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1 **ANISOTROPIC TRIANGULATIONS VIA DISCRETE RIEMANNIAN**
2 **VORONOI DIAGRAMS***

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5 **Abstract.** The construction of anisotropic triangulations is desirable for various applications,
6 such as the numerical solving of partial differential equations and the representation of surfaces in
7 graphics. To solve this notoriously difficult problem in a practical way, we introduce the discrete
8 Riemannian Voronoi diagram, a discrete structure that approximates the Riemannian Voronoi dia-
9 gram. This structure has been implemented and was shown to lead to good triangulations in \mathbb{R}^2 and
10 on surfaces embedded in \mathbb{R}^3 as detailed in our experimental companion paper.

11 In this paper, we study theoretical aspects of our structure. Given a finite set of points \mathcal{P}
12 in a domain Ω equipped with a Riemannian metric, we compare the discrete Riemannian Voronoi
13 diagram of \mathcal{P} to its Riemannian Voronoi diagram. Both diagrams have dual structures called the
14 discrete Riemannian Delaunay and the Riemannian Delaunay complex. We provide conditions that
15 guarantee that these dual structures are identical. It then follows from previous results that the
16 discrete Riemannian Delaunay complex can be embedded in Ω under sufficient conditions, leading
17 to an anisotropic triangulation with curved simplices. Furthermore, we show that, under similar
18 conditions, the simplices of this triangulation can be straightened.

19 **Key words.** Riemannian Geometry, Voronoi diagram, Delaunay triangulation

20 **AMS subject classifications.** 65D18, 68U05

21 **1. Introduction.** *Anisotropic triangulations* are triangulations whose elements
22 are elongated along prescribed directions. Anisotropic triangulations are known to be
23 well suited when solving PDE's [14, 26, 32]. They can also significantly enhance the
24 accuracy of a surface representation if the anisotropy of the triangulation conforms
25 to the curvature of the surface [20].

26 Many methods to generate anisotropic triangulations are based on the notion of
27 Riemannian metric and create triangulations whose elements adapt locally to the size
28 and anisotropy prescribed by the local geometry. The numerous theoretical and prac-
29 tical results [1] of the Euclidean Voronoi diagram and its dual structure, the Delaunay
30 triangulation, have pushed authors to try and extend these well-established concepts
31 to the anisotropic setting. Labelle and Shewchuk [24] and Du and Wang [16] indepen-
32 dently introduced two anisotropic Voronoi diagrams whose anisotropic distances are
33 based on a discrete approximation of the Riemannian metric field. Contrary to their
34 Euclidean counterpart, the fact that the dual of these anisotropic Voronoi diagrams
35 is an embedded triangulation is not immediate, and, despite their strong theoretical
36 foundations, the anisotropic Voronoi diagrams of Labelle and Shewchuk and Du and
37 Wang have only been proven to yield, under certain conditions, a good triangulation
38 in a two-dimensional setting [9, 10, 13, 16, 24].

39 Both these anisotropic Voronoi diagrams are variants of the exact Riemannian

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40 Voronoi diagram, whose cells are defined as

$$41 \quad V_g(p_i) = \{x \in \Omega \mid d_g(p_i, x) \leq d_g(p_j, x), \forall p_j \in \mathcal{P} \setminus \{p_i\}\},$$

42 where $d_g(p, q)$ denotes the geodesic distance. The main advantage of the anisotropic
 43 Voronoi diagrams compared to the Riemannian version is to ease the computation of
 44 the anisotropic diagrams, but at the cost of differences in the intrinsic properties of
 45 the diagrams. However, their theoretical and practical results are rather limited.

46 The exact Riemannian Voronoi diagram comes with the benefit of providing a
 47 more favorable theoretical framework and recent works have provided sufficient condi-
 48 tions for a point set to be an embedded Riemannian Delaunay complex [2, 18, 25].
 49 We approach the Riemannian Voronoi diagram and its dual Riemannian Delaunay
 50 complex with a focus on both practicality and theoretical robustness. We introduce
 51 the discrete Riemannian Voronoi diagram, a discrete approximation of the (exact)
 52 Riemannian Voronoi diagram. Experimental results, presented in our companion pa-
 53 per [31], have shown that this approach leads to good anisotropic triangulations for
 54 two-dimensional domains and surfaces, see Figure 1.



FIG. 1. *The discrete Riemannian Voronoi diagram (colored cells with bisectors in white) and its dual complex (in black) realized with straight simplices of a two-dimensional domain endowed with a hyperbolic shock-based metric field.*

55 We introduce in this paper the theoretical side of this work, showing that our
 56 approach is theoretically sound in all dimensions. We prove that, under sufficient
 57 conditions, the discrete Riemannian Voronoi diagram has the same combinatorial
 58 structure as the (exact) Riemannian Voronoi diagram and that the dual discrete Rie-
 59 manian Delaunay complex can be embedded as a triangulation of the point set, with
 60 either curved or straight simplices. Discrete Voronoi diagrams have been indepen-
 61 dently studied, although in a two-dimensional isotropic setting by Cao et al. [11].

62 **2. Simplices and complexes.** We start by recalling precise definitions of con-
 63 cepts and notions related to simplicial complexes. The following definitions live within
 64 the context of abstract simplices and complexes.

65 A *simplex* σ is a non-empty finite set. The *dimension* of σ is given by $\dim \sigma =$
 66 $\#(\sigma) - 1$, and a j -simplex refers to a simplex of dimension j . The elements of σ are
 67 called the *vertices* of σ . The set of vertices of σ is noted $\text{Vert}(\sigma)$.

68 If a simplex τ is a subset of σ , we say it is a *face* of σ , and we write $\tau \leq \sigma$. A
 69 1-dimensional face is called an *edge*. If τ is a proper subset of σ , we say it is a *proper*
 70 face and we write $\tau < \sigma$. A *facet* of σ is a face τ with $\dim \tau = \dim \sigma - 1$.

71 For any vertex $p \in \text{Vert}(\sigma)$, the face opposite to p is the face determined by the
 72 other vertices of σ , and is denoted by σ_p . If τ is a j -simplex, and p is not a vertex
 73 of σ , we may construct a $(j + 1)$ -simplex $\sigma = p * \tau$, called the *join* of p and τ as the
 74 simplex defined by p and the vertices of τ .

75 The *length* of an edge is the distance between its vertices. The *height* of p in σ
 76 is $D(p, \sigma) = d(p, \text{aff } \sigma_p)$.

77 A *circumscribing ball* for a simplex σ is any n -dimensional ball that contains
 78 the vertices of σ on its boundary. If σ admits a circumscribing ball, then it has a
 79 circumcenter, $C(\sigma)$, which is the center of the unique smallest circumscribing ball
 80 for σ . The radius of this ball is the *circumradius* of σ , denoted by $R(\sigma)$.

81 The *length* of an edge is the distance between its vertices. A *circumscribing ball* for
 82 a simplex σ is any n -dimensional ball that contains the vertices of σ on its boundary.
 83 If σ admits a circumscribing ball, then it has a circumcenter, $C(\sigma)$, which is the
 84 center of the unique smallest circumscribing ball for σ . The radius of this ball is the
 85 *circumradius* of σ , denoted by $R(\sigma)$. The *height* of p in σ is $D(p, \sigma) = d(p, \text{aff}(\sigma_p))$.
 86 The *dihedral angle* between two facets is the angle between their two supporting
 87 planes. If σ is a j -simplex with $j \geq 2$, then for any two vertices $p, q \in \sigma$, the dihedral
 88 angle between σ_p and σ_q defines an equality between ratios of heights (see [Figure 2](#)).

89
$$\sin \angle(\text{aff}(\sigma_p), \text{aff}(\sigma_q)) = \frac{D(p, \sigma)}{D(p, \sigma_q)} = \frac{D(q, \sigma)}{D(q, \sigma_p)}.$$

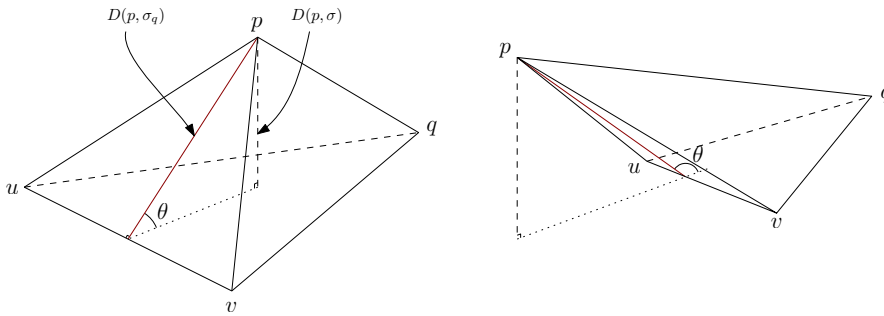


FIG. 2. Acute and obtuse dihedral angles

90 **2.1. Simplicial complexes.** Before defining Delaunay triangulations, we intro-
 91 duce the more general concept of simplicial complexes. Since the standard definition
 92 of a simplex as the convex hull of a set of points does not extend well to the Riemannian
 93 setting (see Dyer et al. [17]), we approach these definitions from a more abstract
 94 point of view.

95 Simplicial complexes form the underlying framework of Delaunay triangulations.
 96 An *abstract simplicial complex* is a set \mathcal{K} of simplices such that if $\sigma \in \mathcal{K}$, then all
 97 the faces of σ also belong to \mathcal{K} , and if $\sigma_1, \sigma_2 \in \mathcal{K}$, then $\sigma_1 \cap \sigma_2 \in \mathcal{K}$. The union of
 98 the vertices of all the simplices of \mathcal{K} is called the *vertex set* of K . The dimension
 99 of a complex is the largest dimension of any of its simplices. A subset $L \subseteq \mathcal{K}$ is a
 100 *subcomplex* of \mathcal{K} if it is also a complex. Two simplices are *adjacent* if they share a
 101 face and *incident* if one is a face of the other. If a simplex in \mathcal{K} is not a face of a
 102 larger simplex, we say that it is *maximal*. If all the maximal simplices in a complex \mathcal{K}
 103 of dimension n have dimension n , then the simplicial complex is *pure*. The *star* of
 104 a vertex p in a complex \mathcal{K} is the subcomplex S formed by set of simplices that are
 105 incident to p . The *link* of a vertex p is the union of the simplices opposite of p in S_p .

106 A *geometric simplicial complex* is an abstract simplicial complex whose simplices
 107 are geometrically realized.

108 **3. Riemannian geometry.** In the main part of the text we consider an (open)
 109 domain Ω , with compact closure, in \mathbb{R}^n endowed with a smooth Riemannian metric
 110 g , which we shall discuss below. We assume that the metric g is Lipschitz continuous.
 111 The construction of a triangulation is local in nature, and in this paper we focus on
 112 local results. By this we mean that we assume that we are sufficiently far from the
 113 boundary. Dealing with the boundary is a topic in itself, on which the authors are
 114 in fact working at the moment. By the assumption that we are far away from the
 115 boundary of Ω , we mean in particular that we always assume that the shortest path
 116 between any two points we consider exists and lies in the interior of Ω .

117 The fact that we basically ignore the boundary may seem very bold, however our
 118 study in \mathbb{R}^m under these assumptions allows us to triangulate compact n -manifolds
 119 (embedded in \mathbb{R}^m) without boundary, see [section 10](#) for details.¹ The structures of
 120 interest will be built from a finite set of points \mathcal{P} , which we call *sites*.

121 **3.1. Riemannian metric.** A *Riemannian metric field* g , defined over Ω , as-
 122 sociates a *metric* $g(p) = G_p$ to any point p of the domain. This means that for
 123 any $v, w \in \mathbb{R}^n$ we associate² an inner product $\langle v, w \rangle_{g(p)} = v^t g(p) w$, in a way that
 124 smoothly depends on p . Using a Riemannian metric, we can associate lengths to
 125 curves and define the geodesic distance d_g as the minimizer of the lengths of all
 126 curves between two points. When the map $g : p \mapsto G$ is constant, the metric field is
 127 said to be *constant*. In this case, the distance between two points x and y in Ω is
 128 $d_g(x, y) = d_G(x, y) = \|x - y\|_G = \sqrt{(x - y)^t G (x - y)}$. The metric field associated to
 129 the Euclidean distance is denoted g_E and thus the Euclidean distance is also denoted
 130 $\|\cdot\|_{g_E}$ or simply $\|\cdot\|$. The geodesic (closed) ball centered on $p \in \Omega$ and of radius r is
 131 defined as $B_g(p, r) = \{x \in \Omega \mid d_g(p, x) \leq r\}$. In the following, we assume that $\Omega \subset \mathbb{R}^n$
 132 is endowed with a Lipschitz continuous metric field g .

133 **4. Distortion.** The concept of distortion was originally introduced by Labelle
 134 and Shewchuk [24] to relate distances with respect to two metrics, but this result can
 135 be (locally) extended to geodesic distances. We first recall the definition from Labelle
 136 and Shewchuk, and then show how to extend it to metric fields.

137 **4.1. Original distortion.** The notion of metric transformation is required to
 138 define the distortion from Labelle and Shewchuk, and we thus recall it now.

¹Treating non-compact manifolds from a computational point of view does not seem feasible, using these techniques.

²We shall always identify any linear space with its tangent space.

139 **4.1.1. Metric transformation.** Given a symmetric positive definite matrix G ,
 140 we denote by F any matrix such that $\det(F) > 0$ and $F^t F = G$. The matrix F is
 141 called a *square root* of G . The square root of a matrix is not uniquely defined: the
 142 Cholesky decomposition provides, for example, an upper triangular F , but it is also
 143 possible to consider the diagonalization of G as $O^T D O$, where O is an orthonormal
 144 matrix and D is a diagonal matrix; the square root is then taken as $F = O^T \sqrt{D} O$.

145 The latter definition is more natural than other decompositions since \sqrt{D} is canonically
 146 defined, in fact (under the assumption that G is symmetric positive definite,
 147 which it clearly is) the square root defined as such is the unique square root which
 148 is positive definite, see [22, Theorem 7.2.6]. This choice is also continuous, which
 149 follows from the fact that it can be defined by a contour integral, see for example [21,
 150 Example 3.16]. We therefore specifically use this root in the following.

151 The square root F offers a linear bijective transformation between the Euclidean
 152 space and a metric space, noting that:

$$153 \quad d_G(x, y) = \sqrt{(x - y)^t F^t F (x - y)} = \|F(x - y)\| = \|Fx - Fy\|,$$

154 where $\|\cdot\|$ stands for the Euclidean norm. Thus, the metric distance between x and y
 155 in Euclidean space can be seen as the Euclidean distance between two *transformed*
 156 points Fx and Fy living in the metric space of G .

157 **4.1.2. Distortion.** The *distortion* between two points p and q of Ω is defined
 158 by Labelle and Shewchuk [24] as $\psi(p, q) = \psi(G_p, G_q) = \max\{\|F_p F_q^{-1}\|, \|F_q F_p^{-1}\|\}$,
 159 where $\|\cdot\|$ is the Euclidean matrix norm, that is $\|M\| = \sup_{x \in \mathbb{R}^n} \frac{\|Mx\|}{\|x\|}$. Observe
 160 that $\psi(G_p, G_q) \geq 1$ and $\psi(G_p, G_q) = 1$ when $G_p = G_q$.

161 A fundamental property of the distortion is to relate the two distances d_{G_p}
 162 and d_{G_q} . Specifically, for any pair x, y of points in U , we have:

$$163 \quad (1) \quad \frac{1}{\psi(p, q)} d_{G_p}(x, y) \leq d_{G_q}(x, y) \leq \psi(p, q) d_{G_p}(x, y).$$

164 Indeed,

$$165 \quad d_p(x, y) = \|F_p(x - y)\| = \|F_p F_q^{-1} F_q(x - y)\| \\ 166 \quad \leq \|F_p F_q^{-1}\| \|F_q(x - y)\| \leq \psi(G_p, G_q) d_q(x, y). \\ 167$$

168 The other inequality is obtained similarly.

169 **4.2. Geodesic distortion.** To quantify the variations of a metric field and to
 170 compare two metric fields, we define a notion of distortion. Firstly, we define the
 171 *metric distortion* $\psi_p(g, g')$ between two distance functions d_g and $d_{g'}$ at a point $p \in \Omega$
 172 as the smallest constant such that

$$173 \quad 1/\psi_p(g, g') \langle v, w \rangle_{g(p)} \leq \langle v, w \rangle_{g'(p)} \leq \psi_p(g, g') \langle v, w \rangle_{g(p)},$$

174 for all $v, w \in T_p \Omega$. This definition thus generalizes the concept of distortion between
 175 two metrics $g(p)$ and $g(q)$, as defined by Labelle and Shewchuk [24]. Further general-
 176 izing the concept of distortion, we define the *metric distortion* between two distance
 177 functions d_g and $d_{g'}$ over a neighborhood $U \subset \Omega$ as

$$178 \quad \psi_U(g, g') = \sup_{p \in U} \psi_p(g, g').$$

179 Observe that $\psi_U(g, g') \geq 1$ and that $\psi_U(g, g') = 1$ when $g = g'$.

180 LEMMA 4.1. Let $U \subset \Omega$ be open, and g and g' be two Riemannian metric fields
 181 on Ω . Let $\psi_0 \geq 1$ be a bound on the metric distortion, in the sense of Labelle and
 182 Shewchuk. Suppose that U is included in a ball $B_g(p_0, r_0)$, with $p_0 \in U$. Moreover
 183 assume that $B_g(p_0, 5r_0) \subset \Omega$ and $5r_0$ is less than the convexity radius with respect
 184 to g , see [12, Section IX.6], and we have $\forall p \in B_g(p_0, 5r_0), \psi_p(g, g') \leq \psi_0$, where we
 185 assume that $\psi_0 < 2$. Then, for all $x, y \in U$,

$$186 \quad \frac{1}{\psi_0} d_g(x, y) \leq d_{g'}(x, y) \leq \psi_0 d_g(x, y),$$

187 where d_g and $d_{g'}$ indicate the geodesic distances with respect to g and g' respectively.

188 *Proof.* Recall that for $p \in B_g(p_0, 5r_0)$ and, for any pair x, y of points, we have

$$189 \quad (2) \quad \frac{1}{\psi_0} d_{g(p)}(x, y) \leq d_{g'(p)}(x, y) \leq \psi_0 d_{g(p)}(x, y),$$

191 where the $d_{g(p)}(x, y)$ should be interpreted as the metric on the tangent spaces.

192 Now, note that any geodesic connecting two points in U with respect to both g and
 193 g' lies in $B_g(p_0, 5r_0)$. For g , this is a consequence of geodesic convexity. For g' , this
 194 follows by a slightly longer argument: the distance from $\partial B_g(p_0, r_0)$ to $\partial B_g(p_0, 5r_0)$,
 195 with respect to g is 4, and thus any curve starting and ending in U leaving $B_g(p_0, 5r_0)$
 196 has length at least $2 \cdot 4/\psi > 4$ with respect to g' . Because of the upper bound on the
 197 distortion and the definition of the geodesic ball, we have that any geodesic connecting
 198 with respect to g connecting two points in U has length at most $2\psi < 4$ with respect
 199 to g' . We conclude that all geodesics remain inside $B_g(p_0, 5r_0)$.

200 Equation (2) now gives that, for any curve $\gamma(t)$ connecting two points in U , we
 201 have that:

$$202 \quad (3) \quad \frac{1}{\psi_0} \int \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_{g(\gamma(t))}} dt \leq \int \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_{g'(\gamma(t))}} dt \leq \psi_0 \int \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_{g(\gamma(t))}} dt.$$

204 Considering the infimum over all paths γ that begin at x and end at y , we obtain the
 205 result. \square

206 Note that we weakened the result in (3), because we do not want to drag along $\sqrt{\psi_0}$
 207 in the notation.

208 Note that this result is independent from the definition of the distortion and is
 209 entirely based on the inequality comparing distances in two metrics (Equation (1)).

210 **4.3. Geodesics.** Let $v \in \mathbb{R}^n$. From the unique geodesic γ satisfying $\gamma(0) = p$
 211 with initial tangent vector $\dot{\gamma}(0) = v$, one defines the *exponential map* at p through
 212 $\exp_p(v) = \gamma(1)$. (Note that, here again, we ignore the boundary.) The *injectivity*
 213 *radius* at a point p of Ω is the largest radius for which the exponential map at p
 214 restricted to a ball of that radius is a diffeomorphism onto its image. The injectivity
 215 radius ι_Ω of Ω is defined as the infimum of the injectivity radii at all points. For
 216 any $p \in \Omega$ and for a two-dimensional linear subspace H of the tangent space at p ,
 217 we define the *sectional curvature* K at p for H as the Gaussian curvature at p of
 218 the surface $\exp_p(D_H)$, where D_H is a disk of radius less than the injectivity radius
 219 centred at p in H .

220 In the theoretical studies of our algorithm, we will assume that the injectivity
 221 radius of Ω is strictly positive and its sectional curvatures are bounded.

222 **4.4. Power protected nets.** Controlling the quality of the Delaunay and
 223 Voronoi structures will be essential in our proofs. For this purpose, we use the notions
 224 of *net* and of *power protection*.

225

226 **Power protection of point sets** Power protection of simplices is a concept formally
 227 introduced by Boissonnat, Dyer and Ghosh [2]. Let σ be a simplex whose vertices
 228 belong to \mathcal{P} , and let $B_g(\sigma) = B_g(c, r)$ denote a circumscribing ball of σ where $r =$
 229 $d_g(c, p)$ for any vertex p of σ . We call c the circumcenter of σ and r its circumradius.

230 For $0 \leq \delta \leq r$, we associate to $B_g(\sigma)$ the dilated ball $B_g^{+\delta}(\sigma) = B_g(c, \sqrt{r^2 + \delta^2})$.
 231 We say that σ is δ -*power protected* if $B_g^{+\delta}(\sigma)$ does not contain any point of $\mathcal{P} \setminus \text{Vert}(\sigma)$
 232 where $\text{Vert}(\sigma)$ denotes the vertex set of σ . The ball $B_g^{+\delta}$ is the *power protected* ball
 233 of σ . Finally, a point set \mathcal{P} is δ -power protected if the Delaunay ball of its simplices
 234 are δ -power protected.

235 **Nets** To ensure that the simplices of the structures that we shall consider are well
 236 shaped, we will need to control the density and the sparsity of the point set. The con-
 237 cept of net conveys these requirements through *sampling* and *separation* parameters.

238 The sampling parameter is used to control the density of a point set: if Ω is
 239 a bounded domain, \mathcal{P} is said to be an ε -*sample set* for Ω with respect to a metric
 240 field g if $d_g(x, \mathcal{P}) < \varepsilon$, for all $x \in \Omega$. The sparsity of a point set is controlled by the
 241 separation parameter: the set \mathcal{P} is said to be μ -*separated* with respect to a metric
 242 field g if $d_g(p, q) \geq \mu$ for all $p, q \in \mathcal{P}$. If \mathcal{P} is an ε -sample that is μ -separated, we
 243 say that \mathcal{P} is an (ε, μ) -*net*.

244 **5. Riemannian Delaunay triangulations.** Given a metric field g , the *Rie-*
 245 *mannian Voronoi diagram* of a point set \mathcal{P} , denoted by $\text{Vor}_g(\mathcal{P})$, is the Voronoi
 246 diagram built using the geodesic distance d_g . Formally, it is a partition of the do-
 247 main in *Riemannian Voronoi cells* $\{V_g(p_i)\}$, where $V_g(p_i) = \{x \in \Omega \mid d_g(p_i, x) \leq$
 248 $d_g(p_j, x), \forall p_j \in \mathcal{P} \setminus p_i\}$.

249 The Riemannian Delaunay complex of \mathcal{P} is an abstract simplicial complex, defined
 250 as the nerve of the Riemannian Voronoi diagram, that is the set of simplices $\text{Del}_g(\mathcal{P}) =$
 251 $\{\sigma \mid \text{Vert}(\sigma) \in \mathcal{P}, \cap_{p \in \text{Vert}(\sigma)} V_g(p) \neq \emptyset\}$. There is a straightforward duality between
 252 the diagram and the complex, and between their respective elements.

253 In this paper, we will consider both abstract simplices and complexes, as well as
 254 their geometric realization in \mathbb{R}^n with vertex set \mathcal{P} . We now introduce two realizations
 255 of a simplex that will be useful, one curved and the other one straight.

256 The *straight realization* of a n -simplex σ with vertices in \mathcal{P} is the convex hull
 257 of its vertices. We denote it by $\bar{\sigma}$. In other words,

$$258 \quad (4) \quad \bar{\sigma} = \{x \in \Omega \subset \mathbb{R}^n \mid x = \sum_{p \in \sigma} \lambda_p(x) p, \lambda_p(x) \geq 0, \sum_{p \in \sigma} \lambda_p(x) = 1\}.$$

259 The *curved realization*, noted $\tilde{\sigma}$ is based on the notion of Riemannian center of
 260 mass [23, 17]. Let y be a point of $\bar{\sigma}$ with barycentric coordinate $\lambda_p(y), p \in \sigma$. We can
 261 associate the energy functional

$$262 \quad \mathcal{E}_y(x) = \frac{1}{2} \sum_{p \in \sigma} \lambda_p(y) d_g(x, p)^2,$$

263 for all $x \in \Omega$. We then define the curved realization of σ as

$$264 \quad (5) \quad \tilde{\sigma} = \{\tilde{x} \in \Omega \subset \mathbb{R}^n \mid \tilde{x} = \text{argmin } \mathcal{E}_{\tilde{x}}(x), x \in \Omega, \tilde{x} \in \bar{\sigma}\}.$$

265 Note that this definition extends sufficiently small neighbourhoods of arbitrary man-
 266 ifolds, see [17]. The edges of $\tilde{\sigma}$ are geodesic arcs between the vertices. Such a curved
 267 realization is well defined provided that the vertices of σ lie in a sufficiently small ball
 268 according to the following theorem of Karcher [23].

