



**HAL**  
open science

# Stabilization of a perturbed chain of integrators in prescribed time

Yacine Chitour, Rosane Ushirobira

► **To cite this version:**

Yacine Chitour, Rosane Ushirobira. Stabilization of a perturbed chain of integrators in prescribed time. CDC 2019 - 58th IEEE Conference on Decision and Control, Dec 2019, Nice, France. 10.1109/CDC40024.2019.9029571 . hal-02418557

**HAL Id: hal-02418557**

**<https://inria.hal.science/hal-02418557>**

Submitted on 18 Dec 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Stabilization of a perturbed chain of integrators in prescribed time

Yacine Chitour and Rosane Ushirobira

**Abstract**—In this work, we study issues of prescribed time stabilization of a chain of integrators of arbitrary length, that can be either pure (i.e. with no disturbance) or perturbed. In the first part, we revisit the feedback law proposed by Song et al. and we show that it can be appropriately recast within the framework of time-varying homogeneity. Since this feedback is not robust with respect to measurement noise, in the second part of the paper, we provide a feedback law inspired by the sliding mode theory. This latter feedback not only stabilizes the pure chain of integrators in prescribed time but also exhibits robustness in the presence of disturbances.

## I. INTRODUCTION

In this paper, we study the following problem: for  $n$  a positive integer and  $T > 0$ , consider the perturbed chain of integrators given by

$$\dot{x}(t) = J_n x(t) + (d(t) + b(t)u(t)) e_n, \quad \forall t \in [0, T], \quad (1)$$

where the state  $x(t) \in \mathbb{R}^n$ , the input  $u(t) \in \mathbb{R}$ ,  $(e_i)_{1 \leq i \leq n}$  denotes the canonical basis of  $\mathbb{R}^n$ ,  $J_n$  denotes the  $n$ th Jordan block (i.e.  $J_n e_i = e_{i-1}$  for  $1 \leq i \leq n$ ,  $e_0 = 0$ ),  $d(t) \in \mathbb{R}$  is the (external) perturbation and  $b(t) \in \mathbb{R}$  is the uncertainty. Also assume that there exists a positive constant  $\underline{b}$  such that

$$b(t) \geq \underline{b}, \quad \forall t \in [0, T]. \quad (2)$$

Our goal is to design a feedback control  $u$  which renders the system (1) fixed-time input-to-state stable in any time  $T > 0$  and convergent to zero (PT-ISS-C) (cf. [1] and Def. 3). Notice that one may ask similar robustness properties in the presence of noise measurement  $d_0$ , for instance, if the feedback control  $u$  is static, it takes the form  $u = F(x + d_0)$ , with the feedback law  $F$  stabilizing the pure chain of integrators.

We start by revisiting the paper by Song et al. [1] where the feedback is (essentially) of the form  $u(t) = \lambda(t)K^\top x(t) + \mu(t)$ ,  $\forall t \in [0, T]$  where  $T > 0$  is prescribed,  $K \in \mathbb{R}^n$  and both functions  $\lambda, \mu : [0, T] \rightarrow \mathbb{R}_+^* := \mathbb{R}_+ \setminus \{0\}$  are sums of integer powers of the rational function  $\frac{1}{T-t}$ , hence blowing up to

Y. Chitour is with Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), CNRS - CentraleSupélec - Université Paris-Sud, 3, rue Joliot Curie, 91192, Gif-sur-Yvette, France, and R. Ushirobira is with Inria, Univ. Lille, CNRS, UMR 9189 - CRISTAL - Centre de Recherche en Informatique Signal et Automatique de Lille, F-59000 Lille, France. Email: yacine.chitour@l2s.centralesupelec.fr, rosane.ushirobira@inria.fr

This research was partially supported by the iCODE Institute, research project of the IDEX Paris-Saclay, and by the Hadamard Mathematics LabEx (LMH) through the grant number ANR-11-LABX-0056-LMH in the "Programme des Investissements d'Avenir".

infinity as  $t$  tends to  $T$ . In the present paper, we show that this choice of feedback actually falls very simply into the realm of homogeneity control systems [2] except that the homogeneity factor is time-varying. This different framework simplifies the presentation and the proofs in [1]. In addition, we also precisely demonstrate here why this approach fails when the feedback is perturbed by noise, a fact that had been already pointed out in [1].

The homogeneity viewpoint allows the use of a feedback of the form  $u(t) = \lambda(t)F(x(t))$  where  $\lambda$  is a time-varying function as before but the state feedback  $F$  comes from a sliding mode design. That allows some perturbations on the system to be handled but yet no measurement noise.

In the last part of the paper, we propose a feedback design that does not involve a time-varying function  $\lambda$  but it is simply based on fixed-time stabilization (cf. [3] and [2]) with a control on the supremum of the convergence time in the case of unperturbed chain of integrators. Some partial positive results in case of measurement noise on the feedback can then be obtained. This feedback design relies on the sliding mode feedback laws proposed by [4] for finite-time stabilization of a pure chain of integrators of length  $n$  (the dimension of the state space). Recall that, in that reference, it is shown that for every homogeneity parameter  $\kappa \in [-\frac{1}{n}, \frac{1}{n}]$ , there exists a control law  $u = \omega_\kappa^H(x)$  which stabilizes  $\dot{x} = J_n x + u e_n$  and there exists a Lyapunov function  $V_\kappa$  for the closed-loop system which satisfies  $\dot{V}_\kappa \leq -CV_\kappa^{\frac{2+2\kappa}{2+\kappa}}$ , for some positive constant  $C$ , independent of  $\kappa$ . One of the main virtues of these feedback and Lyapunov functions is that they admit explicit closed forms formulas computable once the dimension  $n$  is given. Here, to achieve first fixed-time stabilization, we choose as in [3], a feedback law of the type  $u = \omega_{\kappa(x)}^H(x)$ , where the homogeneity parameter is a function of the state and that can be made continuous using the nice idea of [2]. Finally, we use a standard homogeneity trick to pass from fixed-time to prescribed-time stabilization and at once obtain robustness results of ISS type with respect to measurement noise and external disturbance, as in [2].

