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# 3D-2D Stokes-Darcy coupling for the modelling of seepage with an application to fluid-structure interaction with contact

Erik Burman, Miguel A. Fernández, Stefan Frei and Fannie M. Gerosa

**Abstract** In this note we introduce a mixed dimensional Stokes-Darcy coupling where a  $d$  dimensional Stokes' flow is coupled to a Darcy model on the  $d - 1$  dimensional boundary of the domain. The porous layer introduces tangential creeping flow along the boundary and allows for the modelling of boundary flow due to surface roughness. This leads to a new model of flow in fracture networks with reservoirs in an impenetrable bulk matrix. Exploiting this modelling capability, we then formulate a fluid-structure interaction method with contact, where the porous layer allows for mechanically consistent contact and release. Physical seepage in the contact zone due to rough surfaces is modelled by the porous layer. Some numerical examples are reported, both on the Stokes'-Darcy coupling alone and on the fluid-structure interaction with contact in the porous boundary layer.

## 1 Introduction

In numerous environmental or biomedical applications there is a need to model the coupling between a flow in a reservoir and flow in a surrounding porous medium.

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Erik Burman

Department of Mathematics, University College London, Gower Street, London UK, WC1E 6BT.  
e-mail: e.burman@ucl.ac.uk

Miguel A. Fernández

Inria, 75012 Paris, Sorbonne Université & CNRS, UMR 7598 LJLL, 75005 Paris. e-mail:  
miguel.fernandez@inria.fr

Stefan Frei

Department of Mathematics & Statistics, University of Konstanz, Germany. e-mail: stefan.frei@uni-konstanz.de

Fannie M. Gerosa

Inria, 75012 Paris, Sorbonne Université & CNRS, UMR 7598 LJLL, 75005 Paris, France. e-mail:  
fannie.gerosa@inria.fr

This is particularly challenging if the porous medium is fractured and the bulk matrix has very low permeability. Typically the fractures are modelled as  $d - 1$  dimensional manifolds, embedded in a  $d$  dimensional porous bulk matrix. For the modelling of the fractured porous medium we refer to [3]. Observe however that if the bulk permeability is negligible the fluid in the reservoir can not penetrate into the fractures since the  $d - 1$  dimensional manifolds have an intersection of the reservoir boundary of  $d - 1$  measure zero. This means that such a model can not be used for the fluid flow between two reservoirs connected by a fracture in an impenetrable medium. Here we propose to introduce a Darcy equation for the tangential flow on the boundary of the reservoir. Since this equation is set on a  $d - 1$  dimensional manifold it can provide an interface allowing for flow from the reservoir to the cracks. The flow on the boundary communicates with the flow in the cracks through continuity of pressure and conservation expressed by Kirchhoff's law. This gives a cheap and flexible model for flow in reservoirs connected by fractures.

Our original motivation for this model is the particular case of fluid structure interaction with contact where the situation described above occurs when two boundaries enter in contact provoking a change of topology of the fluid domain. It has recently been observed by several authors [1, 4] that the consistent modelling of fluid-structure interaction with contact requires a fluid model, in particular a pressure, also in the contact zone. Indeed, some seepage is expected to occur due to permeability of the contacting bodies or their surface roughness. Otherwise there is no continuous mechanism for the release of contact and non-physical voids can occur. For instance, it was argued in [1] that a consistent modelling of FSI with contact requires a complete modelling of the FSI-poroelastic coupling. Similar ideas were introduced in [4], but for computational reasons. Indeed, in the latter reference an elastic body immersed in a fluid enters in contact with a rigid wall and to allow for a consistent numerical modelling the permeability of the wall is relaxed. This motivates the introduction of an artificial porous medium whose permeability goes to zero with the mesh-size. Both approaches allow for the seepage that appears to be necessary for physical contact and release. However, in case the contacting solids are (modelled as) impenetrable, this seepage must be due to porous media flow in a thin layer in the contact zone due to surface roughness. The complete modelling of the poroelastic interaction of [1] or the bulk porous medium flow of [4] then appears artificial and unnecessarily expensive. For such situations the mixed dimensional modelling suggested above can offer an attractive compromise between model detail and computational cost.

In this note, we will focus exclusively on the modelling aspect. The coupled Stokes-Darcy model is introduced in section 2. Then, in section 3, we show how the ideas of [4] can be used to model FSI with contact together with the mixed-dimensional fluid system. Finally, we illustrate the two model situations numerically in section 4. First, the Stokes'-Darcy reservoir coupling (section 4.1) and then the full FSI with contact (section 4.2). In the latter case, we also give comparisons with the results from [4]. The numerical analysis of the resulting methods will be the subject of future work.