269 **THEOREM 5.1** (Karcher). *Let the sectional curvatures K of Ω be bounded, that*
 270 *is $\Lambda_- \leq K \leq \Lambda_+$. Let us consider the function \mathcal{E}_y on B_ρ , a geodesic ball of radius*
 271 *ρ that contains the vertices $\{p_i\}$ of the simplex σ . Assume that $\rho \in \mathbb{R}^+$ is less than*
 272 *half the injectivity radius and less than $\pi/4\sqrt{\Lambda_+}$ if $\Lambda_+ > 0$. Then \mathcal{E}_y has a unique*
 273 *minimum point in B_ρ , which is called the center of mass.*

274 Given an (abstract) simplicial complex \mathcal{K} with vertices in \mathcal{P} , we define the straight
 275 (resp., curved) realization of \mathcal{K} as the collection of straight (resp., curved) realizations
 276 of its simplices, and we write $\bar{\mathcal{K}} = \{\bar{\sigma}, \sigma \in \mathcal{K}\}$ and $\tilde{\mathcal{K}} = \{\tilde{\sigma}, \sigma \in \mathcal{K}\}$.

277 We will consider the case where \mathcal{K} is $\text{Del}_g(\mathcal{P})$. A simplex of $\widetilde{\text{Del}}_g(\mathcal{P})$ will simply be
 278 called a straight Riemannian Delaunay simplex and a simplex of $\widetilde{\text{Del}}_g(\mathcal{P})$ will be called
 279 a curved Riemannian Delaunay simplex, omitting “realization of”. In the next two
 280 sections, we give sufficient conditions for $\widetilde{\text{Del}}_g(\mathcal{P})$ and $\widetilde{\text{Del}}_g(\mathcal{P})$ to be embedded in Ω ,
 281 in which case we will call them the straight and the curved Riemannian *triangulations*
 282 of \mathcal{P} .

283 **5.1. Sufficient conditions for $\widetilde{\text{Del}}_g(\mathcal{P})$ to be a triangulation of \mathcal{P} .** It is
 284 known that $\widetilde{\text{Del}}_g(\mathcal{P})$ is embedded in Ω under sufficient conditions. We give a short
 285 overview of these results. As in Dyer et al. [17], we define the non-degeneracy of a
 286 simplex $\tilde{\sigma}$ of $\widetilde{\text{Del}}_g(\mathcal{P})$.

287 **DEFINITION 5.2.** *The curved realization $\tilde{\sigma}$ of a Riemannian Delaunay simplex σ*
 288 *is said to be non-degenerate if and only if it is homeomorphic to the standard simplex.*

289 Sufficient conditions for the complex $\widetilde{\text{Del}}_g(\mathcal{P})$ to be embedded in Ω were given
 290 in [17]: a curved simplex is known to be non-degenerate if the Euclidean simplex
 291 obtained by lifting the vertices to the tangent space at one of the vertices via the ex-
 292 ponential map has sufficient quality compared to the bounds on sectional curvature.
 293 Here, good quality means that the simplex is well shaped, which may be expressed ei-
 294 ther through its fatness (volume to longest edge length ratio) or its thickness (smallest
 295 height to longest edge length ratio).

296 Let us assume that, for each vertex p of $\text{Del}_g(\mathcal{P})$, all the curved Delaunay simplices
 297 in a neighborhood of p are non-degenerate and patch together well, conditions for
 298 which can be found in [2], see also [5] and [17]. We will assume throughout that the
 299 conditions of [2] are satisfied. Under these conditions, $\widetilde{\text{Del}}_g(\mathcal{P})$ is embedded in Ω . We
 300 call $\widetilde{\text{Del}}_g(\mathcal{P})$ the *curved Riemannian Delaunay triangulation* of \mathcal{P} .

301 **5.2. Sufficient conditions for $\widetilde{\text{Del}}_g(\mathcal{P})$ to be a triangulation of \mathcal{P} .** As-
 302 suming that the conditions for $\widetilde{\text{Del}}_g(\mathcal{P})$ to be embedded in Ω are satisfied, we now
 303 give conditions such that $\widetilde{\text{Del}}_g(\mathcal{P})$ is also embedded in Ω . The key ingredient will
 304 be a bound on the distance between a point of a simplex $\tilde{\sigma}$ and the corresponding
 305 point on the associated straight simplex $\bar{\sigma}$ (Lemma 5.3). This bound depends on the
 306 properties of the set of sites and on the local distortion of the metric field. When this
 307 bound is sufficiently small, $\widetilde{\text{Del}}_g(\mathcal{P})$ is embedded in Ω as stated in Theorem 5.4.

308 **LEMMA 5.3.** *Let σ be an n -simplex of $\text{Del}_g(\mathcal{P})$ and $U \subset \Omega$ an open neighborhood,*
 309 *as defined in Lemma 4.1, containing σ . Let \bar{x} be a point of $\bar{\sigma}$ and \tilde{x} the associated*
 310 *point on $\tilde{\sigma}$ (as defined in Equation (5)). If the geodesic distance d_g is close to the*

311 Euclidean distance $d_{g_{\mathbb{E}}}$, i.e. the distortion $\psi_U(g, g_{\mathbb{E}})$ is bounded by ψ_0 , then $\|\tilde{x} - \bar{x}\| \leq$
 312 $\sqrt{2 \cdot 4^3(\psi_0 - 1)\varepsilon^2}$.

313 We now apply [Lemma 5.3](#) to the facets of the simplices of $\widetilde{\text{Del}}_g(\mathcal{P})$. The altitude
 314 (height to longest edge length ratio) of the vertex p in a simplex τ is noted $D(p, \tau)$.

315 **THEOREM 5.4.** *Let \mathcal{P} be a δ -power protected (ε, μ) -net with respect to g on Ω .
 316 Let σ be any n -simplex of $\text{Del}_g(\mathcal{P})$, and p be any vertex of σ . Let τ be a facet of σ
 317 opposite of vertex p . If, for all $\tilde{x} \in \tilde{\tau}$, we have $\|\tilde{x} - \bar{x}\| \leq D(p_i, \sigma)$ (\bar{x} is defined in
 318 Equation (4)), then $\overline{\text{Del}}_d(\mathcal{P})$ is embedded in Ω .*

319 The condition $\|\tilde{x} - \bar{x}\| \leq D(p_i, \sigma)$ is achieved for a sufficiently dense sampling
 320 according to [Lemma 5.3](#) and the fact that the distortion $\psi_U(g, g_{\mathbb{E}})$ goes to 1 when the
 321 density increases and the neighborhood U is chosen smaller. The complete proofs
 322 of [Lemma 5.3](#) and [Theorem 5.4](#) can be found in [Appendix D](#).

323 **6. Discrete Riemannian structures.** Although Riemannian Voronoi
 324 diagrams and Delaunay triangulations are appealing from a theoretical point of view,
 325 they are very difficult to compute in practice despite many studies [28]. To circumvent
 326 this difficulty, we introduce the discrete Riemannian Voronoi diagram. This discrete
 327 structure is easy to compute (see our companion paper [31] for details) and, as will be
 328 shown in the following sections, it is a good approximation of the exact Riemannian
 329 Voronoi diagram. In particular, their dual Delaunay structures are identical under
 330 appropriate conditions.

331 We assume that we are given a dense triangulation of a closed and bounded
 332 neighbourhood of the domain Ω we call the *canvas* and denote by \mathcal{C} . The canvas will
 333 be used to approximate geodesic distances between points of Ω and to construct the
 334 discrete Riemannian Voronoi diagram of \mathcal{P} , which we denote by $\text{Vor}_g^d(\mathcal{P})$. This bears
 335 some resemblance to the graph-induced complex of Dey et al. [15]. Notions related
 336 to the canvas will explicitly carry *canvas* in the name (for example, an edge of \mathcal{C} is a
 337 *canvas edge*). In our analysis, we shall assume that the canvas is a dense triangulation,
 338 although weaker and more efficient structures can be used (see [section 11](#) and [31]).

339 **6.1. The discrete Riemannian Voronoi Diagram.** To define the discrete
 340 Riemannian Voronoi diagram of \mathcal{P} , we need to give a unique color to each site of \mathcal{P}
 341 and to color the vertices of the canvas accordingly. Specifically, each canvas vertex is
 342 colored with the color of its closest site.

343 **DEFINITION 6.1** (Discrete Riemannian Voronoi diagram). *Given a metric field g ,
 344 we associate to each site p_i its discrete Voronoi cell³ $V_g^d(p_i)$ defined as the union of
 345 all canvas simplices with at least one vertex of the color of p_i . We call the set of these
 346 cells the discrete Riemannian Voronoi diagram of \mathcal{P} , and denote it by $\text{Vor}_g^d(\mathcal{P})$.*

347 The idea of approximating a Voronoi diagram by its value of a set of points has pre-
 348 viously been explored to obtain quick visualizations of a Voronoi diagram [10, 29, 34].
 349 Our approach however aims to extract a provably correct dual of our discrete struc-
 350 ture. Observe furthermore that contrary to typical Voronoi diagrams, our discrete
 351 Riemannian Voronoi diagram is not a partition of the canvas. Indeed, there is a
 352 one canvas simplex-thick overlapping since each canvas simplex $\sigma_{\mathcal{C}}$ belongs to all the
 353 Voronoi cells whose sites' colors appear in the vertices of $\sigma_{\mathcal{C}}$. This is intentional and

³We do not claim or impose any a priori topological properties of $V_g^d(p_i)$. The name derives from the fact that it is a discrete version of a Voronoi cell. Note that even Riemannian Voronoi cells can have complicated topology if the sampling density of the points is low.

354 allows for a straightforward definition of the complex induced by this diagram, as
 355 shown below.

356 **6.2. The discrete Riemannian Delaunay complex.** We define the *discrete*
 357 *Riemannian Delaunay complex* as the set of simplices $\text{Del}_g^d(\mathcal{P}) = \{\sigma \mid \text{Vert}(\sigma) \subset$
 358 $\mathcal{P}, \cap_{p \in \text{Vert}(\sigma)} V_g^d(p) \neq \emptyset\}$. Using a triangulation as canvas offers a very intuitive way
 359 to construct the discrete complex since each canvas k -simplex σ of \mathcal{C} has $k + 1$
 360 vertices $\{v_0, \dots, v_k\}$ with respective colors $\{c_0, \dots, c_k\}$ corresponding to the sites
 361 $\{p_{c_0}, \dots, p_{c_k}\} \in \mathcal{P}$. Due to the way discrete Voronoi cells overlap, a canvas sim-
 362 plex $\sigma_{\mathcal{C}}$ belongs to each discrete Voronoi cell whose color appears in the vertices of σ .
 363 Therefore, the intersection of the discrete Voronoi cells $\{V_g^d(p_i)\}$ whose colors appear
 364 in the vertices of σ is non-empty and the simplex σ with vertices $\{p_i\}$ thus belongs
 365 to the discrete Riemannian Delaunay complex. In that case, we say that the canvas
 366 simplex $\sigma_{\mathcal{C}}$ *witnesses* (or is a witness of) σ . For example, if the vertices of a canvas
 367 3-simplex $\tau_{\mathcal{C}}$ have colors yellow–blue–blue–yellow, then the intersection of the dis-
 368 crete Voronoi cells of the sites p_{yellow} and p_{blue} is non-empty and the one-simplex σ
 369 with vertices p_{yellow} and p_{blue} belongs to the discrete Riemannian Delaunay complex.
 370 The canvas simplex $\tau_{\mathcal{C}}$ thus witnesses the (abstract, for now) edge between p_{yellow}
 371 and p_{blue} .

372 **Figure 3** illustrates a canvas painted with discrete Voronoi cells, and the witnesses
 373 of the discrete Riemannian Delaunay complex.

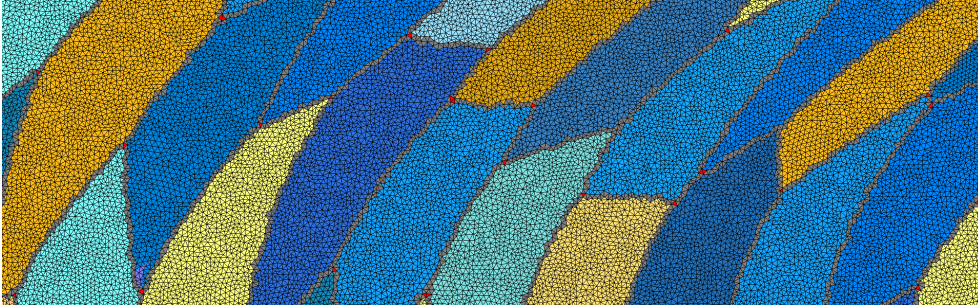


FIG. 3. A canvas (black edges) and a discrete Riemannian Voronoi diagram drawn on it. The canvas simplices colored in red are witnesses of Voronoi vertices. The canvas simplices colored in grey are witnesses of Voronoi edges. Canvas simplices whose vertices all have the same color are colored with that color.

374 *Remark 6.2.* If the intersection $\bigcap_{i=0 \dots k} V_g^d(p_{c_i})$ is non-empty, then the intersec-
 375 tion of any subset of $\{V_g^d(p_{c_i})\}_{i=0 \dots k}$ is non-empty. In other words, if a canvas sim-
 376 plex $\sigma_{\mathcal{C}}$ witnesses a simplex σ , then for each face τ of σ , there exists a face $\tau_{\mathcal{C}}$ of $\sigma_{\mathcal{C}}$
 377 that witnesses τ . This means that the discrete Riemannian Delaunay complex is in-
 378 deed a complex, and since we assume that there is no boundary, this complex is pure.
 379 Furthermore, it is sufficient to only consider canvas n -simplices whose vertices have
 380 all different colors to build $\text{Del}_g^d(\mathcal{P})$.

381 Similarly to the definition of curved and straight Riemannian Delaunay com-
 382 plexes, we can define their discrete counterparts we respectively denote by $\widetilde{\text{Del}}_g^d(\mathcal{P})$
 383 and $\overline{\text{Del}}_g^d(\mathcal{P})$. We will now exhibit conditions such that these complexes are well-
 384 defined and embedded in Ω .

385 **7. Equivalence between the discrete and the exact structures.** We first
 386 give conditions such that $\text{Vor}_g^d(\mathcal{P})$ and $\text{Vor}_g(\mathcal{P})$ have the same combinatorial structure,
 387 or, equivalently, that the dual Delaunay complexes $\text{Del}_g(\mathcal{P})$ and $\text{Del}_g^d(\mathcal{P})$ are identical.
 388 Under these conditions, the fact that $\text{Del}_g^d(\mathcal{P})$ is embedded in Ω will immediately fol-
 389 low from the fact that the exact Riemannian Delaunay complex $\text{Del}_g(\mathcal{P})$ is embedded
 390 (see [subsection 5.1](#) and [subsection 5.2](#)). It thus remains to exhibit conditions under
 391 which $\text{Del}_g^d(\mathcal{P})$ and $\text{Del}_g(\mathcal{P})$ are identical.

392 Requirements will be needed on both the set of sites in terms of density, sparsity
 393 and protection, and on the density of the canvas. The central idea in our analysis
 394 is that power protection of \mathcal{P} will imply a lower bound on the distance separating
 395 two non-adjacent Voronoi objects (and in particular two Voronoi vertices). From this
 396 lower bound, we will obtain an upper bound on the size on the cells of the canvas so
 397 that the combinatorial structure of the discrete diagram is the same as that of the
 398 exact one. The density of the canvas is expressed by e_C , the length of its longest edge.

399 The main result of this paper is the following theorem.

400 **THEOREM 7.1.** *Assume that \mathcal{P} is a δ -power protected (ε, μ) -net in Ω with respect
 401 to g . Assume further that ε is sufficiently small and δ is sufficiently large compared to
 402 the distortion between $g(p)$ and g in an ε -neighborhood of p . Let $\{\lambda_i\}$ be the eigenvalues
 403 of $g(p)$ and ℓ_0 a value that depends on ε and δ . Under conditions on ε, δ and ℓ_0 and
 404 provided that the canvas is dense enough, then $\text{Del}_g^d(\mathcal{P}) = \text{Del}_g(\mathcal{P})$.*

405 The following remark contains some pointers to the appendix for the constants
 406 of [Theorem 7.1](#) and notation.

407 *Remark 7.2.* First, the condition on the density of the canvas is:

$$408 \quad e_C < \min_{p \in \mathcal{P}} \left[\min_i \left(\sqrt{\lambda_i} \right) \min \{ \mu/3, \ell_0/2 \} \right].$$

409 We write ℓ_0 for a lower bound on the distance between two Voronoi vertices, this
 410 bound can be found in [\(15\)](#). A bound on the protection δ is given in [\(17\)](#), which
 411 involves a bound on the distortion of the Voronoi vertices in arbitrary dimension
 412 (χ) and two dimensions (χ_2), see [Lemma C.14](#) and [Lemma C.13](#) respectively. These
 413 bounds use the dihedral angle, see [\(9\)](#). The bound on the dihedral angle in turn
 414 is based on the lower bound on the height of the simplices, see [Lemma B.5](#), [\(8\)](#) in
 415 particular.

416 If we look at denser sets, that is when ε decreases, we can look at smaller neigh-
 417 bourhoods to determine the metric distortion, that is ψ_0 tends to 1. If we would
 418 assume that we have a bound on the derivative of the metric in the coordinate neigh-
 419 bourhood the distortion bound would decrease linearly with the size of the neighbour-
 420 hood. In this paper, as in for example in [\[24\]](#), we consider the distortion ψ_0 to be the
 421 fundamental quantity. This means we don't give convergence bounds on ψ_0 based on
 422 some other assumptions.

423 To be able to interpret the constants in the formulae in the appendix mentioned
 424 above it is important to note that we assume that:

- 425 • The protection parameter δ is proportional to the sampling parameter ε , thus
 426 there exists a positive ι , with $\iota \leq 1$, such that $\delta = \iota\varepsilon$
- 427 • the separation parameter μ is proportional to the sampling parameter ε and
 428 thus there exists a positive λ , with $\lambda \leq 1$, such that $\mu = \lambda\varepsilon$.

429 The rest of the paper will be devoted to the proof of this theorem. Our analysis is
 430 divided into two parts. We first consider in [section 8](#) the most basic case of a domain

431 of \mathbb{R}^n endowed with the Euclidean metric field. The result is given by [Theorem 8.1](#).
 432 The assumptions are then relaxed and we consider the case of an arbitrary metric
 433 field over Ω in [section 9](#). As we shall see, the Euclidean case already contains most of
 434 the difficulties that arise during the proof and the extension to more complex settings
 435 will be deduced from the Euclidean case by bounding the distortion.

436 **8. Equality of the Riemannian Delaunay complexes in the Euclidean**
 437 **setting.** In this section, we restrict ourselves to the case where the metric field is the
 438 Euclidean metric $g_{\mathbb{E}}$. To simplify matters, we initially assume that geodesic distances
 439 are computed exactly on the canvas. The following theorem gives sufficient conditions
 440 to have equality of the complexes.

441 **THEOREM 8.1.** *Assume that \mathcal{P} is a δ -power protected (ε, μ) -net of Ω with respect*
 442 *to the Euclidean metric field $g_{\mathbb{E}}$. Denote by \mathcal{C} the canvas, a triangulation with maximal*
 443 *edge length $e_{\mathcal{C}}$. If $e_{\mathcal{C}} < \min\{\mu/16, \delta^2/64\varepsilon\}$, then $\text{Del}_{\mathbb{E}}^d(\mathcal{P}) = \text{Del}_{\mathbb{E}}(\mathcal{P})$.*

444 We shall now prove [Theorem 8.1](#) by enforcing the two following conditions which,
 445 combined, give the equality between the discrete Riemannian Delaunay complex and
 446 the Riemannian Delaunay complex:

- 447 (1) for every Voronoi vertex in the Riemannian Voronoi diagram $v = \cap_{\{p_i\}} V_g(p_i)$,
 448 there exists at least one canvas simplex with the corresponding colors $\{c_{p_i}\}$;
 449 (2) no canvas simplex witnesses a simplex that does not belong to the Riemannian
 450 Delaunay complex (equivalently, no canvas simplex has vertices whose colors
 451 are those of non-adjacent Riemannian Voronoi cells).

452 [Item \(2\)](#) is a consequence of the separation of Voronoi objects, which in turn
 453 follows from power protection. The separation of Voronoi objects has previously been
 454 studied, for example by Boissonnat et al. [2]. Although the philosophy is the same,
 455 our setting is slightly more difficult and the results using power protection are new
 456 and use a more geometrical approach (see [Appendix A](#)).

457 **8.1. Sperner's lemma.** Rephrasing [Item \(1\)](#), we seek requirements on the density
 458 of the canvas \mathcal{C} and on the nature of the point set \mathcal{P} such that there exists at
 459 least one canvas n -simplex of \mathcal{C} that has exactly the colors c_0, \dots, c_d of the vertices
 460 p_0, \dots, p_d of a simplex σ , for all $\sigma \in \text{Del}_g(\mathcal{P})$. To prove the existence of such a canvas
 461 simplex, we employ Sperner's lemma [33], which is a discrete analog of Brouwer's
 462 fixed point theorem. We recall this result in [Theorem 8.2](#) and illustrate it in a two-
 463 dimensional setting in the figure below.

464 **THEOREM 8.2 (Sperner's lemma).**

465 *Let $\sigma = (p_0, \dots, p_n)$ be an n -simplex and let T_{σ} denote a triangulation of the simplex.*
 466 *Let each vertex $v' \in T_{\sigma}$ be colored such that the following conditions are satisfied:*

- 467
 - *The vertices p_i of σ all have different colors.*
 - *If a vertex p' lies on a k -face $(p_{i_0}, \dots, p_{i_k})$ of σ , then p' has the same color as*
 469 *one of the vertices of the face, that is p_{i_j} .*

470 *Then, there exists an odd number of simplices in T_{σ} whose vertices are colored with*
 471 *all $n + 1$ colors. In particular, there must be at least one.*

472 **Remark 8.3.** Note that Sperner's lemma is a combinatorial result, and it is thus
 473 the topological aspect of the simplex and its triangulation that are important, and
 474 not the geometry.

475 We shall apply Sperner's lemma to the canvas \mathcal{C} and show that for every Voronoi
 476 vertex v in the Riemannian Voronoi diagram, we can find a subset \mathcal{C}_v of the canvas
 477 that fulfills the assumptions of Sperner's lemma, hence obtaining the existence of a

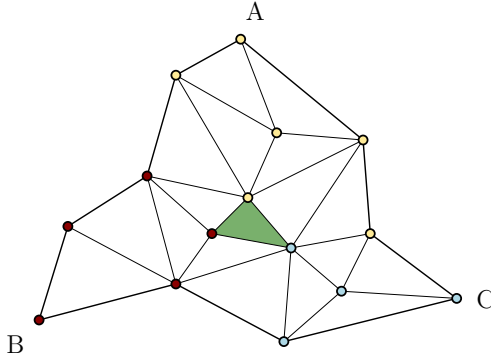


FIG. 4. *Sperner's lemma in 2D: the green triangle is colored with all 3 colors of the initial (topological) triangle ABC.*

478 canvas simplex in \mathcal{C}_v (and therefore in \mathcal{C}) that witnesses σ_v . Concretely, the subset \mathcal{C}_v
 479 is obtained in two steps:

- 480 – We first apply a barycentric subdivision of the Riemannian Voronoi cells
 481 incident to v . From the resulting set of simplices, we extract a triangulation
 482 \mathcal{T}_v composed of the simplices incident to v (subsection 8.2).
 483 – We then construct the subset \mathcal{C}_v by overlaying the boundary of \mathcal{T}_v and the
 484 canvas (subsection 8.3).

485 We then show that if the canvas simplices are small enough –in terms of edge length–
 486 then \mathcal{C}_v is the triangulation of a simplex that satisfies the assumptions of Sperner's
 487 lemma.

488 The construction of \mathcal{C}_v is detailed in the following sections and illustrated in Fig-
 489 ure 5: starting from a colored canvas (left), we subdivide the incident Voronoi cells
 490 of v to obtain \mathcal{T}_v (middle), and deduce the set of canvas simplices \mathcal{C}_v which forms a
 491 triangulation that satisfies the hypotheses of Sperner's lemma, thus giving the exist-
 492 ence of a canvas simplex (in green, right) that witnesses the Voronoi vertex within
 493 the union of the simplices, and therefore in the canvas.

494 **8.2. The triangulation \mathcal{T}_v .** For a given Voronoi vertex v in the Euclidean
 495 Voronoi diagram $\text{Vor}_{\mathbb{E}}(\mathcal{P})$ of the domain Ω , the initial triangulation \mathcal{T}_v is obtained
 496 by applying a combinatorial barycentric subdivision of the Voronoi cells of $\text{Vor}_{\mathbb{E}}(\mathcal{P})$
 497 that are incident to v : to each Voronoi cell V incident to v , we associate to each
 498 face F of V a point c_F in F which is not necessarily the geometric barycenter. We
 499 randomly associate to c_F the color of any of the sites whose Voronoi cells intersect
 500 to give F . For example, in a two-dimensional setting, if the face F is a Voronoi edge
 501 that is the intersection of V_{red} and V_{blue} , then c_F is colored either red or blue. Then,
 502 the subdivision of V is computed by associating to all possible sequences of faces
 503 $\{F_0, F_1, \dots, F_{n-1}, F_n\}$ such that $F_0 \subset F_1 \subset \dots \subset F_n = V$ and $\dim(F_{i+1}) = \dim(F_i) + 1$
 504 the simplex with vertices $\{c_{F_0}, c_{F_1}, \dots, c_{F_{n-1}}, c_{F_n}\}$. These barycentric subdivisions
 505 are allowed since Voronoi cells are convex polytopes.