The structure of the paper goes as follow. In Section II, general stability notions are recalled and homogeneity properties are provided in Section III. Linear time-varying homogeneous feedback are presented in Section IV, as well as corresponding stability results. Section V contains different approaches with a feedback insuring prescribed-

time stabilization, with and without homogeneous notions. Numerical examples illustrate our results in Section VI. Most proofs are omitted for the lack of space.

## II. STABILITY DEFINITIONS

We begin by recalling some stability notions [5].

*Definition 1:* Let us consider a nonlinear system  $\dot{x} = f(x, t)$ . The solution of this system for an initial condition  $x_0$  is denoted by  $X(t, x_0)$ . Let  $\Omega$  be an open neighborhood of a forward invariant set  $\mathcal{A} \subset \mathbb{R}^n$ <sup>1</sup>. At  $\mathcal{A}$ , the system is:

- (a) *Lyapunov stable* if for any  $x_0 \in \Omega$  the solution  $X(t, x_0)$  is defined for all  $t \geq 0$ , and for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any  $x_0 \in \Omega$ , if  $\|x_0\|_{\mathcal{A}} \leq \delta$  then  $\|X(t, x_0)\|_{\mathcal{A}} \leq \varepsilon, \forall t \geq 0$ .
- (b) *asymptotically stable* if it is Lyapunov stable and for any  $\kappa > 0, \varepsilon > 0$ , there exists  $T(\kappa, \varepsilon) \geq 0$  such that for any  $x_0 \in \Omega$ , if  $\|x_0\|_{\mathcal{A}} \leq \kappa$  then  $\|X(t, x_0)\|_{\mathcal{A}} \leq \varepsilon, \forall t \geq T(\kappa, \varepsilon)$ .
- (c) *finite-time converging from  $\Omega$*  if for any  $x_0 \in \Omega$  there exists  $0 \leq T < +\infty$  such that  $X(t, x_0) \in \mathcal{A}$  for all  $t \geq T$ . The function  $\mathcal{T}_{\mathcal{A}}(x_0) = \inf\{T \geq 0 \mid X(t, x_0) \in \mathcal{A}, \forall t \geq T\}$  is called the *settling time* of the system.
- (d) *finite-time stable* if it is Lyapunov stable and finite-time converging from  $\Omega$ .
- (e) *fixed-time stable* if it is finite-time stable and  $\sup_{x_0 \in \Omega} \mathcal{T}_{\mathcal{A}}(x_0) < +\infty$ .

Furthermore, for prescribed-time stability and robustness notions, we consider disturbances  $d : [0, \infty) \rightarrow \mathbb{R}^n$  that are measurable functions where  $\|d\|_{[t_0, t_1]}$  denotes the essential supremum over any time interval  $[t_0, t_1)$  contained in  $[0, \infty)$ . If  $[t_0, t_1) = [0, \infty)$ , then we say that  $d$  is *bounded* if  $\|d\|_{\infty} := \|d\|_{[0, \infty)}$  is finite. We have (see [1] and [2]) the following two definitions

*Definition 2:* A system  $\dot{x} = f(x, t, d)$  is *prescribed-time input-to-state stable in time  $T$*  (PT-ISS) if there exist functions  $\beta \in \mathcal{KL}$ <sup>2</sup> and  $\gamma \in \mathcal{K}$  such that for all  $t \in [t_0, t_0 + T)$  and bounded  $d$ ,  $|x(t)| \leq \beta\left(|x_0|, \frac{t-t_0}{T+t_0-t}\right) + \gamma(\|d\|_{[t_0, t]})$ .

*Definition 3:* A system  $\dot{x} = f(x, t, d)$  is *fixed-time input-to-state stable in time  $T$  and convergent to zero* (PT-ISS-C) if there exist functions  $\beta, \beta_f \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that for all  $t \in [t_0, t_0 + T)$  and bounded  $d$ ,  $|x(t, d, x_0)| \leq \beta_f\left(\beta(|x_0|, t - t_0) + \gamma(\|d\|_{[t_0, t]}), \frac{t-t_0}{T+t_0-t}\right)$ .

*Definition 4:* A system  $\dot{x} = f(x, t, d)$  is *input-to-state practically stable* (ISpS) if for any bounded disturbance  $d$ , there exist functions  $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$  and  $c > 0$  such that, for all

<sup>1</sup>Meaning that if  $x_0 \in \mathcal{A}$ , then the solution  $X(t, x_0) \in \mathcal{A}$  for all  $t \geq 0$ .

<sup>2</sup>A function  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to a class  $\mathcal{K}$  if it is strictly increasing and continuous with  $\gamma(0) = 0$ . A function  $\alpha$  is said to belong to a class  $\mathcal{K}_{\infty}$  if  $\alpha \in \mathcal{K}$  and it increases to infinity. A function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to a class  $\mathcal{KL}$  if for each fixed  $t \in \mathbb{R}_+, \beta(\cdot, t) \in \mathcal{K}_{\infty}$  and if for each fixed  $s \in \mathbb{R}_+, \beta(s, t) \xrightarrow{t \rightarrow \infty} 0$ .

$t \geq 0$  and bounded  $d, |x(t, d, x_0)| \leq \beta(|x_0|, t) + \gamma(\|d\|_{[0, t]}) + c$ . The system is *input-to-state stable* (ISS) if  $c = 0$ .

Note that PT-ISS-C is a much stronger property than ISS.

