence, if  $\hat{\varphi} \equiv 0$ , then  $0 = \|\nabla\varphi\|_{\Omega_i^\delta}^2 / \|\nabla\varphi\|_{\Omega_e^\delta}^2 \leq M_D^\delta$ . If  $\hat{\varphi} \not\equiv 0$ , from (17) and the identity  $\|\nabla\varphi\|_{\Omega}^2 = \|\nabla\hat{\varphi}\|_{\Omega}^2 + \|\nabla\Phi\|_{\Omega}^2$  (see (12)), we deduce that

$$\|\nabla\varphi\|_{\Omega_e^\delta}^2 = \|\nabla\hat{\varphi}\|_{\Omega_e^\delta}^2 + \|\nabla\Phi\|_{\Omega_e^\delta}^2 \geq \|\nabla\hat{\varphi}\|_{\Omega_e^\delta}^2.$$

This implies

$$M_D^\delta \geq \frac{\|\nabla\hat{\varphi}\|_{\Omega_i^\delta}^2}{\|\nabla\hat{\varphi}\|_{\Omega_e^\delta}^2} \geq \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}. \quad (18)$$

Taking the supremum over all  $\varphi \in \mathcal{H}_D^\delta \setminus \{0\}$  in (18), we deduce that (16) is also true with “ $\leq$ ” replaced by “ $\geq$ ”. This shows (15).  $\square$

Finally, thanks to the orthogonality conditions imposed to the elements of  $\hat{\mathcal{H}}_D^\delta$ , one can prove that the constants  $m_D^\delta$ ,  $M_D^\delta$  satisfy  $0 < m_D^\delta \leq M_D^\delta < +\infty$ . To proceed, one can work by contradiction to show the Poincaré-Wirtinger inequality

$$\exists C^\delta > 0 \quad \text{such that} \quad \|\varphi\|_{\Omega_i^\delta} \leq C^\delta \|\nabla\varphi\|_{\Omega_i^\delta}, \quad \forall \varphi \in \hat{\mathcal{H}}_D^\delta$$

and use also the continuity of the traces from  $H^1(\Omega_i^\delta)$  and  $H^1(\Omega_e^\delta)$  to  $H^{1/2}(\partial\Omega_i^\delta)$ . After these considerations, we can now establish the following criterion concerning the invertibility of  $A_\varepsilon^\delta$ .

**Proposition 3.1.** *Assume that  $\kappa_\varepsilon \in (-\infty; -1/m_D^\delta) \cup (-1/M_D^\delta; 0)$  where  $m_D^\delta$  and  $M_D^\delta$  are defined in (14). Then  $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is an isomorphism.*

*Proof.* Define the operator  $T_D^+ : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that for  $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$  with  $\hat{\varphi}_h \in \hat{\mathcal{H}}_D^\delta$ ,  $\Phi_h \in \text{span}_{k \in K^\delta} \{\varphi_D^k\}$  and  $\tilde{\varphi} \in H_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$ , there holds

$$T_D^+\varphi = \begin{cases} \hat{\varphi}_h + \Phi_h + \tilde{\varphi} & \text{in } \Omega_e^\delta \\ \hat{\varphi}_h + \Phi_h - \tilde{\varphi} & \text{in } \Omega_i^\delta. \end{cases} \quad (19)$$

Note that since  $\tilde{\varphi} = 0$  on  $\partial\Omega_i^\delta$ , the operator  $T_D^+$  is indeed valued in  $H_0^1(\Omega)$ . Moreover we have  $T_D^+ \circ T_D^+ = \text{Id}$  which shows that  $T_D^+$  is an isomorphism of  $H_0^1(\Omega)$ . For all  $\varphi \in H_0^1(\Omega)$ , we find

$$(\nabla(A_\varepsilon^\delta(T_D^+\varphi)), \nabla\varphi) = (\varepsilon\nabla\hat{\varphi}_h, \nabla\hat{\varphi}_h) + (\varepsilon_e\nabla\Phi_h, \nabla\Phi_h)_{\Omega_e^\delta} + (|\varepsilon|\nabla\tilde{\varphi}, \nabla\tilde{\varphi}). \quad (20)$$

For the first term of the right hand side of (20), we can write

$$\begin{aligned} (\varepsilon\nabla\hat{\varphi}_h, \nabla\hat{\varphi}_h) &= \varepsilon_e \|\nabla\hat{\varphi}_h\|_{\Omega_e^\delta}^2 - |\varepsilon_i| \|\nabla\hat{\varphi}_h\|_{\Omega_i^\delta}^2 \\ &\geq (\varepsilon_e - |\varepsilon_i| M_D^\delta) \|\nabla\hat{\varphi}_h\|_{\Omega_e^\delta}^2 \\ &\geq \frac{1}{2} (\varepsilon_e - |\varepsilon_i| M_D^\delta) (\|\nabla\hat{\varphi}_h\|_{\Omega_e^\delta}^2 + (M_D^\delta)^{-1} \|\nabla\hat{\varphi}_h\|_{\Omega_i^\delta}^2). \end{aligned} \quad (21)$$

Using this estimate in (20), we deduce that when  $\varepsilon_e > |\varepsilon_i| M_D^\delta \Leftrightarrow \kappa_\varepsilon = \varepsilon_i / \varepsilon_e > -1/M_D^\delta$ , the sesquilinear form  $(\nabla(A_\varepsilon^\delta(T_D^+\cdot)), \nabla\cdot)$  is coercive in  $H_0^1(\Omega)$ . Using Lax-Milgram theorem, we infer that when  $\kappa_\varepsilon > -1/M_D^\delta$ , the operator  $A_\varepsilon^\delta \circ T_D^+$  is an isomorphism of  $H_0^1(\Omega)$  and so is  $A_\varepsilon^\delta$ .

To address the case  $\kappa_\varepsilon < -1/m_D^\delta$ , let us work with the operator  $T_D^- : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that

$$T_D^-\varphi = \begin{cases} -\hat{\varphi}_h + \Phi_h + \tilde{\varphi} & \text{in } \Omega_e^\delta \\ -\hat{\varphi}_h + \Phi_h - \tilde{\varphi} & \text{in } \Omega_i^\delta. \end{cases} \quad (22)$$



We also have  $T_D^- \circ T_D^- = \text{Id}$  which guarantees that  $T_D^-$  is an isomorphism of  $H_0^1(\Omega)$ . And for all  $\varphi \in H_0^1(\Omega)$ , we find

$$(\nabla(A_\varepsilon^\delta(T_D^- \varphi)), \nabla \varphi) = -(\varepsilon \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) + (\varepsilon_e \nabla \Phi_h, \nabla \Phi_h)_{\Omega_e^\delta} + (|\varepsilon| \nabla \tilde{\varphi}, \nabla \tilde{\varphi}). \quad (23)$$

This time, we can write

$$\begin{aligned} -(\varepsilon \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) &= -\varepsilon_e \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + |\varepsilon_i| \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2 \\ &\geq (-\varepsilon_e + |\varepsilon_i| m_D^\delta) \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 \geq \frac{1}{2} (-\varepsilon_e + |\varepsilon_i| m_D^\delta) (\|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + (M_D^\delta)^{-1} \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2). \end{aligned} \quad (24)$$

As a consequence, we see from (23) that when  $|\varepsilon_i| m_D^\delta > \varepsilon_e \Leftrightarrow \kappa_\varepsilon = \varepsilon_i / \varepsilon_e < -1/m_D^\delta$ , the sesquilinear form  $(\nabla(A_\varepsilon^\delta(T_D^- \cdot)), \nabla \cdot)$  is coercive in  $H_0^1(\Omega)$ . We can conclude as above that when  $\kappa_\varepsilon < -1/m_D^\delta$ , the operator  $A_\varepsilon^\delta$  is an isomorphism of  $H_0^1(\Omega)$ .  $\square$

### 3.1.2 Criterion of invertibility for the operator $B_\mu^\delta$

Now we show similar results for the operator  $B_\mu^\delta : H_{\#}^1(\Omega) \rightarrow H_{\#}^1(\Omega)$ . First, define the space

$$H_\diamond^1(\Omega) := \left\{ \varphi \in H^1(\Omega) \mid \int_{\partial\Omega_i^\delta} \varphi \, d\sigma = 0 \right\}.$$

**Lemma 3.3.** *We have the decomposition  $H_\diamond^1(\Omega) = \mathcal{H}_N^\delta \oplus H_{0, \partial\Omega_i^\delta}^1(\Omega)$  where  $\mathcal{H}_N^\delta := \{\varphi \in H_\diamond^1(\Omega) \mid \Delta\varphi = 0 \text{ in } \Omega_e^\delta \cup \Omega_i^\delta, \partial_n \varphi = 0 \text{ on } \partial\Omega\}$  and  $H_{0, \partial\Omega_i^\delta}^1(\Omega) := \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega_i^\delta\} \subset H_\diamond^1(\Omega)$ .*

**Remark 3.3.** *This time, the index  $N$  in the notation  $\mathcal{H}_N^\delta$  stands for Neumann and refers to the homogeneous Neumann boundary condition imposed on  $\partial\Omega$  to the elements of  $\mathcal{H}_N^\delta$ .*

*Proof.* For  $\varphi$  given in  $H_\diamond^1(\Omega)$ , introduce  $\tilde{\varphi} \in H_{0, \partial\Omega_i^\delta}^1(\Omega)$  the function such that

$$(\nabla \tilde{\varphi}, \nabla \varphi') = (\nabla \varphi, \nabla \varphi'), \quad \forall \varphi' \in H_{0, \partial\Omega_i^\delta}^1(\Omega).$$

Note that since the functions of  $H_{0, \partial\Omega_i^\delta}^1(\Omega)$  vanish on  $\partial\Omega_i^\delta$ , one can verify that this problem admits a unique solution. Then we have  $\varphi = (\varphi - \tilde{\varphi}) + \tilde{\varphi}$  and one can check that  $\varphi - \tilde{\varphi}$  belongs to  $\mathcal{H}_N^\delta$ . Finally if  $\varphi_1$  and  $\varphi_2$  are elements of  $\mathcal{H}_N^\delta$  and  $H_{0, \partial\Omega_i^\delta}^1(\Omega)$ , a direct integration by parts gives  $(\nabla \varphi_1, \nabla \varphi_2) = 0$ .  $\square$

In what follows, some particular elements of  $\mathcal{H}_N^\delta$  will play a key role. Let  $k_0$  be a given element of  $K^\delta$  and for  $k \in K^\delta \setminus \{k_0\}$ , define the function  $\varphi_N^k \in \mathcal{H}_N^\delta$  such that

$$\varphi_N^k = \begin{cases} 1 & \text{in } \Omega_{ik}^\delta \\ -1 & \text{in } \Omega_{ik_0}^\delta \\ 0 & \text{in } \Omega_{ik'}^\delta \quad \text{for } k' \in K^\delta \setminus \{k_0, k\}. \end{cases}$$

Then set

$$\hat{\mathcal{H}}_N^\delta := \{\varphi \in \mathcal{H}_N^\delta \mid (\nabla \varphi, \nabla \varphi_N^k) = 0, \quad \forall k \in K^\delta \setminus \{k_0\}\}$$

so that we have

$$H_\diamond^1(\Omega) = \mathcal{H}_N^\delta \oplus \text{span}_{k \in K^\delta \setminus \{k_0\}} \{\varphi_N^k\} \oplus H_{0, \partial\Omega_i^\delta}^1(\Omega).$$

Finally, we define the constants

$$m_N^\delta := \inf_{\varphi \in \hat{\mathcal{H}}_N^\delta \setminus \{0\}} \frac{\|\nabla \varphi\|_{\Omega_i^\delta}^2}{\|\nabla \varphi\|_{\Omega_e^\delta}^2}, \quad M_N^\delta := \sup_{\varphi \in \hat{\mathcal{H}}_N^\delta \setminus \{0\}} \frac{\|\nabla \varphi\|_{\Omega_i^\delta}^2}{\|\nabla \varphi\|_{\Omega_e^\delta}^2}. \quad (25)$$

Using the orthogonality conditions satisfied by the elements of  $\hat{\mathcal{H}}_N^\delta$ , working by contradiction, one can prove the Poincaré-Wirtinger inequality

$$\exists C^\delta > 0 \quad \text{such that} \quad \|\varphi\|_{\Omega_i^\delta} \leq C^\delta \|\nabla \varphi\|_{\Omega_i^\delta}, \quad \forall \varphi \in \hat{\mathcal{H}}_N^\delta.$$

Using also the continuity of the traces from  $H^1(\Omega_i^\delta)$  and  $H^1(\Omega_e^\delta)$  to  $H^{1/2}(\partial\Omega_i^\delta)$ , one can check that there holds  $0 < m_N^\delta \leq M_N^\delta < +\infty$ . As in (14), the functions  $\varphi_N^k$  satisfy

$$\|\nabla \varphi_N^k\|_{\Omega_i^\delta}^2 = 0 \quad \text{and} \quad \|\nabla \varphi_N^k\|_{\Omega_e^\delta}^2 \neq 0$$

so that the infimum of (25) considered over  $\mathcal{H}_N^\delta \setminus \{0\}$  is zero. And working exactly as in the proof of Lemma 3.2, we get the following result.

**Lemma 3.4.** *The constant  $M_N^\delta$  defined in (25) satisfies*

$$M_N^\delta = \sup_{\varphi \in \mathcal{H}_N^\delta \setminus \{0\}} \frac{\|\nabla \varphi\|_{\Omega_i^\delta}^2}{\|\nabla \varphi\|_{\Omega_e^\delta}^2}. \quad (26)$$

Now, we give our criterion of invertibility for the operator  $B_\mu^\delta$ .