506 Denote by Σ_V the set of simplices obtained by barycentric subdivision of V and
 507 $\Sigma_v = \{\cup \Sigma_V \mid v \in V\}$. The triangulation \mathcal{T}_v is defined as the star of v in Σ_v , that is the
 508 set of simplices in Σ_v that are incident to v . \mathcal{T}_v is illustrated in Figure 6 in dimension 3.
 509 As shall be proven in Lemma 8.4, \mathcal{T}_v can be used to define a combinatorial simplex
 510 that satisfies the assumptions of Sperner's lemma.

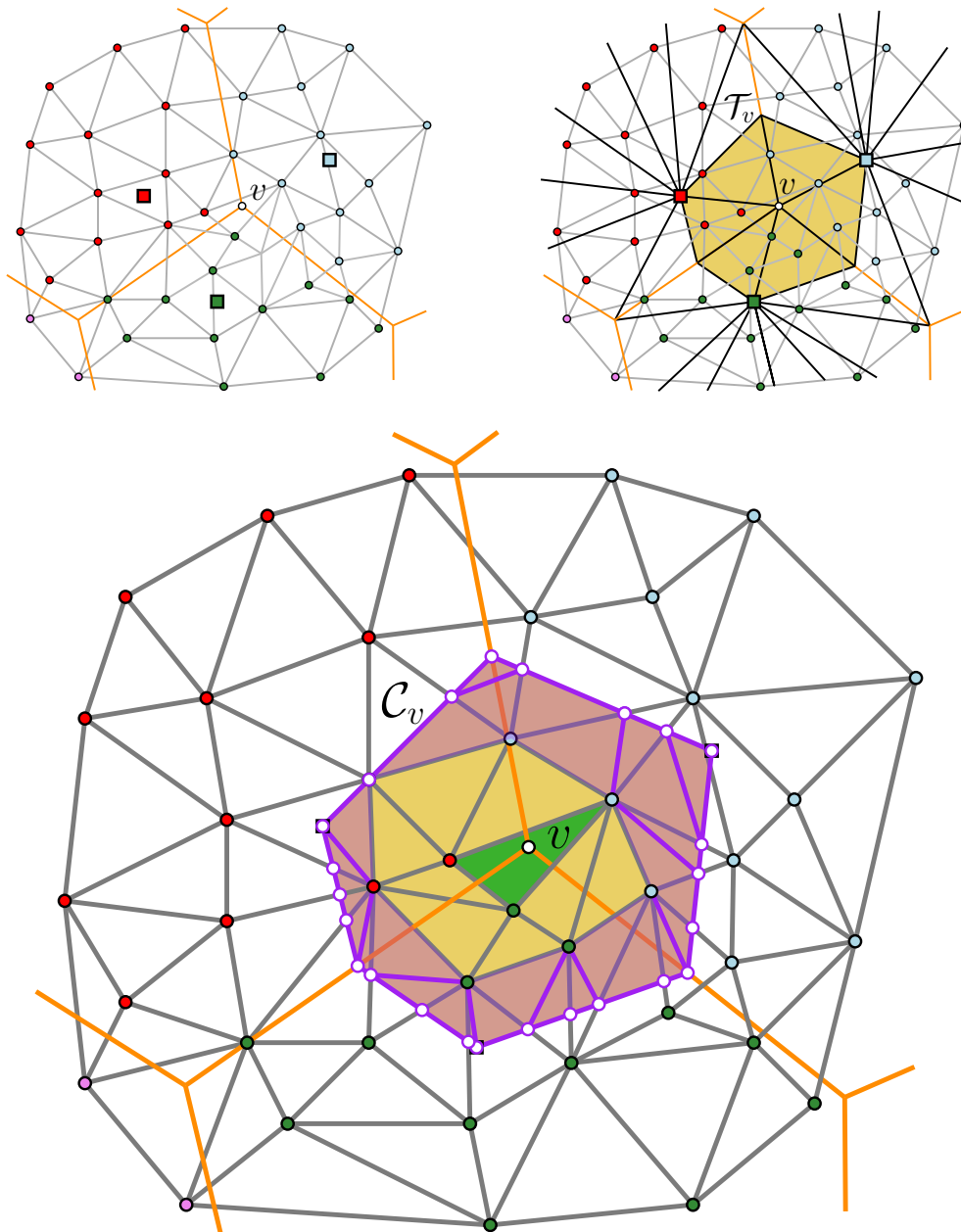


FIG. 5. Illustration of the construction of C_v . The Voronoi diagram is drawn with thick orange edges and the sites are colored squares. The canvas is drawn with thin gray edges and colored circular vertices. The middle frame shows the subdivision of the incident Voronoi cells with thick black edges and the triangulation \mathcal{T}_v is drawn in yellow. On the right frame, the set of simplices C_v is colored in purple (simplices that do not belong to C) and in dark yellow (simplices that belong to C).

511 \mathcal{T}_v as a triangulation of an n -simplex. By construction, the triangulation \mathcal{T}_v is
 512 a triangulation of the (Euclidean) Delaunay simplex σ_v dual of v as follows. We first
 513 perform the standard barycentric subdivision on this Delaunay simplex σ_v . We then

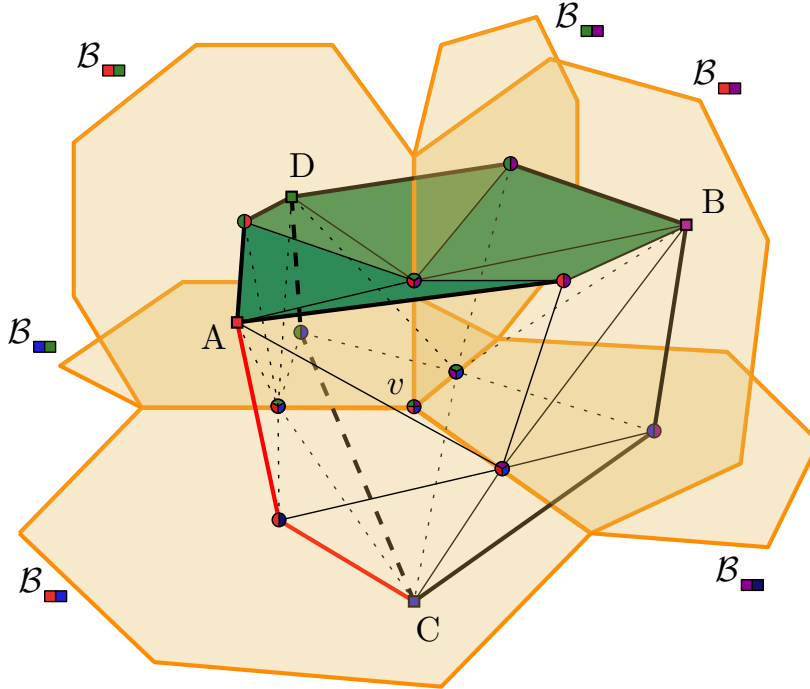


FIG. 6. The triangulation \mathcal{T}_v in 3D with initial tetrahedron $ABCD$. The vertices A , B , C , and D are associated each a unique color; points located on a bisector of these vertices are associated the colors of the cells whose intersection give the bisector. A face (in green) and an edge (in red) of σ_S .

514 map the barycenter of a k -face τ of σ_v to the point c_{F_i} on the Voronoi face F_i , where F_i
 515 is the Voronoi dual of the k -face τ . This gives a piecewise linear homeomorphism from
 516 the Delaunay simplex σ_v to the triangulation \mathcal{T}_v . We call the image of this map the
 517 simplex σ_S and refer to the images of the faces of the Delaunay simplex as the faces
 518 of σ_S . We can now apply Sperner's lemma.

519 **LEMMA 8.4.** *Let \mathcal{P} be a δ -power protected (ε, μ) -net. Let v be a Voronoi vertex*
 520 *in the Euclidean Voronoi diagram, $\text{Vor}_{\mathbb{E}}(\mathcal{P})$, and let Σ_v be defined as above. Then*
 521 *the simplex σ_S and the triangulation \mathcal{T}_v satisfy the assumptions of Sperner's lemma*
 522 *in dimension n .*

523 *Proof.* By the piecewise linear map that we have described above, \mathcal{T}_v is a triangulation
 524 of the simplex σ_S . Because by construction the vertices c_{F_i} lie on the Voronoi
 525 duals F_i of the corresponding Delaunay face τ , c_{F_i} has the one of the colors of the
 526 Delaunay vertices of τ . Therefore, σ_S satisfies the assumptions of Sperner's lemma
 527 and there exists an n -simplex in \mathcal{T}_v that witnesses v and its corresponding simplex σ_v
 528 in $\text{Del}_g(\mathcal{P})$. \square

529 **8.3. Building the triangulation \mathcal{C}_v .** Let p_i be the vertices of the k -face τ_S
 530 of σ_S . In this section we shall assume not only that τ_S is contained in the union of
 531 the Voronoi cells of $V(p_i)$, but in fact that τ_S is a distance $8e_C$ removed from the
 532 boundary of $\cup V(p_i)$, where e_C is the longest edge length of a simplex in the canvas.
 533 We will now construct a triangulation \mathcal{C}_v of σ_S such that:

- 534
 - σ_S and its triangulation \mathcal{C}_v satisfy the conditions of Sperner's lemma,

- the simplices of \mathcal{C}_v that have no vertex that lies on the boundary $\partial\sigma_S$ are simplices of the canvas \mathcal{C} .

The construction goes as follows. We first intersect the canvas \mathcal{C} with σ_S and consider the canvas simplices $\sigma_{\mathcal{C},i}$ such that the intersection of σ_S and $\sigma_{\mathcal{C},i}$ is non-empty. These simplices $\sigma_{\mathcal{C},i}$ can be subdivided into two sets, namely those that lie entirely in the interior of σ_S , which we denote by $\sigma_{\mathcal{C},i}^{\text{int}}$, and those that intersect the boundary, denoted by $\sigma_{\mathcal{C},i}^{\partial}$.

The simplices $\sigma_{\mathcal{C},i}^{\text{int}}$ are added to the set \mathcal{C}_v . We intersect the simplices $\sigma_{\mathcal{C},i}^{\partial}$ with σ_S and triangulate the intersection. Note that $\sigma_{\mathcal{C},i}^{\partial} \cap \sigma_S$ is a convex polyhedron and thus triangulating it is not a difficult task. The vertices of the simplices in the triangulation of $\sigma_{\mathcal{C},i}^{\partial} \cap \sigma_S$ are colored according to which Voronoi cell they belong to. Finally, the simplices in the triangulation of $\sigma_{\mathcal{C},i}^{\partial} \cap \sigma_S$ are added to the set \mathcal{C}_v .

Since \mathcal{T}_v is a triangulation of σ_S , the set \mathcal{C}_v is by construction also a triangulation of σ_S . This triangulation trivially gives a triangulation of the faces τ_S . Because we assume that τ_S is contained in the union of its Voronoi cells, with a margin of $8\epsilon_C$ we now can draw two important conclusions:

- The vertices of the triangulation of each face τ_S have the colors of the vertices p_i of τ_S .
- None of the simplices in the triangulation of $\sigma_{\mathcal{C},i}^{\partial} \cap \sigma_S$ can have $n + 1$ colors, because every such simplex must be close to one face τ_S , which means that it must be contained in the union of the Voronoi cells $V(p_i)$ of the vertices of τ_S .

We can now invoke Sperner's lemma; \mathcal{C}_v is a triangulation of the simplex σ_S whose every face has been colored with the appropriate colors (since σ_S triangulated by \mathcal{T}_v satisfies the assumptions of Sperner's lemma, see [Lemma 8.4](#)). This means that there is a simplex \mathcal{C}_v that is colored with $n + 1$ colors. Because of our second observation above, the simplex with these $n + 1$ colors must lie in the interior of σ_S and is thus a canvas simplex.

We summarize by the following lemma:

LEMMA 8.5. *If every face τ_S of σ_S with vertices p_i is at distance $8\epsilon_C$ from the boundary of the union of its Voronoi cells $\partial(\cup V(p_i))$, then there exists a canvas simplex in \mathcal{C}_v such that it is colored with the same vertices as the vertices of σ_S .*

The key task that we now face is to guarantee that faces τ_S indeed lie well inside of the union of the appropriate Voronoi regions. This requires first and foremost power protection. Indeed, if a point set is power protected, the distance between a Voronoi vertex c and the Voronoi faces that are not incident to c , which we will refer to from now on as *foreign* Voronoi faces, can be bounded, as shown in the following Lemma:

LEMMA 8.6. *Suppose that c is the circumcenter of a δ -power protected simplex σ of a Delaunay triangulation built from an ϵ -sample, then all foreign Voronoi faces are at least $\delta^2/8\epsilon$ far from c .*

The proof of this Lemma is given in the appendix ([Appendix A.2](#)).

In almost all cases, this result gives us the distance bound we require: we can assume that vertices $\{c_{F_0}, c_{F_1}, \dots, c_{F_{n-1}}, c_{F_n}\}$ which we used to construct \mathcal{T}_v , are well placed, meaning that there is a minimum distance between these vertices and foreign Voronoi objects. However it can still occur that foreign Voronoi objects are close to a face τ_S of σ_S . This occurs even in two dimensions, where a Voronoi vertex v' can be very close to a face τ_S because of obtuse angles, as illustrated in [Figure 7](#).

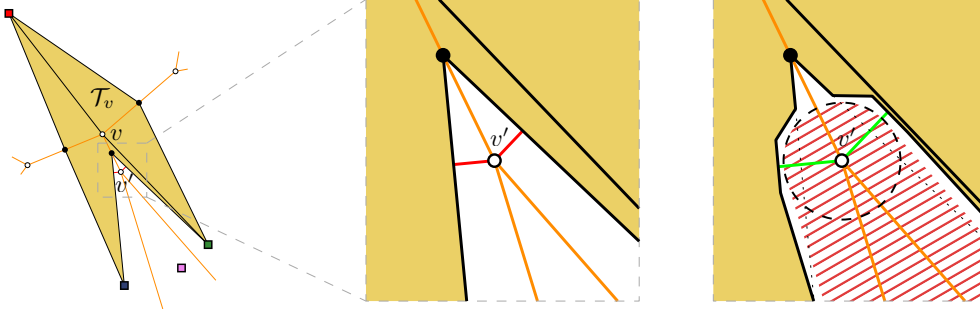


FIG. 7. The point v' can be arbitrarily close to \mathcal{T}_v , as shown by the red segments (left and center). After piecewise linear deformation, this issue is resolved, as seen by the green segments (right).

582 Thanks to power protection, we know that v' is removed from foreign Voronoi
 583 objects. This means that we can deform $\sigma_{\mathcal{S}}$ (in a piecewise linear manner) in a neigh-
 584 borhood of v' such that the distance between v' and all the faces of the deformed $\sigma_{\mathcal{S}}$
 585 is lower bounded.

586 In general the deformation of $\sigma_{\mathcal{S}}$ is performed by “radially pushing” simplices
 587 away from the foreign Voronoi faces of v with a ball of radius $r = \min \{ \mu/16, \delta^2/64\varepsilon \}$.
 588 The value $\mu/16$ is chosen so that we do not move any vertex of σ_v (the dual of v): in-
 589 deed, \mathcal{P} is μ -separated and thus $d_{g_{\mathbb{E}}}(p_i, p_j) > \mu$. The value $\delta^2/64\varepsilon$ is chosen so that $\sigma_{\mathcal{S}}$
 590 and its deformation stay isotopic (no “pinching” can happen), using Lemma 8.6. In
 591 fact it is advisable to use a piecewise linear version of “radial pushing”, to ensure that
 592 the deformation of $\sigma_{\mathcal{S}}$ is a polyhedron. This guarantees that we can triangulate the
 593 intersection, see Chapter 2 of Rourke and Sanderson [30]. After this deformation we
 594 can follow the steps we have given above to arrive at a well-colored simplex.

595 LEMMA 8.7. Let \mathcal{P} be a δ -power protected (ε, μ) -net. Let v be a Voronoi vertex of
 596 the Euclidean Voronoi diagram $\text{Vor}_{\mathbb{E}}(\mathcal{P})$, and \mathcal{T}_v as defined above. If the length e_C of
 597 the longest canvas edge is bounded as follows: $e_C < r = \min \{ \mu/16, \delta^2/64\varepsilon \}$, then there
 598 exists a canvas simplex that witnesses v and the corresponding simplex σ_v in $\text{Del}_{\mathbb{E}}(\mathcal{P})$.

599 Conclusion. So far, we have only proven that $\text{Del}_g(\mathcal{P}) \subseteq \text{Del}_g^d(\mathcal{P})$. The other
 600 inclusion, which corresponds to Item (2) mentioned above, is much simpler: as long
 601 as a canvas edge is shorter than the smallest distance between a Voronoi vertex and a
 602 foreign face of the Riemannian Voronoi diagram, then no canvas simplex can witness
 603 a simplex that is not in $\text{Del}_g(\mathcal{P})$. Such a bound is already given by Lemma 8.6
 604 and thus, if $e_C < \delta^2/8\varepsilon$ then $\text{Del}_g^d(\mathcal{P}) \subseteq \text{Del}_g(\mathcal{P})$. Observe that this requirement is
 605 weaker than the condition imposed in Lemma 8.7 and it was thus already satisfied.
 606 It follows that $\text{Del}_g^d(\mathcal{P}) = \text{Del}_g(\mathcal{P})$ if $e_C < \min \{ \mu/16, \delta^2/64\varepsilon \}$, which concludes the
 607 proof of Theorem 8.1.

608 Remark 8.8. Assuming that the point set is a δ -power protected (ε, μ) -net might
 609 seem like a strong assumption. However, it should be observed that any non-
 610 degenerate point set can be seen as a δ -power protected (ε, μ) -net, for a sufficiently
 611 large value of ε and sufficiently small values of δ and μ . Our results are therefore
 612 always applicable but the necessary canvas density increases as the quality of the
 613 point set worsens (Lemma 8.7). In our companion practical paper [31, Section 7], we
 614 showed how to generate δ -power protected (ε, μ) -nets for given values of ε , μ and δ .

615 **9. Extension to more complex settings.** In the previous section, we have
 616 placed ourselves in the setting of an (open) domain endowed with the Euclidean
 617 metric field. To prove [Theorem 7.1](#), we need to generalize [Theorem 8.1](#) to more
 618 general metrics, which will be done in the two following subsections.

619 The common path to prove $\text{Del}_g^d(\mathcal{P}) = \text{Del}_g(\mathcal{P})$ in all settings is to assume that \mathcal{P}
 620 is a power protected net with respect to the metric field. We then use the stabil-
 621 ity of entities under small metric perturbations to take us back to the now solved
 622 case of the domain Ω endowed with an Euclidean metric field. Separation and sta-
 623 bility of Delaunay and Voronoi objects has previously been studied by Boissonnat et
 624 al. [[2](#), [4](#)], but our work lives in a slightly more complicated setting. Moreover, our
 625 proofs are generally more geometrical and sometimes simpler. For completeness, the
 626 extensions of these results to our context are detailed in [Appendix A](#) for separation,
 627 and in [Appendix C](#) for stability.

628 We now detail the different intermediary settings. For completeness, the full
 629 proofs are included in the appendices.

630 **9.1. Constant metric field.** We first consider the rather easy case of a non-
 631 Euclidean but constant metric field over an (open) domain. The square root of a
 632 metric gives a linear transformation between the base space where distances are con-
 633 sidered in the metric and a *metric space* where the Euclidean distance is used (see [sub-](#)
 634 [section 4.1.1](#)). Additionally, we show that a δ -power protected (ε, μ) -net with respect
 635 to the constant metric is, after transformation, still a δ -power protected (ε, μ) -net but
 636 with respect to the Euclidean setting ([Lemma C.1](#) in [Appendix C](#)), bringing us back
 637 to the setting we have solved in [section 8](#). Bounds on the power protection, sampling
 638 and separation coefficients, and on the canvas edge length can then be obtained from
 639 the result for the Euclidean setting, using [Lemma 8.7](#). These bounds can be trans-
 640 ported back to the case of constant metric fields by scaling these values according to
 641 the smallest eigenvalue of the metric ([Theorem F.1](#) in [Appendix F](#)).

642 **9.2. Arbitrary metric field.** The case of an arbitrary metric field over Ω is
 643 handled by observing that an arbitrary metric field is locally well-approximated by a
 644 constant metric field. It is then a matter of controlling the distortion.

645 We first show that, for any point $p \in \Omega$, density separation and power protec-
 646 tion are locally preserved in a neighborhood U_p around p when the metric field g is
 647 approximated by the constant metric field $g' = g(p)$ ([Lemma C.2](#) and [Lemma C.16](#)
 648 in [Appendix C](#)): if \mathcal{P} is a δ -power protected (ε, μ) -net with respect to g , then \mathcal{P} is a
 649 δ' -power protected (ε', μ') -net with respect to g' . Previous results can now be applied
 650 to obtain conditions on $\delta', \varepsilon', \mu'$ and on the (local) maximal length of the canvas such
 651 that $\text{Del}_g^d(\mathcal{P}) = \text{Del}_g(\mathcal{P})$ (see [Lemma F.3](#) in [Appendix F](#)).

652 These local triangulations can then be stitched together to form a triangulation
 653 embedded in Ω because they are compatible by construction, see [[5](#)]. The (global)
 654 bound on the maximal canvas edge length is given by the minimum of the local
 655 bounds, each computed through the results of the previous sections. This ends the
 656 proof of [Theorem 7.1](#).

657 Once the equality between the complexes is obtained, conditions giving the em-
 658 beddability of the discrete Karcher Delaunay triangulation and the discrete straight
 659 Delaunay triangulation are given by previous results that we have established in [sub-](#)
 660 [section 5.1](#) and [subsection 5.2](#) respectively.

661 **10. Extensions of the main result. Approximate geodesic computa-**
 662 **tions** Approximate geodesic distance computations can be incorporated in the analy-

663 sis of the previous section by observing that computing inaccurately geodesic distances
 664 in a domain Ω endowed with a metric field g can be seen as computing exactly geodesic
 665 distances in Ω with respect to a metric field g' that is close to g (Appendix F.3 in Ap-
 666 pendix F).

667

668 **General manifolds** The previous section may also be generalized to an arbitrary
 669 smooth n -manifold \mathcal{M} embedded in \mathbb{R}^m . We shall assume that, apart from the metric
 670 induced by the embedding of the domain in Euclidean space, there is a second met-
 671 ric g defined on \mathcal{M} . Let $\pi_p : \mathcal{M} \rightarrow T_p\mathcal{M}$ be the orthogonal projection of points of \mathcal{M}
 672 on the tangent space $T_p\mathcal{M}$ at p . For a sufficiently small neighborhood $U_p \subset T_p\mathcal{M}$, π_p
 673 is a local diffeomorphism (see Niyogi et al. [27]).

674 Denote by \mathcal{P}_{T_p} the point set $\{\pi_p(p_i), p_i \in \mathcal{P}\}$ and \mathcal{P}_{U_p} the restriction of \mathcal{P}_{T_p}
 675 to U_p . Assuming that the conditions of Niyogi et al. [27] are satisfied (which are
 676 simple density constraints on ε compared to the reach of the manifold), the pullback
 677 of the metric with the inverse projection $(\pi_p^{-1})^*g$ defines a metric g_p on U_p such that
 678 for all $q, r \in U_p$, $d_{g_p}(q, r) = d_g(\pi_p^{-1}(q), \pi_p^{-1}(r))$. This implies immediately that if \mathcal{P} is
 679 a δ -power protected (ε, μ) -net on \mathcal{M} with respect to g then \mathcal{P}_{U_p} is a δ -power protected
 680 (ε, μ) -net on U_p . We have thus a metric on a subset of a n -dimensional space, in this
 681 case the tangent space, giving us a setting that we have already solved. It is left
 682 to translate the sizing field requirement from the tangent plane to the manifold \mathcal{M}
 683 itself. Note that the transformation π_p is completely independent of g . Boissonnat
 684 et al. [6] give bounds on the metric distortion of the projection on the tangent space.
 685 This result allows to carry the canvas sizing field requirement from the tangent space
 686 to \mathcal{M} . We stress that the local constructions are compatible, see [5].

687 **11. Implementation.** The construction of the discrete Riemannian Voronoi
 688 diagram and of the discrete Riemannian Delaunay complex has been implemented
 689 for $n = 2, 3$ and for surfaces of \mathbb{R}^3 . An in-depth description of our structure and its
 690 construction as well as an empirical study can be found in our practical paper [31].
 691 We simply make a few observations here.

692 The theoretical bounds on the canvas edge length provided by Theorem 7.1
 693 and Theorem 8.1 are far from tight and thankfully do not need to be honored in
 694 practice. A canvas whose edge length are about a tenth of the distance between
 695 two seeds suffices. This creates nevertheless unnecessarily dense canvasses since the
 696 density does not in fact need to be equal everywhere at all points and even in all
 697 directions. This issue is resolved by the use of anisotropic canvasses.