Next, basic definitions of homogeneity are recollected.

*Definition 5:*

- (i) A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be *homogeneous* of degree  $m \in \mathbb{R}$  with respect to the weights  $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$  if for all  $x \in \mathbb{R}^n$  and  $\varepsilon \in \mathbb{R}_{>0}, f(D_{\varepsilon}^{\mathbf{r}} x) = \varepsilon^m f(x)$ , where  $D_{\varepsilon}^{\mathbf{r}} = \text{diag}(\varepsilon^{r_i})_{i=1}^n$  defines a family of dilations. We say also that  $f$  is  $\mathbf{r}$ -homogeneous of degree  $m$ .
- (b) A vector field  $\Phi = (f_1, \dots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be *homogeneous* of degree  $m \in \mathbb{R}$  if for all  $1 \leq k \leq n$ , for all  $x \in \mathbb{R}^n$  and  $\varepsilon \in \mathbb{R}_{>0}, f_k(D_{\varepsilon}^{\mathbf{r}} x) = \varepsilon^{m+r_k} f_k(x)$ . In other words, if the coordinate functions  $f_k$  are homogeneous of degree  $m + r_k$ . We say also that  $F$  is  $\mathbf{r}$ -homogeneous of degree  $m$ .
- (c) Let  $\Phi$  be a continuous vector field. If  $\Phi$  is  $\mathbf{r}$ -homogeneous of degree  $m$ , then the system  $\dot{x} = \Phi(x), x \in \mathbb{R}^n$  is  $\mathbf{r}$ -homogeneous of degree  $m$ .

The next lemma is important in the proof of our results in Section V (see for instance [2]).

*Lemma 1:* [6] Let  $\dot{x} = f(x, t)$  be a  $\mathbf{r}$ -homogeneous system of degree  $\kappa$  asymptotically stable at the origin. Then at the origin, the system is globally finite-time stable if  $\kappa < 0$ , globally exponentially stable if  $\kappa = 0$  and globally fixed-time stable with respect to any open set containing the origin if  $\kappa > 0$ .

## III. TIME-VARYING HOMOGENEITY

Recall that  $(e_i)_{1 \leq i \leq n}$  denotes the canonical basis of  $\mathbb{R}^n$  and  $J_n$  the  $n$ th Jordan block. For  $\lambda > 0$ , using the notation for  $D_{\lambda}^{\mathbf{r}}$  in Def. 5, we have (see also [7]):

$$D_{\lambda}^{\mathbf{r}} = \text{diag}(\lambda^{r_i})_{i=1}^n, D_{\lambda}^{\mathbf{r}} J_n (D_{\lambda}^{\mathbf{r}})^{-1} = \lambda J_n, D_{\lambda}^{\mathbf{r}} e_n = \lambda e_n \quad (3)$$

where we set  $r_i = n - i + 1$  ( $i = 1, \dots, n$ ).

In the literature devoted to prescribed-time stabilization (see [1] and references therein) and as clearly stated in Definitions 2 and 3, the quantity  $\frac{t-t_0}{T+t_0-t}$  can be interpreted as a new time scale that tends to infinity as  $t$  tends to the prescribed convergence time  $T$ . This fact suggests to consider the homogeneity parameter  $\lambda$  depending on the time  $t$  such that with the new time

$$s : [0, T) \rightarrow \mathbb{R}_+^*, \quad s(t) = \int_0^t \lambda(\xi) d\xi, \quad (4)$$

then  $s(t) \xrightarrow{t \rightarrow T} \infty$ . In that case, consider the change of coordinates and time given by

$$y(s) = D_{\lambda(t)}^{\mathbf{r}} x(t), \quad \forall t \in [0, T). \quad (5)$$

To analyze the dynamics of  $y$ , we denote by  $y'$  its derivative with respect to the new time  $s$ . Taking derivatives with respect to  $s$  to simplify the formulas and using

(1), (3) and (4), we obtain  $\lambda y' = \dot{y} = \dot{\lambda} \frac{\partial D_\lambda^r}{\partial \lambda} (D_\lambda^r)^{-1} y + \lambda (J_n y + b u e_n + d e_n)$ .

Since for every  $\mu > 0$ , it is immediate that  $\frac{\partial D_\mu^r}{\partial \mu} (D_\mu^r)^{-1} = \frac{1}{\mu} D^*$  with  $D^* = \text{diag}(n, n-1, \dots, 1)$ , it follows:

$$y' = \left( \frac{\dot{\lambda}}{\lambda^2} D^* + J_n \right) y + (b(s) u(s) + d(s)) e_n. \quad (6)$$

Remark that here we consider the control  $u$  and both  $b$  and  $f$  as functions of the new time  $s$ .

Let  $a : [0, T] \rightarrow \mathbb{R}$  be a non-negative continuous function so that the  $\mathcal{C}^1$ -function  $A : [0, T] \rightarrow \mathbb{R}$  defined by  $A(t) = \int_t^T a(v) dv$  is positive on  $[0, T]$ . Setting

$$\lambda : [0, T] \rightarrow \mathbb{R}_+^*, t \mapsto \frac{1}{A(t)}, \quad (7)$$

we obtain  $\frac{\dot{\lambda}(t)}{\lambda^2(t)} = a(t)$ ,  $t \in [0, T]$ .

The function  $\lambda$  is then strictly increasing and it tends to infinity as  $t$  tends to  $T$ . As a consequence, the time  $s$  defined in (4) realizes an increasing  $\mathcal{C}^1$  bijection from  $[0, T]$  to  $[0, \infty)$ .

With this choice, (6) becomes

$$y' = (a(s)D^* + J_n)y + (b(s) u(s) + d(s))e_n. \quad (8)$$

Our goal is to design a feedback control  $u$  that renders this system PT-ISS-C in time  $T > 0$ . The idea is to set

$$u = F(y(s)), \quad (9)$$

where  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function to be chosen later.