**Proposition 3.2.** *Assume that  $\kappa_\mu \in (-\infty; -1/m_N^\delta) \cup (-1/M_N^\delta; 0)$  where  $m_N^\delta$  and  $M_N^\delta$  are defined in (25). Then  $B_\mu^\delta : H_{\#}^1(\Omega) \rightarrow H_{\#}^1(\Omega)$  is an isomorphism.*

*Proof.* Introduce the mappings  $\ell_\diamond : H_{\#}^1(\Omega) \rightarrow H_\diamond^1(\Omega)$  and  $\ell_{\#} : H_\diamond^1(\Omega) \rightarrow H_{\#}^1(\Omega)$  such that

$$\ell_\diamond(\varphi) = \varphi - \frac{1}{|\partial\Omega_i^\delta|} \int_{\partial\Omega_i^\delta} \varphi \, d\sigma, \quad \ell_{\#}(\varphi) = \varphi - \frac{1}{|\Omega|} \int_{\Omega} \varphi \, dx.$$

Here and in what follows, for an open set  $\mathcal{O} \subset \mathbb{R}^3$ , we denote  $|\mathcal{O}| = \int_{\mathcal{O}} 1 \, dx$  and  $|\partial\mathcal{O}| = \int_{\partial\mathcal{O}} 1 \, d\sigma$ . Then define the operators  $\tilde{\mathbf{T}}_N^\pm : H_\diamond^1(\Omega) \rightarrow H_\diamond^1(\Omega)$  such that for  $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$  with  $\hat{\varphi}_h \in \hat{\mathcal{H}}_N^\delta$ ,  $\Phi_h \in \text{span}_{k \in K^\delta \setminus \{k_0\}} \{\varphi_N^k\}$  and  $\tilde{\varphi} \in H_{0, \partial\Omega_i^\delta}^1(\Omega)$ , there holds

$$\tilde{\mathbf{T}}_N^\pm \varphi = \begin{cases} \pm \hat{\varphi}_h + \Phi_h + \tilde{\varphi} & \text{in } \Omega_e^\delta \\ \pm \hat{\varphi}_h + \Phi_h - \tilde{\varphi} & \text{in } \Omega_i^\delta. \end{cases} \quad (27)$$

Finally, we define the operators

$$\mathbf{T}_N^\pm := \ell_{\#} \circ \tilde{\mathbf{T}}_N^\pm \circ \ell_\diamond.$$

For  $\psi \in H_{\#}^1(\Omega)$ , we set  $\varphi := \ell_\diamond(\psi) \in H_\diamond^1(\Omega)$  and we use the notation  $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$  with  $\hat{\varphi}_h \in \hat{\mathcal{H}}_N^\delta$ ,  $\Phi_h \in \text{span}_{k \in K^\delta \setminus \{k_0\}} \{\varphi_N^k\}$  and  $\tilde{\varphi} \in H_{0, \partial\Omega_i^\delta}^1(\Omega)$ . We find

$$(\nabla(B_\mu^\delta(\mathbf{T}_N^\pm \psi)), \nabla \psi) = \pm(\mu \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) + (\mu_e \nabla \Phi_h, \nabla \Phi_h)_{\Omega_e^\delta} + (|\mu| \nabla \tilde{\varphi}, \nabla \tilde{\varphi}). \quad (28)$$

For the first term of the right hand side of (28), we can write

$$\begin{aligned} (\mu \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) &= \mu_e \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 - |\mu_i| \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2 \geq (\mu_e - |\mu_i| M_N^\delta) \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 \\ &\geq \frac{1}{2} (\mu_e - |\mu_i| M_N^\delta) (\|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + (M_N^\delta)^{-1} \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2) \end{aligned}$$

and

$$\begin{aligned} -(\mu \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) &= -\mu_e \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + |\mu_i| \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2 \\ &\geq (-\mu_e + |\mu_i| m_N^\delta) \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 \geq \frac{1}{2} (-\mu_e + |\mu_i| m_N^\delta) (\|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + (M_N^\delta)^{-1} \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2). \end{aligned} \quad (29)$$

Observing that  $\nabla \varphi = \nabla \psi$  ( $\varphi$  and  $\psi$  differ from each other by an additive constant), we deduce from the first estimate of (28) that when  $\mu_e > |\mu_i| M_N^\delta \Leftrightarrow \kappa_\mu = \mu_i/\mu_e > -1/M_N^\delta$ , the sesquilinear form  $(\nabla(B_\mu^\delta(\mathbf{T}_N^\pm \cdot)), \nabla \cdot)$  is coercive in  $H_{\#}^1(\Omega)$ . With the Lax-Milgram theorem, we infer that when  $\kappa_\mu > -1/M_N^\delta$ , the operator  $B_\mu^\delta \circ \mathbf{T}_N^\pm$  is an isomorphism of  $H_{\#}^1(\Omega)$ . Since  $B_\mu^\delta$  is selfadjoint (because it is bounded and symmetric), this implies that  $B_\mu^\delta$  is an isomorphism. Working similarly with  $\mathbf{T}_N^-$ , from (29) one finds that when  $|\mu_i| m_N^\delta > \mu_e \Leftrightarrow \kappa_\mu = \mu_i/\mu_e < -1/m_N^\delta$ , the operator  $B_\mu^\delta$  is an isomorphism. Note that with additional few lines, one can check that we have  $\mathbf{T}_N^\pm \circ \mathbf{T}_N^\pm = \text{Id}$ .  $\square$

### 3.2 Comparison between the criteria of invertibility

In this section, we compare the constants involved in the criteria ensuring the invertibility of the operators  $A_\varepsilon^\delta$  (Dirichlet) and  $B_\mu^\delta$  (Neumann).

**Proposition 3.3.** *For all  $\delta > 0$ , the constants  $m_D^\delta$ ,  $M_D^\delta$  defined in (14) and the constants  $m_N^\delta$ ,  $M_N^\delta$  defined in (25) satisfy*

$$m_D^\delta \leq m_N^\delta \quad \text{and} \quad M_D^\delta \leq M_N^\delta. \quad (30)$$

*Proof.* We start by proving the second inequality of (30). Let  $\varphi$  be an element of  $\hat{\mathcal{H}}_D^\delta \setminus \{0\}$ . Define the function  $\zeta \in \mathcal{H}_N^\delta$  such that  $\zeta = \varphi - c$  on  $\partial\Omega_i^\delta$  where  $c = |\partial\Omega_i^\delta|^{-1} \int_{\partial\Omega_i^\delta} \varphi \, d\sigma$ . In particular, we have  $\Delta\zeta = 0$  in  $\Omega_e^\delta \cup \Omega_i^\delta$  and  $\partial_n\zeta = 0$  on  $\partial\Omega$ . Note that necessarily, there holds  $\zeta \not\equiv 0$ . Then we have  $\zeta = \varphi - c$  in  $\Omega_i^\delta$  and so

$$\|\nabla\zeta\|_{\Omega_i^\delta} = \|\nabla\varphi\|_{\Omega_i^\delta}. \quad (31)$$

On the other hand, integrating by parts, we find

$$\begin{aligned} & (\nabla\zeta, \nabla(\zeta - \varphi))_{\Omega_e^\delta} \\ &= (\nabla\zeta, \nabla(\zeta - (\varphi - c)))_{\Omega_e^\delta} = \int_{\partial\Omega} \frac{\partial\zeta}{\partial n} (\zeta - (\varphi - c)) \, d\sigma + \int_{\partial\Omega_i^\delta} \frac{\partial\zeta}{\partial n_e} (\zeta - (\varphi - c)) \, d\sigma = 0. \end{aligned}$$

We deduce that

$$\|\nabla\zeta\|_{\Omega_e^\delta}^2 \leq \|\nabla\varphi\|_{\Omega_e^\delta}^2. \quad (32)$$

Gathering (31), (32) and using Lemma 3.4, we infer that

$$\frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2} \leq \frac{\|\nabla\zeta\|_{\Omega_i^\delta}^2}{\|\nabla\zeta\|_{\Omega_e^\delta}^2} \leq M_N^\delta. \quad (33)$$

Taking the supremum over all  $\varphi \in \hat{\mathcal{H}}_D^\delta \setminus \{0\}$  in (33), we obtain that  $M_D^\delta \leq M_N^\delta$ .

Now we show the first inequality of (30). Let  $\varphi$  be an element of  $\hat{\mathcal{H}}_N^\delta \setminus \{0\}$ . Define the function  $\zeta \in \mathcal{H}_D^\delta$  such that  $\zeta = \varphi$  on  $\partial\Omega_i^\delta$ . Then decompose  $\zeta$  as  $\zeta = \hat{\zeta} + Z$  with  $\hat{\zeta} \in \hat{\mathcal{H}}_D^\delta$  and  $Z \in \text{span}_{k \in K^\delta} \{\varphi_D^k\}$ . Since  $Z$  is constant in each of the  $\Omega_{ik}^\delta$ ,  $k \in K^\delta$ , we have

$$\|\nabla\hat{\zeta}\|_{\Omega_i^\delta} = \|\nabla\varphi\|_{\Omega_i^\delta}. \quad (34)$$

On the other hand, integrating by parts, we find

$$(\nabla\varphi, \nabla(\hat{\zeta} - \varphi))_{\Omega_e^\delta} = \int_{\partial\Omega} \frac{\partial\varphi}{\partial n} (\hat{\zeta} - \varphi) \, d\sigma + \int_{\partial\Omega_i^\delta} \frac{\partial\varphi}{\partial n_e} (\hat{\zeta} - \varphi) \, d\sigma = - \int_{\partial\Omega_i^\delta} \frac{\partial\varphi}{\partial n_e} Z \, d\sigma. \quad (35)$$

Since the function  $\varphi$  is in  $\hat{\mathcal{H}}_N^\delta$ , for all  $k \in K^\delta \setminus \{k_0\}$ , we have  $(\nabla\varphi, \nabla\varphi_N^k) = 0$ . Integrating by parts, this implies

$$\int_{\partial\Omega_{ik}^\delta} \frac{\partial\varphi}{\partial n_e} \, d\sigma = \int_{\partial\Omega_{ik_0}^\delta} \frac{\partial\varphi}{\partial n_e} \, d\sigma.$$

But we also have  $\int_{\partial\Omega_i^\delta} \partial_{n_e}\varphi \, d\sigma = 0$ . As a consequence, we must have, for all  $k \in K^\delta$ ,

$$\int_{\partial\Omega_{ik}^\delta} \frac{\partial\varphi}{\partial n_e} \, d\sigma = 0.$$

Since  $Z$  is constant on each of the  $\partial\Omega_{ik}^\delta$ , we deduce that the terms of the equalities of (35) are equal to zero. Hence, there holds

$$\|\nabla\varphi\|_{\Omega_e^\delta}^2 \leq \|\nabla\hat{\zeta}\|_{\Omega_e^\delta}^2. \quad (36)$$

Gathering (34) and (36) leads to

$$m_D^\delta \leq \frac{\|\nabla \hat{\zeta}\|_{\Omega_i^\delta}^2}{\|\nabla \hat{\zeta}\|_{\Omega_e^\delta}^2} \leq \frac{\|\nabla \varphi\|_{\Omega_i^\delta}^2}{\|\nabla \varphi\|_{\Omega_e^\delta}^2}. \quad (37)$$

Taking the infimum over all  $\varphi \in \hat{\mathcal{H}}_N^\delta \setminus \{0\}$  in (37), we obtain that  $m_D^\delta \leq m_N^\delta$ .  $\square$

### 3.3 Uniform criterion of invertibility

The bounds on the contrasts  $\kappa_\varepsilon, \kappa_\mu$  that we obtained in Propositions 3.1, 3.2 which ensure the invertibility of the scalar operators  $A_\varepsilon^\delta$  and  $B_\mu^\delta$ , depend on  $\delta$ . In this paragraph, we wish to get bounds which are uniform with respect to  $\delta$ .

Introduce the Hilbert spaces of functions defined in the reference cell  $Y$

$$\begin{aligned} \mathcal{H}_0 &:= \{\varphi \in H_0^1(Y) \mid \Delta \varphi = 0 \text{ in } Y_e \cup Y_i\} \\ \mathcal{H}_\diamond &:= \{\varphi \in H_\diamond^1(Y) \mid \Delta \varphi = 0 \text{ in } Y_e \cup Y_i\} \end{aligned}$$

where  $H_\diamond^1(Y) := \{\varphi \in H^1(Y) \mid \int_{\partial Y_i} \varphi \, d\sigma = 0\}$ . Define the function  $\varphi_D \in \mathcal{H}_0$  such that  $\varphi_D = 1$  in  $Y_i$  and set

$$\begin{aligned} \hat{\mathcal{H}}_0 &:= \{\varphi \in \mathcal{H}_0 \mid (\nabla \varphi, \nabla \varphi_D) = 0\} \\ \hat{\mathcal{H}}_\diamond &:= \{\varphi \in \mathcal{H}_\diamond \mid \partial_n \varphi = 0 \text{ on } \partial Y\}. \end{aligned} \quad (38)$$

Then we introduce the constants

$$m := \inf_{\varphi \in \hat{\mathcal{H}}_0 \setminus \{0\}} \frac{\|\nabla \varphi\|_{Y_i}^2}{\|\nabla \varphi\|_{Y_e}^2}, \quad M := \sup_{\varphi \in \hat{\mathcal{H}}_\diamond \setminus \{0\}} \frac{\|\nabla \varphi\|_{Y_i}^2}{\|\nabla \varphi\|_{Y_e}^2}. \quad (39)$$

We emphasize that  $m$  and  $M$  are independent of  $\delta$ . Adapting the proof of Lemma 3.2, we establish the following result.

**Lemma 3.5.** *The constant  $M$  defined in (39) satisfies*

$$M = \sup_{\varphi \in \hat{\mathcal{H}}_\diamond \setminus \{0\}} \frac{\|\nabla \varphi\|_{Y_i}^2}{\|\nabla \varphi\|_{Y_e}^2} \quad (40)$$

(here the sup is considered over  $\hat{\mathcal{H}}_\diamond \setminus \{0\}$  and not  $\hat{\mathcal{H}}_0 \setminus \{0\}$ ).

**Lemma 3.6.** *For all  $\delta > 0$ , we have the relations*

$$m \leq m_D^\delta \leq m_N^\delta \quad \text{and} \quad M_D^\delta \leq M_N^\delta \leq M, \quad (41)$$

where  $m_D^\delta, M_D^\delta$  are defined in (14),  $m_N^\delta, M_N^\delta$  are defined in (25) and  $m, M$  are defined in (39).

*Proof.* From Proposition 3.3, we know that we have  $m_D^\delta \leq m_N^\delta$  and  $M_D^\delta \leq M_N^\delta$ . Now we show that we have  $M_N^\delta \leq M$ . Let  $\varphi$  be a non zero element of  $\hat{\mathcal{H}}_N^\delta$ . For all  $k \in K^\delta$ , we define the function  $\varphi_k^\delta \in H^1(Y)$  such that  $\varphi_k^\delta(y) = \varphi(\delta(k+y))$  for  $y \in Y$ . And we set  $c_k := |\partial Y|^{-1} \int_{\partial Y} \varphi_k^\delta \, d\sigma$ . Since  $\varphi_k^\delta - c_k \in \mathcal{H}_\diamond$ , using Lemma 3.5, we can write

$$\begin{aligned} \|\nabla \varphi\|_{\Omega_{ik}^\delta}^2 &= \delta \|\nabla \varphi_k^\delta\|_{Y_i}^2 = \delta \|\nabla(\varphi_k^\delta - c_k)\|_{Y_i}^2 \\ &\leq \delta M \|\nabla(\varphi_k^\delta - c_k)\|_{Y_e}^2 \leq \delta M \|\nabla \varphi_k^\delta\|_{Y_e}^2 \leq M \|\nabla \varphi\|_{\Omega_{ek}^\delta}^2. \end{aligned}$$

Summing these estimates over all  $k \in K^\delta$ , we get (recall that  $\mathcal{U}^\delta = \Omega \setminus \overline{\Omega^\delta}$ )

$$\|\nabla \varphi\|_{\Omega_i^\delta}^2 \leq M \|\nabla \varphi\|_{\Omega_e^\delta \setminus \overline{\mathcal{U}^\delta}}^2 \leq M \|\nabla \varphi\|_{\Omega_e^\delta}^2. \quad (42)$$

Taking the supremum in (42) over all  $\varphi \in \hat{\mathcal{H}}_N^\delta$ , we deduce that  $M_N^\delta \leq M$ .