698 Our analysis was based on the assumption that all canvas vertices are painted
 699 with the color of the closest site. In our implementation, we color the canvas using
 700 a multiple-front vector Dijkstra algorithm [8], which empirically does not suffer from
 701 the same convergence issues as the traditional Dijkstra algorithm, starting from all the
 702 sites. It should be noted that any geodesic distance computation method can be used,
 703 as long as it converges to the exact geodesic distance when the canvas becomes denser.
 704 The Riemannian Delaunay complex is built on the fly during the construction of the
 705 discrete Riemannian Voronoi diagram: when a canvas simplex is first fully colored,
 706 its combinatorial information is extracted and the corresponding simplex is added
 707 to $\text{Del}_g(\mathcal{P})$.

708 **12. Conclusion and future work.** We introduce a discrete Voronoi diagram
 709 of point sets in a subdomain of \mathbb{R}^n endowed with a smooth metric. Based on this
 710 Voronoi diagram we can define (in a constructive way) its dual structure, the Riemannian
 711 Delaunay complex. In [31], we have shown that this approach yields a practical

712 way to construct triangulations of surfaces. By closely analyzing the geometry of the
 713 Riemannian Voronoi diagram, we detail conditions under which the discrete Riemannian
 714 Delaunay complex is combinatorially equivalent to the Riemannian Delaunay
 715 complex. Note that previous work establishes conditions under which the discrete
 716 Riemannian Delaunay complex is a triangulation. We also give conditions under
 717 which the discrete Riemannian Delaunay complex can be embedded using simplices
 718 that are either straight in \mathbb{R}^n or curved, by which we mean that they conform to the
 719 Riemannian metric. Finally, we show how to extend these results for domains in \mathbb{R}^n
 720 to submanifolds of Euclidean space.

721 Combining the assumptions of protection and separation of a point set yields a
 722 powerful tool that can be applied to other variants of Voronoi diagrams. Labelle and
 723 Shewchuk [24] and Du and Wang [16] introduced such variants, as we discussed in the
 724 introduction. The authors are working on extending the results of this paper to these
 725 other variants of Voronoi diagrams.

726 Proofs and technical details are comprehensively described in the appendix.

727 **Overview of the appendices.** The topics of the appendices are as follows:

728 **Appendix A** We discuss the separation of Voronoi objects and prove [Lemma 8.6](#). The
 729 main differences between this appendix and [2] are in our definition of metric
 730 distortion, the use of power protection, and the more geometrical nature of
 731 the proofs.

732 **Appendix B** This appendix is related to the previous one and focuses on dihedral angles
 733 of Delaunay simplices. These results are intermediary steps used in Appen-
 734 dices [Appendix C](#) and [Appendix F](#).

735 **Appendix C** Here we built upon the previous two sections and discuss the stability of nets
 736 and Voronoi cells. This section distinguishes itself by the elementary and
 737 geometrical nature of the proofs.

738 **Appendix D** We prove that the Delaunay simplices can be straightened, under sufficient
 739 conditions ([Theorem 5.4](#)). The proof of the stability of the center of mass,
 740 which forms the core of this appendix, is also remarkable in the sense that it
 741 generalizes trivially to a far more general setting.

742 **Appendix E** We illustrate a degenerate case of [subsection 8.3](#).

743 **Appendix F** This appendix gives the proofs for our main result, [Theorem 7.1](#), in the general
 744 setting of an arbitrary metric. We naturally rely heavily on Appendices [Ap-](#)
 745 [pendix A](#), [Appendix B](#), [Appendix C](#), and [Appendix D](#).

746 **Appendix A. Separation of Voronoi objects.** Power protected point sets
 747 were introduced to create quality bounds for the simplices of Delaunay triangulations
 748 built using such point sets [2]. We will show that power protection allows to deduce
 749 additional useful results for Voronoi diagrams. In this section we show that when a
 750 Voronoi diagram is built using a power protected sample set, its non-adjacent Voronoi
 751 faces, and specifically its Voronoi vertices are separated. This result is essential to our
 752 proofs in [section 8](#) and [section 9](#) where we approximate complicated Voronoi cells with
 753 simpler Voronoi cells without creating inversions in the dual, which is only possible
 754 because we know that Voronoi vertices are far from one another.

755 We assume that the protection parameter δ is proportional to the sampling para-
 756 meter ε , thus there exists a positive ι , with $\iota \leq 1$, such that $\delta = \iota\varepsilon$. We assume
 757 that the separation parameter μ is proportional to the sampling parameter ε and thus
 758 there exists a positive λ , with $\lambda \leq 1$, such that $\mu = \lambda\varepsilon$.

759 **A.1. Separation of Voronoi vertices.** The main result provides a bound on
 760 the Euclidean distance between Voronoi vertices of the Euclidean Voronoi diagram of
 761 a point set and is given by [Lemma A.4](#). The following three lemmas are intermediary
 762 results needed to prove [Lemma A.4](#).

763 **LEMMA A.1.** *Let $B = B(c, R)$ and $B' = B(c', R')$ be two n -balls whose bounding
 764 spheres ∂B and $\partial B'$ intersect, and let H be the bisecting hyperplane of B and B' , i.e.
 765 the hyperplane that contains the $(n - 2)$ -sphere $S = \partial B \cap \partial B'$. Let θ be the angle of
 766 the cone (c, S) . Writing $\rho = \frac{R'}{R}$ and $\|c - c'\| = \lambda R$, we have*

$$767 \quad (6) \quad \cos(\theta) = \frac{1 + \lambda^2 - \rho^2}{2\lambda}.$$

769 *If $R \geq R'$, we have $\cos(\theta) \geq \frac{\lambda}{2}$.*

770 *Proof.* Let $q \in S$; applying the cosine rule to the triangle $\triangle cc'q$ gives

$$771 \quad (7) \quad \lambda^2 R^2 + R^2 - 2\lambda R^2 \cos(\theta) = R'^2,$$

773 which proves Equation (6). If $R \geq R'$, then $\rho \leq 1$, and $\cos(\theta) \geq \lambda/2$ immediately
 774 follows from Equation (6). \square

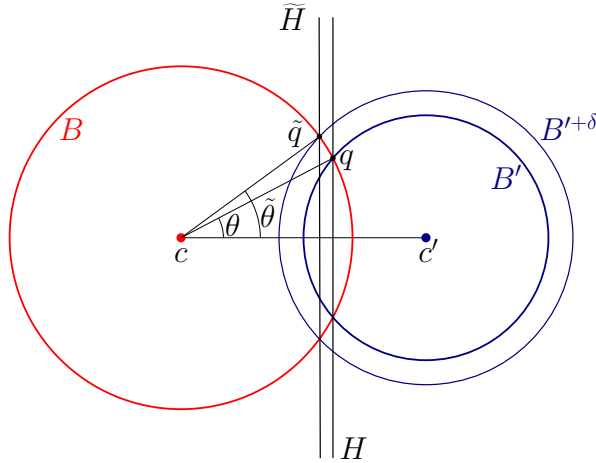


FIG. 8. Construction used in [Lemma A.1](#) and [Lemma A.2](#).

775 **LEMMA A.2.** *Let $B = B(c, R)$ and $B' = B(c', R')$ be two n -balls whose bounding
 776 spheres ∂B and $\partial B'$ intersect, and let $\tilde{\theta}$ be the angle of the cone (c, \tilde{S}) where $\tilde{S} =$
 777 $\partial B \cap \partial B'^{\delta}$. Writing $\|c - c'\| = \lambda R$, we have*

$$778 \quad \cos(\tilde{\theta}) = \cos(\theta) - \frac{\delta^2}{2R^2\lambda}$$

780 *Proof.* Let $\tilde{q} \in \tilde{S}$, applying the cosine rule to the triangle $\triangle cc'\tilde{q}$ gives

$$781 \quad \lambda^2 R^2 + R^2 - 2\lambda R^2 \cos(\tilde{\theta}) = R'^2 + \delta^2.$$

783 Subtracting Equation (7) from the previous equality yields $\delta^2 = 2\lambda R^2(\cos(\theta) - \cos(\tilde{\theta}))$,
 784 which proves the lemma. \square

785 The altitude of the vertex p_i in the simplex σ is denoted by $D(p_i, \sigma)$.

786 LEMMA A.3. *Let $\sigma = p * \tau$ and $\sigma' = p' * \tau$ be two Delaunay simplices sharing a*
 787 *common facet τ . (Here $*$ denotes the join operator.) Let $B(\sigma) = B(c, R)$ and $B(\sigma') =$*
 788 *$B(c', R')$ be the circumscribing balls of σ and σ' respectively. Then σ' is δ -power*
 789 *protected with respect to p , that is $p \notin B(\sigma')^{+\delta}$ if and only if $\|c - c'\| \geq \frac{\delta^2}{2D(p, \sigma)}$.*

790 *Proof.* The spheres ∂B and $\partial B'^{+\delta}$ intersect in a $(n - 2)$ -sphere \tilde{S} which is con-
 791 tained in a hyperplane \tilde{H} parallel to the hyperplane $H = \text{aff}(\tau)$. For any $\tilde{q} \in \tilde{S}$ we
 792 have

$$793 \quad d(\tilde{H}, H) = d(\tilde{q}, H) = R(\cos(\theta) - \cos(\tilde{\theta})) = \frac{\delta^2}{2\|c - c'\|},$$

794 where the last equality follows from Lemma A.2 and $d(\tilde{H}, H)$ denotes the distance
 795 between the two parallel hyperplanes. See Figure 8 for a sketch. Since $p \in \partial B$, p
 796 belongs to $B(\sigma')^{+\delta}$ if and only if p lies in the strip bounded by H and \tilde{H} , which is
 797 equivalent to

$$798 \quad d(p, H) = D(p, \sigma) < \frac{\delta^2}{2\|c - c'\|}.$$

799 The result now follows. □

800 We can make this bound independent of the simplices considered, as shown
 801 in Lemma A.4.

802 LEMMA A.4. *Let \mathcal{P} be a δ -power protected (ε, μ) -sample. The protection param-*
 803 *eter ι is given by $\delta = \iota\varepsilon$. For any two adjacent Voronoi vertices c and c' of $\text{VD}(\mathcal{P})$,*
 804 *we have*

$$805 \quad \|c - c'\| \geq \frac{\delta^2}{4\varepsilon} = \frac{\iota^2\varepsilon}{4}.$$

806 *Proof.* For any simplex σ , we have $D(p, \sigma) \leq 2R_\sigma$ for all $p \in \text{Vert}(\sigma)$, where R_σ
 807 denotes the radius of the circumsphere of σ . For any σ in the triangulation of an
 808 ε -net, we have $R_\sigma \leq \varepsilon$. Thus $D(p, \sigma) \leq 2\varepsilon$, and Lemma A.3 yields $\|c - c'\| \geq \delta^2/4\varepsilon$. □

809 *Remark A.5.* In this section, we have computed a lower bound on the distance
 810 between two (adjacent) Voronoi vertices. In Appendix C, we shall show that Voronoi
 811 vertices are stable under metric perturbations, meaning that when a metric field is
 812 slightly modified, the position of a Voronoi vertex does not move too much. The
 813 combination of this separation and stability will then be the basis of many proofs in
 814 this paper.

815 **A.2. Separation of Voronoi faces (Proof of Lemma 8.6).** Similar results
 816 can be obtained on the distance between a Voronoi vertex c and faces that are not
 817 incident to c , also referred to as *foreign* faces. Note that we are still in the context of
 818 an Euclidean metric.

819 The following lemma can be found in [3, Lemma 3.3] for ordinary protection
 820 instead of power protection.

821 LEMMA A.6. *Suppose that c is the circumcenter of a δ -power protected simplex σ*
 822 *of a Delaunay triangulation built from an ε -sample, then all foreign Voronoi faces are*
 823 *at least $\delta^2/8\varepsilon$ removed from c .*

824 *Proof.* We denote by p_0 an (arbitrary) vertex of σ , and by q a vertex that is not
 825 in σ but is adjacent to p_0 in $\text{Del}(\mathcal{P})$. Let x be a point in $B(c, r) \cap V_{\mathbb{E}}(p_0)$, with
 826 $0 < r < \delta^2/4\varepsilon$. The upper bound for r is chosen with Lemma A.3 in mind: we

827 are trying to find a condition such that $x \in V_{\mathbb{E}}(p_0)$. The notations are illustrated
 828 in Figure 9.

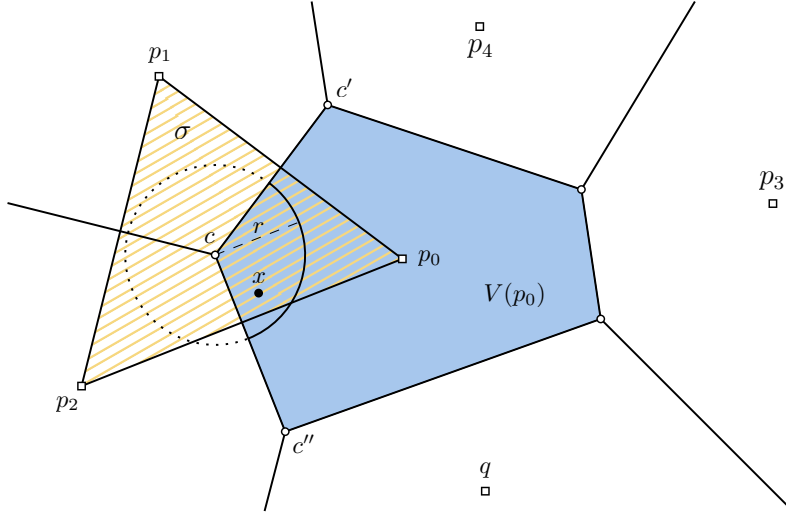


FIG. 9. Illustration of the notations for the proof of Lemma 8.6. The simplex σ is dashed in yellow and has vertices $p_0 p_1 p_2$. The distances cc' and cc'' are lower bounded by $\delta^2/4\varepsilon$.

829 Because of the triangle inequality, we have that

$$\begin{aligned} 830 \quad & |d(c, q) - d(x, q)| \leq d(x, c) \\ 831 \quad & |d(c, p_i) - d(x, p_i)| \leq d(x, c). \end{aligned}$$

833 Furthermore, we have by power protection that $d(c, q)^2 \geq d(c, p_i)^2 + \delta^2$. Therefore,

$$\begin{aligned} 834 \quad & (d(x, q) + r)^2 \geq (d(x, p_i) - r)^2 + \delta^2 \\ 835 \quad & d(x, q)^2 + 2rd(x, q) \geq d(x, p_i)^2 - 2rd(x, p_i) + \delta^2 \\ 836 \quad & d(x, q)^2 \geq d(x, p_i)^2 - 2r(d(x, p_i) + d(x, q)) + \delta^2. \end{aligned}$$

838 Without loss of generality, we can assume that q is the site closest to c and thus
 839 $d(x, q) < d(x, c)$. If \mathcal{P} is an ε -net, we have

$$840 \quad d(x, p_i) + d(x, q) \leq d(x, p_i) + d(c, q) \leq \varepsilon + 3\varepsilon = 4\varepsilon,$$

841 thus

$$842 \quad d(x, q)^2 \geq d(x, p_i)^2 - 8r\varepsilon + \delta^2.$$

844 This implies that as long $r < \delta^2/8\varepsilon$, x lies in a Voronoi object associated to the
 845 vertices p_i of σ . \square

846 Further progressing, we can show that Voronoi faces are thick, using Lemma A.7.
 847 This property is useful to construct a triangulation that satisfies the hypotheses of
 848 Sperner's lemma.

849 LEMMA A.7. Let \mathcal{P} be a δ -power protected (ε, μ) -net. Let V_0 be the Voronoi cell
 850 of the site $p_0 \in \mathcal{P}$ in the Euclidean Voronoi diagram $\text{VD}_{\mathbb{E}}(\mathcal{P})$. Then for any k -face F_0

851 of V_0 with $k \in [1, \dots, n]$, there exists $x \in F_0$ such that

$$852 \quad d(x, \partial F_0) > \frac{\delta^2}{16\varepsilon},$$

853 where ∂F_0 denotes the boundary of the face F_0 .

854 *Proof.* All the vertices of F_0 are circumcenters of $\text{VD}_{\mathbb{E}}(\mathcal{P})$. Consider the erosion
855 of the face F_0 by a ball of radius $\frac{\delta^2}{16\varepsilon}$ and denote it F_0^- . If F_0^- is empty, we contradict
856 the separation Lemma (Lemma 8.6). \square

857 **Appendix B. Bounds on dihedral angles.** The use of nets allows us to
858 deduce bounds on the dihedral angles of faces of the Delaunay triangulation, as well
859 as on the dihedral angles between adjacent faces of a Voronoi diagram. Those bounds
860 are frequently used throughout this paper, and specifically to prove stability of Voronoi
861 vertices (see Appendix C). Since we are interested in dihedral bounds in the Euclidean
862 setting, the point set is first assumed to be a net with respect to the Euclidean metric
863 field. We complicate matters slightly with Lemma B.5 by assuming that the point set
864 is a power protected net with respect to an arbitrary metric field that is not too far
865 from the Euclidean metric field (in terms of distortion), and still manage to expose
866 bounds with respect to the Euclidean distance.

867 **B.1. Bounds on the dihedral angles of Euclidean Voronoi cells.** Assum-
868 ing that a point set is an (ε, μ) -net allows us to deduce lower and upper bounds on the
869 dihedral angles between adjacent Voronoi faces when the metric field is Euclidean.

870 **LEMMA B.1.** *Let $\Omega = \mathbb{R}^n$ and \mathcal{P} be an (ε, μ) -net with respect to the Euclidean
871 distance on Ω . Let $p \in \mathcal{P}$ and $V(p)$ be the Voronoi cell of $p \in \mathcal{P}$. Let $q, r \in \mathcal{P}$ be two
872 sites such that $V(q)$ and $V(r)$ are adjacent to $V(p)$ in the Euclidean Voronoi diagram
873 of \mathcal{P} . Let θ be the dihedral angle between $\text{BS}(p, q)$ and $\text{BS}(p, r)$. Then*

$$874 \quad 2 \arcsin\left(\frac{\mu}{2\varepsilon}\right) \leq \theta \leq \pi - \arcsin\left(\frac{\mu}{2\varepsilon}\right).$$

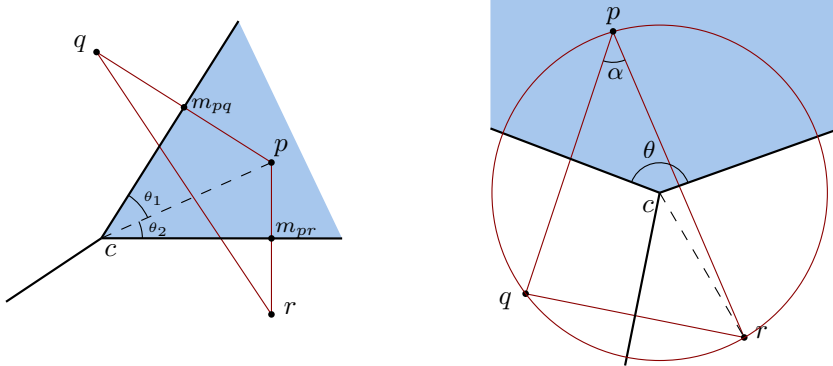


FIG. 10. Construction and notations used in Lemma B.1.

875 *Proof.* We consider the plane \mathcal{H} that passes through the sites p, q and r . Notations
876 used below are illustrated in Figure 10.

877 **Lower bound.** Let m_{pq} and m_{pr} be the projections of the site p on respectively
878 the bisectors $\text{BS}(p, q)$ and $\text{BS}(p, r)$. Since \mathcal{P} is an (ε, μ) -net, we have that $l_q =$

879 $\|p - m_{pq}\| \geq \mu/2$, $l_r = \|p - m_{pr}\| \geq \mu/2$ and $L = \|p - c\| \leq \varepsilon$. Thus

$$880 \quad \theta = \arcsin\left(\frac{l_q}{L}\right) + \arcsin\left(\frac{l_r}{L}\right) \geq 2 \arcsin\left(\frac{\mu/2}{\varepsilon}\right) = 2 \arcsin\left(\frac{\mu}{2\varepsilon}\right).$$

881 Note that since $0 < \mu < 2\varepsilon$, we have $0 < \mu/2\varepsilon < 1$.

882 **Upper bound.** To obtain an upper bound on θ , we compute a lower bound on the
 883 angle $\alpha = \widehat{qpr}$ at p , noting that $\theta = \pi - \alpha$. Let $l_{qr} = \|q - r\|$, and $R = \|c - r\|$. By
 884 the law of sines, we have

$$885 \quad \frac{l_{qr}}{\sin(\alpha)} = 2R.$$

886 Since \mathcal{P} is an (ε, μ) -net, we have $l_{qr} \geq \mu$ and $R \leq \varepsilon$. Finally,

$$887 \quad \alpha \geq \arcsin\left(\frac{\mu}{2\varepsilon}\right) \implies \theta \leq \pi - \arcsin\left(\frac{\mu}{2\varepsilon}\right). \quad \square$$

889 B.2. Bounds on the dihedral angles of Euclidean Delaunay simplices.

890 Bounds on the dihedral angles of simplices guarantee the thickness – the smallest
 891 height of any vertex – of simplices, and thus their quality. Additionally, they can be
 892 used to show that circumcenters of adjacent simplices are far from one another, thus
 893 proving the stability of circumcenters and of Delaunay simplices.

894 **B.2.1. Using power protection with respect to the Euclidean metric**
 895 **field.** We first assume that the metric field g is the Euclidean metric field $g_{\mathbb{E}}$
 896 and show that the simplices of an Euclidean Delaunay triangulation constructed from a
 897 power protected net are thick.

898 Recall that the dihedral angle can be expressed through heights as

$$899 \quad \sin \angle(\text{aff}(\sigma_p), \text{aff}(\sigma_q)) = \frac{D(p, \sigma)}{D(p, \sigma_q)} = \frac{D(q, \sigma)}{D(q, \sigma_p)}.$$

900 The bound on dihedral angles is obtained by bounding the height of vertices in a
 901 simplex. An obvious upper bound on the height of a vertex p in σ is $D(p, \sigma) <$
 902 2ε . A lower bound is already obtained in [Lemma A.3](#): we have that $D(p, \sigma) \geq$
 903 $\delta^2/2 \|c - c'\| = \delta^2/4\varepsilon$. We can thus bound the dihedral angles as follows:

904 **LEMMA B.2.** *Let \mathcal{P} be a δ -power protected (ε, μ) -net with respect to the Euclidean*
 905 *metric field g_0 . Let φ be the dihedral angle between two facets τ_1 and τ_2 of a simplex σ*
 906 *of $\text{Del}_{g_0}(\mathcal{P})$. Then*

$$907 \quad \arcsin(s_0) \leq \varphi \leq \pi - \arcsin(s_0),$$

908 *with $s_0 = \frac{A_{\lambda, \iota}}{2}$ and $A_{\lambda, \iota}$ defined as in the previous Lemma.*

909 *Proof.* Recall that

$$910 \quad \sin(\varphi) = \sin \angle(\text{aff}(\sigma_p), \text{aff}(\sigma_q)) = \frac{D(p, \sigma)}{D(p, \sigma_q)} = \frac{D(q, \sigma)}{D(q, \sigma_p)}.$$

911 From previous remarks, we have that

$$912 \quad D(q, \sigma_p) \geq D(q, \sigma) \geq \frac{\delta^2}{4\varepsilon},$$

913 and $D(q, \sigma) \leq 2\varepsilon$. Thus, if $\varphi = \angle(\text{aff}(\sigma_p), \text{aff}(\sigma_q))$, then

$$914 \quad \sin \angle(\text{aff}(\sigma_p), \text{aff}(\sigma_q)) \geq \frac{\delta^2}{4\varepsilon} \frac{1}{2\varepsilon} = \frac{\iota^2}{2} =: s_0$$

915

916 Note that $0 < s_0 < 1$ and thus

$$917 \quad \arcsin(s_0) \leq \varphi \leq \pi - \arcsin(s_0). \quad \square$$

918 **B.2.2. Using power protection with respect to an arbitrary metric field.**

919 When considering a Voronoi diagram built using the geodesic distance induced by an
 920 arbitrary metric field g , the assumption of a power protected net is made with respect
 921 to this geodesic distance. To prove the stability of the power protected assumption
 922 under metric perturbation, we will however need to deduce lower and upper bounds
 923 on the dihedral angles between faces of the simplices of the Riemannian Delaunay
 924 complex with respect to the Euclidean metric field. We prove here that if the point
 925 set \mathcal{P} is a δ -power protected (ε, μ) -net with respect to an arbitrary metric field g and if
 926 the distortion between g and the Euclidean metric field $g_{\mathbb{E}}$ is small, then the dihedral
 927 angles of the simplices of the Euclidean Delaunay triangulation of \mathcal{P} can be bounded.

928 We first give a result on the stability of Delaunay balls which expresses that if
 929 two metric fields have low distortion, the Delaunay balls of a simplex with respect to
 930 each metric field are close. One of these metric fields is assumed to be the Euclidean
 931 metric field. A similar result can be found in the proof of Lemma 5 in the theoretical
 932 analysis of locally constant anisotropic meshes of Boissonnat et al. [7].