#### IV. LINEAR FEEDBACK

In this section, we revisit the results obtained in [1] in the light of time-varying homogeneity. To establish the connection with that reference, we compare their change of variables to ours defined in (7). At once, we see that the function  $\mu$  of [1] defined by  $\mu(t) = \frac{T^{n+m}}{(T-t)^{n+m}}$ ,  $\forall t \in [0, T]$  for  $m, n \in \mathbb{N}$ , corresponds to the time-varying homogeneity parameter  $\lambda$ , up to a positive constant, where  $a$  is chosen as  $a(t) = (T-t)^{m-1}$  ( $m \in \mathbb{N}$ ), for all  $t \in [0, T]$ . In opposite to [1], our approach does not involve time derivatives of  $\lambda$  (or equivalently of  $\mu$ ), hence our computations are much simpler.

As for the feedback control in [1], it is given by  $u = -\frac{1}{b}(d + L_0 + L_1 + kz)$ , where  $L_0$  is a linear combination of successive derivatives of  $\mu$  and the state components,  $L_1$  contains a gain matrix  $K_{n-1}$ ,  $k$  is a gain and  $z$  is a change of variable of the  $n$ th state. This expression shows that their choice of feedback can be essentially reduced to a linear one (realized by the constant  $k$  and the  $\mathbb{R}^{n-1}$  vector  $K_{n-1}$  in [1]). This justifies, in our case, the setting of  $F(y) = -K^\top y$  for some vector  $K \in \mathbb{R}^n$ . After replacing  $u$  in (8), it follows

$$y' = \left( a(s)D^* + J_n - b(s)e_n K^\top \right) y + d(s) e_n, \quad (10)$$

i.e.,  $y' = M(s)y + d(s) e_n$  where  $M(s) = a(s)D^* + J_n - b(s) e_n K^\top$  with  $b$  satisfying (2).

Such systems were considered in [7] (without the term  $a(s)D^*$ ) where the authors proved that there exist a positive constant  $\mu > 0$ , a real symmetric positive definite  $n \times n$  matrix  $S > 0$  and a vector  $K \in \mathbb{R}^n$  such that

$$\left( J_n - b e_n K^\top \right)^\top S + S \left( J_n - b e_n K^\top \right) \leq -\mu \text{Id}_n, \quad \forall \underline{b} \leq b \leq \bar{b}, \quad (11)$$

where  $\text{Id}_n$  denotes the  $n \times n$  identity matrix and  $S, K$  and  $\mu$  depend on some positive constants  $\underline{b}$  and  $\bar{b}$ . A careful examination of the argument shows that actually the upper bound on the parameter  $b$  can be removed. We thus obtain a slightly stronger result, whose proof is omitted:

*Proposition 1:* Let  $n \in \mathbb{N}$  and  $\underline{b} \in \mathbb{R}_+^*$ . Then there exist a positive constant  $\mu > 0$ , a real symmetric positive definite  $n \times n$  matrix  $S > 0$  and a vector  $K \in \mathbb{R}^n$  so that

$$\left( J_n - b e_n K^\top \right)^\top S + S \left( J_n - b e_n K^\top \right) \leq -\mu \text{Id}_n, \quad \forall b \geq \underline{b}. \quad (12)$$

With an obvious perturbation argument, we immediately derive the following corollary.

*Proposition 2:* Let  $n \in \mathbb{N}$  and  $\underline{b} \in \mathbb{R}_+^*$ . Then there exist  $\mu, C \in \mathbb{R}_+^*$ , a real symmetric positive definite  $n \times n$  matrix  $S > 0$  and  $K \in \mathbb{R}^n$  so that  $\forall a \in [-C, C]$  and  $b \geq \underline{b}$

$$\left( aD^* + J_n - b e_n K^\top \right)^\top S + S \left( aD^* + J_n - b e_n K^\top \right) \leq -\mu \text{Id}_n \quad (13)$$

Applying Proposition 2 to the ISS properties of (10) allow us to prove the following proposition.

*Proposition 3:* Consider the dynamics in (8) and  $D_\eta^r$  in (3). Then there exists  $K \in \mathbb{R}^n$  such that for every  $\eta > 0$ , the state feedback  $u = -K^\top D_\eta^r y$  provides the following estimate: for all  $1 \leq i \leq n$ ,  $|y_i(s)| \leq \frac{\max(1, \eta^{n-1})}{\eta^{n-i}} \exp(-C\mu \eta s) \|y(0)\| + \frac{C}{\eta^{n-i+1}} \max_{r \in [0, s]} |d(r)|$ , for all  $s \geq 0$ , where  $C$  and  $\mu$  are positive constants depending only on the lower bound  $\underline{b}$ .

The previous argument can be rewritten using an LMI formulation. For that purpose, one needs a result similar to Proposition 2, which involves the extra parameter  $\eta$ . More precisely, one easily shows the following proposition.

*Proposition 4:* Let  $n \in \mathbb{N}$  and  $\underline{b} \in \mathbb{R}_+^*$ . Then there exist positive constants  $\mu > 0$ , a real symmetric positive definite  $n \times n$ -matrix  $S > 0$  and a vector  $K \in \mathbb{R}^n$  such that, for every  $C > 0$  and  $\eta > 0$  large enough, one has for  $b \geq \underline{b}$  and  $a \in [-C, C]$ ,  $\left( aD^* + J_n - b e_n K_\eta^\top \right)^\top S_\eta + S_\eta \left( aD^* + J_n - b e_n K_\eta^\top \right) \leq -\mu_* \eta (D_\eta^r)^2$ , where  $S_\eta = D_\eta^r S D_\eta^r$  and  $K_\eta = D_\eta^r K \in \mathbb{R}^n$ .