To establish (41), it remains to show that  $m \leq m_D^\delta$ . For  $\varphi$  given in  $\hat{\mathcal{H}}_D^\delta$ , introduce the function  $v \in H_0^1(\Omega)$  such that for all  $k \in K^\delta$ ,

$$\begin{cases} v = \varphi & \text{in } \Omega_{ik}^\delta \\ \Delta v = 0 & \text{in } \Omega_{ek}^\delta \\ v = 0 & \text{on } \partial\Omega_{ek}^\delta. \end{cases}$$

We also impose  $v = 0$  in  $\mathcal{U}^\delta = \Omega \setminus \overline{\Omega^\delta}$ . For all  $k \in K^\delta$ , define the function  $\phi_D^k \in H_0^1(\Omega)$  such that

$$\begin{cases} \phi_D^k = 1 & \text{in } \Omega_{ik}^\delta \\ \Delta \phi_D^k = 0 & \text{in } \Omega_{ek}^\delta \\ \phi_D^k = 0 & \text{in } \Omega \setminus \Omega_k^\delta. \end{cases}$$

Then set

$$\tilde{v} := v - \sum_{k \in K^\delta} a_k \phi_D^k \quad \text{with } a_k := (\nabla v, \nabla \phi_D^k) / \|\nabla \phi_D^k\|^2.$$

Integrating by parts, we find

$$(\nabla \varphi, \nabla(\varphi - \tilde{v}))_{\Omega_e^\delta} = \int_{\partial\Omega} \frac{\partial \varphi}{\partial n} (\varphi - \tilde{v}) \, d\sigma + \int_{\partial\Omega_i^\delta} \frac{\partial \varphi}{\partial n_e} (\varphi - \tilde{v}) \, d\sigma = \sum_{k \in K^\delta} a_k \int_{\partial\Omega_{ik}^\delta} \frac{\partial \varphi}{\partial n_e} \phi_D^k \, d\sigma. \quad (43)$$

Since the function  $\varphi$  is in  $\hat{\mathcal{H}}_D^\delta$ , for all  $k \in K^\delta$ , we have  $(\nabla \varphi, \nabla \phi_D^k) = 0$ . Integrating by parts, this implies

$$\int_{\partial\Omega_{ik}^\delta} \frac{\partial \varphi}{\partial n_e} \, d\sigma = 0.$$

Using that  $\phi_D^k$  is constant on the  $\partial\Omega_{ik'}^\delta$ , we deduce from (43) that  $(\nabla \varphi, \nabla(\varphi - \tilde{v}))_{\Omega_e^\delta} = 0$ . Hence, we have

$$\|\nabla \varphi\|_{\Omega_e^\delta}^2 \leq \|\nabla \tilde{v}\|_{\Omega_e^\delta}^2 = \sum_{k \in K^\delta} \|\nabla \tilde{v}\|_{\Omega_{ek}^\delta}^2. \quad (44)$$

For  $k \in K^\delta$ , define the function  $\tilde{v}_k^\delta \in H^1(Y)$  such that  $\tilde{v}_k^\delta(y) = \tilde{v}(\delta(k + y))$  for  $y \in Y$ . Observe that we have  $\tilde{v}_k^\delta \in \hat{\mathcal{H}}_0$  so that we can write

$$\|\nabla \tilde{v}\|_{\Omega_{ek}^\delta}^2 = \delta \|\nabla \tilde{v}_k^\delta\|_{Y_e}^2 \leq \delta m^{-1} \|\nabla \tilde{v}_k^\delta\|_{Y_i}^2 \leq \delta m^{-1} \|\nabla \varphi_k^\delta\|_{Y_i}^2 \leq m^{-1} \|\nabla \varphi\|_{\Omega_{ik}^\delta}^2. \quad (45)$$

As a consequence, inserting (45) in (44), we obtain

$$\|\nabla \varphi\|_{\Omega_e^\delta}^2 \leq m^{-1} \|\nabla \varphi\|_{\Omega_i^\delta}^2. \quad (46)$$

Taking the infimum in (46) over all  $\varphi \in \hat{\mathcal{H}}_D^\delta$ , we deduce that  $m \leq m_D^\delta$ .  $\square$

Finally, we deduce a criterion of uniform invertibility for the operators  $A_\varepsilon^\delta$  and  $B_\mu^\delta$ .

**Theorem 3.1.** *Let  $m, M$  be the constants defined in (39).*

*When  $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$ ,  $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is uniformly invertible as  $\delta \rightarrow 0$ .*

*When  $\kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ ,  $B_\mu^\delta : H_{\#}^1(\Omega) \rightarrow H_{\#}^1(\Omega)$  is uniformly invertible as  $\delta \rightarrow 0$ .*

*Proof.* Let us show the result for  $A_\varepsilon^\delta$ , the proof is completely similar for  $B_\mu^\delta$ . From the decomposition of the space  $H_0^1(\Omega)$  in (13), one observes that the operators  $\mathsf{T}_D^\pm = (\mathsf{T}_D^\pm)^{-1}$  defined in (19) and (22) are uniformly continuous. And from the estimate (21) (resp. (24)) together with the result of Lemma 3.6, one infers that as  $\delta \rightarrow 0$ ,  $(\nabla(A_\varepsilon^\delta(\mathsf{T}_D^+ \cdot)), \nabla \cdot)$  (resp.  $(\nabla(A_\varepsilon^\delta(\mathsf{T}_D^- \cdot)), \nabla \cdot)$ ) is uniformly coercive in  $H_0^1(\Omega)$  when  $\kappa_\varepsilon > -1/M$  (resp. when  $\kappa_\varepsilon < -1/m$ ). Since  $A_\varepsilon^\delta$  is also uniformly continuous, this is enough to guarantee that  $A_\varepsilon^\delta$  is uniformly invertible as  $\delta$  tends to zero.  $\square$

### 3.4 Optimality of the criterion and connection to the Neumann-Poincaré operator

Let us discuss the criterion we have obtained above. We focus our attention on the analysis for the operator  $A_\varepsilon^\delta$ , similar comments can be made for the operator  $B_\mu^\delta$ . We assume in this section that  $\partial Y_i$ , and so  $\partial\Omega_i^\delta$ , is of class  $\mathcal{C}^2$ . Note that this assumption is important to ensure that the spectrum of Problem (48) below is discrete. It has been proved in [7] that in this case,  $A_\varepsilon^\delta$  is Fredholm of index zero when  $\kappa_\varepsilon \neq -1$ . Therefore when  $\kappa_\varepsilon \neq -1$ , the operator  $A_\varepsilon^\delta$  is an isomorphism if and only if it is injective. As it has been observed in different works (see in particular [14]), and as we recall below, the question of the injectivity of  $A_\varepsilon^\delta$  is directly linked to the spectrum of the so-called Neumann-Poincaré operator. The latter has been widely studied when  $\Omega$  is the whole space  $\mathbb{R}^d$ . For this problem, among the references, let us cite [36, 1, 37, 38, 30, 28, 27, 35, 15, 16, 26].

Set  $\Sigma^\delta := \partial\Omega_i^\delta$  and introduce the two Dirichlet-to-Neumann operators  $\Lambda_e : \mathbf{H}^{1/2}(\Sigma^\delta) \rightarrow \mathbf{H}^{-1/2}(\Sigma^\delta)$ ,  $\Lambda_i : \mathbf{H}^{1/2}(\Sigma^\delta) \rightarrow \mathbf{H}^{-1/2}(\Sigma^\delta)$  such that for all  $\varphi \in \mathbf{H}^{1/2}(\Sigma^\delta)$ , we have  $\Lambda_e\varphi = \partial_{n_e}u_e$ ,  $\Lambda_i\varphi = \partial_{n_i}u_i$  where  $u_e, u_i$  solve respectively the problems

$$\left\{ \begin{array}{l} \Delta u_e = 0 \quad \text{in } \Omega_e^\delta \\ u_e = 0 \quad \text{on } \partial\Omega \\ u_e = \varphi \quad \text{on } \Sigma^\delta \end{array} \right. \quad \left\{ \begin{array}{l} \Delta u_i = 0 \quad \text{in } \Omega_i^\delta \\ u_i = \varphi \quad \text{on } \Sigma^\delta. \end{array} \right. \quad (47)$$

Define also the lifting operator  $\mathcal{R} : \mathbf{H}^{1/2}(\Sigma^\delta) \rightarrow \mathbf{H}_0^1(\Omega)$  such that  $\mathcal{R}\varphi = u_e$  in  $\Omega_e^\delta$ ,  $\mathcal{R}\varphi = u_i$  in  $\Omega_i^\delta$ , where  $u_e, u_i$  are the solutions to (47).

If  $u$  belongs to  $\ker A_\varepsilon^\delta \setminus \{0\}$ , then  $\varphi := u|_{\Sigma^\delta} \in \mathbf{H}^{1/2}(\Sigma^\delta) \setminus \{0\}$  satisfies  $\Lambda_e\varphi = -\kappa_\varepsilon\Lambda_i\varphi$ . And by a straightforward computation, we find that the pair  $(\alpha, \varphi)$ , with  $\alpha := (\kappa_\varepsilon + 1)/(\kappa_\varepsilon - 1) \in (-1; 1)$ , is a solution to the problem

$$\left\{ \begin{array}{l} \text{Find } (\alpha, \varphi) \in \mathbb{R} \times (\mathbf{H}^{1/2}(\Sigma^\delta) \setminus \{0\}) \text{ such that:} \\ \Lambda_-\varphi = \alpha\Lambda_+\varphi \end{array} \right. \quad (48)$$

with  $\Lambda_\pm := \Lambda_e \pm \Lambda_i$ . Reciprocally, assume that  $(\alpha, \varphi)$  is a solution to (48) with  $\alpha \in (-1; 1)$ . Then,  $\mathcal{R}\varphi \in \mathbf{H}_0^1(\Omega)$  is an element of  $\ker A_\varepsilon^\delta \setminus \{0\}$  for  $\kappa_\varepsilon = (\alpha - 1)/(\alpha + 1) \in (-\infty; 0)$ . This shows that it is sufficient to work on Problem (48) to study the injectivity of  $A_\varepsilon^\delta$ .

**Theorem 3.2.** *The spectrum of Problem (48) is discrete and coincides with two sequences of real numbers*

$$-1 < \alpha_1^- \leq \alpha_2^- \leq \dots \leq 0 \quad \text{and} \quad 1 = \alpha_1^+ = \dots = \alpha_{\text{card}(K^\delta)}^+ > \alpha_{\text{card}(K^\delta)+1}^+ \geq \dots \geq 0$$

such that  $\lim_{n \rightarrow +\infty} \alpha_n^\pm = 0$ . Here  $\text{card}(K^\delta)$  is the cardinal of the set  $K^\delta$  defined after (3).

*Proof.* First, we show that  $\Lambda_+ : \mathbf{H}^{1/2}(\Sigma^\delta) \rightarrow \mathbf{H}^{-1/2}(\Sigma^\delta)$  is an isomorphism. Consider some  $\psi \in \mathbf{H}^{-1/2}(\Sigma^\delta)$ . If  $\varphi \in \mathbf{H}^{1/2}(\Sigma^\delta)$  verifies  $\Lambda_+\varphi = \psi$ , then  $\mathcal{R}\varphi$  is a solution to

$$\left\{ \begin{array}{l} \text{Find } u \in \mathbf{H}_0^1(\Omega) \text{ such that} \\ (\nabla u, \nabla v) = \langle \psi, v \rangle_{\Sigma^\delta}, \quad \forall v \in \mathbf{H}_0^1(\Omega). \end{array} \right. \quad (49)$$

Reciprocally, assume that  $u$  is a solution to (49). Then the function  $\varphi := u|_{\Sigma^\delta}$  satisfies  $\Lambda_+\varphi = \psi$ . According to the Lax-Milgram theorem, Problem (49) admits a unique solution for all  $\psi \in \mathbf{H}^{-1/2}(\Sigma^\delta)$ . We infer that  $\Lambda_+ : \mathbf{H}^{1/2}(\Sigma^\delta) \rightarrow \mathbf{H}^{-1/2}(\Sigma^\delta)$  is indeed an isomorphism.

Now, remarking that  $\Lambda_e, \Lambda_i$  have the same principal symbol and using standard arguments of pseudo-differential operators theory (work as in the proof of [27, Theorem 1]), we can show that  $\Lambda_- = \Lambda_e - \Lambda_i : \mathbf{H}^{1/2}(\Sigma^\delta) \rightarrow \mathbf{H}^{-1/2}(\Sigma^\delta)$  is compact. We emphasize that the assumption of

smoothness of  $\Sigma^\delta$  here is important.

Using the Riesz representation theorem, define the operator  $K : H^{1/2}(\Sigma^\delta) \rightarrow H^{1/2}(\Sigma^\delta)$  such that

$$(K\varphi, \varphi')_{\Sigma^\delta} = \langle \Lambda_- \varphi, \varphi' \rangle_{\Sigma^\delta} \quad \text{for all } \varphi, \varphi' \in H^{1/2}(\Sigma^\delta). \quad (50)$$

Here, we use the notation  $(\cdot, \cdot)_{\Sigma^\delta} := \langle \Lambda_+ \cdot, \cdot \rangle_{\Sigma^\delta}$ . Note that according to the features of  $\Lambda_+$ , the latter form is an inner product in  $H^{1/2}(\Sigma^\delta)$  equivalent to the usual one. Remark that  $(\alpha, \varphi)$  is an eigenpair for (48) if and only if we have  $K\varphi = \alpha\varphi$ . But due to the properties of  $\Lambda_-$ ,  $K$  is a selfadjoint and compact operator. Therefore, the spectrum of (48) coincides with a sequence of eigenvalues which accumulate at zero. We can use the *min-max* principle (see [45, chapter 3]) to characterize these eigenvalues. We have

$$\alpha_1^+ = \sup_{\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}. \quad (51)$$

By the *min-max* principle, we know that this sup is attained for some  $\varphi_1^+$ . By induction, for  $k \geq 2$ , we define

$$\alpha_k^+ = \sup_{\substack{\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}, \\ \varphi \perp \{\varphi_1^+, \dots, \varphi_{k-1}^+\}}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}. \quad (52)$$

Here, if  $\varphi, \varphi'$  are two elements of  $H^{1/2}(\Sigma^\delta)$ , we write  $\varphi \perp \varphi'$  when  $(\varphi, \varphi')_{\Sigma^\delta} = \langle \Lambda_+ \varphi, \varphi' \rangle_{\Sigma^\delta} = (\nabla(\mathcal{R}\varphi), \nabla(\mathcal{R}\varphi')) = 0$ . Similarly, we define

$$\alpha_1^- = \inf_{\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}, \quad (53)$$

and, by induction, for  $k \geq 2$ ,

$$\alpha_k^- = \inf_{\substack{\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}, \\ \varphi \perp \{\varphi_1^-, \dots, \varphi_{k-1}^-\}}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}. \quad (54)$$

Observing that for all  $\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}$  we have

$$\frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}} = \frac{1-a}{1+a}, \quad \text{with } a = \langle \Lambda_i \varphi, \varphi \rangle_{\Sigma^\delta} / \langle \Lambda_e \varphi, \varphi \rangle_{\Sigma^\delta} \geq 0,$$

we deduce that there holds  $\alpha_k^\pm \in [-1; 1]$  for all  $k \in \mathbb{N}^* := \{1, 2, \dots\}$ . Taking  $\varphi = \varphi_D^k|_{\Sigma^\delta}$  with  $\varphi_D^k$  defined in (11), we find  $a = 0$  and so  $\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta} / \langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta} = 1$ . This allows one to prove that  $\alpha_1^+ = \dots = \alpha_{\text{card}(K^\delta)}^+ = 1$ . Now, if  $\alpha_{\text{card}(K^\delta)+1}^+ = 1$ , then there is  $\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}$  such that  $\langle \Lambda_i \varphi, \varphi \rangle_{\Sigma^\delta} = 0$  and  $\mathcal{R}\varphi \in \hat{\mathcal{H}}_D^\delta \setminus \{0\}$ . This is impossible and therefore there holds  $\alpha_{\text{card}(K^\delta)+1}^+ < 1$ . Similarly, if  $\alpha_1^- = -1$ , then there exists  $\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}$  such that  $\langle \Lambda_e \varphi, \varphi \rangle_{\Sigma^\delta} = 0$ . This can not happen, which implies that  $\alpha_1^- > -1$ .  $\square$

From the discussion preceding the statement of Theorem 3.2, we deduce the following result.