933 **LEMMA B.3.** *Let $U \subset \Omega$ be open, and g and g' be two Riemannian metric fields on*
 934 *U . Let $\psi_0 = \psi_U(g, g')$ be the bound on the metric distortion of the fields g and g' an*
 935 *assume furthermore that U is small enough such that $\psi_0 < 2$, and the assumptions of*
 936 *Lemma 4.1 are satisfied. To be self-contained, we suppose that U is included in a ball*
 937 *$B_g(p_0, r_0)$, with $p_0 \in U$ and $r_0 \in \mathbb{R}^+$, such that $\forall p \in B(p_0, 5r_0), \psi(g(p), g'(p)) \leq \psi_0$*
 938 *and the ball $B(p_0, 5r_0)$ is geodesically convex. Assume furthermore that g' is the*
 939 *Euclidean distance (thus $d_{g'} = d_{g_{\mathbb{E}}}$). Let $B = B_g(c, r)$ be the geodesic ball with respect*
 940 *to the metric field g , centered on $c \in U$ and of radius r . Assume that $B_{\mathbb{E}}(c, \psi_0 r) \subset U$.*
 941 *Then B can be encompassed by two Euclidean balls $B_{\mathbb{E}}(c, r - \psi_0)$ and $B_{\mathbb{E}}(c, r + \psi_0)$ with*
 942 *$r - \psi_0 = r/\psi_0$ and $r + \psi_0 = \psi_0 r$, that is $B_{\mathbb{E}}(c, r - \psi_0) \subset B \subset B_{\mathbb{E}}(c, r + \psi_0)$.*

943 *Proof.* This is a straight consequence from Lemma 4.1. Indeed, we have for all
 944 $x \in U$ that

$$945 \quad \frac{1}{\psi_0} d_{g_{\mathbb{E}}}(c, x) \leq d_g(c, x) \leq \psi_0 d_{g_{\mathbb{E}}}(c, x),$$

946 and similarly

$$947 \quad \frac{1}{\psi_0} d_g(c, x) \leq d_{g_{\mathbb{E}}}(c, x) \leq \psi_0 d_g(c, x).$$

948 Thus,

$$949 \quad x \in B_{\mathbb{E}}\left(c, \frac{r}{\psi_0}\right),$$

950 which is equivalent to

$$951 \quad d_{g_{\mathbb{E}}}(c, x) \leq \frac{r}{\psi_0},$$

952 implying:

$$953 \quad \frac{1}{\psi_0} d_g(c, x) \leq \frac{r}{\psi_0}$$

954

957 giving us $B_{\mathbb{E}}(c, r_{-\psi_0}) \subset B$.

958 On the other hand, we have

$$959 \quad x \in B_g(c, r).$$

961 Thus,

$$962 \quad d_g(c, x) \leq r$$

964 and

$$965 \quad \frac{1}{\psi_0} d_{g_{\mathbb{E}}}(c, x) \leq r,$$

967 giving us $B \subset B_{\mathbb{E}}(c, r_{+\psi_0})$. \square

968 Note that $r_{-\psi_0}$ and $r_{+\psi_0}$ go to r as ψ_0 goes to 1.

969 We now use this stability result to provide Euclidean dihedral angle bounds as-
 970 suming power protection with an arbitrary metric field that is close to $g_{\mathbb{E}}$. We first
 971 require the intermediary result given by [Lemma B.4](#).

972 **LEMMA B.4** (Whitney's lemma). *Let H be a hyperplane in Euclidean n -space*
 973 *and τ an $n - 1$ -simplex whose vertices lie at most η from the H and whose minimum*
 974 *height is h_{\min} . Then the angle ξ between $\text{aff}(\tau)$ and H is bounded from above by*

$$975 \quad \sin(\xi) \leq \frac{\eta n}{h_{\min}}.$$

976 *Proof.* By definition, the barycenter of a $(n - 1)$ -simplex has barycentric coordi-
 977 nates $\lambda_i = 1/n$. This means that it has distance a h_{\min}/n to each of its faces. So the
 978 ball centered on the barycenter with radius h_{\min}/n is contained in the simplex. This
 979 means that for any direction in $\text{aff}(\tau)$ there exists a line segment of length $2h_{\min}/n$
 980 that lies within τ . Moreover the end points of this line segments lie at most η from H
 981 because the vertices of the τ do. This means that the angle ξ is bounded by

$$982 \quad \sin(\xi) \leq \frac{2\eta}{2h_{\min}/n} = \frac{\eta n}{h_{\min}}. \quad \square$$

984 We can now give the main result which bounds *Euclidean* dihedral angles, assum-
 985 ing power protection with respect to the arbitrary metric field.

986 **LEMMA B.5.** *Suppose that U is as in [Lemma 4.1](#). In particular we let ψ_0 be a*
 987 *bound on the metric distortion of the fields g and g' . Let \mathcal{P}_U be a δ -power protected*
 988 *(ε, μ) -net over U , with respect to g . Let $B = B_g(c, r)$ and $B' = B_g(c', r')$ be the*
 989 *geodesic Delaunay balls of $\sigma = \tau * p$ and $\sigma' = \tau * p'$, with $\sigma, \sigma' \in \text{Del}_g(\mathcal{P}_U)$. Assume*
 990 *that \mathcal{P}_U is sufficiently dense such that U contains B and B' . Then the minimum*
 991 *height of the simplex satisfies*

$$992 \quad h_{\min} = \sqrt{1 - \left(n \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right)}{8\varepsilon h_{\min}} \right)^2}$$

$$993 \quad \left(\frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right) + \frac{\delta^2}{\psi_0^2}}{4\varepsilon} - \frac{n \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right)}{8\varepsilon h_{\min}}}{\sqrt{1 - \left(n \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right)}{8\varepsilon h_{\min}} \right)^2}} \right) (r + r')$$

994

995 *Note that this is the height of p in σ with respect to the Euclidean metric.*

996 We proceed in a similar fashion as the proof Lemma (Lemma A.1). However, a signif-
 997 icant difference is that we are interested here in only proving that power protection
 998 with respect to the generic metric field provides a height bound in the Euclidean
 999 setting (rather than an equivalence).

1000 *Proof.* We use the following notations, illustrated in Figure 11:

- 1001 • $B_{\mathbb{E}}^{\pm\psi_0}$ and $B'_{\mathbb{E}}^{\pm\psi_0}$ are the two sets of (Euclidean) enclosing balls of respectively
- 1002 B and B' defined as in Lemma B.3.
- 1003 • $B'^{+\delta}$ is the power protected ball of σ' , given by $B'^{+\delta} = B(c', \sqrt{r'^2 + \delta^2})$.
- 1004 • $B'^{+\delta, \pm\psi_0}$ are the two Euclidean balls enclosing $B'^{+\delta}$.
- 1005 • $S = \partial B^{-\psi_0} \cap \partial B'^{+\psi_0}$, $\tilde{S} = \partial B^{+\psi_0} \cap \partial B'^{+\delta, -\psi_0}$ and $S' = \partial B^{+\psi_0} \cap \partial B'^{-\psi_0}$.
- 1006 • q is a point on S , \tilde{q} is a point on \tilde{S} and q' is a point on S' .
- 1007 • H is the geodesic supporting plane of τ , that is $\{\operatorname{argmin}(\sum_{v_i \in \tau} \lambda_i d_g(x, v_i))\}$.
- 1008 • $H_{\mathbb{E}}$, $\tilde{H}_{\mathbb{E}}$ and $H'_{\mathbb{E}}$ are the two Euclidean hyperplanes orthogonal to $[c']$ passing
- 1009 through q , \tilde{q} and q' .
- 1010 • $\theta = c'cq$, $\tilde{\theta} = c'c\tilde{q}$, and $\lambda_c = \|c - c'\| / r$.

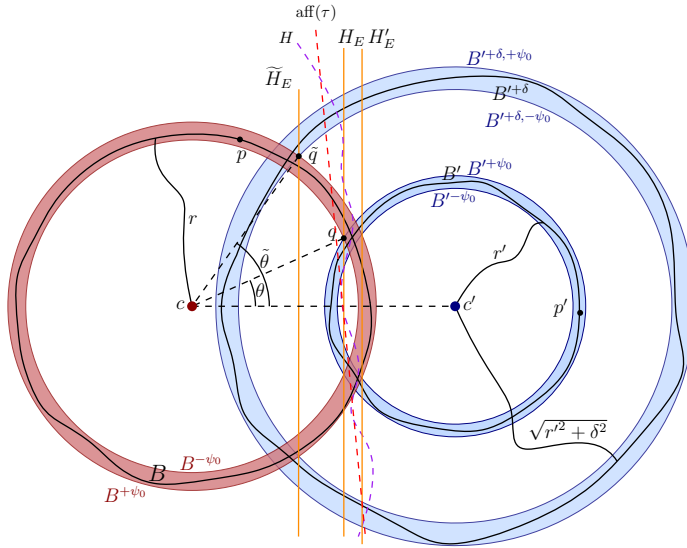


FIG. 11. Construction and notations used in the proof of Lemma B.5

1011 While the vertices of τ live on H , $\operatorname{aff}_{\mathbb{E}}(\tau)$ is not necessarily orthogonal to $[c']$ and
 1012 consequently

$$1013 \quad d_{g_{\mathbb{E}}}(p, H_{\mathbb{E}}) \leq d_{g_{\mathbb{E}}}(p, \tau).$$

1014 The separation between the hyperplanes $H_{\mathbb{E}}$ and $\tilde{H}_{\mathbb{E}}$ provides a lower bound on the dis-
 1015 tance $d_{g_{\mathbb{E}}}(p, \tau)$, thus on the height $D_{\mathbb{E}}(p, \sigma)$. We therefore seek to bound $d_{g_{\mathbb{E}}}(H_{\mathbb{E}}, \tilde{H}_{\mathbb{E}})$.

1016 By definition of the enclosing Euclidean balls, we have that

$$1017 \quad \|c - q\| = \frac{r}{\psi_0}, \quad \|c - \tilde{q}\| = \psi_0 r, \quad \|c' - q\| = \psi_0 r', \quad \text{and} \quad \|c' - \tilde{q}\| = \frac{1}{\psi_0} \sqrt{r'^2 + \delta^2}$$

1018 Using the law of cosines in the triangles $\triangle cc'q$ and $\triangle cc'\tilde{q}$, we find

$$1019 \quad \frac{r'^2 + \delta^2}{\psi_0^2} = \lambda_c^2 r^2 + \psi_0^2 r^2 - 2\lambda_c \psi_0 r^2 \cos(\tilde{\theta})$$

$$1020 \quad \psi_0^2 r'^2 = \lambda_c^2 r^2 + \frac{r^2}{\psi_0^2} - 2\frac{\lambda_c r^2}{\psi_0} \cos(\theta),$$

1022 where $\lambda_c r = \|c - c'\|$. Subtracting one from the other, we obtain

$$1023 \quad \psi_0^2 r'^2 - \frac{r'^2 + \delta^2}{\psi_0^2} = \frac{r^2}{\psi_0^2} - \psi_0^2 r^2 + 2\lambda_c \psi_0 r^2 \cos(\tilde{\theta}) - 2\frac{\lambda_c r^2}{\psi_0} \cos(\theta)$$

$$1024 \quad \frac{r}{\psi_0} \cos(\theta) - \psi_0 r \cos(\tilde{\theta}) = \frac{\frac{r^2}{\psi_0^2} - \psi_0^2 r^2 + \frac{r'^2 + \delta^2}{\psi_0^2} - \psi_0^2 r^2}{2\lambda_c r}$$

$$1025 \quad \frac{r}{\psi_0} \cos(\theta) - \psi_0 r \cos(\tilde{\theta}) = \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right) + \frac{\delta^2}{\psi_0^2}}{2\lambda_c r},$$

1026

1028 so that

$$1029 \quad d(H_{\mathbb{E}}, \tilde{H}_{\mathbb{E}}) = \frac{r}{\psi_0} \cos(\theta) - \psi_0 r \cos(\tilde{\theta}) = \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right) + \frac{\delta^2}{\psi_0^2}}{2\lambda_c r}.$$

1030 Similarly, we can calculate the distance $d_{\mathbb{E}}(H_{\mathbb{E}}, H'_{\mathbb{E}})$ to be:

$$1031 \quad d_{\mathbb{E}}(H_{\mathbb{E}}, H'_{\mathbb{E}}) = \frac{\frac{1}{\psi^2}(r')^2 + \|c - c'\|^2 - \psi^2 r^2}{2\|c - c'\|} - \frac{\psi^2(r')^2 + \|c - c'\|^2 - \frac{1}{\psi^2} r^2}{2\|c - c'\|}$$

$$1032 \quad = \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right)}{2\|c - c'\|}.$$

$$1033 \quad \leq \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right)}{4\varepsilon}$$

1035 **Lemma B.4** gives us that the angle ξ between $H_{\mathbb{E}}$ and $\text{aff}(\tau)$ is bounded by

$$1036 \quad \sin(\xi) \leq \frac{n d_{\mathbb{E}}(H_{\mathbb{E}}, H'_{\mathbb{E}})}{2h_{\min}}.$$

1037 The vertices of τ lie in between $H_{\mathbb{E}}$ and $H'_{\mathbb{E}}$ and inside $B_{\mathbb{E}}(c', \psi_0 r')$. Thus, if we
 1038 restrict to the $\tilde{q}cc'$ plane, distance between the point where $\text{aff}(\tau)$ intersects $H_{\mathbb{E}}$ and
 1039 the orthogonal projection $\pi_{H_{\mathbb{E}}}(\tilde{q})$ of \tilde{q} on $H_{\mathbb{E}}$ is at most $\psi_0(r + r')$. This in turn implies
 1040 that the line connecting $\pi_{H_{\mathbb{E}}}(\tilde{q})$ and \tilde{q} intersects $\text{aff}(\tau)$ at most distance $(r + r') \tan(\xi)$
 1041 from $H_{\mathbb{E}}$. This also gives us that the distance from \tilde{q} to its orthogonal projection
 1042 $\pi_{\text{aff}(\tau)}(\tilde{q})$ on $\text{aff}(\tau)$ is bounded by

$$1043 \quad \cos(\xi)(d(\tilde{H}_{\mathbb{E}}, H_{\mathbb{E}}) - \tan(\xi)(r + r')) = \sqrt{1 - \left(\frac{n d_{\mathbb{E}}(H_{\mathbb{E}}, H'_{\mathbb{E}})}{2h_{\min}} \right)^2}.$$

$$1044 \quad \left(d(\tilde{H}_{\mathbb{E}}, H_{\mathbb{E}}) - \frac{\frac{n d_{\mathbb{E}}(H_{\mathbb{E}}, H'_{\mathbb{E}})}{2h_{\min}}}{\sqrt{1 - \left(\frac{n d_{\mathbb{E}}(H_{\mathbb{E}}, H'_{\mathbb{E}})}{2h_{\min}} \right)^2}} (r + r') \right).$$

1045

1046 Here we note that h_{\min} originally referred to the minimum height of a face, but
 1047 because the height of a face is always greater than the height in the simplex we may
 1048 read this in a general way, that is we regard h_{\min} as a universal lower bound on the
 1049 height. Because $\|\tilde{q} - \pi_{\text{aff}(r)}(\tilde{q})\|$ bounds the height of the simplex we get the following
 1050 relation:

$$1051 \quad h_{\min} = \sqrt{1 - \left(\frac{n d_{\mathbb{E}}(H_{\mathbb{E}}, H'_{\mathbb{E}})}{2h_{\min}} \right)^2} \left(d(\tilde{H}_{\mathbb{E}}, H_{\mathbb{E}}) - \frac{\frac{n d_{\mathbb{E}}(H_{\mathbb{E}}, H'_{\mathbb{E}})}{2h_{\min}}}{\sqrt{1 - \left(\frac{n d_{\mathbb{E}}(H_{\mathbb{E}}, H'_{\mathbb{E}})}{2h_{\min}} \right)^2}} (r + r') \right)$$

$$1052 \quad h_{\min} = \sqrt{1 - \left(n \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right)}{8\varepsilon h_{\min}} \right)^2} \cdot$$

$$1053 \quad \left(\frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right) + \frac{\delta^2}{\psi_0^2}}{4\varepsilon} - \frac{n \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right)}{8\varepsilon h_{\min}}}{\sqrt{1 - \left(n \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right)}{8\varepsilon h_{\min}} \right)^2}} (r + r') \right)$$

1055 To make the expression a bit more readable, we introduce

$$1056 \quad s_1 = \frac{(r^2 + r'^2) \left(\frac{1}{\psi_0^2} - \psi_0^2 \right)}{4\varepsilon}$$

1057 so that

$$1058 \quad h_{\min} = \sqrt{1 - \left(n \frac{s_1}{2h_{\min}} \right)^2} \left(\left(s_1 + \frac{\delta^2}{4\varepsilon\psi_0^2} \right) - \frac{n \frac{s_1}{2h_{\min}}}{\sqrt{1 - \left(n \frac{s_1}{2h_{\min}} \right)^2}} (r + r') \right)$$

$$1059 \quad h_{\min} + n \frac{s_1}{2h_{\min}} (r + r') = \sqrt{1 - \left(n \frac{s_1}{2h_{\min}} \right)^2} \left(s_1 + \frac{\delta^2}{4\varepsilon\psi_0^2} \right).$$

1061 Multiplying with h_{\min} and squaring we find:

$$1062 \quad (8) \quad \left(h_{\min}^2 + n \frac{s_1}{2} (r + r') \right)^2 = \left(h_{\min}^2 - \left(n \frac{s_1}{2} \right)^2 \right) \left(s_1 + \frac{\delta^2}{4\varepsilon\psi_0^2} \right)^2$$

1064 We note that as U becomes smaller, ψ_0 goes to 1 and s_1 tends to zero. We therefore
 1065 expand h_{\min} in terms of s_1 . Using a computer algebra system, we find:

$$1066 \quad h_{\min}^2 = \frac{\delta^4}{2^{10}\psi_0^4\varepsilon^2} + \left(\frac{\delta^2}{128\psi_0^2\varepsilon} - n(r + r') \right) s_1$$

$$1067 \quad - \frac{\delta^4 (16n^2 - 1) + 2^{14}n^2\psi_0^4\varepsilon^2(r + r')^2}{64\delta^4} s_1^2 + \mathcal{O}(s_1^3)$$

1069 We emphasize that this equation gives $h_{\min} = \delta^2/4\varepsilon$ as ψ_0 goes to 1. This means
 1070 that for a sufficiently small metric distortion the height of a protected simplex will be
 1071 strictly positive. \square

1072 LEMMA B.6. Let g be a metric field and \mathcal{P} be a point set defined over a neigh-
 1073 borhood $U \subset B_g(p, r_0) \subset \mathbb{R}^n$, for some p , as in Lemma 4.1. Let ψ_0 be the bound on
 1074 the metric distortion of the fields g and $g_{\mathbb{E}}$, in the ball $B_g(p, 5r_0)$. Assume that \mathcal{P} is
 1075 a δ -power protected (ε, μ) -net over \mathbb{R}^n . Let ϕ be the dihedral angle between two faces
 1076 of a simplex $\tau \in \text{Del}_{\mathbb{E}}(\mathcal{P})$. Then

$$1077 \quad \arcsin(s_0) \leq \phi \leq \pi - \arcsin(s_0),$$

1078 with s_0 detailed in the proof.

1079 *Proof.* Denote h_{\min} the lower bound on $D(q, \sigma)$ obtained in the previous lemma.
 1080 We also immediate have that

$$1081 \quad D(q, \sigma) \leq 2\varepsilon.$$

1082 Let φ be the dihedral angle between $\text{aff}(\sigma_p)$ and $\text{aff}(\sigma_q)$. Recall that

$$1083 \quad \sin(\varphi) = \sin \angle(\text{aff}(\sigma_p), \text{aff}(\sigma_q)) = \frac{D(p, \sigma)}{D(p, \sigma_q)} = \frac{D(q, \sigma)}{D(q, \sigma_p)}$$

1084 Thus

$$1085 \quad (9) \quad \sin(\varphi) \geq \frac{h_{\min}}{2\varepsilon} =: s_0.$$

1087 For s_0 to make sense, we want $s_0 > 0$, which is bound to happen when U is
 1088 made smaller and ψ_0 goes to 1, as shown in Lemma B.5: h_{\min} goes to $\delta^2/4\varepsilon$ and thus
 1089 $h_{\min}/4\varepsilon$ goes to $\delta^2/4 > 0$. \square

1090 **Appendix C. Stability.** The notion of stability designates the conservation of
 1091 a property despite changes of other parameters. In our context, the main assumptions
 1092 concern the nature of point sets: we assume that point sets are power protected nets
 1093 and wish to preserve these hypotheses despite (small) metric perturbations. The
 1094 stability is important both from a theoretical and a practical point of view. Indeed,
 1095 if an assumption is stable under perturbation, we can simplify matters without losing
 1096 information. For example, we will prove that if a point set is a net with respect
 1097 to a metric field g , then it is also a net (albeit with slightly different sampling and
 1098 separation parameters) for a metric field g' that is close to g (in terms of distortion)
 1099 In a practical context, the stability of an assumption provides robustness with respect
 1100 to numerical errors (see, for example, the work of Funke et al. [19] on the stability of
 1101 Delaunay simplices).

1102 **C.1. Stability of the protected net hypothesis under metric transfor-**
 1103 **mation.** It is rather immediate that the power protected net property is preserved
 1104 when the point set is transformed (see subsection 4.1.1) between these spaces, as
 1105 shown by the following lemma.

1106 LEMMA C.1. Let \mathcal{P} be a δ -power protected (ε, μ) -net in Ω . Let $g = g_0$ be a
 1107 constant Riemannian metric field and F_0 a square root of g_0 . If \mathcal{P} is a δ -power
 1108 protected (ε, μ) -net with respect to g_0 then $\mathcal{P}' = \{F_0 p_i, p_i \in \mathcal{P}\}$ is a δ -power protected
 1109 (ε, μ) -net with respect to the Euclidean metric.

1110 *Proof.* This results directly from the observation that

$$1111 \quad d_0(x, y)^2 = (x - y)^t g_0(x - y) = \|F_0(x - y)\|^2 = d(F_0 x, F_0 y)^2. \quad \square$$

1112 **C.2. Stability of the protected net hypothesis under metric perturba-**
 1113 **tion.** Metric field perturbations are small modifications of a metric field in terms of
 1114 distortion. Since a generic Riemannian metric field g is difficult to study, we will gen-
 1115 erally consider a constant approximation g_0 of g within a small neighborhood, such
 1116 that the distortion between both metric fields is small over that neighborhood. In
 1117 that context, the perturbation of g is the act of bringing g onto g_0 . Stability of the
 1118 assumption of power protection was previously investigated by Boissonnat et al. [4]
 1119 in the context of manifold reconstructions.

1120 **C.2.1. Stability of the net property.** The following lemma shows that the
 1121 “net” property is preserved when the metric field is perturbed: a point set that is a
 1122 net with g is also a net with respect to g' , a metric field that is close to g .

1123 **LEMMA C.2.** *Suppose that U is as in Lemma 4.1. In particular we write ψ_0 for*
 1124 *a bound on the metric distortion of the fields g and g' . Suppose that \mathcal{P}_U is an (ε, μ) -*
 1125 *net of U with respect to g . Then \mathcal{P}_U is an $(\varepsilon_{g'}, \mu_{g'})$ -net of U with respect to g' with*
 1126 *$\varepsilon_{g'} = \psi_0\varepsilon$ and $\mu_{g'} = \mu/\psi_0$.*

1127 *Proof.* By Lemma 4.1, we have that

$$1128 \quad \forall x, y \in U, \frac{1}{\psi_0}d_{g'}(x, y) \leq d_g(x, y) \leq \psi_0d_{g'}(x, y).$$

1129 Therefore

$$1130 \quad \forall x \in U, \exists p \in \mathcal{P}_U, d_g(x, p) \leq \varepsilon \Rightarrow \forall x \in U, \exists p \in \mathcal{P}_U, d_{g'}(x, p) \leq \psi_0\varepsilon,$$

1131 and

$$1132 \quad \forall p, q \in \mathcal{P}_U, d_g(p, q) \geq \mu \Rightarrow \forall p, q \in \mathcal{P}_U, d_{g'}(p, q) \geq \frac{\mu}{\psi_0}.$$

1133 And, with $\varepsilon_{g'} = \psi_0\varepsilon$ and $\mu_{g'} = \mu/\psi_0$, \mathcal{P} is an $(\varepsilon_{g'}, \mu_{g'})$ -net of U . \square

1134 **Remark C.3.** We assumed that $\mu = \lambda\varepsilon$. By Lemma C.2, we have $\varepsilon' = \psi_0\varepsilon$ and
 1135 $\mu' = \frac{\mu}{\psi_0}$. Therefore

$$1136 \quad \mu' = \frac{\lambda\varepsilon}{\psi_0} = \frac{\lambda}{\psi_0^2}\varepsilon'.$$

1137 **C.2.2. Stability of the power protection property.** It is more complex to
 1138 show that the assumption of power protection is preserved under metric perturbation.
 1139 Previously, we have only considered two similar but arbitrary Riemannian metric
 1140 fields g and g' on a neighborhood U . We now restrict ourselves to the case where g' is a
 1141 constant metric field. We shall always compare the metric field g in a neighborhood U
 1142 with the constant metric field $g' = g_0 = g(p_0)$ where $p_0 \in U$. Because g_0 and the
 1143 Euclidean metric field differ only by a linear transformation, we simplify matters and
 1144 assume that g_0 is the Euclidean metric field.

1145 We now give conditions such that the point set is also protected with respect
 1146 to g_0 . A few intermediary steps are needed to prove the main result:

- 1147 • Given two sites, we prove that the bisectors of these two sites in the Voronoi
 1148 diagrams built with respect to g and with respect to $g' = g_{\mathbb{E}}$ are close
 1149 (Lemma C.5).
- 1150 • We prove that the Voronoi cell of a point p_0 with respect to g , $V_g(p_0)$ can
 1151 be encompassed by two scaled versions (one larger and one smaller) of the
 1152 Euclidean Voronoi cell $V_{\mathbb{E}}(p_0)$ (Lemma C.12).