Using Proposition 3 and the fact that  $\lambda(t)^{n-i+1} |x_i(t)| \leq |y_i(s)|$ ,  $1 \leq i \leq n$ ,  $t \geq 0$ , we deduce the PT-ISS-C property for  $x$  in any time  $T > 0$ .

*Theorem 1:* Consider the dynamics in (1). Let  $a : [0, T] \rightarrow \mathbb{R}$  be any non-negative continuous function such that

$\int_t^\top a(\xi)d\xi > 0$  for  $t \in [0, T)$ . Set

$$\lambda(t) = \frac{1}{\int_t^\top a(\xi)d\xi}, \quad s(t) = \int_0^t \lambda(\xi)d\xi, \quad 0 \leq t < T. \quad (14)$$

Then there exist  $K \in \mathbb{R}^n$  and  $\mu > 0$  such that for every  $C > 0$   $\eta$  large enough, the state feedback

$$u = -K^\top D_{\eta\lambda(t)}^\top x(t), \quad t \geq 0, \quad (15)$$

provides the following estimate, for every  $t \geq 0$ ,  $1 \leq i \leq n$ , and every measurable disturbance  $d$ :

$$|x_i(t)| \leq \frac{\eta^{-(n-i+1)}}{\lambda^{n-i+1}(t)} \left( \eta \max(1, \eta^{n-1}) \exp(-C\mu\eta s(t)) \|x(0)\| + C \max_{r \in [0, t]} |d(r)| \right) \max_{t \in [0, T']} |x_i(t)| \leq \eta \|x(0)\| + C_{T', T} \max_{t \in [0, T']} |d(t)|, \quad \forall t \geq 0, \quad 1 \leq i \leq n, \quad (16)$$

( $C$  and  $\mu$  are positive constants depending only on the lower bound  $\underline{b}$ ).

*Remark 1:* The shape of our disturbance is simpler than that of [1], it is bounded there by  $|d(t)|\psi(x)$ , with  $d$  any measurable function on  $[0, \infty)$  and  $\psi \geq 0$  a known scalar-valued continuous function. To lighten the presentation, we did not consider the function  $\psi$  since the analysis in this case is similar to the above by using (25) in [1].

*Remark 2:* Let us compare our results with those obtained in [1]. First of all, we recover at once the main result of that reference (Theorem 2 and (79)) by choosing the function  $a$  appearing in Theorem 1 to be equal to  $C(T-t)^{m-1}$  where  $C$  is a positive constant and  $m$  a positive integer. Our results are though slightly better since we can prescribe the rate of exponential decay as well as the estimate on the error term modeled by  $d$  thanks to the occurrence of the parameter  $\eta$  in our findings. Another advantage of our presentation is the more transparent structure of the feedback as well, since we relate the approach proposed by [1] to the weighted-homogeneity approach (see for instance [2]) which is classically used to handle sliding mode issues. Indeed the choice of the function  $\lambda$  in [1] (called  $\mu$  in that paper) must be specific since there is a need to express its time derivatives as polynomials in  $\lambda$ . In our presentation instead, there is a greater freedom in the choice of  $\lambda$ . Finally and more importantly, our framework yields a simpler proof with a unique time scaling for variables and everything boiling down to an LMI.

*Remark 3:* As noticed in [1], the linear feedback defined in (24) is not suitable if it is subject to measurement noise on  $x$ . More precisely, this amounts to have instead of (24) a feedback  $\tilde{u}$  given by  $\tilde{u}(t) = -K^\top D_{\eta\lambda(t)}^\top (x(t) + d(t)) = u(t) - K^\top D_{\eta\lambda(t)}^\top d(t)$ ,  $t \geq 0$ . That is, with a disturbance  $|d|$  of the form  $\eta\lambda(t) \max(1, (\eta\lambda)^{n-1}(t))|d(t)|$  in (17). From Theorem 1, we can only derive the following estimate: for  $t \geq 0$  and  $1 \leq i \leq n$ ,  $|x_i(t)| \leq \frac{\eta^{-(n-i+1)}}{\lambda^{n-i+1}(t)} (\eta \max(1, \eta^{n-1}) \exp(-C\mu\eta s(t)) \|x(0)\| +$

$C\eta\lambda(t) \max(1, \eta^{n-1}\lambda^{n-1}(t)) \max_{r \in [0, t]} |d(r)|$ ). The right-hand side blows up as  $t$  tends to  $T$ , except for  $i = 1$ , with a loss of regulation accuracy (we do not have anymore convergence to zero but to an arbitrary small neighborhood). On the other hand, by choosing  $\eta$  of the amplitude of  $\lambda(t)$  as  $t$  tends to  $T$ , we deduce at once from the above the following corollary.

*Corollary 2:* With the notations of Theorem 1, consider the dynamics given in (1) with the perturbed feedback  $\tilde{u}(t) = -K^\top D_{\eta\lambda(t)}^\top (x(t) + d(t))$ . Then for every time  $T' < T$ , there exists  $\eta > 0$  such that:

$$\max_{t \in [0, T']} |x_i(t)| \leq \eta \|x(0)\| + C_{T', T} \max_{t \in [0, T']} |d(t)|, \quad \forall t \geq 0, \quad 1 \leq i \leq n, \quad (17)$$

where  $C_{T', T}$  is a positive constant, tending to infinity as  $T'$  tends to  $T$ .

The previous result of semi-global nature has been suggested in [1] and it has been obtained here thanks to the extra parameter  $\eta$ . In particular, it follows the idea that to obtain estimates for prescribed-time control in time  $T'$ , one can use the previous strategy of prescribed-time control in a time  $T > T'$  and then use (17). This estimate is not satisfactory and is a direct result of the use of the time-varying function  $\lambda$ .