**Theorem 3.3.** *For  $\kappa_\varepsilon \in (-\infty; 0) \setminus \{-1\}$ , the operator  $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is an isomorphism if and only if*

$$\kappa_\varepsilon \notin \left\{ \frac{\alpha_k^- + 1}{\alpha_k^- - 1}, k \geq 1 \right\} \cup \left\{ \frac{\alpha_k^+ + 1}{\alpha_k^+ - 1}, k \geq \text{card}(K^\delta) + 1 \right\},$$

where the  $\alpha_k^\pm$  are defined in (52)–(54).

Observing that the map  $\alpha \mapsto (\alpha + 1)/(\alpha - 1)$  is decreasing on  $(-1; 1)$ , we deduce in particular from Theorem 3.3 that  $A_\varepsilon^\delta$  is an isomorphism for

$$\kappa_\varepsilon \in \left( -\infty; \frac{\alpha_1^- + 1}{\alpha_1^- - 1} \right) \cup \left( \frac{\alpha_{\text{card}(K^\delta)+1}^+ + 1}{\alpha_{\text{card}(K^\delta)+1}^+ - 1}; 0 \right). \quad (55)$$

But one can verify that we have

$$\frac{\alpha_1^- + 1}{\alpha_1^- - 1} = -1/M_D^\delta \quad \text{and} \quad \frac{\alpha_{\text{card}(K^\delta)+1}^+ + 1}{\alpha_{\text{card}(K^\delta)+1}^+ - 1} = -1/m_D^\delta$$

where  $m_D^\delta, M_D^\delta$  are the constants defined in (14). As a consequence, the invertibility condition for  $A_\varepsilon^\delta$  obtained in Proposition 3.1 is the same as (55). This shows that the result of Proposition 3.1 is optimal in a certain sense. This is the first remark of this section.

In previous articles (see in particular [12] and [18]), authors have worked with the operator  $\mathbf{T} : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that

$$\mathbf{T}\varphi = \begin{cases} \varphi & \text{in } \Omega_e^\delta \\ -\varphi + 2P\varphi & \text{in } \Omega_i^\delta \end{cases} \quad (56)$$

where, setting  $H_{0,\partial\Omega}^1(\Omega_e^\delta) := \{\varphi|_{\Omega_e^\delta}, \varphi \in H_0^1(\Omega)\}$ ,  $P : H_{0,\partial\Omega}^1(\Omega_e^\delta) \rightarrow H^1(\Omega_i^\delta)$  denotes the harmonic extension operator, i.e. the operator such that  $P\varphi$  solves the problem

$$\begin{cases} \Delta(P\varphi) = 0 & \text{in } \Omega_i^\delta \\ P\varphi = \varphi & \text{on } \partial\Omega_i^\delta. \end{cases} \quad (57)$$

We have  $\mathbf{T} \circ \mathbf{T} = \text{Id}$  which shows that  $\mathbf{T}$  is an isomorphism of  $H_0^1(\Omega)$ . On the other hand, for all  $\varphi \in H_0^1(\Omega)$ , we find

$$(\nabla(A_\varepsilon^\delta(\mathbf{T}\varphi)), \nabla\varphi) = \varepsilon_e \|\nabla\varphi\|_{\Omega_e^\delta}^2 + |\varepsilon_i| \|\nabla\varphi\|_{\Omega_i^\delta}^2 + 2\varepsilon_i (\nabla(P\varphi), \nabla\varphi)_{\Omega_i^\delta}. \quad (58)$$

Set

$$\tilde{M}_D^\delta := \sup_{\varphi \in H_{0,\partial\Omega}^1(\Omega_e^\delta) \setminus \{0\}} \frac{\|\nabla(P\varphi)\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}. \quad (59)$$

Using Young's inequality, from (58) we infer that for all  $\tau > 0$ , there holds

$$|(\nabla(A_\varepsilon^\delta(\mathbf{T}\varphi)), \nabla\varphi)| \geq (\varepsilon_e - \tau^{-1}|\varepsilon_i|\tilde{M}_D^\delta) \|\nabla\varphi\|_{\Omega_e^\delta}^2 + |\varepsilon_i|(1 - \tau) \|\nabla\varphi\|_{\Omega_i^\delta}^2. \quad (60)$$

As a consequence, we deduce that when  $\varepsilon_e > |\varepsilon_i|\tilde{M}_D^\delta \Leftrightarrow \kappa_\varepsilon = \varepsilon_i/\varepsilon_e > -1/\tilde{M}_D^\delta$ , the operator  $A_\varepsilon^\delta$  is an isomorphism of  $H_0^1(\Omega)$ . Let us compare this operator  $\mathbf{T}$  introduced in (56) with the  $\mathbf{T}_D^+$  defined in (19). Clearly in  $\Omega_e^\delta$ , we have  $\mathbf{T}\varphi = \mathbf{T}_D^+\varphi$ . In  $\Omega_i^\delta$ , for  $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$  with  $\hat{\varphi}_h \in \hat{\mathcal{H}}_D^\delta$ ,  $\Phi_h \in \text{span}_{k \in K^\delta} \{\varphi_D^k\}$  and  $\tilde{\varphi} \in H_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$ , we have

$$\mathbf{T}_D^+\varphi = \hat{\varphi}_h + \Phi_h - \tilde{\varphi}.$$

But one observes that

$$P\varphi = P(\hat{\varphi}_h + \Phi_h + \tilde{\varphi}) = P(\hat{\varphi}_h + \Phi_h) = \hat{\varphi}_h + \Phi_h.$$

Therefore, we have  $-\varphi + 2P\varphi = \hat{\varphi}_h + \Phi_h - \tilde{\varphi} = \mathbf{T}_D^+\varphi$  in  $\Omega_i^\delta$  which shows that the operator  $\mathbf{T}$  defined in (56) coincides with  $\mathbf{T}_D^+$ . Moreover, using Lemma 3.2, it is an exercise to prove that  $\tilde{M}_D^\delta$  is equal to the constant  $M_D^\delta$  defined in (14). Therefore, the simple operator  $\mathbf{T}$  in (56) is already very efficient. This is the second remark of this section.



Finally, we explain how to construct an operator of T-coercivity for contrasts  $\kappa_\varepsilon$  as in the statement of Theorem 3.3, in particular for contrasts in  $(-1/m_D^\delta; -1/M_D^\delta) \setminus \{-1\}$ , this case being not covered by Proposition 3.1. First, we reindex the eigenvalues  $\{\alpha_n^-\}_{n \geq 1}$ ,  $\{\alpha_n^+\}_{n \geq \text{card}(K^\delta)+1}$  and denote them  $\{\alpha_n\}_{n \geq 1}$ . Let  $(\varphi_n)$  be a family of eigenfunctions of the operator  $K$  introduced in (50) associated with the eigenvalues  $\alpha_n$ . We choose them so that the functions  $\mathcal{R}\varphi_n$ ,  $n \geq 1$ , form an orthonormal basis of  $\hat{\mathcal{H}}_D^\delta$ . Now we define the operator  $T_D : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that for  $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$  with  $\hat{\varphi}_h = \sum_{n \in \mathbb{N}^*} \gamma_n \mathcal{R}\varphi_n \in \hat{\mathcal{H}}_D^\delta$ ,  $\Phi_h \in \text{span}_{k \in K^\delta} \{\varphi_D^k\}$  and  $\tilde{\varphi} \in H_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$ , there holds

$$T_D \varphi = \begin{cases} \sum_{n \in \mathbb{N}^*} t_n \gamma_n \mathcal{R}\varphi_n + \Phi_h + \tilde{\varphi} & \text{in } \Omega_e^\delta \\ \sum_{n \in \mathbb{N}^*} t_n \gamma_n \mathcal{R}\varphi_n + \Phi_h - \tilde{\varphi} & \text{in } \Omega_i^\delta. \end{cases} \quad (61)$$

Here we take  $t_n = 1$  for  $n$  such that  $\kappa_\varepsilon > \kappa_n := (\alpha_n + 1)/(\alpha_n - 1)$  and  $t_n = -1$  otherwise. The operator  $T_D$  is valued in  $H_0^1(\Omega)$  and we have  $T_D \circ T_D = \text{Id}$  which guarantees that  $T_D$  is an isomorphism of  $H_0^1(\Omega)$ .

**Proposition 3.4.** *Assume that  $\kappa_\varepsilon \neq -1$  is such that for all  $n \in \mathbb{N}^*$ , we have  $\kappa_\varepsilon \neq \kappa_n$  with*

$$\kappa_n = \frac{\alpha_n + 1}{\alpha_n - 1}. \quad (62)$$

Let  $T_D : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  denote the isomorphism defined in (61). Then  $(\nabla(A_\varepsilon^\delta(T_D \cdot)), \nabla \cdot)$  is coercive in  $H_0^1(\Omega)$ . As a consequence,  $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is an isomorphism.

*Proof.* For all  $\varphi \in H_0^1(\Omega)$ , we find

$$(\varepsilon \nabla \varphi, \nabla(T_D \varphi)) = \sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\varepsilon \nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n)) + (\varepsilon_\varepsilon \nabla \Phi_h, \nabla \Phi_h)_{\Omega_e^\delta} + (|\varepsilon| \nabla \tilde{\varphi}, \nabla \tilde{\varphi}). \quad (63)$$

But by the definition of the  $\kappa_n$ , we have, for all  $n \in \mathbb{N}^*$ ,

$$(\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_e^\delta} = -\kappa_n (\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_i^\delta}.$$

This allows us to write

$$\begin{aligned} \sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\varepsilon \nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n)) &= \varepsilon_\varepsilon \sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\kappa_\varepsilon - \kappa_n) (\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_i^\delta} \\ &= \varepsilon_\varepsilon \sum_{n \in \mathbb{N}^*} |\gamma_n|^2 |\kappa_\varepsilon - \kappa_n| (\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_i^\delta}. \end{aligned} \quad (64)$$

Observing that we have  $\|\nabla(\mathcal{R}\varphi_n)\|_{\Omega_i^\delta}^2 \geq \inf_{m \in \mathbb{N}^*} |\kappa_m|^{-1} \|\nabla(\mathcal{R}\varphi_n)\|_{\Omega_e^\delta}^2$  (note that the sequence  $(|\kappa_m|)$  is bounded), from (64) we obtain

$$\sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\varepsilon \nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n)) \geq C \inf_{n \in \mathbb{N}^*} |\kappa_\varepsilon - \kappa_n| \sum_{n \in \mathbb{N}^*} |\gamma_n|^2 \|\nabla(\mathcal{R}\varphi_n)\|^2. \quad (65)$$

Using (65) into (63), we get  $(\varepsilon \nabla \varphi, \nabla(T_D \varphi)) \geq C \inf_{n \in \mathbb{N}^*} |\kappa_\varepsilon - \kappa_n| \|\nabla \varphi\|^2$  for all  $\varphi \in H_0^1(\Omega)$ .  $\square$

**Remark 3.4.** *In the following, we will not work with the operator  $T_D$  defined in (61) to investigate what happens for contrasts in  $(-1/m; -1/M) \setminus \{-1\}$ . The reason is that the value of the  $\kappa_n$  defined in (62) depends on  $\delta$  and the operator  $T_D$  is useful to prove a result of uniform invertibility of  $A_\varepsilon^\delta$  only if we know that there is a segment of  $(-1/m; -1/M) \setminus \{-1\}$  of non empty interior which is uniformly free of the  $\kappa_n$  as  $\delta$  tends to zero. And it is an open question to find conditions on the geometry such that this occurs.*

## 4 Analysis of the cell problem and properties of the homogenized tensors

In this section, we study a scalar problem set in the reference cell (supplemented with periodic boundary conditions) and the associated homogenized tensor. These quantities, which appear in the homogenization of Maxwell's equations considered in Section 5, are the same as the ones in [18] and [14], so that the results below complement and improve those obtained therein.

### 4.1 Cell problem

Denote  $\mathcal{C}_{\text{per}}^\infty(\bar{Y})$  the subset of functions of  $\mathcal{C}^\infty(\bar{Y})$  satisfying periodic boundary conditions on  $\partial Y$ . Let  $\mathbf{H}_{\text{per}}^1(Y)$  be the closure of  $\mathcal{C}_{\text{per}}^\infty(\bar{Y})$  for the norm of  $\mathbf{H}^1(Y)$ . Then set

$$\mathbf{H}_{\text{per},\diamond}^1(Y) := \{\varphi \in \mathbf{H}_{\text{per}}^1(Y) \mid \int_{\partial Y_i} \varphi \, d\sigma = 0\}.$$

We endow this space with the inner product  $(\nabla \cdot, \nabla \cdot)_Y$ . For  $\eta$  equal to  $\varepsilon$  or  $\mu$  as defined in (2), the problem we are interested in writes

$$\left| \begin{array}{l} \text{Find } \varphi \in \mathbf{H}_{\text{per},\diamond}^1(Y) \text{ such that:} \\ (\eta \nabla \varphi, \nabla \varphi')_Y = \ell(\varphi'), \quad \forall \varphi' \in \mathbf{H}_{\text{per},\diamond}^1(Y), \end{array} \right. \quad (66)$$

where  $\ell$  is a continuous linear functional on  $\mathbf{H}_{\text{per},\diamond}^1(Y)$ . In order to study this problem, we introduce the closed subspace of  $\mathbf{H}_{\text{per},\diamond}^1(Y)$

$$\mathbf{H}_{\text{per},0,\partial Y_i}^1(Y) := \{\varphi \in \mathbf{H}_{\text{per},\diamond}^1(Y) \mid \varphi = 0 \text{ on } \partial Y_i\}.$$

Then we define the space  $\hat{\mathcal{H}}_b$  such that

$$\mathbf{H}_{\text{per},\diamond}^1(Y) = \hat{\mathcal{H}}_b \oplus \mathbf{H}_{\text{per},0,\partial Y_i}^1(Y). \quad (67)$$