1153 • We combine this encompassing with bounds on the dihedral angles of Eu-
 1154 cludean Delaunay simplices given by Lemma B.6 to compute a stability region
 1155 around Voronoi vertices where the same combinatorial Voronoi vertex of lives
 1156 for both $V_g(p_0)$ and $V_{\mathbb{E}}(p_0)$ in 2D (Lemma C.13). We then extend it to any
 1157 dimension by induction (Lemma C.14).

1158 The main result appears in Lemma C.16 and gives the stability of power protection
 1159 under metric perturbation.

1160 We first define the scaled version of a Voronoi cell more rigorously.

1161 DEFINITION C.4 (Relaxed Voronoi cell). *Let $\omega \in \mathbb{R}$. The relaxed Voronoi cell of*
 1162 *the site p_0 is*

$$1163 \quad V_g^{+\omega}(p_0) = \{x \in U \mid d_g(p_0, x)^2 \leq d_g(p_i, x)^2 + \omega \text{ for all } i \neq 0\}.$$

1164 The following lemma expresses that two Voronoi cells computed in similar metric
 1165 fields are close.

1166 LEMMA C.5. *Suppose that U is as in Lemma 4.1. In particular we let ψ_0 be a*
 1167 *bound on the metric distortion of the fields g and g' . Let $\mathcal{P}_U = \{p_i\}$ be a point set in*
 1168 *U . Let $V_{p_0, g}$ denote a Voronoi cell with respect to the Riemannian metric field g .*

1169 *Suppose that the Voronoi cell $V_{g'}^{2\rho^2(\psi_0^2-1)}(p_0)$ lies in a ball of radius ρ with respect*
 1170 *to the metric g' , which lies completely in U . Let $\omega_0 = 2\rho^2(\psi_0^2 - 1)$. Then $V_g(p_0)$ lies*
 1171 *in $V_{g'}^{+\omega_0}(p_0)$ and contains $V_{g'}^{-\omega_0}(p_0)$.*

1172 *Proof.* Let $BS_g(p_0, p_i)$ be the bisector between p_0 and p_i with respect to g . Let $y \in$
 1173 $BS_g(p_0, p_i) \cap B_{g'}(p_0, \rho)$, where $B_{g'}(p_0, \rho)$ denotes the ball centered at p_0 of radius ρ
 1174 with respect to g' . Now $d_g(y, p_0) = d_g(y, p_i)$, and thus

$$1175 \quad |d_{g'}(y, p_0)^2 - d_{g'}(y, p_i)^2| = |d_{g'}(y, p_0)^2 - d_g(y, p_0)^2 + d_g(y, p_i)^2 - d_{g'}(y, p_i)^2|$$

$$1176 \quad \leq |d_{g'}(y, p_0)^2 - d_g(y, p_0)^2| + |d_{g'}(y, p_i)^2 - d_g(y, p_i)^2|$$

$$1177 \quad \leq (\psi_0^2 - 1)(d_{g'}(y, p_0)^2 + d_{g'}(y, p_i)^2)$$

$$1178 \quad \leq 2\rho^2(\psi_0^2 - 1).$$

1180 Thus $d_{g'}(y, p_0)^2 \leq d_{g'}(y, p_i)^2 + \omega$ and $d_{g'}(y, p_0)^2 \geq d_{g'}(y, p_i)^2 - \omega$ with $\omega = 2\rho^2(\psi_0^2 - 1)$,
 1181 which gives us the expected result. \square

1182 Remark C.6. Lemma C.5 does not require g' to be a constant metric field.

1183 We clarify in the next lemma that the bisectors of a Voronoi diagram with respect
 1184 to a constant metric field are affine hyperplanes.

1185 LEMMA C.7. *Suppose that U is as in Lemma 4.1. In particular we let ψ_0 be a*
 1186 *bound on the metric distortion of the fields g and g' . Let $\mathcal{P}_U = \{p_i\}$ be a point set in*
 1187 *U . Let g' be a constant metric field. We refer to g' as g_0 to emphasis its constancy.*
 1188 *Let $p_0 \in \mathcal{P}_U$. The bisectors of $V_{g_0}^{\pm\omega_0}(p_0)$ are hyperplanes.*

1189 *Proof.* The bisectors of $V_{g_0}^{\pm\omega_0}(p_0)$ are given by

$$1190 \quad BS_{g_0}^{\pm\omega_0}(p_0, p_i) = \{x \in \Omega \mid d_{g_0}(p_0, x)^2 = d_{g_0}(p_i, x)^2 \pm \omega_0\}.$$

1191 For $x \in BS_{g_0}^{\pm\omega_0}(p_0, p_i)$, we have by definition that

$$1192 \quad \|x - p_0\|_{g_0}^2 = \|x - p_i\|_{g_0}^2 \pm \omega_0$$

$$1193 \quad (x - p_0)^t g_0 (x - p_0) = (x - p_i)^t g_0 (x - p_i) \pm \omega_0$$

$$1194 \quad 2x^t g_0 (p_i - p_0) = p_i^t g_0 p_i - p_0^t g_0 p_0 \pm \omega_0.$$

1196 which is the equation of an hyperplane since g_0 is constant. \square

1197 The cells $V_{g_0}^{\pm\omega_0}(p_0)$ are unfortunately impractical to manipulate as we do not have
 1198 an explicit distance between the boundaries $\partial V_{g_0}(p_0)$ and $\partial V_{g_0}^{\pm\omega_0}(p_0)$. However that
 1199 distance can be bounded; this is the purpose of the following lemma.

1200 LEMMA C.8. *Suppose that U is as in Lemma 4.1. In particular we let ψ_0 be a*
 1201 *bound on the metric distortion of the fields g and $g' = g_0$. Let $\mathcal{P}_U = \{p_i\}$ be a point*
 1202 *set in U . Assume furthermore that g_0 is the Euclidean metric field. Let $p_0 \in \mathcal{P}_U$. We*
 1203 *have*

$$1204 \quad d_{g_0}(\partial V_{g_0}(p_0), \partial V_{g_0}^{\pm\omega_0}(p_0)) := \min_{\substack{x \in \partial V_{g_0}(p_0) \\ y \in \partial V_{g_0}^{\pm\omega_0}(p_0)}} d_{g_0}(x, y) \leq \frac{\rho_0^2(\psi_0^2 - 1)}{\mu_0},$$

1205 with ρ_0 defined as ρ is in Lemma C.5.

1206 *Proof.* Let m_{ω_0} be the intersection of the segment $[p_0, p_i]$ with the bisector
 1207 $BS_{g_0}^{-\omega_0}(p_0, p_i)$, for $i \neq 0$. Let m be the intersection of the segment $[p_0, p_i]$ and
 1208 $\partial V_{g_0}(p_0)$, for $i \neq 0$. We have

$$1209 \quad \begin{cases} m_{\omega_0} \in BS_{g_0}^{-\omega_0}(p_0, p_i) & \iff 2m_{\omega_0}^T(p_i - p_0) = p_i^T p_i - p_0^T p_0 - \omega_0 \\ m \in \partial V_{g_0}(p_0) & \iff 2m^T(p_i - p_0) = p_i^T p_i - p_0^T p_0 \end{cases}$$

1210 Therefore

$$1211 \quad 2(m - m_{\omega_0})^T(p_i - p_0) = \omega_0.$$

1212 Since $(m - m_{\omega_0})$ and $(p_i - p_0)$ are linearly dependent,

$$1213 \quad \omega_0 = 2 \|m - m_{\omega_0}\| \|p_i - p_0\|.$$

1214 \mathcal{P} is μ_0 -separated, which implies that

$$1215 \quad \omega_0 \geq 2 \|m - m_{\omega_0}\| \mu_0,$$

1216 and

$$1217 \quad \|m - m_{\omega_0}\| \leq \frac{\omega_0}{2\mu_0} = \frac{\rho_0^2(\psi_0^2 - 1)}{\mu_0}. \quad \square$$

1218 DEFINITION C.9. *In the following, we use $\rho = 2\varepsilon_0$, and therefore $\omega_0 = 8\varepsilon_0^2(\psi_0^2 -$
 1219 $1)$. We show that this choice is reasonable in Lemma C.15. Additionally, we define*

$$1220 \quad \eta_0 = \frac{\rho_0^2(\psi_0^2 - 1)}{\mu_0}.$$

1221 We are now ready to encompass the Riemannian Voronoi cell of p_0 with respect
 1222 to an arbitrary metric field g with two scaled versions of the Euclidean Voronoi cell
 1223 of p_0 . The notions of *dilated* and *eroded* Voronoi cells will serve the purpose of defining
 1224 precisely these scaled cells.

1225 DEFINITION C.10 (Eroded Voronoi cell). *Let $\omega \in \mathbb{R}$. The eroded Voronoi cell*
 1226 *of p_0 is the morphological erosion of $V_g(p_0)$ by a ball of radius ω :*

$$1227 \quad EV_g^{-\omega}(p_0) = \{x \in V_g(p_0) \mid d_g(x, \partial V_g(p_0)) > \omega\}.$$

1228 DEFINITION C.11 (Dilated Voronoi cell). Let $\omega \in \mathbb{R}$. The dilated Voronoi cell
 1229 of p_0 is:

$$1230 \quad DV_g^{+\omega}(p_0) = \bigcap_{i \neq 0} H^\omega(p_0, p_i),$$

1231 where $H^\omega(p_0, p_i)$ is the half-space containing p_0 and that is delimited by the bisec-
 1232 tor $\mathcal{BS}(p_0, p_i)$ translated away from p_0 by ω_0 (see Figure 12).

1233 The second important step on our path towards the stability of power protection
 1234 is the encompassing of the Riemannian Voronoi cell, and is detailed below.

1235 LEMMA C.12. Suppose that U is as in Lemma 4.1. In particular we let ψ_0 be a
 1236 bound on the metric distortion of the fields g and $g' = g_0$. Let $\mathcal{P}_U = \{p_i\}$ be a point
 1237 set in U . We have

$$1238 \quad EV_{g_0}^{-\eta_0}(p_0) \subseteq V_{g_0}^{-\omega_0}(p_0) \subseteq V_{g_0}(p_0) \subseteq V_{g_0}^{+\omega_0}(p_0) \subseteq DV_{g_0}^{+\eta_0}(p_0).$$

1239 These inclusions are illustrated in Figure 12.

1240 Proof. Using the notations and the result of Lemma C.8, we have

$$1241 \quad \|m - m_{\omega_0}\| \leq \eta_0.$$

1242 Since the bisectors $\mathcal{BS}_{g'}^{\omega_0}(p_0, p_i)$ are hyperplanes, the result follows. \square

1243 In Figure 12 and Figure 13, $DV_{g_0}^{+\eta_0}$ and $EV_{g_0}^{-\eta_0}$ are shown in green and V_{g_0} in
 1244 yellow.

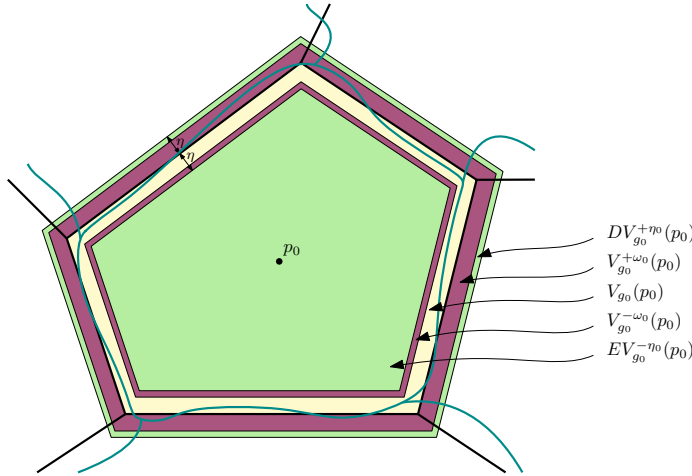


FIG. 12. The different encompassing cells around p_0 . The Riemannian Voronoi diagrams with respect to g and g_0 are traced in dark cyan and black respectively. The Voronoi cell $V_{g_0}(p_0)$ is colored in yellow. The cells $DV_{g_0}^{+\eta_0}$ and $EV_{g_0}^{-\eta_0}$ are colored in green, and the cells $V_{g_0}^{\pm\omega_0}(p_0)$ are colored in purple.

1245 The next step is to prove that we have stability of the Voronoi vertices of $V_g(p_0)$,
 1246 meaning that a small perturbation of the metric only creates a small displacement of
 1247 any Voronoi vertex of the Voronoi cell of p_0 . The following lemma shows that Voronoi
 1248 vertices are close if the distortion between the metric fields g and g_0 is small. The
 1249 approach is to use Lemma C.5 and Lemma C.12: we know that each $(n - 1)$ -face of
 1250 the Riemannian Voronoi cell $V_g(p_0)$ and shared by a Voronoi cell $V_g(q)$ is enclosed

1251 within the bisectors of $DV_{g_0}(p_0)$ and $EV_{g_0}(p_0)$ (which are translations of the bisectors
 1252 of $V_{g_0}(p_0)$) for the sites p_0 and q . These two bisectors create a “band” that contains
 1253 the bisector $BS_g(p_0, q)$. Given a Voronoi vertex c in $V_g(p_0)$, c can be obtained as the
 1254 intersection of $n + 1$ Voronoi cells, but also as the intersection of n Voronoi $(n - 1)$ -
 1255 faces of $V_g(p_0)$. The intersection of the bands associated to those n $(n - 1)$ -faces is
 1256 a parallelotopic-shaped region which by definition contains the same (combinatorially
 1257 speaking) Voronoi vertex, but for $V_{g_0}(p_0)$. [Lemma C.13](#) and [Lemma C.14](#) express this
 1258 reasoning, which is illustrated in [Figure 13](#) for 2D and [Figure 14](#) for any dimension.

1259 **LEMMA C.13.** *We consider here $\Omega = \mathbb{R}^2$. Suppose that U is as in [Lemma 4.1](#). In
 1260 particular we let ψ_0 be a bound on the metric distortion of the fields g and $g' = g_0$. Let
 1261 $\mathcal{P}_U = \{p_i\}$ be a point set in U . Assume furthermore that \mathcal{P}_U is a δ_0 -power protected
 1262 (ε_0, μ_0) -net with respect to g_0 (the Euclidean metric). Let $p_0 \in \mathcal{P}_U$. Let c and c_0 be
 1263 the same Voronoi vertex in respectively $V_g(p_0)$ and $V_{g_0}(p_0)$. Then $d_{g_0}(c, c_0) \leq \chi_2$
 1264 with*

$$1265 \quad \chi_2 = \frac{2\eta_0}{\sqrt{1 + \frac{\mu_0}{2\varepsilon_0}} - \sqrt{1 - \frac{\mu_0}{2\varepsilon_0}}} = \frac{2\eta_0}{\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}}.$$

1266 where λ is given by $\mu_0 = \frac{\lambda}{\psi_0^2}\varepsilon_0$ (see [Remark C.3](#)).

1267 *Proof.* We use [Lemma C.12](#). $V_g(p_0)$ lies in $DV_{g_0}^{+\eta_0}$ and contains $EV_{g_0}^{-\eta_0}$. The
 1268 circumcenters c and c_0 lie in a parallelogrammatic region centered on c_0 , itself included
 1269 in the ball centered on c_0 and with radius χ . The radius χ is given by half the length
 1270 of the longest diagonal of the parallelogram (see [Figure 13](#)). By [Lemma C.2](#), \mathcal{P} is an
 1271 (ε_0, μ_0) -net with respect to g_0 . Let θ be the angle of the Voronoi corner of $V_{g_0}(p_0)$
 1272 at c_0 . By [Lemma B.1](#), that angle is bounded:

$$1273 \quad \theta_m = 2 \arcsin\left(\frac{\mu_0}{2\varepsilon_0}\right) \leq \theta \leq \pi - \arcsin\left(\frac{\mu_0}{2\varepsilon_0}\right) = \theta_M.$$

1274 Since $\pi - \theta_M < \theta_m$, χ is maximal when $\theta > \pi/2$. We thus assume $\theta > \pi/2$, and
 1275 compute a bound on χ as follows:

$$1276 \quad \sin\left(\frac{\pi - \theta}{2}\right) = \frac{\eta_0}{\chi} \implies \chi = \frac{\eta_0}{\sin\left(\frac{\pi - \theta}{2}\right)}$$

$$1277 \quad \leq \frac{\eta_0}{\sin\left(\frac{1}{2} \arcsin\left(\frac{\mu_0}{2\varepsilon_0}\right)\right)}$$

$$1278 \quad \leq \frac{2\eta_0}{\sqrt{1 + \frac{\mu_0}{2\varepsilon_0}} - \sqrt{1 - \frac{\mu_0}{2\varepsilon_0}}} =: \chi_2, \quad \square$$

$$1279$$

1280 using $\sin(\frac{1}{2} \arcsin(\alpha)) = \frac{1}{2}(\sqrt{1 + \alpha} - \sqrt{1 - \alpha})$.

1281 The result obtained in [Lemma C.13](#) can be extended to any dimension using
 1282 induction and the stability of the Voronoi vertices of facets.

1283 **LEMMA C.14.** *We consider here $\Omega = \mathbb{R}^n$. Let $U \subset \Omega$ be open, and g and $g' = g_0$ be
 1284 two Riemannian metric fields on Ω . Let $\mathcal{P}_U = \{p_i\}$ be a δ_0 -power protected (ε_0, μ_0) -
 1285 net with respect to g_0 (the Euclidean metric) and let $p_0 \in \mathcal{P}_U$. Suppose that U is
 1286 included in a ball $B_g(p_0, r_0)$, with $p_0 \in U$ and $r_0 \in \mathbb{R}^+$, such that*

$$1287 \quad \forall p \in B(p_0, 5r_0), \psi(g(p), g'(p)) \leq \psi_0$$

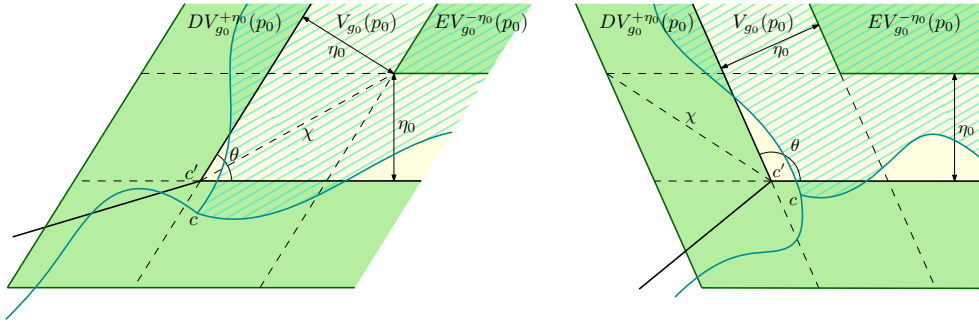


FIG. 13. Black lines trace Vor_{g_0} and cyan lines trace Vor_g . The cell $V_{g_0}(p_0)$ is colored in yellow and the cell $V_g(p_0)$ is dashed. The green regions correspond to $DV_{g_0}^{+\eta_0}(p_0)$ and $EV_{g_0}^{-\eta_0}(p_0)$. On the left, the configuration where $\theta < \pi/2$; on the right, $\theta > \pi/2$.

1288 and $B_g(p_0, 5r_0) \subset \Omega$ is geodesically convex. Let c and c_0 be the same Voronoi vertex
 1289 in respectively $V_g(p_0)$ and $V_{g_0}(p_0)$. Then $d_{g_0}(c, c_0) \leq \chi$ with

1290
$$\chi = \frac{\chi_2}{\sin^{n-2}\left(\frac{\varphi_0}{2}\right)}$$

1291 where χ_2 is defined as in Lemma C.13, and φ_0 is the maximal dihedral angle between
 1292 two faces of a simplex.

1293 *Proof.* We know from Lemma C.5 that $V_g(p_0)$ lies in $DV_{g_0}^{+\eta_0}$ and contains $EV_{g_0}^{-\eta_0}$.
 1294 The circumcenters c and c_0 lie in a parallelotopic region centered on c_0 defined by the
 1295 intersection of n Euclidean thickened Voronoi faces. This parallelotope and its circum-
 1296 scribing sphere are difficult to compute. However, it can be seen as the intersection
 1297 of two parallelotopic tubes defined by the intersection of $n - 1$ Euclidean thickened
 1298 Voronoi faces. From another point of view, this is the computation of the intersection
 1299 of the thickened duals of two facets τ_1 and τ_2 incident to p_0 of the simplex $\sigma \in \text{Del}(\mathcal{P})$,
 1300 dual of c (and c_0), see Figure 14 (left).

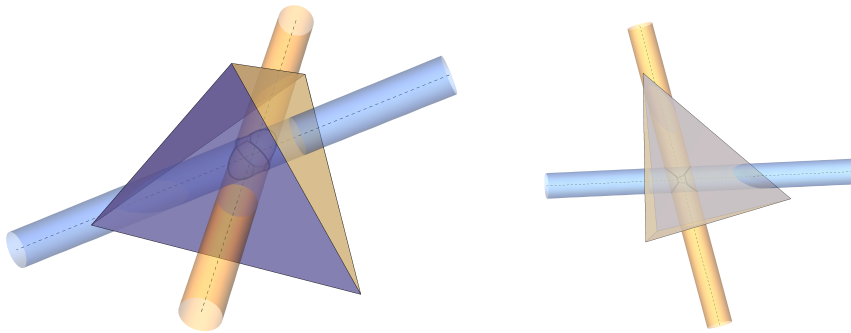


FIG. 14. Left, a simplex, the duals of two faces and their respective thickened duals (cylinders); the orange thickened dual is orthogonal to the purple face (and inversely). Right, the intersection of the two tubes, seen from above, illustrates the proof of Lemma C.14.

1301 The stability radius χ is computed incrementally by increasing the dimension

1302 and proving stability of the circumcenters of the faces of the simplices. We prove the
1303 formula by induction.

1304 The radius of each tube is given by the stability of the radius of the circumcenter
1305 in the lower dimension of the facet. The base case, $\mathbb{R}^n = \mathbb{R}^3$, is solved by [Lemma C.13](#),
1306 and gives

$$1307 \quad \chi_2 = \frac{2\eta_0}{\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}}.$$

1308 We now consider two facets τ_1 and τ_2 that are incident to p_0 . Denote \mathcal{D}_1 and \mathcal{D}_2
1309 their respective duals. By [Lemma C.5](#), \mathcal{D}_1 and \mathcal{D}_2 lie in two cylinders C_1 and C_2 of
1310 radius χ_2 . C_1 and C_2 are also orthogonal to τ_1 and τ_2 and c and c_0 lie in $C_1 \cap C_2$.
1311 The angle φ between C_1 and C_2 is exactly the dihedral angle between τ_1 and τ_2 .
1312 By [Lemma B.6](#), we have

$$1313 \quad \arcsin(s_0) \leq \varphi \leq \pi - \arcsin(s_0) \text{ with } s_0 = \frac{1}{2} \left[\frac{\iota^2}{4\psi_0^2} - \frac{1}{2} \left(\psi_0^2 - \frac{1}{\psi_0^2} \right) \right]$$

1314 Let $\varphi_0 = \arcsin(s_0) = \pi - \arcsin(s_0)$. We encompass the intersection of the cylin-
1315 ders, difficult to compute, with a sphere whose radius can be computed as follows
1316 (see [Figure 14](#), right):

$$1317 \quad \chi_3 = \frac{\chi_2}{\cos(\alpha)} \text{ with } \alpha = \frac{\pi}{2} - \frac{\varphi_0}{2}.$$

1318 Thus,

$$1319 \quad \chi_3 = \frac{\chi_2}{\sin\left(\frac{\varphi_0}{2}\right)}.$$

1320 Recursively,

$$1321 \quad \chi = \frac{\chi_2}{\sin^{n-2}\left(\frac{\varphi_0}{2}\right)}. \quad \square$$

1322 We have assumed in different lemmas that we could pick values of ρ_0 or ω_0 that
1323 fit our need. The following lemma shows that these assumptions were reasonable.