Looking back at (24), the most natural choice is a linear feedback and it has been (essentially) first addressed in [1] and revisited here. One can also use other feedback laws, especially those providing finite-time stability (in the scale  $s$ ). In any case, the the issue of non robustness with respect to noisy measurement will have to be solved.

## V. FIXED-TIME DESIGN FEEDBACK

In the previous section, a linear feedback  $u = K^\top y$  was considered but this choice faces a pernicious problem as soon as there is some noisy measurement on the state.

In this part, we propose a solution to prescribed-time stabilize a chain of integrators that avoids the use of time-varying homogeneity (and hence the problems described previously) and which is robust to perturbations. Here, perturbations do include noisy measurements and hence robustness must be understood in the ISS setting of Def. 4. The strategy consists in first solving a fixed-time stabilization, which turns out to be robust to perturbations and then to use a simple trick to extend that solution to prescribed-time stabilization.

Our solution to fixed-time stabilization is based on the use of sliding mode feedback with state-dependent homogeneity degree. This idea was first considered in [3] with a very explicit feedback law. However the latter bears a serious drawback since it is discontinuous. This defect has been removed in a subsequent work in [2], relying on a nice perturbation argument, nevertheless the proposed solution does not bear an explicit character and requires an important extra work for practical implementations.

Here we revisit the solution of [3] and use the perturbation idea of [2] to provide an explicit and continuous feedback law. Consider (1) in the unperturbed case

$$\dot{x} = J_n x + u e_n, \quad (18)$$

and refer to it as the *pure  $n$ -chain of integrators*. Our aim is to stabilize (18) with a static feedback law  $u = F(x)$ , in a robust manner with respect to measurement noise and external disturbances. The corresponding perturbed  $n$ -chain of integrators is given by

$$\dot{x} = J_n x + F(x + d_1) e_n + d_2, \quad (19)$$

where  $d_1 \in \mathbb{R}^n$  is the measurement noise and  $d_2 \in \mathbb{R}^n$  the external perturbation. We call  $d := (d_1, d_2) \in \mathbb{R}^{2n}$  the perturbation.

The necessary material to describe the solution of [3] is provided in the sequel. The following construction has been given first in [4] and we will adapt it to the present situation.

*Definition 6:* Let  $\ell_j > 0$ ,  $j = 1, \dots, n$  be positive constants. For  $\kappa \in [-\frac{1}{n}, \frac{1}{n}]$ , define the weights  $\mathbf{r}(\kappa) = (r_1, \dots, r_n)$  by  $r_j = 1 + (j-1)\kappa$ ,  $j = 1, \dots, n$ . Define the feedback control law

$$u = \omega_\kappa^H(x) := v_n, \quad (20)$$

where the  $v_j = v_j(x)$  are defined inductively by:

$$v_0 = 0, \quad v_j = -\ell_j \left[ |x_j|^{\beta_{j-1}} - |v_{j-1}|^{\beta_{j-1}} \right]^{\frac{r_j + \kappa}{r_j \beta_{j-1}}}, \quad (21)$$

and where the  $\beta_i$ 's are defined by  $\beta_0 = r_2$ ,  $(\beta_j + 1)r_{j+1} = \beta_0 + 1 > 0$ ,  $j = 1, \dots, n-1$ .

Consider the union of the homogeneous unit spheres  $S^j = \bigcup_{\kappa \in [-\frac{1}{2n}, \frac{1}{2n}]} \left\{ x \in \mathbb{R}^j \mid |x_1|^{\frac{2}{r_1}} + \dots + |x_n|^{\frac{2}{r_n}} = 1 \right\}$ , for  $1 \leq j \leq n$ . Then  $S^j$  is clearly a compact subset of  $\mathbb{R}^j$  and dealing with this set constitutes the main difference with [4].

We have then the following proposition.

*Proposition 5:* There exist positive constants  $\ell_j > 0$ ,  $j = 1, \dots, n$  such that for every  $\kappa \in [-\frac{1}{2n}, \frac{1}{2n}]$ , the feedback law  $u = \omega_\kappa^H(x)$  defined in (20) stabilizes the system (18). Moreover, there exists a homogeneous  $\mathcal{C}^1$ -function  $V_\kappa: \mathbb{R}^n \rightarrow \mathbb{R}_+$  given by

$$V_\kappa(x) = \sum_{j=1}^n \frac{\left( |x_j|^{\beta_{j-1}+1} - |v_{j-1}|^{\beta_{j-1}+1} \right)}{\beta_{j-1} + 1} - |v_{j-1}|^{\beta_{j-1}} (x_j - v_{j-1}), \quad (22)$$

which is a Lyapunov function for the closed-loop system (18) with the state feedback  $\omega_\kappa^H$ , and it satisfies

$$\dot{V}_\kappa \leq -C V_\kappa^{1+\alpha(\kappa)}, \quad \alpha(\kappa) := \frac{\kappa}{2+\kappa} \quad (23)$$

for some positive constant  $C$ , independent of  $\kappa$ . Moreover,  $V_\kappa$  is  $\mathbf{r}(\kappa)$ -homogeneous of degree  $(2+\kappa)$  with respect to the family of dilations  $(D_{\mathbf{r}(\kappa)}^\varepsilon)_{\varepsilon>0}$ .