We will not look for an exact characterization of  $\hat{\mathcal{H}}_b$ . Let us simply remark that if  $\varphi \in \hat{\mathcal{H}}_b$ , then for all  $\zeta \in \mathcal{C}_0^\infty(Y_e \cup Y_i) \subset \mathbf{H}_{\text{per},0,\partial Y_i}^1(Y)$ , we have  $0 = (\nabla \varphi, \nabla \zeta)_Y$ . This implies that the elements of  $\hat{\mathcal{H}}_b$  are harmonic in  $Y_e \cup Y_i$ . Then we introduce the constants

$$m_b := \inf_{\varphi \in \hat{\mathcal{H}}_b \setminus \{0\}} \frac{\|\nabla \varphi\|_{Y_i}^2}{\|\nabla \varphi\|_{Y_e}^2}, \quad M_b := \sup_{\varphi \in \hat{\mathcal{H}}_b \setminus \{0\}} \frac{\|\nabla \varphi\|_{Y_i}^2}{\|\nabla \varphi\|_{Y_e}^2}. \quad (68)$$

**Theorem 4.1.** *Assume that  $\kappa_\varepsilon$  (resp.  $\kappa_\mu$ )  $\in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (39). Then the problem (66) with  $\eta = \varepsilon$  (resp.  $\eta = \mu$ ) admits a unique solution which depends continuously on  $\ell$ .*

*Proof.* To set ideas, we take  $\eta = \varepsilon$ , the proof is the same for  $\eta = \mu$ . With the Riesz representation theorem, define the operator  $D_\varepsilon : \mathbf{H}_{\text{per},\diamond}^1(Y) \rightarrow \mathbf{H}_{\text{per},\diamond}^1(Y)$  such that

$$(\nabla(D_\varepsilon \varphi), \nabla \varphi')_Y = (\varepsilon \nabla \varphi, \nabla \varphi')_Y, \quad \forall \varphi, \varphi' \in \mathbf{H}_{\text{per},\diamond}^1(Y). \quad (69)$$

Let us show that  $D_\varepsilon$  is an isomorphism when  $\kappa_\varepsilon = \varepsilon_i/\varepsilon_e \in (-\infty; -1/m) \cup (-1/M; 0)$ . For  $\varphi \in \mathbf{H}_{\text{per},\diamond}^1(Y)$ , consider the decomposition  $\varphi = \varphi_h + \tilde{\varphi}$  with  $\varphi_h \in \hat{\mathcal{H}}_b$  and  $\tilde{\varphi} \in \mathbf{H}_{\text{per},0,\partial Y_i}^1(Y)$ . And define the operators  $\mathbf{T}_b^\pm$  such that

$$\mathbf{T}_b^\pm \varphi = \begin{cases} \pm \varphi_h + \tilde{\varphi} & \text{in } Y_e \\ \pm \varphi_h - \tilde{\varphi} & \text{in } Y_i. \end{cases}$$

Working as in the proof of Proposition 3.1 with the operators  $\mathbf{T}^\pm$  replaced by  $\mathbf{T}_b^\pm$ , one establishes that  $D_\varepsilon$  is an isomorphism when  $\kappa_\varepsilon \in (-\infty; -1/m_b) \cup (-1/M_b; 0)$ . To obtain the desired result, it remains to show that  $m \leq m_b$  and  $M_b \leq M$ . Since  $\hat{\mathcal{H}}_b \subset \mathcal{H}_\diamond$ , from Lemma 3.5, we clearly have  $M_b \leq M$ . Now let  $\varphi$  be an element of  $\hat{\mathcal{H}}_b \setminus \{0\}$ . Denote  $\zeta \in \mathcal{H}_0$  the function such that  $\zeta = \varphi$  on  $\partial Y_i$ . The function  $\zeta$  decomposes as  $\zeta = \hat{\zeta} + \alpha \varphi_D$  with  $\hat{\zeta} \in \hat{\mathcal{H}}_0$  and  $\alpha \in \mathbb{R}$  ( $\varphi_D$  is defined before (38)). Note that  $\hat{\zeta} \neq 0$  otherwise we would have  $\alpha = 0$  (because  $\varphi_D = 1$  on  $\partial Y_i$  and  $\int_{\partial Y_i} \zeta \, d\sigma = 0$ ) and so  $\zeta \equiv 0$ . Observing that  $\varphi - \hat{\zeta} - \alpha$  is in  $\mathbf{H}_{\text{per},0,\partial Y_i}^1(Y)$ , due to the decomposition (67), we can write

$$(\nabla \varphi, \nabla(\varphi - \hat{\zeta}))_Y = (\nabla \varphi, \nabla(\varphi - \hat{\zeta} - \alpha))_Y = 0. \quad (70)$$

But on the other hand, since we have  $\nabla \varphi = \nabla \hat{\zeta} = \nabla \zeta$  in  $Y_i$ , so that in particular there holds

$$\|\nabla \varphi\|_{Y_i}^2 = \|\nabla \hat{\zeta}\|_{Y_i}^2, \quad (71)$$

we infer from (70) that

$$(\nabla \varphi, \nabla(\varphi - \hat{\zeta}))_{Y_e} = 0. \quad (72)$$

This implies

$$\|\nabla \varphi\|_{Y_e}^2 \leq \|\nabla \hat{\zeta}\|_{Y_e}^2. \quad (73)$$

Gathering (71) and (73), we deduce that

$$m \leq \frac{\|\nabla \hat{\zeta}\|_{Y_i}^2}{\|\nabla \hat{\zeta}\|_{Y_e}^2} \leq \frac{\|\nabla \varphi\|_{Y_i}^2}{\|\nabla \varphi\|_{Y_e}^2}. \quad (74)$$

Taking the infimum over all  $\varphi \in \hat{\mathcal{H}}_b \setminus \{0\}$  in (74), we obtain that  $m \leq m_b$ .  $\square$

## 4.2 Homogenized tensors

Assume that the contrasts  $\kappa_\varepsilon$  and  $\kappa_\mu$  are located in  $(-\infty; -1/m) \cup (-1/M; 0)$ . For  $\eta = \varepsilon$  or  $\mu$  and  $j = 1, 2, 3$ , we define the function  $\chi_j^\eta \in \mathbf{H}_{\text{per},\diamond}^1(Y)$  such that

$$(\eta \nabla \chi_j^\eta, \nabla \xi)_Y = (\eta \nabla y_j, \nabla \xi)_Y, \quad \forall \xi \in \mathbf{H}_{\text{per},\diamond}^1(Y). \quad (75)$$

Note that the right hand side of (75) simply writes

$$(\eta \nabla y_j, \nabla \xi)_Y = \int_Y \eta \frac{\partial \xi}{\partial y_j} \, dy$$

and that Theorem 4.1 ensures that the functions  $\chi_j^\eta$  are well-defined. It is also worth noticing that by setting  $\boldsymbol{\chi}^\eta := (\chi_1^\eta, \chi_2^\eta, \chi_3^\eta)^\top$ , we have for all  $\boldsymbol{\lambda} \in \mathbb{R}^3$ :

$$(\eta \nabla(\boldsymbol{\lambda} \cdot \boldsymbol{\chi}^\eta), \nabla \xi)_Y = (\eta \nabla(\boldsymbol{\lambda} \cdot \mathbf{y}), \nabla \xi)_Y = \int_Y \eta \boldsymbol{\lambda} \cdot \nabla \xi \, dy, \quad \forall \xi \in \mathbf{H}_{\text{per},\diamond}^1(Y). \quad (76)$$

Denoting by  $\nabla \boldsymbol{\chi}^\eta$  the jacobian matrix of  $\boldsymbol{\chi}^\eta$ :

$$\nabla \boldsymbol{\chi}^\eta = \left( \frac{\partial \chi_j^\eta}{\partial y_k} \right)_{1 \leq j, k \leq 3},$$

the homogenized tensor associated with  $\eta$  is classically defined as the  $3 \times 3$  symmetric matrix  $\mathcal{H}(\eta) = (\mathcal{H}_{jk}(\eta))_{1 \leq j, k \leq 3}$  given by (see, for instance, identity (6.35) in [22])

$$\mathcal{H}(\eta) = \frac{1}{|Y|} \int_Y \eta(y) [\text{Id} - (\nabla \boldsymbol{\chi}^\eta)^\top] \, dy, \quad (77)$$

or equivalently (see (6.37) in [22]):

$$\mathcal{H}_{jk}(\eta) = \frac{1}{|Y|} \int_Y \eta \nabla(y_j - \chi_j^\eta) \cdot \nabla(y_k - \chi_k^\eta) \, dy. \quad (78)$$

**Proposition 4.1.** *Assume that  $\kappa_\varepsilon$  (resp.  $\kappa_\mu$ )  $\in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (39). Then the matrix  $\mathcal{H}(\varepsilon)$  (resp.  $\mathcal{H}(\mu)$ ) is definite positive.*

*Proof.* The proofs for  $\mathcal{H}(\varepsilon)$  and  $\mathcal{H}(\mu)$  are the same and to set ideas, we choose to work with  $\varepsilon$ . According to formula (6.44) in [22], for all  $\xi = (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{R}^3$ , we have

$$\mathcal{H}(\varepsilon) \xi \cdot \xi = \int_Y \varepsilon |\nabla \varphi_\xi|^2 dy$$

where the function  $\varphi_\xi$  is defined by

$$\varphi_\xi(y) = \sum_{j=1}^3 \xi_j (y_j - \chi_j^\varepsilon(y)).$$

Note that if  $\varphi_\xi$  is constant in  $Y$ , then evaluating  $\varphi_\xi$  on  $\partial Y$  and using the fact that the functions  $\chi_j^\varepsilon$  satisfy periodic boundary conditions, we find that  $\xi = 0$  and so  $\varphi_\xi \equiv 0$ . Now, we assume that  $\xi \neq 0$ . Subtracting the mean value of the test functions on  $\partial Y_i$ , we see from (75) that  $\chi_j^\varepsilon$  satisfy the slightly more general variational equality (the variational space is not the same as in (75))

$$(\varepsilon \nabla \chi_j^\varepsilon, \nabla \varphi')_Y = (\varepsilon \nabla y_j, \nabla \varphi')_Y, \quad \forall \varphi' \in H_{\text{per}}^1(Y).$$

Taking  $\varphi' \in \mathcal{C}_0^\infty(Y)$ , this implies that we have

$$\operatorname{div}(\varepsilon \nabla \varphi_\xi) = 0 \quad \text{in } Y. \quad (79)$$

i) Introduce the function  $\hat{\varphi}_\xi$  such that

$$\hat{\varphi}_\xi = \varphi_\xi - \frac{1}{|\partial Y_i|} \int_{\partial Y_i} \varphi_\xi d\sigma \in H_\diamond^1(Y).$$

From (79), we deduce that  $\hat{\varphi}_\xi$  is harmonic in  $Y_e \cup Y_i$ . Therefore, we have  $\hat{\varphi}_\xi \in \mathcal{H}_\diamond$  and from Lemma 3.5, we can write

$$\|\nabla \varphi_\xi\|_{Y_i}^2 = \|\nabla \hat{\varphi}_\xi\|_{Y_i}^2 \leq M \|\nabla \hat{\varphi}_\xi\|_{Y_e}^2 = M \|\nabla \varphi_\xi\|_{Y_e}^2.$$

This allows us to write

$$\mathcal{H}(\varepsilon) \xi \cdot \xi = \varepsilon_e \|\nabla \varphi_\xi\|_{Y_e}^2 - |\varepsilon_i| \|\nabla \varphi_\xi\|_{Y_i}^2 \geq (\varepsilon_e - |\varepsilon_i| M) \|\nabla \varphi_\xi\|_{Y_e}^2.$$

And when  $\varepsilon_e > |\varepsilon_i| M \Leftrightarrow \kappa_\varepsilon = \varepsilon_i/\varepsilon_e > -1/M$ , we deduce that the matrix  $\mathcal{H}(\varepsilon)$  is definite-positive. Note that we have  $\nabla \varphi_\xi \neq 0$  in  $Y_e$  otherwise we would have  $\nabla \varphi_\xi = 0$  in  $Y$  (because  $\varphi_\xi \in H^1(Y)$  is harmonic in  $Y_e$ ) which is impossible when  $\xi \neq 0$  (see the discussion above).

ii) Now, we consider the case  $\kappa_\varepsilon \in (-\infty; -1/m)$ . The proof is a bit less straightforward and we divide it in two steps. Define the quadratic form  $q(\cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}$  such that

$$q(\xi) = \mathcal{H}(\varepsilon) \xi \cdot \xi.$$

*Step 1.* First, assume that  $\xi \in \mathbb{R}^3 \setminus \{0\}$  is such that

$$q(\xi) \leq 0 \quad \Leftrightarrow \quad \int_Y \varepsilon |\nabla \varphi_\xi|^2 dy \leq 0.$$

Then from identity (79), we infer that we must have

$$\int_{\partial Y} \varepsilon_e \frac{\partial \varphi_\xi}{\partial n} \varphi_\xi d\sigma \leq 0. \quad (80)$$

Now, introduce  $\zeta \in \mathcal{H}_0$  the function such that  $\zeta = \varphi_\xi$  on  $\partial Y_i$ . The function  $\zeta$  decomposes as  $\zeta = \hat{\zeta} + \alpha \varphi_D$  with  $\hat{\zeta} \in \hat{\mathcal{H}}_0$  and  $\alpha \in \mathbb{R}$  ( $\varphi_D$  is defined before (38)). Observe that we have  $\hat{\zeta} \not\equiv 0$  otherwise  $\varphi_\xi$  would be constant in  $Y_e$  and so in  $Y$  by (79). According to the discussion above, this is impossible when  $\xi \neq 0$ . Observing that  $\varphi_\xi - (\hat{\zeta} + \alpha) = 0$  on  $\partial Y_i$ , integrating by parts, we can write

$$\begin{aligned} (\nabla \varphi_\xi, \nabla(\varphi_\xi - \hat{\zeta}))_{Y_e} &= (\nabla \varphi_\xi, \nabla(\varphi_\xi - (\hat{\zeta} + \alpha)))_{Y_e} = \int_{\partial Y} \frac{\partial \varphi_\xi}{\partial n} \varphi_\xi \, d\sigma + \int_{\partial Y} \frac{\partial \varphi_\xi}{\partial n} \alpha \, d\sigma \\ &= \int_{\partial Y} \frac{\partial \varphi_\xi}{\partial n} \varphi_\xi \, d\sigma \leq 0. \end{aligned} \quad (81)$$

The last equality above has been obtained using (80) and identity (79) multiplied by  $\alpha$ . From (81) and the Cauchy-Schwarz inequality, we infer that

$$\|\nabla \varphi_\xi\|_{Y_e}^2 \leq \|\nabla \hat{\zeta}\|_{Y_e}^2.$$

Since on the other hand there holds  $\nabla \varphi_\xi = \nabla \hat{\zeta}$  in  $Y_i$  so that  $\|\nabla \varphi_\xi\|_{Y_i}^2 = \|\nabla \hat{\zeta}\|_{Y_i}^2$ , we deduce that

$$m \leq \frac{\|\nabla \hat{\zeta}\|_{Y_i}^2}{\|\nabla \hat{\zeta}\|_{Y_e}^2} \leq \frac{\|\nabla \varphi_\xi\|_{Y_i}^2}{\|\nabla \varphi_\xi\|_{Y_e}^2}. \quad (82)$$

But then, when  $\kappa_\varepsilon = \varepsilon_i/\varepsilon_e < -m^{-1} \Leftrightarrow \varepsilon_e < |\varepsilon_i|m$ , we can write

$$q(\xi) = \mathcal{H}(\varepsilon) \xi \cdot \xi = \int_Y \varepsilon |\nabla \varphi_\xi|^2 \, dy = \varepsilon_e \|\nabla \varphi_\xi\|_{Y_e}^2 - |\varepsilon_i| \|\nabla \varphi_\xi\|_{Y_i}^2 \leq (\varepsilon_e - |\varepsilon_i|m) \|\nabla \varphi_\xi\|_{Y_e}^2 < 0.$$

This shows in particular that if  $q(\xi) = 0$ , then necessarily  $\xi = 0$ . In other words, the quadratic form  $q(\cdot)$  is definite in  $\mathbb{R}^3$ . From classical results concerning quadratic forms, therefore we know that for each  $\kappa_\varepsilon \in (-\infty; -1/m)$ ,  $q(\cdot)$  is either positive definite or negative definite.