1324 **LEMMA C.15.** *[Lemma C.12](#) and [Lemma C.14](#) allow us to characterize the pa-
1325 rameter ρ_0 more precisely. Indeed, an assumption of [Lemma C.5](#) was that $V_{g_0}^{\omega_0}(p_0)$
1326 is included in a ball $B_{g_0}(p_0, \rho_0)$. If the sampling of \mathcal{P} is sufficiently dense, such an
1327 assumption is reasonable.*

1328 *Proof.* By definition, the Voronoi cell $V_{g_0}^{\omega_0}(p_0)$ is included in the dilated cell
1329 $DV_{g_0}^{+\eta_0}(p_0)$. Since the point set is an ε -sample, we have $d_{g_0}(p_0, x) \leq \varepsilon_0$ for $x \in V_{g_0}(p_0)$.
1330 By [Lemma C.14](#), we have for $x \in DV_{g_0}^{+\eta_0}(p_0)$

$$1331 \quad d_{g_0}(p_0, x) \leq \varepsilon_0 + \chi.$$

1332 Recall from [Lemma C.14](#) that

$$1333 \quad \chi = \frac{\chi_2}{\left[\sin\left(\frac{1}{2} \arcsin(s_0)\right)\right]^{n-2}}.$$

$$1334 \quad \text{with } \eta_0 = \frac{\rho_0^2(\psi_0^2 - 1)}{\mu_0} \text{ and } s_0 = \frac{1}{2} \left[\frac{\iota^2}{4\psi_0^2} - \frac{1}{2} \left(\psi_0^2 - \frac{1}{\psi_0^2} \right) \right].$$

1335 We require $V_{g_0}^{\omega_0}(p_0) \subset B_{g_0}(p_0, \rho_0)$, which is verified if $DV_{g_0}^{+\eta_0}(p_0) \subset B_{g_0}(p_0, \rho_0)$,
 1336 that is if

$$\begin{aligned}
 1337 \quad & \varepsilon_0 + \frac{\lambda 2}{\left[\sin\left(\frac{1}{2} \arcsin(s_0)\right)\right]^{n-2}} \leq \rho_0 \\
 1338 \quad & \frac{4\eta_0}{\left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) \left(\sqrt{1 + s_0} - \sqrt{1 - s_0}\right)^{n-2}} \leq \rho_0 - \varepsilon_0 \\
 1339 \quad & \frac{4\rho_0^2(\psi_0^2 - 1)}{\mu_0 \left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) \left(\sqrt{1 + s_0} - \sqrt{1 - s_0}\right)^{n-2}} \leq \rho_0 - \varepsilon_0 \\
 1340 \quad & \frac{4(\psi_0^2 - 1)}{\left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) \left(\sqrt{1 + s_0} - \sqrt{1 - s_0}\right)^{n-2}} \leq \frac{\mu_0(\rho_0 - \varepsilon_0)}{\rho_0^2} \\
 1341 \quad & \frac{4\psi_0(\psi_0^2 - 1)}{\left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) \left(\sqrt{1 + s_0} - \sqrt{1 - s_0}\right)^{n-2}} \leq \frac{\lambda\varepsilon(\rho_0 - \varepsilon_0)}{\rho_0^2}, \\
 1342 \quad &
 \end{aligned}$$

1343 using [Remark C.3](#).

1344 The parameter ρ_0 can be chosen arbitrarily as long as it is greater than ε_0 , and
 1345 we have taken $\rho_0 = 2\varepsilon_0$ (see [Definition C.9](#)), which imposes

$$\begin{aligned}
 1346 \quad (10) \quad & \frac{\psi_0^2(\psi_0^2 - 1)}{\left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) \left(\sqrt{1 + s_0} - \sqrt{1 - s_0}\right)^{n-2}} \leq \frac{\lambda}{16}. \\
 1347 \quad &
 \end{aligned}$$

1348 Recall that the parameter λ is fixed. By continuity of the metric field, when ε goes
 1349 0, U can be chosen smaller and $\psi_0 = 1$, therefore the left hand side goes to 0 and
 1350 Inequality (10) is eventually satisfied as the sampling is made denser. \square

1351 Finally, we can now show the main result: the power protection property is
 1352 preserved when the metric field is perturbed.

1353 **LEMMA C.16.** *Let $U \subset \Omega$ be open, and g and $g' = g_0$ be two Riemannian metric
 1354 fields on Ω . Suppose that U is included in a ball $B_g(p_0, r_0)$, with $p_0 \in U$ and $r_0 \in \mathbb{R}^+$,
 1355 such that*

$$1356 \quad \forall p \in B(p_0, 5r_0), \psi(g(p), g'(p)) \leq \psi_0$$

1357 and $B_g(p_0, 5r_0) \subset \Omega$ is geodesically convex. Let $U \subset \Omega$ be open, and g and g' be two
 1358 Riemannian metric fields on U . Assume that \mathcal{P}_U is a δ -power protected (ε, μ) -net in
 1359 U with respect to g . If δ is well chosen, then \mathcal{P}_U is a δ_0 -power protected net with
 1360 respect to g_0 , with

$$1361 \quad \delta_0^2 = \left(\frac{1}{\psi_0^2} - \psi_0^2\right) (\varepsilon + \chi)^2 - \frac{4\varepsilon\chi}{\psi_0^2} + \frac{\delta^2}{\psi_0^2}.$$

1362 *Proof.* By [Lemma C.2](#), we know that \mathcal{P}_U is (ε_0, μ_0) -net with respect to g_0 . Let $q \in$
 1363 \mathcal{P}_U , with q not a vertex in the dual of c . Let c_0 be the combinatorial equivalent of c
 1364 in $V_{g_0}(\mathcal{P})$. Since \mathcal{P} is a δ -power protected net with respect to g , we have $d_g(c, q) >$

1365 $\sqrt{r^2 + \delta^2}$, where $r = d_g(c, p)$. On the one hand, we have

$$\begin{aligned}
 1366 \quad d_{g_0}(c_0, q) &\geq d_{g_0}(q, c) - d_{g_0}(c, c_0) \\
 1367 \quad &\geq \frac{1}{\psi_0} d_g(q, c) - \chi \\
 1368 \quad &\geq \frac{1}{\psi_0} \sqrt{r^2 + \delta^2} - \chi. \\
 1369
 \end{aligned}$$

1370 by [Lemma C.13](#). On the other hand, for any $p \in \mathcal{P}_U$ such that p is a vertex of the
1371 dual of c , we have

$$\begin{aligned}
 1372 \quad r_0 = d_{g_0}(c_0, p) &\leq d_{g_0}(c, p) + d_{g_0}(c_0, c) \\
 1373 \quad &\leq \psi_0 d_g(c, p) + \chi \\
 1374 \quad &\leq \psi_0 r + \chi. \\
 1375
 \end{aligned}$$

1376 Thus δ_0 -power protection of \mathcal{P}_U with respect to g_0 requires

$$\begin{aligned}
 1377 \quad \frac{1}{\psi_0} \sqrt{r^2 + \delta^2} - \chi &> \chi + \psi_0 r, \\
 1378
 \end{aligned}$$

1379 which is equivalent to requiring

$$\begin{aligned}
 1380 \quad \sqrt{r^2 + \delta^2} &> \psi_0(2\chi + \psi_0 r). \\
 1381
 \end{aligned}$$

1382 This is verified if

$$\begin{aligned}
 1383 \quad \delta^2 &> (\psi_0(2\chi + \psi_0 r))^2 - r^2 = 4\chi^2\psi_0^2 + 4\chi\psi_0^3 r + \psi_0^4 r^2 - r^2 \\
 1384 \quad &= 4\chi^2\psi_0^2 + 4\chi\psi_0^3 r + (\psi_0^4 - 1)r^2,
 \end{aligned}$$

1386 for all $r \in [\mu/2, \varepsilon]$. This gives us

$$\begin{aligned}
 1387 \quad (11) \quad \delta^2 &> 4\chi^2\psi_0^2 + 4\chi\psi_0^3\varepsilon + (\psi_0^4 - 1)\varepsilon^2.
 \end{aligned}$$

1388 This condition on δ is only reasonable if the right hand side is not too large.
1389 Indeed, since \mathcal{P} is an ε -sample, we must have $d_g(c, q) < 2\varepsilon$. However, we have that
1390 $d_g(c, q)^2 > d_g(c, p)^2 + \delta^2$ by δ -power protection of \mathcal{P} with respect to g . Because
1391 $d_g(c, p) < \varepsilon$, it suffices that $\delta < \varepsilon$. We will now show that this is reasonable by
1392 examining the limit of the right hand side of Inequality (11).

1393 We note, see [Lemma C.14](#), that

$$\begin{aligned}
 1394 \quad \chi &= \frac{\chi_2}{\sin^{n-2}\left(\frac{\varphi}{2}\right)} = \frac{4\eta}{\left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) \left(\sqrt{1 + s_0} - \sqrt{1 - s_0}\right)^{n-2}}, \\
 1395
 \end{aligned}$$

1396 where $\varepsilon_0 = \psi_0\varepsilon$ and $\mu_0 = \mu/\psi_0 = \lambda\varepsilon/\psi_0$ (see [Remark C.3](#)). So that

$$\begin{aligned}
 1397 \quad 4\chi^2\psi_0^2 + 4\chi\psi_0^3\varepsilon + (\psi_0^4 - 1)\varepsilon^2 &= \\
 1398 \quad &4 \left(\frac{16\varepsilon\psi_0^4(\psi_0^2 - 1)}{\lambda \left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) \left(\sqrt{1 + s_0} - \sqrt{1 - s_0}\right)^{n-2}} \right)^2 \\
 1399 \quad &+ 4 \frac{16\varepsilon\psi_0^3(\psi_0^2 - 1)}{\lambda \left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) \left(\sqrt{1 + s_0} - \sqrt{1 - s_0}\right)^{n-2}} \psi_0^3 \\
 1400 \quad (12) \quad &+ (\psi_0^2 - 1)(\psi_0^2 + 1)\varepsilon^2.
 \end{aligned}$$

1402 This means that the right hand side of (11) is of the form $f(\psi_0)(\psi_0^2 - 1)\varepsilon^2$, where
 1403 $f(\psi_0)$ is a function that tends to a constant as ψ_0 goes to 1:

$$\begin{aligned}
 1404 \quad f(\psi_0) &= 4 \left(\frac{16\psi_0^4}{\lambda \left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}} \right) \left(\sqrt{1 + s_0} - \sqrt{1 - s_0} \right)^{n-2}} \right)^2 \\
 1405 \quad &+ 4 \frac{16\psi_0^6}{\lambda \left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}} \right) \left(\sqrt{1 + s_0} - \sqrt{1 - s_0} \right)^{n-2}} \\
 1406 \quad &+ (\psi_0^2 + 1) \\
 1407 \quad &\xrightarrow{\psi_0 \rightarrow 1} 4 \left(\frac{16}{\lambda \left(\sqrt{1 + \frac{\lambda}{2}} - \sqrt{1 - \frac{\lambda}{2}} \right) \left(\sqrt{1 + \frac{\lambda^2}{4}} - \sqrt{1 - \frac{\lambda^2}{4}} \right)^{n-2}} \right)^2 \\
 1408 \quad &+ 4 \frac{16}{\lambda \left(\sqrt{1 + \frac{\lambda}{2}} - \sqrt{1 - \frac{\lambda}{2}} \right) \left(\sqrt{1 + \frac{\lambda^2}{4}} - \sqrt{1 - \frac{\lambda^2}{4}} \right)^{n-2}} \\
 1409 \quad &+ 2 \\
 1410
 \end{aligned}$$

1411 So the bound given in Equality (11) may be easily satisfied if the metric distortion is
 1412 sufficiently small.

1413 We now provide an explicit value for δ_0 in terms of δ . Let $\xi = d_g(c, q)$ and $\xi_0 =$
 1414 $d_{g_0}(c_0, q_0)$. We have the following bounds on r_0 and ξ_0 :

$$\begin{aligned}
 1415 \quad &\frac{1}{\psi_0}(r - \xi) \leq r_0 \leq \psi_0(r + \xi) \\
 1416 \quad &\frac{1}{\psi_0} \sqrt{(r - \chi)^2 + \delta^2} \leq \xi_0 \leq \psi_0 \sqrt{(r + \chi)^2 + \delta^2}. \\
 1417
 \end{aligned}$$

1418 If we had $\tilde{\delta}$ -power protection, we would have

$$\begin{aligned}
 1419 \quad r_0^2 + \tilde{\delta}^2 \leq \xi_0^2 &\iff \tilde{\delta}^2 \leq \xi_0^2 - r_0^2 \\
 1420 \quad &\iff \tilde{\delta}^2 \leq \frac{1}{\psi_0^2} \left((r - \chi)^2 + \delta^2 \right) - \psi_0^2 (r + \chi)^2 \\
 1421 \quad &\iff \tilde{\delta}^2 \leq \frac{1}{\psi_0^2} (r + \chi)^2 - \frac{4r\chi}{\psi_0^2} + \frac{\delta^2}{\psi_0^2} - \psi_0^2 (r + \chi)^2, \\
 1422
 \end{aligned}$$

1423 which is verified if

$$\begin{aligned}
 1424 \quad \tilde{\delta}^2 &\leq \left(\frac{1}{\psi_0^2} - \psi_0^2 \right) (\varepsilon + \chi)^2 - \frac{4\varepsilon\chi}{\psi_0^2} + \frac{\delta^2}{\psi_0^2}. \\
 1425
 \end{aligned}$$

1426 Therefore we can take $\delta_0^2 = \frac{\delta^2}{\psi_0^2} + \left(\frac{1}{\psi_0^2} - \psi_0^2 \right) (\varepsilon + \chi)^2 - \frac{4\varepsilon\chi}{\psi_0^2}$. Note that with this
 1427 definition, δ_0 goes to δ as the sampling increases and the neighborhood U is chosen
 1428 smaller (giving $\psi_0 \rightarrow 1$), which proves that our value of δ_0 is legitimate. \square

1429 **Appendix D. Embeddability of the straight Delaunay triangulation**
 1430 **(Proofs of subsection 5.2).** We first prove Lemma 5.3, recalled below, which
 1431 bounds the distance between the same point on the Karcher and the straight simplex.

1432 LEMMA D.1. Let \mathcal{P} be an ε -sample with respect to g on Ω . Let $\{p_i\}$ be a set
 1433 of $n + 1$ vertices in \mathcal{M} such that $N = \bigcap_{p_i \in \mathcal{P}} V(p_i) \neq \emptyset$. Moreover assume that
 1434 $\{p_i\} \subset U$. Suppose that U is included in a ball $B_g(p_0, r_0)$, with $p_0 \in U$ and $r_0 \in \mathbb{R}^+$,
 1435 such that the distortion compared to the Euclidean metric field satisfies

$$1436 \quad \forall p \in B(p_0, 5r_0), \psi(g(p), g_{\mathbb{E}}) \leq \psi_0$$

1437 and $B_g(p_0, 5r_0) \subset \Omega$ is geodesically convex. Let $\bar{\sigma}$ and $\tilde{\sigma}$ be the straight and Karcher
 1438 simplices that realize N . Let \tilde{x} be a point on the Karcher simplex $\tilde{\sigma}$ determined by
 1439 the barycentric coordinates $\{\lambda_i\}$ (see Equation (5)). Let x_e be the point uniquely
 1440 determined by $\{\lambda_i\}$ as $x_e = \sum_i \lambda_i p_i$. Then, $\|\tilde{x} - x_e\| \leq \sqrt{2 \cdot 4^3 (\psi_0 - 1) \varepsilon^2}$.

1441 *Proof.* The key observation is that given a convex function f and a function f'
 1442 that is close ($f - f' < \alpha$ with α small), then the minimum value of f' is at most of
 1443 $\min f + \alpha$. If we observe that at any point x where $f(x) > \min f + 2\alpha$, we also have
 1444 $f'(x) > \min f + \alpha$ so x is not a minimum of f' , we see that the minima of f and f' can
 1445 not be far apart. In particular, we have that if $x_{f', \min}$ is the point where f' attains
 1446 its minimum, then $f'(x_{f', \min}) \leq \min f + \alpha$. The precise argument goes as follows.

1447 We again assume that (possibly after a linear transformation) the metric is close
 1448 to the Euclidean one, that is:

$$1449 \quad d_g(x, y) = \|x - y\| + \delta d_g(x, y),$$

1450 with $|\delta d_g(x, y)| \leq (\psi_0 - 1) \|x - y\|$. If we assume that $\|x - y\| \leq 4\varepsilon$ and $\psi_0 \leq 2$, it
 1451 follows that

$$1452 \quad d_g(x, y)^2 = \|x - y\|^2 + \delta d_g^2(x, y),$$

1453 with

$$1454 \quad \delta d_g^2(x, y) \leq 4^3 (\psi_0 - 1) \varepsilon^2.$$

1455 Recall that \tilde{x} is the point where the functional

$$1456 \quad \mathcal{E}_\lambda(x) = \sum_i \lambda_i d_g(x, p_i)^2$$

1457 attains its minimum.

1458 Using the bounds above, we find that

$$1459 \quad \sum_i \lambda_i d_g(x, p_i)^2 = \sum_i \lambda_i \|x - p_i\|^2 + \sum_i \lambda_i \delta d_g^2(x, p_i),$$

1460 where $\sum_i \lambda_i \delta d_g^2(x, p_i) \leq 4^3 (\psi_0 - 1) \varepsilon^2$. We also see that

$$1461 \quad \left| \sum_i \lambda_i d_g(\tilde{x}, p_i)^2 - \sum_i \lambda_i \|\tilde{x} - p_i\|^2 \right| \leq 4^3 (\psi_0 - 1) \varepsilon^2.$$

1462 Taking f' to be $\sum_i \lambda_i d_g(\tilde{x}, p_i)^2$ in the explanation above, we find that

$$1463 \quad \left| \sum_i \lambda_i d_g(\tilde{x}, p_i)^2 - \sum_i \lambda_i \left\| \sum_j \lambda_j v_j - p_i \right\|^2 \right| \leq 4^3 (\psi_0 - 1) \varepsilon^2,$$

1464 because the Euclidean barycenter $x_e = \sum_i \lambda_i p_i$ is where the function $\sum_i \lambda_i \|\tilde{x} - p_i\|^2$
 1465 attains its minimum. Combining these results yields

$$1466 \quad (13) \quad \left| \sum_i \lambda_i \|\tilde{x} - p_i\|^2 - \sum_i \lambda_i \left\| \sum_j \lambda_j v_j - p_i \right\|^2 \right| \leq 2 \cdot 4^3 (\psi_0 - 1) \varepsilon^2.$$

1467 This bounds the distance between \tilde{x} and x_e . An explicit bound can be found by
 1468 observing that

$$\begin{aligned} 1469 \quad \sum_i \lambda_i \|x - p_i\|^2 &= \sum_i \lambda_i \left((x^1 - p_i^1)^2 + \dots + (x^n - p_i^n)^2 \right) \\ 1470 &= \sum_i \lambda_i (x^1 - p_i^1)^2 + \sum_i \lambda_i (x^2 - p_i^2)^2 + \dots + \sum_i \lambda_i (x^n - p_i^n)^2 \\ 1471 &= \left(x^1 - \sum_i \lambda_i p_i^1 \right)^2 - \left(\sum_i \lambda_i p_i^1 \right)^2 + \sum_i \lambda_i (p_i^1)^2 + \dots \\ 1472 &\quad + \left(x^n - \sum_i \lambda_i p_i^n \right)^2 - \left(\sum_i \lambda_i p_i^n \right)^2 + \sum_i \lambda_i (p_i^n)^2 \\ 1473 \quad (14) \quad &= \left\| x - \sum_i \lambda_i p_i \right\|^2 + \sum_j \left[- \left(\sum_i \lambda_i p_i^j \right)^2 + \sum_i \lambda_i (p_i^j)^2 \right]. \\ 1474 \end{aligned}$$

1475 Then, applying Equation (14) for both both $x = \tilde{x}$ and $x = x_e = \sum_j \lambda_j p_j$ in Equa-
 1476 tion (13), we obtain:

$$\begin{aligned} 1477 \quad &\left\| \left\| \tilde{x} - \sum_i \lambda_i p_i \right\|^2 + \sum_j \left[- \left(\sum_i \lambda_i p_i^j \right)^2 + \sum_i \lambda_i (p_i^j)^2 \right] \right. \\ 1478 \quad &\left. - \left(\left\| \sum_j \lambda_j p_j - \sum_i \lambda_i p_i \right\|^2 + \sum_j \left[- \left(\sum_i \lambda_i p_i^j \right)^2 + \sum_i \lambda_i (p_i^j)^2 \right] \right) \right\| \\ 1479 \quad &= \left\| \tilde{x} - \sum_i \lambda_i p_i \right\|^2 \leq 2 \cdot 4^3 (\psi_0 - 1) \varepsilon^2. \\ 1480 \end{aligned}$$

1481 which yields a distance bound of $\sqrt{2 \cdot 4^3 (\psi_0 - 1) \varepsilon^2}$. \square

1482 Although we have formulated this metric distortion result for simplices, the same
 1483 proof extends almost verbatim to continuous distributions. By this we mean that
 1484 the barycenter with respect to a metric g of a continuous distribution is close to the
 1485 barycenter with respect to the Euclidean metric, if g is close to the Euclidean metric.
 1486 Furthermore, note that the proof does not depend on the weights being positive.

1487 We now prove [Theorem 5.4](#), recalled below.

1488 **THEOREM D.2.** *Let \mathcal{P} be a δ -power protected (ε, μ) -net with respect to g on Ω .
 1489 Let $\{p_i\}$ be a set of $n + 1$ vertices in Ω such that $\cap_{p_i \in \mathcal{P}} V(p_i) \neq \emptyset$. Let $\bar{\sigma}$ and $\tilde{\sigma}$ be
 1490 the straight and Karcher simplices with vertices $\{p_i\}$. Let $\tilde{\tau}$ be a facet of $\tilde{\sigma}$, opposite
 1491 of the vertex p_i . If for all $\tilde{x} \in \tilde{\tau}$, we have $\|\tilde{x} - x_e\|$ smaller than the lower bound*

1492 on $D(p_i, \sigma)$, where x_e is the corresponding point on $\bar{\sigma}$ (as defined in [Lemma 5.3](#)),
 1493 then there is no inversion created when $\tilde{\sigma}$ is straightened onto $\bar{\sigma}$. Furthermore, if this
 1494 condition is fulfilled for all $\tilde{\sigma} \in \widetilde{\text{Del}}_g(\mathcal{P})$, then $\overline{\text{Del}}_g(\mathcal{P})$ is embedded.

1495 *Proof.* The lower bound on $D(p, \sigma)$ given in [Appendix B](#) is proportional to ε . The
 1496 proximity (upper) bound from [Lemma 5.3](#) is proportional to $\sqrt{(\psi_0 - 1)\varepsilon}$, therefore
 1497 going to 0 much faster. The embeddability is thus satisfied once

$$1498 \quad \sqrt{2 \cdot 4^3 (\psi_0 - 1) \varepsilon^2} < \frac{\delta^2}{4\varepsilon},$$

1500 that is once

$$1501 \quad \psi_0 < 1 + \frac{\iota^4}{32 \cdot 4^3} \quad \square$$

1503 **Appendix E. Deforming the triangulation \mathcal{T}_v .** An extreme configuration
 1504 can have a sphere in [Figure 15](#) separating parts of \mathcal{T}_v from the Voronoi vertex v . In
 1505 that case, we can chose different spheres to “push away” faces. [Figure 15](#) shows the
 1506 construction. We do not detail the computations.

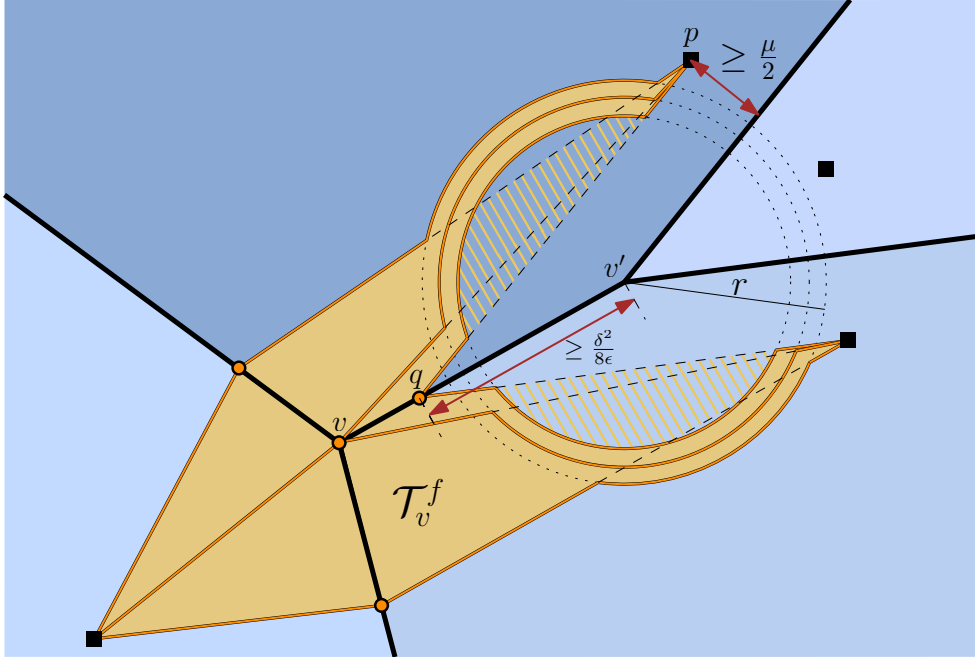


FIG. 15. *Limit case of the deformation of \mathcal{T}_v into \mathcal{T}_v^f .*

1507 **Appendix F. Equality of the Riemannian Delaunay complexes in ad-**
 1508 **vanced geometric settings.** In this section, we explain precisely how to obtain
 1509 conditions on the maximal length of a canvas edge and the quality of the sample set
 1510 such that the $\text{Del}_g^d(\mathcal{P})$ and the $\text{Del}_g(\mathcal{P})$ are equal.

1511 **F.1. Constant metric field.** We first investigate the setting of a subdomain
 1512 of \mathbb{R}^n endowed with a constant metric field.

1513 If, for all $\sigma \in \text{Del}_g^d(\mathcal{P})$, there exists $\sigma_C \in \mathcal{C}$ such that σ_C witnesses σ , and $\forall \sigma_C \in \mathcal{C}$
 1514 the simplex witnessed by σ_C belongs to $\text{Del}_g^d(\mathcal{P})$, then we say that \mathcal{C} captures $\text{Del}_g^d(\mathcal{P})$.
 1515 A Voronoi cell is said to be *captured* if all its Voronoi vertices are witnessed by the
 1516 canvas.