*Remark 4:*

- (i) The previous proposition is essentially Theorem 3.1 of [4], except that the gains  $\ell_i$  are uniform with respect to  $\kappa \in [-\frac{1}{2n}, \frac{1}{2n}]$ .
- (ii) The critical exponent  $1 + \alpha(\kappa)$  appearing in (23) is bigger than 1 if  $\kappa > 0$  and smaller than 1 if  $\kappa < 0$ .
- (iii) For  $\kappa = 0$ , a linear feedback is obtained, and also  $V_0$  is a positive definite quadratic form, hence there exists a real symmetric positive definite  $n \times n$  matrix  $P$  such that  $V_0(x) = x^\top P x$ ,  $\forall x \in \mathbb{R}^n$ . The time derivative of  $V_0$  is the quadratic form associated with the  $n \times n$  matrix  $L^\top P + P L$  where  $L$  is the companion matrix associated with the coefficients  $\ell_1, \dots, \ell_n$ . We deduce at once that  $L$  is Hurwitz since, for  $\kappa = 0$ , (23) is equivalent to the LMI  $A^\top P + P A \leq -C P$ .

We next consider a state varying homogeneity degree.

*Definition 7:* For  $m \in (0, 1)$  and  $\kappa_0 \in (0, \frac{1}{n})$ , define the following continuous function  $\kappa: \mathbb{R}^n \rightarrow [-\kappa_0, \kappa_0]$  by

$$\kappa(x) = \begin{cases} \kappa_0, & \text{if } V_0(x) > 1 + m, \\ \kappa_0 \left( 1 + \frac{V_0(x) - (1+m)}{m} \right), & \text{if } 1 - m \leq V_0(x) \leq 1 + m, \\ -\kappa_0, & \text{if } V_0(x) < 1 - m. \end{cases} \quad (24)$$

For  $\kappa \in [-\frac{1}{2n}, \frac{1}{2n}]$  and  $a, b \in \mathbb{R}_+$ , define  $B_{a,b}^\kappa := \{x \in \mathbb{R}^n \mid a \leq V_\kappa(x) \leq b\}$ ,  $B_{<a}^\kappa := \{x \in \mathbb{R}^n \mid V_\kappa(x) < a\}$ ,  $B_{>b}^\kappa := \{x \in \mathbb{R}^n \mid b < V_\kappa(x)\}$ ,  $B_a^\kappa := \{x \in \mathbb{R}^n \mid V_\kappa(x) = a\} \subset \mathbb{R}^n$ .

In the spirit of [2], we now introduce the feedback which will ultimately yield prescribed time stability:

*Theorem 3:* Assume that the uncertainty  $b$  is bounded ( $\bar{b} \geq b(t) \geq \underline{b}$ ,  $t \geq 0$ , for some positive constants  $\bar{b}, \underline{b}$ ). Then, there exist  $m \in (0, 1)$  and  $\kappa_0 \in (0, \frac{1}{2n})$  such that, the undisturbed  $n$ th order chain of integrators defined by

$$\dot{x}(t) = J_n x(t) + b(t)u(t), \quad \bar{b} \geq b(t) \geq \underline{b}, \quad t \geq 0, \quad (25)$$

together with a feedback law  $\omega_{\kappa(x)}^H(x)$ , with  $\kappa$  defined in (24), is globally fixed-time stable at the origin at most time  $T(m, \kappa_0)$  upper bounded as

$$T(m, \kappa_0) \leq \frac{1}{C} \left( \frac{r(m, \kappa_0)^{-\alpha(\kappa_0)}}{\alpha(\kappa_0)} - 2 \ln(2m) + \frac{r(m, -\kappa_0)^{-\alpha(-\kappa_0)}}{-\alpha(-\kappa_0)} \right), \quad (26)$$

where  $r(m, \kappa_0) > 0$  (and  $r(m, -\kappa_0) > 0$ ) is the largest (smallest)  $r > 0$  such that  $B_{<r}^{\kappa_0}$  ( $B_{<r}^{-\kappa_0}$ ) is contained in (contains)  $B_{<1+m}^0$  ( $B_{<1-m}^0$ ) and the constant  $C$  as in (23).

It remains to use a standard time re-scaling technique with homogeneity (cf. [3] for instance) to extend from the result of fixed-time stability contained in Theorem 3 to a result about prescribed-time stability.

*Theorem 4:* Let  $m \in (0, 1)$ ,  $\kappa_0 \in (0, \frac{1}{n})$  defined in Theorem 3 and the feedback law  $\omega_{\kappa(x)}^H(x)$  defined in (24) which renders the system (18) globally fixed-time stable at the origin in settling time less than or equal to  $T(m, \kappa_0)$ . Let  $D_\lambda^\varepsilon$  be defined in (5). Then, given any  $T > 0$ , the feedback law  $\omega_{\kappa(D_\lambda^\varepsilon x)}^H(D_\lambda^\varepsilon x)$  renders the system (18) globally fixed-time stable at the origin in settling time less than or equal to  $T$  as soon as  $\lambda \geq \frac{T(m, \kappa_0)}{T}$ .

We now provide a result on robust properties of the perturbed system (19) stabilized with  $k(x) = \omega_{\kappa(x)}^H(x)$ :

*Theorem 5:* Under the assumptions of Theorem 4, the system

$$\dot{x} = J_n x + \omega_{\kappa(x+d_1)}^H(x) e_n + d_2, \quad x, d_1, d_2 \in \mathbb{R}^n.$$

is ISpS for any bounded  $d = (d_1, d_2)$ .

*Remark 5:* Theorem 5 holds true if one needs to use the feedback  $k(x) = \omega_{\kappa(D_\lambda^r x)}^H(D_\lambda^r x)$  for some appropriate  $\lambda > 0$  instead of  $k(x) = \omega_{\kappa(x)}^H(x)$  to stabilize (19). However the gain functions  $\beta, \gamma$  in Definition 4 will be modified.