*Step 2.* Now consider some  $\xi \in \mathbb{R}^3 \setminus \{0\}$ . Corollary 5.6 of [14] guarantees that  $q(\xi)$  is positive definite for  $\kappa_\varepsilon$  small enough. Using the fact that  $\kappa_\varepsilon \mapsto q(\xi)$  is continuous and that  $q(\cdot)$  is always definite for  $\kappa_\varepsilon \in (-\infty; -1/m)$ , we infer that  $q(\cdot)$  is positive definite for all  $\kappa_\varepsilon \in (-\infty; -1/m)$ . Let us present an alternative proof which is a bit more direct than the Corollary 5.6 of [14]. For  $\xi \in \mathbb{R}^3 \setminus \{0\}$ , assume that we have  $q(\xi) < 0$ . Then estimate (82) is valid for all  $\kappa_\varepsilon \in (-\infty; -1/m)$ . Define the function

$$\check{\varphi}_\xi = \varphi_\xi - \frac{1}{|\partial Y|} \int_{\partial Y} \varphi_\xi \, d\sigma.$$

From (79), we can write

$$\begin{aligned} |\varepsilon_i| \|\nabla \varphi_\xi\|_{Y_i}^2 &= |\varepsilon_i| \|\nabla \check{\varphi}_\xi\|_{Y_i}^2 = \varepsilon_e \|\nabla \check{\varphi}_\xi\|_{Y_e}^2 - \int_{\partial Y} \varepsilon_e \frac{\partial \check{\varphi}_\xi}{\partial n} \check{\varphi}_\xi \, d\sigma \\ &\leq C \varepsilon_e \|\nabla \check{\varphi}_\xi\|_{Y_e}^2 = C \varepsilon_e \|\nabla \varphi_\xi\|_{Y_e}^2. \end{aligned} \quad (83)$$

The last inequality in (83) is a consequence of the continuity of the mappings  $\varphi \mapsto \varphi|_{\partial Y}$  and  $\varphi \mapsto \partial_n \varphi|_{\partial Y}$  from  $\{\varphi \in H^1(Y_e) \mid \Delta \varphi = 0 \text{ in } Y_e\}$  to  $H^{1/2}(\partial Y)$  and  $H^{-1/2}(\partial Y)$  respectively. Note that since the mean of  $\check{\varphi}_\xi$  over  $\partial Y$  is null, a classical Poincaré type inequality allows one to prove that the  $H^1$  norm of  $\check{\varphi}_\xi$  in  $Y_e$  is controlled by  $\|\nabla \check{\varphi}_\xi\|_{Y_e}$ . From (83), we get

$$\frac{\|\nabla \varphi_\xi\|_{Y_i}^2}{\|\nabla \varphi_\xi\|_{Y_e}^2} \leq \frac{C}{|\kappa_\varepsilon|} \quad (84)$$

where  $C > 0$  is independent of  $\kappa_\varepsilon$ . Taking the limit  $\kappa_\varepsilon \rightarrow -\infty$  in (84), we obtain a contradiction with (82) because  $m > 0$  is independent of  $\kappa_\varepsilon$ . Therefore we must have  $q(\xi) > 0$  for contrasts small enough. Then arguing as above, we infer that  $q(\cdot)$  is positive definite for all  $\kappa_\varepsilon \in (-\infty; -1/m)$ .  $\square$

### 4.3 Numerical illustrations

Proposition 4.1 guarantees that if  $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ , the matrices  $\mathcal{H}(\varepsilon), \mathcal{H}(\mu)$  are definite positive. This may seem a bit surprising and when one looks at the definition in (78), this is far from being obvious. The goal of this paragraph is to present some numerics to illustrate this property. To set ideas we compute  $\mathcal{H}(\varepsilon)$  and to simplify we work in 2D. In this case,  $\mathcal{H}(\varepsilon)$  is a  $2 \times 2$  symmetric matrix. We do not expect particular differences between 2D and 3D settings. Numerically, we approximate the solutions of the problems (75) using a P2 finite element method. To proceed, we use the library *FreeFem++*<sup>1</sup>. And then we compute the matrix  $\mathcal{H}(\varepsilon)$  using the formula (78). Admittedly the numerical analysis of problems (75) is not standard because of the sign-changing  $\varepsilon$ . However in general, at least for contrasts  $\kappa_\varepsilon$  “not too close” to  $-1$  when  $\partial Y$  is smooth, we obtain a reasonable numerical solution. We refer the reader to [34, 20, 10] for more details concerning these aspects. In Figures 2 and 3 below, we display the two real eigenvalues of  $\mathcal{H}(\varepsilon)$  with respect to the contrast  $\kappa_\varepsilon \in (-10; 0)$  (we take  $\varepsilon_i = -1$  and  $\varepsilon_e$  varies) for two different geometries of  $Y_i$ . For the numerics of Figure 2, the inclusion  $Y_i$  is an ellipse while for Figure 3, it is a rectangle. We emphasize that in the latter case, problem (75) is not well-posed in the Fredholm sense for  $\kappa_\varepsilon \in (-3, -1/3)$  (see [11, 9]). As a consequence, for this range of contrasts, our numerical solutions have no sense. But for both settings, we observe that for contrasts large enough or small enough, the matrix  $\mathcal{H}(\varepsilon)$  is definite positive as expected. Interestingly, at least in the case of the ellipse where we know that the numerical solution is meaningful except for  $\kappa_\varepsilon \neq -1$ , we also note that  $\mathcal{H}(\varepsilon)$  is not positive definite for all contrasts. We emphasize however that we do not investigate these regimes in our analysis below.

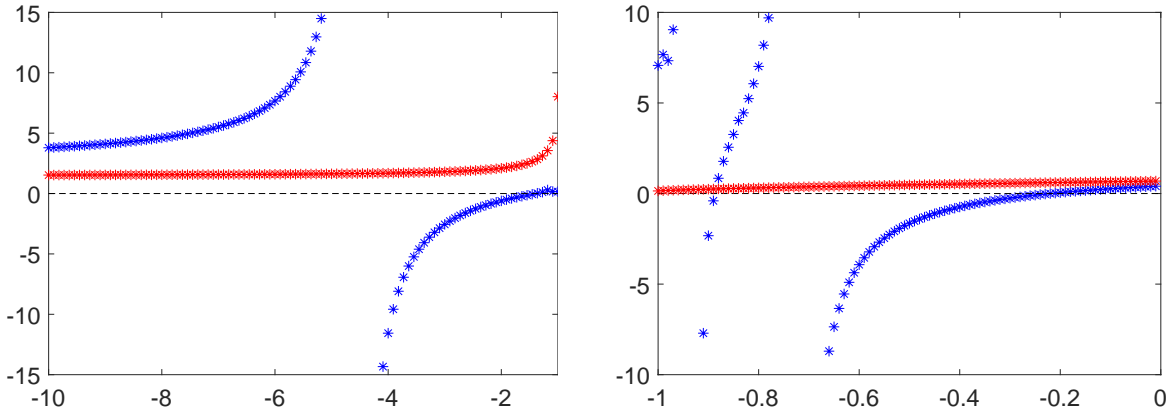


Figure 2: Representation of the two eigenvalues of  $\mathcal{H}(\varepsilon)$  with respect to  $\kappa_\varepsilon$  varying in  $(-10; -1)$  (left) and  $(-1; 0)$  (right). Here the inclusion  $Y_i$  coincides with the interior of the ellipse  $\{(x = 0.5 + 0.4 \cos \theta, y = 0.5 + 0.2 \cos \theta), \theta \in [0; 2\pi)\}$ .

## 5 Homogenization of Maxwell’s equations

We come back to Maxwell’s problem ( $\mathcal{P}^\delta$ ) for the electric field (see (9)). We define the bilinear form  $a_\omega^\delta(\cdot, \cdot)$  associated with (9) such that

$$a_\omega^\delta(\mathbf{E}, \mathbf{E}') = ((\mu^\delta)^{-1} \mathbf{curl} \mathbf{E}, \mathbf{curl} \mathbf{E}') - \omega^2 (\varepsilon^\delta \mathbf{E}, \mathbf{E}'), \quad \forall \mathbf{E}, \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}).$$

Let  $m, M$  be the constants defined in (39). When  $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ , the matrices  $\mathcal{H}(\varepsilon)$  and  $\mathcal{H}(\mu)$  are well-defined according to Theorem 4.1. Moreover, according to Proposition 4.1, these matrices are positive definite. Hence, we can introduce the homogenized problem

$$(\mathcal{P}^{\text{eff}}) \quad \left| \begin{array}{l} \text{Find } \mathbf{E}^{\text{eff}} \in \mathbf{H}_N(\mathbf{curl}) \text{ such that} \\ \mathbf{curl} (\mathcal{H}(\mu)^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}}) - \omega^2 \mathcal{H}(\varepsilon) \mathbf{E}^{\text{eff}} = i\omega \mathbf{J} \end{array} \right. \quad (85)$$

<sup>1</sup>*FreeFem++*, <http://www.freefem.org/ff++/>.

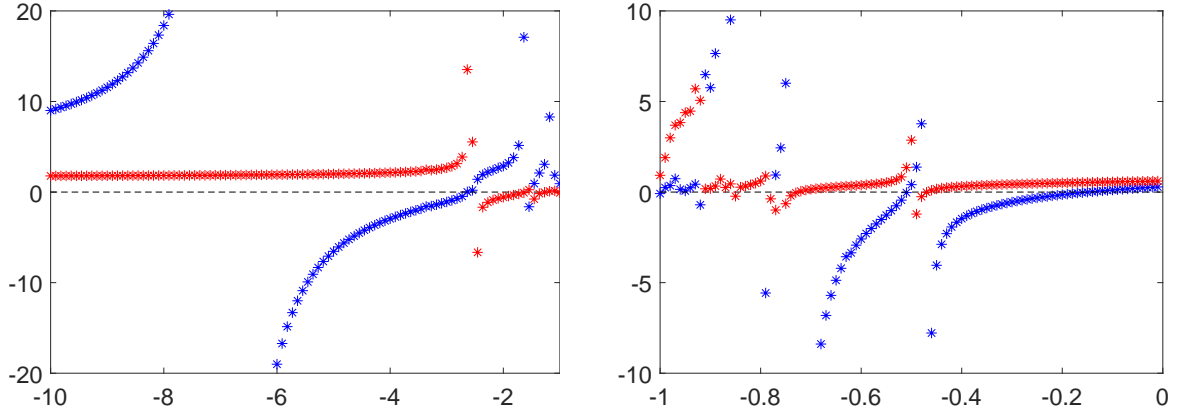


Figure 3: Representation of the two eigenvalues of  $\mathcal{H}(\varepsilon)$  with respect to  $\kappa_\varepsilon$  varying in  $(-10; -1)$  (left) and  $(-1; 0)$  (right). Here the inclusion  $Y_i$  coincides with the rectangle  $(0.1; 0.9) \times (0.3; 0.7)$ .

whose variational formulation writes

$$\begin{cases} \text{Find } \mathbf{E}^{\text{eff}} \in \mathbf{H}_N(\mathbf{curl}) \text{ such that for all } \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}) \\ a_\omega^{\text{eff}}(\mathbf{E}^{\text{eff}}, \mathbf{E}') = i\omega(\mathbf{J}, \mathbf{E}'). \end{cases} \quad (86)$$

Here  $a_\omega^{\text{eff}}(\cdot, \cdot)$  is the bilinear form defined in the space  $\mathbf{H}_N(\mathbf{curl})$  such that

$$a_\omega^{\text{eff}}(\mathbf{E}, \mathbf{E}') = (\mathcal{H}(\mu)^{-1} \mathbf{curl} \mathbf{E}, \mathbf{curl} \mathbf{E}') - \omega^2(\mathcal{H}(\varepsilon) \mathbf{E}, \mathbf{E}').$$

It is worth noticing that the above homogenized problem (which has exactly the same form as the one obtained for classical (positive) Maxwell's equations) involves the homogenized tensors of the scalar problems studied in the previous sections. This fact will be used in a crucial way in the sequel to prove our homogenization result for Maxwell's system. Classically, one can easily prove that  $(\mathcal{P}^{\text{eff}})$  admits a unique solution for all

$$\omega^2 \in \mathbb{C} \setminus \Lambda^{\text{eff}} \quad (87)$$

where  $\Lambda^{\text{eff}}$  is a discrete subset of  $[0; +\infty)$ .

The proof of a homogenization result for Maxwell's equations without sign-changing coefficients is by now quite classical (see for instance [6, 48, 43, 21]). It may be achieved by using, for instance, a notion of convergence specific to periodic homogenization, namely the two-scale convergence, which was introduced by G. Nguetseng in [32] and further developed by G. Allaire [2]. Using this notion, a typical proof for such a homogenization result relies on three main ingredients. First, a uniform energy estimate is obtained for the sequence of solutions of  $(\mathcal{P}^\delta)$ . Next, one shows that this uniformly bounded sequence has a (two-scale) limit that solves a two-scale limit problem. Finally, this limit problem is decoupled, yielding the homogenized problem which is proved to be well-posed. Due to the sign-changing coefficients and the presence of the non sign-definite  $L^2$  term involving  $\omega^2$ , achieving the first step is far from being obvious. In particular, the strategy proposed for instance in [21] does not apply anymore (as the spectral decomposition available in the strongly elliptic case fails). Instead, we proceed as follows. First, we prove a homogenization result for solutions of  $(\mathcal{P}^\delta)$  under a uniform energy estimate condition. Using this result, we prove by contradiction the needed uniform energy estimate for the solutions of  $(\mathcal{P}^\delta)$ . This leads to the main result of the paper (Theorem 5.1), namely the homogenization result for Maxwell's equations with sign-changing coefficients.

## 5.1 Homogenization result under uniform energy estimate condition

Let  $\mathbf{J}$  be a given field of  $\mathbf{L}^2(\Omega)$ . The aim of this section is to obtain a homogenization result for a sequence of functions  $(\mathbf{E}^\delta)$  solving  $(\mathcal{P}^\delta)$  and satisfying the uniform energy estimate

$$\exists C > 0, \forall \delta \in (0; 1], \quad \|\mathbf{E}^\delta\|^2 + \|\mathbf{curl} \mathbf{E}^\delta\|^2 \leq C \|\mathbf{J}\|^2. \quad (88)$$

As it was already observed in [18] for the Dirichlet scalar problem, the presence of sign-changing coefficients does not affect the two-scale convergence result. However, for the sake of completeness, we give here a proof of this convergence result following [6, 48, 43] and in particular [21]. We start by recalling the definition of the two-scale convergence (see [2]). Here we set  $\mathcal{C}_{\text{per}}^\infty(\bar{Y}) := (\mathcal{C}_{\text{per}}^\infty(\bar{Y}))^3$ .