1517 By [Lemma C.1](#), if a point set \mathcal{P} in \mathbb{R}^n is a δ -power protected (ε, μ) -net with
 1518 respect to a constant metric field g_0 then the point set $\mathcal{P}' = \{F_0 p_i, p_i \in \mathcal{P}\}$, where F_0
 1519 is the square root of g_0 , is also a δ -power protected (ε, μ) -net, but with respect to the
 1520 Euclidean metric. We can deduce an upper bound on the maximal length of an edge
 1521 of \mathcal{C} for a constant metric field using the results of [Theorem 8.1](#) for the Euclidean
 1522 setting. The main result is given by the following theorem:

1523 **THEOREM F.1.** *Let \mathcal{P} be a point set in Ω . Let g be a constant Riemannian metric
 1524 field on Ω ($\forall x \in \Omega, g(x) = g_0$). Let \mathcal{C} be the canvas and let e_C be the length of its
 1525 longest edge. If*

$$1526 \quad e_C < \left(\min_i \sqrt{\lambda_i} \right) \min \left\{ \frac{\mu}{3}, \frac{\delta^2}{32\varepsilon} \right\},$$

1527 then $\text{Del}_g^d(\mathcal{P}) = \text{Del}_g(\mathcal{P})$.

1528 *Proof.* Consider $\mathcal{R} = \text{Vor}_{g_0}^d(\mathcal{P})$ over \mathcal{C} . Let F_0 be a square root of G_0 . The ma-
 1529 trix F_0 provides a stretching operator between the Euclidean and the metric spaces.
 1530 Let $\mathcal{P}_0 = \{F_0 p, p \in \mathcal{P}\}$ be the transformed point set and \mathcal{C}_0 be a canvas (a dense tri-
 1531 angulation) of the transformed space. Denote by \mathcal{R}_0 the discrete Riemannian Voronoi
 1532 diagram of \mathcal{P}_0 with respect to $g_{\mathbb{E}}$ over \mathcal{C}_0 . Let $e_{C,0}$ be the upper bound on the canvas
 1533 edge length of \mathcal{C}_0 provided by [Theorem 8.1](#) such that \mathcal{D}_0 is captured by \mathcal{C}_0 .

1534 Since \mathcal{P}_0 is a δ -power protected net with respect to $g_{\mathbb{E}}$, we can invoke [Theorem 8.1](#)
 1535 and we must thus have

$$1536 \quad e_{C,0} < \min \left\{ \frac{\mu}{3}, \frac{\delta^2}{32\varepsilon} \right\},$$

1537 for the canvas \mathcal{C}_0 to capture $\text{Del}_{\mathbb{E}}(\mathcal{P}_0)$.

1538 Let \mathcal{C}'_0 be the image of \mathcal{C} by F_0 . Note that \mathcal{C}'_0 and \mathcal{C}_0 are two different triangu-
 1539 lations of the same space. If any edge of \mathcal{C}'_0 is smaller (with respect to the Euclidean
 1540 metric) than $e_{C,0}$, then \mathcal{C}'_0 satisfies

$$1541 \quad e_{C',0} < \min \left\{ \frac{\mu}{3}, \frac{\delta^2}{32\varepsilon} \right\}$$

1542 and thus \mathcal{C}'_0 captures $\text{Del}_{\mathbb{E}}(\mathcal{P}_0)$.

1543 Recall that given an eigenvector v_i of G_0 with corresponding eigenvalue λ_i , a unit
 1544 length in the direction v_i in the metric space has length $1/\sqrt{\lambda_i}$ in the Euclidean space.
 1545 Therefore, if the bound e_C of \mathcal{C} is smaller than $\alpha e_{C,0}$, with

$$1546 \quad \alpha = \frac{1}{\max_i \left(\frac{1}{\sqrt{\lambda_i}} \right)} = \min_i \sqrt{\lambda_i},$$

1547 then every edge of \mathcal{C}'_0 is smaller than $e_{C,0}$. This implies that \mathcal{C}'_0 captures $\text{Del}_{\mathbb{E}}(\mathcal{P}_0)$
 1548 and therefore that \mathcal{C} captures $\text{Del}_{g_0}(\mathcal{P})$. \square

1549 This settles the case of a constant metric field.

1550 **F.2. Arbitrary metric field.** We now consider an arbitrary metric field g over
 1551 the domain \mathbb{R}^n . The key to proving the equality of the discrete Riemannian Delaunay
 1552 complex and the Riemannian Delaunay complex in this setting is to locally approxi-
 1553 mate the arbitrary metric field with a constant metric field, a configuration that we
 1554 have dealt with in the previous section. We shall always compare the metric field g in
 1555 a neighborhood U with the constant metric field $g' = g(p_0)$ where $p_0 \in U$. Because g'
 1556 and the Euclidean metric field differ by a linear transformation, we can simplify mat-
 1557 ters and assume that g' is the Euclidean metric field. The main argument of the proof
 1558 will be once again that a power protected has stable and separated Voronoi vertices.
 1559 We recall the main result of this section, [Theorem 7.1](#).

1560 **THEOREM F.2.** *Let g be an arbitrary metric field on Ω . Assume that \mathcal{P} is a δ -*
 1561 *power protected (ε, μ) -net in Ω with respect to g . Denote by \mathcal{C} the canvas, and $e_{\mathcal{C}}$ the*
 1562 *length of its longest edge. If*

$$1563 \quad e_{\mathcal{C}} < \min_{p \in \mathcal{P}} e_{\mathcal{C}, p},$$

1564 *where $e_{\mathcal{C}, p}$ is given by [Lemma F.3](#), and if ε is sufficiently small and δ is sufficiently*
 1565 *large (both values will be detailed in the proof), then $\text{Del}_g^d(\mathcal{P}) = \text{Del}_g(\mathcal{P})$.*

1566 We prove [Theorem 7.1](#) by computing for each point $p \in \mathcal{P}$ the maximal edge length
 1567 of the canvas such that the Voronoi cell $V_g(p)$ is captured correctly. Conditions on ε
 1568 and δ shall emerge from the intermediary results on the stability of power protected
 1569 nets.

1570 **LEMMA F.3.** *Let U be an open neighborhood of $\Omega \subset \mathbb{R}^n$ and g be a Riemannian*
 1571 *metric field on Ω . Assume that U is included in a ball $B_g(p_0, r_0)$, with $p_0 \in U$*
 1572 *and $r_0 \in \mathbb{R}^+$ such that $\forall p \in B(p_0, 5r_0), \psi(g(p), g_{\mathbb{E}}(p)) \leq \psi_0$ and $B(p_0, 5r_0) \subset \Omega$ is*
 1573 *geodesically convex. Here we remind ourselves that we write $g_{\mathbb{E}}$ for the Euclidean*
 1574 *metric field. Let \mathcal{P}_U be a point set in U and let $p_0 \in \mathcal{P}_U$.*

1575 *Suppose that \mathcal{P}_U is a δ -power protected (ε, μ) -net of with respect to g . Let $V_g(p_0)$*
 1576 *be the Voronoi cell of p_0 in $\text{Vor}_g^d(\mathcal{P}_U)$. If*

$$1577 \quad e_{\mathcal{C}, p_0} < \min_i \left(\sqrt{\lambda_i} \right) \min \left\{ \frac{\mu}{3}, \frac{\ell_0}{2} \right\},$$

1578 *with $\{\lambda_i\}$ the eigenvalues of g_0 and ℓ_0 that is made explicit in the proof, and if ε is*
 1579 *sufficiently small and δ is sufficiently large (both values will also be detailed in the*
 1580 *proof), then $\text{Del}_g^d(\mathcal{P}) = \text{Del}_g(\mathcal{P})$.*

1581 **F.2.1. Approach.** The many intermediary results needed to prove [Theorem 7.1](#)
 1582 are presented in [Appendix A](#), [Appendix B](#) and [Appendix C](#). We refer to them at
 1583 appropriate times.

1584 We use the fact that a Riemannian Voronoi cell $V_g(p_0)$ can be encompassed into
 1585 two Euclidean Voronoi cells $DV_{\mathbb{E}}^{+\eta}(p_0)$ and $EV_{\mathbb{E}}^{-\eta}(p_0)$ that are scaled up and down
 1586 versions of $V_{\mathbb{E}}(p_0)$ ([Lemma C.12](#)). Specifically, $EV_{\mathbb{E}}^{-\eta}(p_0)$ and $DV_{\mathbb{E}}^{+\eta}(p_0)$ are defined
 1587 by

$$1588 \quad EV_{\mathbb{E}}^{-\omega}(p_0) = \{x \in V_{\mathbb{E}}(p_0) \mid d_{g_{\mathbb{E}}}(x, \partial V_{\mathbb{E}}(p_0)) > \omega\},$$

1589 and

$$1590 \quad DV_{\mathbb{E}}^{+\omega}(p_0) = \bigcap_{i \neq 0} H^{\omega}(p_0, p_i),$$

1591 where $H^{\omega}(p_0, p_i)$ is the half-space containing p_0 and that is delimited by the bisector
 1592 $\text{BS}(p_0, p_i)$ translated away from p_0 by ω_0 . The constant η is the thickness of this

1593 encompassing and depends on the bound on the distortion ψ_0 in the neighborhood,
 1594 and on the sampling and separation parameters ε and μ . We have that η goes to 0
 1595 as ψ_0 goes to 1 (in other words, as U becomes smaller).

1596 The (local) stability of the power protected nets assumption is proved here again
 1597 (Lemma C.2 and Lemma C.16). From this observation, we can deduce that the
 1598 Voronoi vertices of the Euclidean Voronoi cell $V_{\mathbb{E}}(p_0)$ are separated, and thus that
 1599 the Voronoi vertices of $EV_{\mathbb{E}}(p_0)$ are separated. A bound on the maximal length of a
 1600 canvas edge can then be computed such that $EV_{\mathbb{E}}(p_0)$ is captured and thus $V_g(p_0)$ is
 1601 captured.

1602 *Difficulties.* The difficulty almost entirely comes from proving the stability of the
 1603 assumption of power protection under metric perturbation in any dimension, that is
 1604 proving that if we assume that \mathcal{P} is δ -power protected with respect to the arbitrary
 1605 metric field g , then, in a small neighborhood around p_0 , the point set \mathcal{P} is δ_0 -power
 1606 protected with respect to $g_0 = g(p)$. Assuming a power protected net does give us
 1607 some bounds, but creates a tricky circular dependency as the coefficient δ appears in
 1608 the dihedral angles (see Lemma B.2). We remedy this issue by proving that Euclidean
 1609 dihedral angles are bounded assuming power protection with respect to the arbitrary
 1610 metric field, with Lemma B.5 and Lemma B.6.

1611 **F.2.2. Proof of Lemma F.3.** Using Lemma C.5, we have that $V_g(p_0)$ lies
 1612 in $DV_{\mathbb{E}}^{+\eta}(p_0)$ and contains $EV_{\mathbb{E}}^{-\eta}(p_0)$. Since $V_g(p_0)$ contains $EV_{\mathbb{E}}^{-\eta}(p_0)$, if e_C is suffi-
 1613 ciently small such that $EV_{\mathbb{E}}^{-\eta}(p_0)$ is captured, then $V_g(p_0)$ is also captured. Proving
 1614 that $EV_{\mathbb{E}}^{-\eta}(p_0)$ is captured is done similarly to the Euclidean setting. While we do not
 1615 explicitly have the power protected net property for the relaxed Voronoi cells (and
 1616 specifically, $EV_{\mathbb{E}}^{-\eta}(p_0)$), we can still extract the critical property that the Voronoi
 1617 vertices are separated, as shown by the next lemma.

1618 LEMMA F.4. Assume U , g , and ψ_0 as in Lemma F.3. Assume that the point
 1619 set \mathcal{P}_U is a δ_0 -power protected (ε, μ) -net with respect to the Riemannian metric field g .
 1620 Then the Voronoi vertices of $EV_{\mathbb{E}}^{-\eta}(p_0)$ are separated.

1621 *Proof.* By Lemma C.2 and Lemma C.16, we have local stability of the power
 1622 protection and net properties. Hence, \mathcal{P} is δ_0 -power protected (ε_0, μ_0) -net with respect
 1623 to $g_{\mathbb{E}}$ in U .

1624 Let $L_0 = \delta_0^2/4\varepsilon_0$ be the separation bound induced by the δ_0 -power protection
 1625 property of \mathcal{P}_U (see Lemma A.4). Let l be the distance between any two adjacent
 1626 Voronoi vertices of $EV_{\mathbb{E}}^{-\eta}(p_0)$. We know by Lemma C.14 that the parallelotopic region
 1627 around a Voronoi vertex is included in a ball centered on the Voronoi vertex and of
 1628 radius χ . The protection parameter ι is given by $\delta = \iota\varepsilon$. We have that

$$\begin{aligned}
 1629 \quad l &\geq L - 2\chi \\
 1630 \quad &\geq \frac{\delta_0^2}{4\varepsilon_0} - 2 \frac{\chi_2}{\sin^{n-2}\left(\frac{\varphi}{2}\right)} \\
 1631 \quad (15) \quad &\geq \frac{\delta_0^2}{4\varepsilon_0} - 2 \frac{4\eta}{\left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) (\sqrt{1 + s_0} - \sqrt{1 - s_0})^{n-2}} =: \ell_0, \\
 1632
 \end{aligned}$$

1633 where φ represents the dihedral angle and $s_0 = \frac{1}{2} \left[\frac{\iota^2}{4\psi_0^2} - \frac{1}{2} \left(\psi_0^2 - \frac{1}{\psi_0^2} \right) \right]$. For the
 1634 stability regions not to intersect, we require l to be positive. This can be ensured by

1635 enforcing that the lower bound is positive:

$$1636 \quad \frac{\delta_0^2}{4\varepsilon_0} > \frac{8\eta}{\left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) (\sqrt{1 + s_0} - \sqrt{1 - s_0})^{n-2}}$$

1637 Recall that $\varepsilon_0 = \psi_0\varepsilon$, $\mu_0 = \lambda\varepsilon/\psi_0$ and $\rho_0 = 2\psi_0\varepsilon$. Using these notations, we see
1638 that $l > 0$ if

$$1639 \quad \delta_0^2 > \frac{32\varepsilon_0\rho_0^2(\psi_0^2 - 1)}{\mu_0 \left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) (\sqrt{1 + s_0} - \sqrt{1 - s_0})^{n-2}}$$

$$1640 \quad (16) \quad \delta_0^2 > \frac{128\varepsilon^2\psi_0^4(\psi_0^2 - 1)}{\lambda \left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) (\sqrt{1 + s_0} - \sqrt{1 - s_0})^{n-2}}.$$

1642 This condition is easy to satisfy when U becomes smaller and ψ_0 goes to 1 because
1643 the right hand side of Inequality (16) is proportional to $(\psi_0^2 - 1)\varepsilon^2$.

1644 The intermediary results that we use already impose some conditions on δ and ε ,
1645 and we thus would like to give the condition in Equation (16) in terms of δ , so that
1646 it may be compared with Inequality (11). In Lemma C.16, we have seen that

$$1647 \quad \delta_0^2 = \frac{\delta^2}{\psi_0^2} + \left(\frac{1}{\psi_0^2} - \psi_0^2\right) (\varepsilon + \chi)^2 - \frac{4\varepsilon\chi}{\psi_0^2},$$

1648 Thus

$$1649 \quad \frac{\delta^2}{\psi_0^2} + \left(\frac{1}{\psi_0^2} - \psi_0^2\right) (\varepsilon + \chi)^2 - \frac{4\varepsilon\chi}{\psi_0^2} >$$

$$1650 \quad \frac{128\varepsilon^2\psi_0^4(\psi_0^2 - 1)}{\lambda \left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) (\sqrt{1 + s_0} - \sqrt{1 - s_0})^{n-2}},$$

1652 which is equivalent to

$$1653 \quad \delta^2 > \frac{128\varepsilon^2\psi_0^6(\psi_0^2 - 1)}{\lambda \left(\sqrt{1 + \frac{\lambda}{2\psi_0^2}} - \sqrt{1 - \frac{\lambda}{2\psi_0^2}}\right) (\sqrt{1 + s_0} - \sqrt{1 - s_0})^{n-2}}$$

$$1654 \quad + 4\varepsilon\chi$$

$$1655 \quad + (\psi_0^4 - 1)(\varepsilon + \chi)^2.$$

1657 which is equivalent to

$$1658 \quad (17) \quad \delta^2 > 8\varepsilon\psi_0^3\chi + 4\varepsilon\chi + (\psi_0^4 - 1)(\varepsilon + \chi)^2. \quad \square$$

1660 This bound is again proportional to $(\psi_0^2 - 1)\varepsilon$ and is very similar to the bound given
1661 by Inequality (11), made explicit in Inequality (12), but Inequality (17) provides the
1662 tougher bound due to the $(\varepsilon + \chi)$ coefficient.

1663 We can now provide an upper bound on the length of any canvas edge so that
1664 it captures $\text{EV}_{\mathbb{E}}^{-\eta}(p_0)$ and prove Lemma F.3. From Theorem 8.1, we have that if
1665 the canvas edge length is bounded as: $e_C < \min\{\mu_0/16, \delta_0^2/64\varepsilon_0\}$, then $V_{\mathbb{E}}(p_0)$ is
1666 captured as \mathcal{P} is a δ_0 -power protected (ε_0, μ_0) -net with respect to the Euclidean
1667 metric field. As we want to capture $\text{EV}_{\mathbb{E}}(p_0)$, we cannot directly use this result.

1668 We have nevertheless obtained the separation between the Voronoi vertices of the
 1669 eroded Voronoi cell (Equation (15)). It is then straightforward to modify the result
 1670 of Lemma 8.7 by using the separation bound provided in Lemma F.4 instead of the
 1671 one provided by Lemma A.4. We thus choose

$$1672 \quad e_{\mathcal{C},p_0}^0 = \min \left\{ \frac{\mu}{3}, \frac{\ell_0}{2} \right\}.$$

1673 *Remark F.5.* We here ignore the consequences of Appendix E as it only compli-
 1674 cates formulas without changing the logical steps.

1675 We should not forget that we have assumed that g_0 is the Euclidean metric field,
 1676 which is generally not the case, we must in fact proceed like for the case of a constant
 1677 metric field (Theorem F.1):

$$1678 \quad e_{\mathcal{C},p_0} < \min_j \left(\sqrt{\lambda_j} \right) (e_{\mathcal{C},p_0}^0),$$

1679 with $\{\lambda_i\}$ the eigenvalues of g_0 .

1680 Therefore, if the site set satisfies the previous conditions on ε and δ and the
 1681 canvas is enough for all of its edges to have a length smaller than $e_{\mathcal{C},i}$, then $\text{EV}_{\mathbb{E}}^{-\eta}(p_0)$
 1682 is captured, and thus $V_g(p_0)$ is captured, which proves Lemma F.3.

1683 Taking the minimum of all the bounds $e_{\mathcal{C}}$ over all $p_i \in \mathcal{P}$, we obtain an upper
 1684 bound on the length of the longest canvas edge,

$$1685 \quad e_{\mathcal{C}} = \min_{p_i \in \mathcal{P}} e_{\mathcal{C},p_i},$$

1686 such that all the Voronoi cells are captured.

1687 In all the Lemmas necessary to obtain the local results, we have imposed condi-
 1688 tions on ε and δ . Similarly to the bound $e_{\mathcal{C}}$, the domain-wide bounds on ε and δ
 1689 are computed from the local values of ε and δ .

1690 Finally, this proves that $\text{Del}_g^d(\mathcal{P}) = \text{Del}_g(\mathcal{P})$ in the general setting (Theorem 8.1),
 1691 when geodesics are exactly computed.

1692 **F.3. Approximate geodesic distance computations.** We have so far as-
 1693 sumed that geodesics are computed exactly, which is generally not the case in prac-
 1694 tice. Nevertheless, once the error in the approximation of the geodesic distances is
 1695 sufficiently small, the computation of the discrete Riemannian Voronoi diagram with
 1696 approximate geodesic distances can be equivalently seen as the computation of a sec-
 1697 ond discrete Riemannian Voronoi diagram using exact geodesic distances but for a
 1698 slightly different metric field.

1699 Denote by \tilde{d}_g the geodesic approximation and d_g the exact geodesic distance with
 1700 respect to the metric field g . Assume that in a sufficiently small neighborhood U
 1701 (see Lemma 4.1), the distances can be related as

$$1702 \quad \left| d_g(p_0, x) - \tilde{d}_g(p_0, x) \right| \leq \xi d_g(p_0, x),$$

1703 where ξ is a function of x that goes to 0 as the sampling parameter ε goes to 0. We
 1704 can formulate a lemma similar to Lemma C.5 to bound the distance between the same
 1705 bisectors between sites for the exact and the approximate diagrams.

1706 **LEMMA F.6.** *Let U be an open neighborhood of $\Omega \subset \mathbb{R}^2$ and g and g' be two*
 1707 *Riemannian metric fields on U . Assume that U is as in Lemma 4.1 and write ψ_0 for*

1708 the metric distortion. Let \mathcal{P}_U be a point set in U and let $p_0 \in \mathcal{P}_U$. Let $\mathcal{P}_U = \{p_i\}$
 1709 be a point set in U . Let $\widetilde{V}_{p_0, g}$ denote a Voronoi cell with respect to the approximate
 1710 geodesic distance.

1711 Suppose that the Voronoi cell $V_g^{+2\xi\tilde{\rho}}(p_0)$ lies in a ball of radius $\tilde{\rho}$ with respect
 1712 to the metric g , which lies completely in U . Then $\widetilde{V}_{p_0, g}$ lies in $V_g^{+2\xi\tilde{\rho}}(p_0)$ and con-
 1713 tains $V_g^{-2\xi\tilde{\rho}}(p_0)$.

1714 *Proof.* Let $\widetilde{BS}_g(p_0, p_i)$ be the bisector between p_0 and p_i with respect to the ap-
 1715 proximate geodesic distance. Let $y \in \widetilde{BS}_g(p_0, p_i) \cap B_g(p_0, \rho)$, where $B_g(p_0, \rho)$ denotes
 1716 the ball centered at p_0 of radius ρ with respect to the exact geodesic distance. Now
 1717 $\tilde{d}_g(y, p_0) = \tilde{d}_g(y, p_i)$, and thus

$$\begin{aligned}
 1718 \quad |d_g(y, p_0)^2 - d_g(y, p_i)^2| &= |d_g(y, p_0)^2 - \tilde{d}_g(y, p_0)^2 + \tilde{d}_g(y, p_i)^2 - d_g(y, p_i)^2| \\
 1719 &\leq |d_g(y, p_0)^2 - \tilde{d}_g(y, p_0)^2| + |d_g(y, p_i)^2 - \tilde{d}_g(y, p_i)^2| \\
 1720 &\leq 2\xi(d_g(y, p_0)^2 + d_g(y, p_i)^2) \\
 1721 &\leq 2\xi\tilde{\rho}^2.
 \end{aligned}$$

1723 Thus $d_g(y, p_0)^2 \leq d_g(y, p_i)^2 + \tilde{\omega}$ and $d_g(y, p_0)^2 \geq d_g(y, p_i)^2 - \tilde{\omega}$ with $\tilde{\omega} = 2\xi\tilde{\rho}^2$, which
 1724 gives us the expected result. \square

1725 Denote by $\widetilde{V}(p_0)$ the Voronoi cell of p_0 with the approximate metric. We can
 1726 then incorporate [Lemma F.6](#) to obtain a result similar to [Lemma C.12](#).

1727 **LEMMA F.7.** *Let U , ψ_0 , g , g_0 and \mathcal{P}_U be defined as in [Theorem 7.1](#). Then we
 1728 can find η'_0 and ω'_0 such that*

$$1729 \quad EV_{g_0}^{-\eta'_0}(p_0) \subseteq V_{g_0}^{-\omega'_0}(p_0) \subseteq V_g^{-\tilde{\omega}}(p_0) \subseteq \widetilde{V}_g(p_0) \subseteq V_g^{+\tilde{\omega}}(p_0) \subseteq V_{g_0}^{+\omega'_0}(p_0) \subseteq DV_{g_0}^{+\eta'_0}(p_0).$$

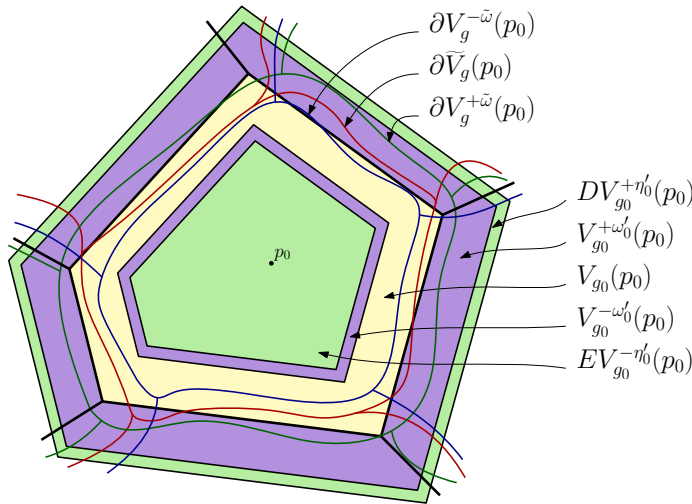


FIG. 16. Illustration of the different encompassing cells around p_0 in the context of an approximate geodesic distance. The RVDs with respect to g and g_0 are respectively traced in red and black. The dilated Voronoi cells are traced in blue and green. The Voronoi cell $V_{g_0}(p_0)$ is colored in yellow. The cells $DV_{g_0}^{+\eta'_0}$ and $EV_{g_0}^{-\eta'_0}$ are colored in green, and the cells $V_{g_0}^{\pm\omega'_0}(p_0)$ are colored in purple.

1730 These inclusions are illustrated in [Figure 16](#).

1731 The subsequent lemmas and proofs are similar to what was done in the case of
1732 exact geodesic computations and we do not explicit them.

1733

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