## VI. NUMERICAL EXAMPLE

A numerical example is provided for the feedback law considered in Section IV with a chain of integrators of length 3, i.e.  $\dot{x}_1 = x_2$ ,  $\dot{x}_2 = x_3$ ,  $\dot{x}_3 = d(t) + b(t)u$ , where the uncertainty  $b$  on the control is given by  $b(t) = 2 + \sin(t)$  (hence the lower  $\underline{b}$  is equal to 1), the disturbance  $d$  is chosen equal to  $d(t) = 10\cos(3t)$  and the prescribed time  $T$  is taken equal to  $\pi$ . We make two choices for the time varying homogeneity function  $\lambda$ , namely  $\lambda_1(t) = \frac{3}{(\pi-t)^3}$ ,  $\lambda_2(t) = \frac{3}{(\pi-t+1+\cos(t))^3}$ ,  $t \in [0, \pi)$ , which correspond to  $a_1(t) = (\pi-t)^2$  and  $a_2(t) = (\pi-t+1+\cos(t))^2(1+\sin(t))$ . The bounds on  $a_i$  are  $\pi^2$  and  $2(1+\pi)^2$ .

One first determines  $S, K$  and  $\mu_0$  so that Prop. 4 holds true with  $\underline{b} = 1$ . Note that one can always take  $\mu_0 = 1$ , up to changing  $S$ . One finds  $K = (10 \ 40 \ 100)^\top$  and  $S = \begin{pmatrix} 9 & 14 & 0.1 \\ 14 & 72 & 0.5 \\ 0.1 & 0.5 & 0.6 \end{pmatrix}$ . Then, one determines  $\eta > 0$  so that Prop. 4 holds true with  $\mu = \frac{1}{2}$  and  $C_1 = \pi^2$  and then  $C_2 = 2(1+\pi)^2$ . One finds that  $\eta_1 = 60$  and  $\eta_2 = 200$  do the job respectively. Hence the feedback are  $u_1(t) = -K^\top D_{\eta_1 \lambda_1(t)}^r x(t)$  and  $u_2(t) = -K^\top D_{\eta_2 \lambda_2(t)}^r x(t)$ .

For the function  $\lambda_1$ , we pick  $x(0) = (0, 1, 10)$  on Fig. 1 and for the function  $\lambda_2$ , we pick  $x(0) = (-0.5, 0, 50)$  on Fig. 2. Fig. 1 and 2 illustrate a very fast rate of convergence, however with an overshoot phenomenon that can be easily handled with a more appropriate choice of parameters.

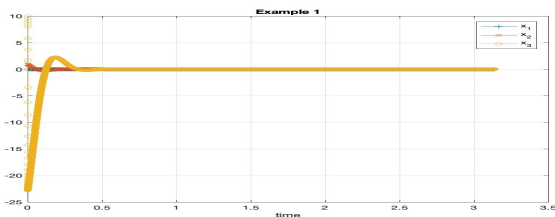


Fig. 1.  $\lambda_1(t) = \frac{3}{(\pi-t)^3}, t \in [0, \pi)$

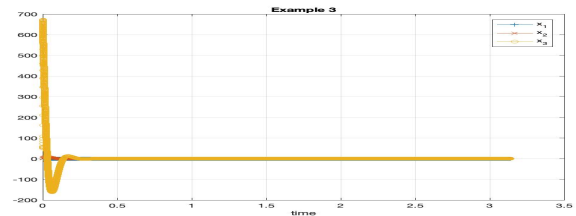


Fig. 2.  $\lambda_2(t) = \frac{3}{(\pi-t+1+\cos(t))^3}, t \in [0, \pi)$

## VII. CONCLUSION

In this paper, we have addressed the issue of prescribed-time stabilization of an  $n$ -chain of integrators,  $n \geq 1$ , either pure or perturbed. We have first recast the results obtained in [1] within the framework of time-varying homogeneity and hence provided simpler proofs. As noticed in [1], the feedback laws (linear or finite time) arising from this time-varying approach do not perform well when the  $n$ -chain of integrators is subject to perturbations, even if one stops before the prescribed settling time. We have proposed instead sliding mode types of feedback laws to handle fixed-time stabilization and to apply a standard trick of time-scale reparametrisation and homogeneity to render the modified stabilizers fit for prescribed-time stabilization of a perturbed  $n$ -chain of integrators. In this paper, we did not address the fundamental issue of tuning the several parameters involved in these stabilizers, in a similar way as made in [2]. This is the object of ongoing work and it will rely on the explicit formulas (21) and (22). Also ISS conditions might be obtained rather than just ISpS.

## REFERENCES

- [1] Y. Song, Y. Wang, J. Holloway, and M. Krstic, "Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time," *Automatica*, vol. 83, 2017.
- [2] F. Lopez-Ramirez, D. Efimov, A. Polyakov, and W. Perruquetti, "Fixed-time output stabilization and fixed-time estimation of a chain of integrators," *Int J Robust Nonlinear Control*, vol. 28, pp. 4647–4665, 2018.
- [3] M. Harmouche, S. Laghrouche, Y. Chitour, and M. Hamerlain, "Stabilization of perturbed chains of integrators using lyapunov-based homogeneous controllers," *Int. Journal of Cont.*, vol. 90, no. 12, pp. 2631–2640, 2017.
- [4] Y. Hong, "Finite-time stabilization and stabilizability of a class of controllable systems," *Systems and Control Letters*, vol. 46, no. 4, pp. 231–236, 2002.
- [5] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, New Jersey: Prentice Hall, 3rd ed., 2002.
- [6] H. Nakamura, Y. Yamashita, and H. Nishitani, "Smooth lyapunov functions for homogeneous differential inclusions," in *Proceedings of the 41st SICE Annual Conference. SICE 2002.*, vol. 3, pp. 1974–1979 vol.3, Aug 2002.
- [7] Y. Chitour and M. Sigalotti, "Persistence excitation for linear systems," *SIAM J. Control Optim.*, vol. 48, no. 6, pp. 4032–4055, 2010.