**Definition 5.1.** *A sequence  $(\mathbf{E}^\delta)$  in  $\mathbf{L}^2(\Omega)$  two-scale converges to  $\mathbf{E}^0 \in \mathbf{L}^2(\Omega \times Y)$  if we have*

$$\lim_{\delta \rightarrow 0} (\mathbf{E}^\delta, \mathbf{v}(\cdot, \cdot/\delta)) = \int_{\Omega} (\mathbf{E}^0(x, \cdot), \mathbf{v}(x, \cdot))_Y dx$$

for all  $\mathbf{v} \in \mathcal{C}_0^\infty(\Omega; \mathcal{C}_{\text{per}}^\infty(\bar{Y}))$ . Then we denote  $\mathbf{E}^\delta \xrightarrow{2s} \mathbf{E}^0$ .

The notion of two-scale convergence is interesting due to the following compactness result (see for instance [43, Proposition 2.5]). It was first obtained by N. Wellander in [46] and then by V. Tiep Chu and V.H. Hoang in [43]. Here,  $\mathbf{H}_{\text{per}}(\mathbf{curl}; Y)$  denotes the closure of  $\mathcal{C}_{\text{per}}^\infty(\bar{Y})$  for the norm  $(\|\cdot\|_Y^2 + \|\mathbf{curl} \cdot\|_Y^2)^{1/2}$ .

**Proposition 5.1.** *Let  $(\mathbf{E}^\delta)$  be a bounded sequence in  $\mathbf{H}(\mathbf{curl})$ . Then, there exist a sub-sequence, still denoted  $(\mathbf{E}^\delta)$ , and functions  $\mathbf{E}^{\text{eff}} \in \mathbf{H}(\mathbf{curl})$ ,  $\Theta \in \mathbf{L}^2(\Omega; \mathbf{H}_{\text{per}}^1(Y))$ ,  $\mathbf{E}^1 \in \mathbf{L}^2(\Omega; \mathbf{H}_{\text{per}}(\mathbf{curl}, Y))$  such that the following two-scale convergence results hold as  $\delta \rightarrow 0$ :*

$$\mathbf{E}^\delta \xrightarrow{2s} \mathbf{E}^{\text{eff}} + \nabla_y \Theta, \quad \mathbf{curl} \mathbf{E}^\delta \xrightarrow{2s} \mathbf{curl} \mathbf{E}^{\text{eff}} + \mathbf{curl}_y \mathbf{E}^1.$$

Moreover, we also have the following weak convergence results in  $\mathbf{L}^2(\Omega)$ :

$$\mathbf{E}^\delta \rightharpoonup \mathbf{E}^{\text{eff}} \quad \text{in } \mathbf{L}^2(\Omega), \quad \mathbf{curl} \mathbf{E}^\delta \rightharpoonup \mathbf{curl} \mathbf{E}^{\text{eff}} \quad \text{in } \mathbf{L}^2(\Omega).$$

We are now in position to prove the main result of this section, namely the convergence of a sequence of solutions of problem  $(\mathcal{P}^\delta)$  satisfying the energy estimate (88) to a solution of  $(\mathcal{P}^{\text{eff}})$  when  $\delta \rightarrow 0$ .

**Proposition 5.2.** *Assume that  $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (39). Let  $(\mathbf{E}^\delta)$  be a sequence of solutions of  $(\mathcal{P}^\delta)$  satisfying the uniform estimate (88). Then as  $\delta \rightarrow 0$ , we have*

$$\mathbf{E}^\delta \rightharpoonup \mathbf{E}^{\text{eff}} \quad \text{and} \quad \mathbf{curl} \mathbf{E}^\delta \rightharpoonup \mathbf{curl} \mathbf{E}^{\text{eff}} \quad \text{in } \mathbf{L}^2(\Omega)$$

where  $\mathbf{E}^{\text{eff}}$  solves the homogenized problem  $(\mathcal{P}^{\text{eff}})$ .

*Proof.* We take in  $(\mathcal{P}^\delta)$  (see (9)) a test function of the form

$$\mathbf{E}'(x) = \varphi(x) + \delta \varphi^1 \left( x, \frac{x}{\delta} \right) + \delta \nabla \psi \left( x, \frac{x}{\delta} \right),$$

with  $\varphi \in \mathcal{C}_0^\infty(\Omega)$ ,  $\varphi^1 \in \mathcal{C}_0^\infty(\Omega; \mathcal{C}_{\text{per}}^\infty(Y))$ ,  $\psi \in \mathcal{C}_0^\infty(\Omega; \mathcal{C}_{\text{per}}^\infty(Y))$ . By taking the limit as  $\delta \rightarrow 0$  thanks to Proposition 5.1, we get as in [43, Proposition 2.5] the following two-scale limit problem:

$$\begin{aligned} & \int_{\Omega \times Y} (\mu(y))^{-1} \left( \mathbf{curl} \mathbf{E}^{\text{eff}}(x) + \mathbf{curl}_y \mathbf{E}^1(x, y) \right) \cdot \left( \mathbf{curl} \varphi(x) + \mathbf{curl}_y \varphi^1(x, y) \right) dx dy \\ & - \omega^2 \int_{\Omega \times Y} \varepsilon(y) \left( \mathbf{E}^{\text{eff}}(x) + \nabla_y \Theta(x, y) \right) \cdot \left( \varphi(x) + \nabla_y \psi(x, y) \right) dx dy \\ & = i\omega \int_{\Omega} \mathbf{J} \cdot \varphi dx + i\omega \int_{\Omega \times Y} \mathbf{J}(x) \cdot \nabla_y \psi(x, y) dx dy. \end{aligned} \quad (89)$$



Since  $\psi(x, \cdot)$  is  $Y$ -periodic, the second integral of the right hand side vanishes and hence, setting

$$\mathbf{R}(x, y) := (\mu(y))^{-1} \left( \mathbf{curl} \mathbf{E}^{\text{eff}}(x) + \mathbf{curl}_y \mathbf{E}^1(x, y) \right) \quad (90)$$

and

$$\mathbf{S}(x, y) := \varepsilon(y) \left( \mathbf{E}^{\text{eff}} + \nabla_y \Theta \right), \quad (91)$$

relation (89) reads

$$\begin{aligned} & \int_{\Omega \times Y} \mathbf{R}(x, y) \cdot \left( \mathbf{curl} \varphi(x) + \mathbf{curl}_y \varphi^1(x, y) \right) dx dy \\ & - \omega^2 \int_{\Omega \times Y} \mathbf{S}(x, y) \cdot (\varphi(x) + \nabla_y \psi(x, y)) dx dy = i\omega \int_{\Omega} \mathbf{J} \cdot \varphi dx. \end{aligned} \quad (92)$$

In order to prove that  $\mathbf{E}^{\text{eff}}$  solves the homogenized problem ( $\mathcal{P}^{\text{eff}}$ ), it suffices to show that the two terms of the left hand side in the above equation can also be written as follows:

$$\int_{\Omega \times Y} \mathbf{R}(x, y) \cdot \left( \mathbf{curl} \varphi(x) + \mathbf{curl}_y \varphi^1(x, y) \right) dx dy = \int_{\Omega} (\mathcal{H}(\mu))^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}} \cdot \mathbf{curl} \varphi dx \quad (93)$$

$$\int_{\Omega \times Y} \mathbf{S}(x, y) \cdot (\varphi(x) + \nabla_y \psi(x, y)) dx dy = \int_{\Omega} \mathcal{H}(\varepsilon) \mathbf{E}^{\text{eff}} \cdot \varphi dx. \quad (94)$$

Indeed, once these two last relations proved, the conclusion follows immediately since problem (92) writes then

$$\int_{\Omega} (\mathcal{H}(\mu))^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}} \cdot \mathbf{curl} \varphi dx - \omega^2 \int_{\Omega} \mathcal{H}(\varepsilon) \mathbf{E}^{\text{eff}} \cdot \varphi dx = i\omega \int_{\Omega} \mathbf{J} \cdot \varphi dx,$$

which is exactly the weak formulation of the homogenized problem ( $\mathcal{P}^{\text{eff}}$ ).

*Step 1: proof of relation (93).* Taking in (92) test functions  $\varphi = 0$  and  $\psi = 0$ , we obtain

$$\int_{\Omega \times Y} \mathbf{R}(x, y) \cdot \mathbf{curl}_y \varphi^1(x, y) dx dy = 0. \quad (95)$$

Choosing in (95) test functions with separate variables, we easily obtain that for almost every  $x \in \Omega$  and for all  $\varphi^1 \in \mathcal{C}_{\text{per}}^{\infty}(Y)$ , one has

$$\int_Y \mathbf{R}(x, y) \cdot \mathbf{curl} \varphi^1(y) dy = 0.$$

For the rest of the proof, we follow the ideas of [6] and [21]. Considering  $x$  as a parameter and applying [25, Theorem 2.9], there exists a function  $\rho \in L^2(\Omega; H_{\text{per}}^1(Y))$  and  $\mathbf{c} \in \mathbf{L}^2(\Omega)$  such that

$$\mathbf{R}(x, y) = \nabla_y \rho(x, y) + \mathbf{c}(x).$$

Integrating the above relation on  $Y$  shows that  $\mathbf{c}(x) = \int_Y \mathbf{R}(x, y) dy$  and hence

$$\mathbf{R}(x, y) = \nabla_y \rho(x, y) + \int_Y \mathbf{R}(x, \hat{y}) d\hat{y}. \quad (96)$$

From the definition (90) of  $\mathbf{R}$  and direct calculation, one has for  $\xi \in H_{\text{per}}^1(Y)$ :

$$\int_Y \mu(y) \mathbf{R}(x, y) \cdot \nabla \xi(y) dy = \int_Y (\mathbf{curl} \mathbf{E}^{\text{eff}}(x) + \mathbf{curl}_y \mathbf{E}^1(x, y)) \cdot \nabla \xi(y) dy = 0. \quad (97)$$

Combining (96) and (97) we get that

$$\int_Y \mu(y) \nabla_y \rho(x, y) \cdot \nabla \xi(y) \, dy = \int_Y \mu(y) \boldsymbol{\lambda} \cdot \nabla \xi(y) \, dy,$$

where we have set  $\boldsymbol{\lambda} = -\int_Y \mathbf{R}(x, y) \, dy \in \mathbb{R}^3$  (here,  $x$  is fixed and can be considered as a parameter). Comparing with (76), we immediately obtain that  $\rho = \boldsymbol{\lambda} \cdot \boldsymbol{\chi}^\mu = \sum_{j=1}^3 \lambda_j \cdot \chi_j^\mu$ , where  $\boldsymbol{\chi}^\mu = (\chi_1^\mu, \chi_2^\mu, \chi_3^\mu)^\top$  solve the cell problems (75) with  $\eta = \mu$ . Consequently, we have  $\nabla_y \rho = \sum_{j=1}^3 \lambda_j \cdot \nabla \chi_j^\mu = (\nabla \boldsymbol{\chi}^\mu)^\top \boldsymbol{\lambda}$ , and hence

$$\mathbf{R}(x, y) = \nabla_y \rho(x, y) + \int_Y \mathbf{R}(x, \hat{y}) \, d\hat{y} = [\text{Id} - (\nabla \boldsymbol{\chi}^\mu)^\top] \int_Y \mathbf{R}(x, \hat{y}) \, d\hat{y},$$

Using the above formula and expression (77) of  $\mathcal{H}(\mu)$ , we get that

$$\int_Y \mu(y) \mathbf{R}(x, y) \, dy = \mathcal{H}(\mu) \int_Y \mathbf{R}(x, y) \, dy.$$

But on the other hand, we also have from definition (90) of  $\mathbf{R}(x, y)$  that

$$\int_Y \mu(y) \mathbf{R}(x, y) \, dy = \int_Y (\mathbf{curl} \mathbf{E}^{\text{eff}}(x) + \mathbf{curl}_y \mathbf{E}^1(x, y)) \, dy = \mathbf{curl} \mathbf{E}^{\text{eff}}(x).$$

Since  $\mathcal{H}(\mu)$  is definite positive for  $\kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ , we obtain by combining the last two relations, that

$$\int_Y \mathbf{R}(x, y) \, dy = (\mathcal{H}(\mu))^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}}(x),$$

which also reads (due to the definition of  $\mathbf{R}$ )

$$\int_Y (\mu(y))^{-1} (\mathbf{curl} \mathbf{E}^{\text{eff}}(x) + \mathbf{curl}_y \mathbf{E}^1(x, y)) \, dy = (\mathcal{H}(\mu))^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}}(x).$$

The claimed relation (93) simply follows by multiplying the above equation by  $\mathbf{curl} \boldsymbol{\varphi}$ , integrating over  $\Omega$  and adding (95).

*Step 2: proof of relation (94).* Taking  $\boldsymbol{\varphi} = \boldsymbol{\varphi}^1 = 0$  in (92), we obtain that (since  $\omega \neq 0$ ):

$$\int_{\Omega \times Y} \mathbf{S}(x, y) \cdot \nabla_y \psi(x, y) \, dx \, dy = 0. \quad (98)$$

Since  $\psi$  is arbitrary in  $\mathcal{C}_0^\infty(\Omega; \mathcal{C}_{\text{per}}^\infty(Y))$ , this implies in particular that for almost every  $x \in \Omega$  and for all  $\xi \in \mathcal{C}_{\text{per}}^\infty(Y)$ :

$$\int_Y \mathbf{S}(x, y) \cdot \nabla \xi(y) \, dy = \int_Y \varepsilon(y) (\nabla_y \Theta(x, y) + \mathbf{E}^{\text{eff}}(x)) \cdot \nabla \xi(y) \, dx \, dy = 0. \quad (99)$$

Hence

$$\int_Y \varepsilon(y) \nabla_y \Theta(x, y) \cdot \nabla \xi(y) \, dy = \int_Y \varepsilon(y) \boldsymbol{\lambda}' \cdot \nabla \xi(y) \, dy,$$

where we have set  $\boldsymbol{\lambda}' = -\mathbf{E}^{\text{eff}}(x) \in \mathbb{R}^3$  (for a fixed value of  $x$ ). Comparing the above relation with (76) for  $\eta = \varepsilon$ , we get that  $\Theta(x, y) = \boldsymbol{\lambda}' \cdot \boldsymbol{\chi}^\varepsilon$  and hence  $\nabla_y \Theta = (\nabla \boldsymbol{\chi}^\varepsilon)^\top \boldsymbol{\lambda}' = -(\nabla \boldsymbol{\chi}^\varepsilon)^\top \mathbf{E}^{\text{eff}}(x)$ . Using expression (77) of the homogenized matrix, we obtain that for every  $\boldsymbol{\varphi} \in \mathcal{C}_0^\infty(\Omega)$ :

$$\begin{aligned} \int_{\Omega \times Y} \mathbf{S}(x, y) \cdot \boldsymbol{\varphi}(x) \, dx \, dy &= \int_{\Omega \times Y} \varepsilon(y) \left( \mathbf{E}^{\text{eff}}(x) + \nabla_y \Theta(x, y) \right) \cdot \boldsymbol{\varphi}(x) \, dx \, dy \\ &= \int_{\Omega} \mathcal{H}(\varepsilon) \mathbf{E}^{\text{eff}} \cdot \boldsymbol{\varphi} \, dx. \end{aligned} \quad (100)$$

Relation (94) follows immediately by adding (98) and (100).  $\square$

## 5.2 Proof of the uniform energy estimate

This section is devoted to the proof of the uniform estimate (88) for solutions of  $(\mathcal{P}^\delta)$ . More precisely, we have the following proposition.

**Proposition 5.3.** *Assume that  $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (39). Assume that  $\omega^2 \in \mathbb{C} \setminus \Lambda^{\text{eff}}$  where  $\Lambda^{\text{eff}}$  appears in (87). Then, there exists  $\delta_0 > 0$  such that for all  $\delta \in (0; \delta_0]$ , problem  $(\mathcal{P}^\delta)$  admits a unique solution  $\mathbf{E}^\delta$ . Moreover we have the estimate*

$$\|\mathbf{E}^\delta\| + \|\mathbf{curl} \mathbf{E}^\delta\| \leq C \|\mathbf{J}\| \quad (101)$$

where  $C > 0$  is independent of  $\delta \in (0; \delta_0]$ .

*Proof.* When  $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ , according to Theorem 3.1, we know that  $A_\varepsilon^\delta : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  and  $B_\mu^\delta : \mathbf{H}_\#^1(\Omega) \rightarrow \mathbf{H}_\#^1(\Omega)$  are isomorphisms. From the Theorem 6.1 of [8], we infer that  $\mathcal{A}_N^\delta(\omega) : \mathbf{H}_N(\mathbf{curl}) \rightarrow \mathbf{H}_N(\mathbf{curl})$  is an isomorphism if it is injective. Therefore, we have to prove that  $\mathcal{A}_N^\delta(\omega)$  is injective for  $\delta$  small enough. To proceed we work by contradiction. Slightly more generally, for a given  $\mathbf{J} \in \mathbf{L}^2(\Omega)$ , assume that there is a sequence of values of  $\delta$  denoted  $(\delta_k)_{k \in \mathbb{N}}$ , with  $\delta_k \rightarrow 0$ , such that if we set  $\varepsilon_k := \varepsilon^{\delta_k}$ ,  $\mu_k := \mu^{\delta_k}$ ,  $\mathbf{E}_k := \mathbf{E}^{\delta_k} \in \mathbf{H}_N(\mathbf{curl})$ , we have

$$a_\omega^k(\mathbf{E}_k, \mathbf{E}') := ((\mu_k)^{-1} \mathbf{curl} \mathbf{E}_k, \mathbf{curl} \mathbf{E}') - \omega^2 (\varepsilon_k \mathbf{E}_k, \mathbf{E}') = i\omega (\mathbf{J}, \mathbf{E}'), \quad \forall \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}),$$

as well as

$$\|\mathbf{E}_k\|^2 + \|\mathbf{curl} \mathbf{E}_k\|^2 > k.$$

Then set  $\tilde{\mathbf{E}}_k := \mathbf{E}_k / (\|\mathbf{E}_k\|^2 + \|\mathbf{curl} \mathbf{E}_k\|^2)$  and  $\tilde{\mathbf{J}}_k := \mathbf{J}_k / (\|\mathbf{E}_k\|^2 + \|\mathbf{curl} \mathbf{E}_k\|^2)$ . We have

$$a_\omega^k(\tilde{\mathbf{E}}_k, \mathbf{E}') = i\omega (\tilde{\mathbf{J}}_k, \mathbf{E}'), \quad \forall \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}) \quad (102)$$

and

$$\|\tilde{\mathbf{E}}_k\|^2 + \|\mathbf{curl} \tilde{\mathbf{E}}_k\|^2 = 1, \quad \lim_{k \rightarrow +\infty} \|\tilde{\mathbf{J}}_k\| = 0.$$

Since  $(\tilde{\mathbf{E}}_k)$  is bounded in  $\mathbf{H}_N(\mathbf{curl})$ , we can extract a subsequence, still denoted  $(\tilde{\mathbf{E}}_k)$ , such that  $(\tilde{\mathbf{E}}_k)$  converges weakly in  $\mathbf{H}_N(\mathbf{curl})$  to some  $\mathbf{E}_0 \in \mathbf{H}_N(\mathbf{curl})$ . Thanks to Proposition 5.2, we can pass to the limit in (102) to get

$$a_\omega^{\text{eff}}(\mathbf{E}_0, \mathbf{E}') = 0, \quad \forall \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}). \quad (103)$$

Since  $\omega \in \mathbb{C} \setminus \Lambda^{\text{eff}}$ , this implies that  $\mathbf{E}_0 = 0$ . In order to obtain a contradiction, it remains to show that  $(\tilde{\mathbf{E}}_k)$  strongly converges to zero in  $\mathbf{H}_N(\mathbf{curl})$ . To proceed, we have to establish some sort of compactness result using the fact that when  $\omega \neq 0$ , we have  $\text{div}(\varepsilon_k \tilde{\mathbf{E}}_k) = 0$  in  $\Omega$  which implies that  $\tilde{\mathbf{E}}_k \in \mathbf{V}_N(\varepsilon_k)$ . For each  $k \geq 1$ , from Theorem 5.1 of [8], we know that when  $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$ ,  $\mathbf{V}_N(\varepsilon_k)$  is compactly embedded in  $\mathbf{L}^2(\Omega)$ . But here we need some uniform result with respect to  $k$ . To proceed, we will take in (102) a well-chosen test function. Let us mention that a similar difficulty appears in the justification of the approximation of Maxwell's equations with finite elements methods, the mesh size  $h$  replacing the parameter  $\delta$  (see [31, §7.3.2] and the references therein). First, introduce the unique function  $\psi_k \in \mathbf{H}_\#^1(\Omega)$  such that

$$(\mu_k \nabla \psi_k, \nabla \psi') = (\mu_k \mathbf{curl} \tilde{\mathbf{E}}_k, \nabla \psi'), \quad \forall \psi' \in \mathbf{H}_\#^1(\Omega).$$

When  $\kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ , from Theorem 3.1, we know that  $\psi_k$  is well-defined. Moreover, we have  $\|\nabla \psi_k\| \leq C \|\mathbf{curl} \tilde{\mathbf{E}}_k\| \leq C$  where  $C > 0$  is independent of  $\delta$ . Then  $\mu_k (\mathbf{curl} \tilde{\mathbf{E}}_k - \nabla \psi_k)$  is divergence free in  $\Omega$  and satisfies  $\mu_k (\mathbf{curl} \tilde{\mathbf{E}}_k - \nabla \psi_k) \cdot \mathbf{n}$  on  $\partial\Omega$ . From [4, Theorem 3.17], we know that there exists a unique  $\mathbb{P}_k \tilde{\mathbf{E}}_k \in \mathbf{V}_N(1)$  such that

$$\mathbf{curl} (\mathbb{P}_k \tilde{\mathbf{E}}_k) = \mu_k (\mathbf{curl} \tilde{\mathbf{E}}_k - \nabla \psi_k). \quad (104)$$

Since in  $\mathbf{V}_N(1)$ ,  $\|\mathbf{curl} \cdot\|_\Omega$  is a norm which is equivalent to  $\|\cdot\|_{\mathbf{curl}}$  (Proposition 2.1), we infer that  $(\mathbb{P}_k)$  is a sequence of operators which are uniformly bounded from  $\mathbf{H}_N(\mathbf{curl})$  to  $\mathbf{V}_N(1)$ . Testing in (102) with  $\mathbf{E}' = \mathbb{P}_k \tilde{\mathbf{E}}_k$ , using (104) and integrating by parts, we get

$$\begin{aligned} (\tilde{\mathbf{J}}_k, \mathbb{P}_k \tilde{\mathbf{E}}_k) + \omega^2(\varepsilon_k \tilde{\mathbf{E}}_k, \mathbb{P}_k \tilde{\mathbf{E}}_k) &= ((\mu_k)^{-1} \mathbf{curl} \tilde{\mathbf{E}}_k, \mathbf{curl}(\mathbb{P}_k \tilde{\mathbf{E}}_k)) \\ &= (\mathbf{curl} \tilde{\mathbf{E}}_k, \mathbf{curl} \tilde{\mathbf{E}}_k - \nabla \psi_k) = \|\mathbf{curl} \tilde{\mathbf{E}}_k\|^2. \end{aligned} \quad (105)$$

Using that  $\mathbb{P}_k : \mathbf{H}_N(\mathbf{curl}) \rightarrow \mathbf{V}_N(1)$  are uniformly bounded,  $(\tilde{\mathbf{E}}_k)$  converges weakly to zero in  $\mathbf{H}_N(\mathbf{curl})$  and  $\mathbf{V}_N(1)$  is compactly embedded in  $\mathbf{L}^2(\Omega)$  (Proposition 2.1), we deduce that we can extract a subsequence, still denoted  $(\tilde{\mathbf{E}}_k)$ , such that  $(\mathbb{P}_k \tilde{\mathbf{E}}_k)$  converges strongly to zero in  $\mathbf{L}^2(\Omega)$ . Then from (105), we deduce that the sequence  $(\mathbf{curl} \tilde{\mathbf{E}}_k)$  converges strongly to zero in  $\mathbf{L}^2(\Omega)$ . Using the result of Proposition 5.4 below which guarantees that  $\|\tilde{\mathbf{E}}_k\| \leq C \|\mathbf{curl} \tilde{\mathbf{E}}_k\|$  with some  $C > 0$  which is independent of  $k$ , we deduce that  $(\tilde{\mathbf{E}}_k)$  converges to zero in  $\mathbf{H}_N(\mathbf{curl})$ . This contradicts the initial assumption. As a consequence, taking first  $\mathbf{J} = 0$  above, we deduce that  $(\mathcal{P}^\delta)$  is injective and so uniquely solvable for  $\delta$  small enough. Then for a given non zero  $\mathbf{J} \in \mathbf{L}^2(\Omega)$ , the above lines imply the uniform estimate (101).  $\square$

**Proposition 5.4.** *Assume that  $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (39). Then there is a constant  $C > 0$  independent of  $\delta$  such that*

$$\|\mathbf{E}\| \leq C \|\mathbf{curl} \mathbf{E}\|, \quad \forall \mathbf{E} \in \mathbf{V}_N(\varepsilon^\delta). \quad (106)$$

*Proof.* If  $\mathbf{E} \in \mathbf{V}_N(\varepsilon^\delta)$ , according to [4, Theorem 3.12], we know that there is a unique  $\mathbf{u} \in \mathbf{V}_T(1)$  such that  $\mathbf{E} = (\varepsilon^\delta)^{-1} \mathbf{curl} \mathbf{u}$ . Then integrating by parts, we find

$$((\varepsilon^\delta)^{-1} \mathbf{curl} \mathbf{u}, \mathbf{curl} \mathbf{u}') = (\mathbf{curl} \mathbf{E}, \mathbf{u}'), \quad \forall \mathbf{u}' \in \mathbf{V}_T(1). \quad (107)$$

Introduce the function  $\varphi \in \mathbf{H}_0^1(\Omega)$  such that

$$(\varepsilon^\delta \nabla \varphi, \nabla \varphi') = (\varepsilon^\delta \mathbf{curl} \mathbf{u}, \nabla \varphi'), \quad \forall \varphi' \in \mathbf{H}_0^1(\Omega).$$

Since  $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$ , from Theorem 3.1, we know that  $\varphi$  is well-defined. Moreover, we have  $\|\nabla \varphi\| \leq C \|\mathbf{curl} \mathbf{u}\|$  where  $C > 0$  is independent of  $\delta$ . Then  $\varepsilon^\delta (\mathbf{curl} \mathbf{u} - \nabla \varphi)$  is divergence free in  $\Omega$  and again from [4, Theorem 3.12], we know that there is a unique  $\mathbf{Tu} \in \mathbf{V}_T(1)$  such that  $\mathbf{curl}(\mathbf{Tu}) = \varepsilon^\delta (\mathbf{curl} \mathbf{u} - \nabla \varphi)$ . Since in  $\mathbf{V}_T(1)$ ,  $\|\mathbf{curl} \cdot\|_\Omega$  is a norm which is equivalent to  $\|\cdot\|_{\mathbf{curl}}$  (Proposition 2.1), we infer that  $\mathbf{T} : \mathbf{V}_T(1) \rightarrow \mathbf{V}_T(1)$  is a uniformly bounded operator. Choosing  $\mathbf{u}' = \mathbf{Tu}$  in (107) and integrating by parts, we obtain

$$(\mathbf{curl} \mathbf{E}, \mathbf{Tu}) = ((\varepsilon^\delta)^{-1} \mathbf{curl} \mathbf{u}, \mathbf{curl}(\mathbf{Tu})) = \|\mathbf{curl} \mathbf{u}\|^2 - (\mathbf{curl} \mathbf{u}, \nabla \varphi) = \|\mathbf{curl} \mathbf{u}\|^2.$$

Using the Cauchy-Schwarz inequality, this gives  $\|\mathbf{curl} \mathbf{u}\| \leq C \|\mathbf{curl} \mathbf{E}\|$  where  $C > 0$  is independent of  $\delta$ . This yields the desired estimate (106).  $\square$

### 5.3 Final result

Gathering Propositions 5.2 and 5.3, we can state the final result of this article.

**Theorem 5.1.** *Assume that  $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (39). Assume that  $\omega \in \mathbb{C} \setminus \Lambda^{\text{eff}}$  where  $\Lambda^{\text{eff}}$  appears in (87). Then, there exists  $\delta_0 > 0$  such that for  $\delta \in (0, \delta_0]$ , the solution  $\mathbf{E}^\delta$  of problem  $(\mathcal{P}^\delta)$ , which is well-defined according to Proposition 5.3, satisfies*

$$\mathbf{E}^\delta \rightharpoonup \mathbf{E}^{\text{eff}} \quad \text{and} \quad \mathbf{curl} \mathbf{E}^\delta \rightharpoonup \mathbf{curl} \mathbf{E}^{\text{eff}} \quad \text{weakly in } \mathbf{L}^2(\Omega)$$

where  $\mathbf{E}^{\text{eff}}$  is the unique solution of problem  $(\mathcal{P}^{\text{eff}})$  given by (85).

Let us conclude this paper with two comments. Firstly, in this work, we only prove weak convergence results. Strong convergence results (using correctors) for Maxwell’s equations with positive materials have been obtained in [41, 42]. It would be interesting to understand if we can adapt the approach proposed in these two articles to our setting. Secondly, the obtained bounds for the contrasts (involving  $m$  and  $M$ ) to ensure the homogenization process are probably not optimal. Improving them would require a sharp analysis of the asymptotic behavior of the critical contrasts given by (62) as  $\delta$  tends to zero (see Remark 3.4). Is it possible that the two scalar problems with Dirichlet and Neumann boundary conditions be uniformly well-posed as  $\delta$  tends to zero, even when some cell problems have a non zero kernel or when the homogenized tensors are not positive definite? This has still to be clarified.

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