## 2 The coupled Stokes-Darcy system

We consider the coupling of a Darcy system in a thin-walled domain  $\Omega_l = \Sigma_l \times (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \in \mathbb{R}^d$  for  $d = 2, 3$  with a Stokes equation in the bulk domain  $\Omega_f$ . The Darcy problem on  $\Omega_l$  writes

$$\begin{cases} u_l + K \nabla p_l = 0 \\ \nabla \cdot u_l = 0 \end{cases} \quad \text{in } \Omega_l, \quad (1)$$

where  $u_l$  denotes the Darcy velocity,  $p_l$  the Darcy pressure and  $K$  is a  $d \times d$  matrix that allows for the decomposition

$$K \nabla p_l = K_\tau \nabla_\tau p_l + K_n \partial_n p_l.$$

We denote the upper boundary of  $\Omega_l$  which couples to  $\Omega_f$  by  $\gamma_f$  and the outer boundary by  $\gamma_o$ . The normal vector  $n$  of the middle surface  $\Sigma_l$  of  $\Omega_l$  is chosen in such a way that it points towards  $\gamma_o$ .

By averaging across the thickness  $\epsilon$ , Martin, Jaffré and Roberts derived in [3] an effective equation for the averaged pressure across the thickness

$$P_l := \frac{1}{\epsilon} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} p_l.$$

Under the modelling assumption that the average pressure is equal to the mean of the pressures on the upper and lower boundary

$$P_l = \frac{1}{2} (p_l|_{\gamma_f} + p_l|_{\gamma_o}) \quad \text{in } \Sigma_l, \quad (2)$$

the authors derived the system

$$\begin{cases} -\nabla_\tau \cdot (\epsilon K_\tau \nabla_\tau P_l) = u_{l,n}|_{\gamma_f} - u_{l,n}|_{\gamma_o} \\ p_l|_{\gamma_f} = P_l + \frac{\epsilon K_n^{-1}}{4} (u_{l,n}|_{\gamma_o} + u_{l,n}|_{\gamma_f}) \end{cases} \quad \text{in } \Sigma_l. \quad (3)$$

Here,  $u_{l,n} = u_l \cdot n$  denotes the normal component of the velocity and  $\tau$  is a tangential vector of  $\Sigma_l$ . We will couple (3) to Stokes flow in  $\Omega_f$

$$\begin{cases} \rho_f \partial_t u_f - \nabla \cdot \sigma_f(u_f, p_f) = 0 \\ \nabla \cdot u_f = 0 \end{cases} \quad \text{in } \Omega_f, \quad (4)$$

where  $u_f$  denotes the fluid velocity,  $p_f$  the pressure,  $\rho^f$  the fluid density,

$$\sigma_f(u_f, p_f) := \mu(\nabla u_f + \nabla u_f^T) - p_f I,$$

the fluid Cauchy stress tensor and  $\mu$  the dynamic viscosity. We assume that the coupling to the Darcy system (1) on  $\gamma_f$  takes place via the interface conditions

$$\begin{cases} \sigma_{f,nn} = -p_l \\ \tau^T \sigma_f n = 0 \\ u_{f,n} = u_{l,n} \end{cases} \quad \text{on } \gamma_f, \quad (5)$$

where  $\sigma_f = \nabla u_f - p_f I$  and  $\sigma_{f,nn} = n^T \sigma_f n$ . In the lower porous wall  $\gamma_o$  we assume for simplicity that  $u_{l,n} = 0$ . Then, the relations (3) can be written as

$$\begin{cases} -\nabla_\tau \cdot (\epsilon K_\tau \nabla_\tau P_l) = u_{f,n} \\ \sigma_{f,nn} = -P_l - \frac{\epsilon K_n^{-1}}{4} u_{f,n} \end{cases} \quad \text{in } \Sigma_l.$$

Note that the only remaining porous medium variable is the averaged pressure  $P_l$ . In the limit of permeability  $K_n \rightarrow 0$ , the system converges to a pure Stokes system with slip conditions on  $\gamma_f$  with an extension of the fluid forces into the porous medium pressure  $P_l$ .

We have the following coupled variational problem for  $(u_f, p_f, P_l)$ :

$$\begin{cases} \rho_f (\partial_t u_f, v_f)_{\Omega_f} + (\sigma_f(u_f, p_f), \nabla v_f)_{\Omega_f} + (q_f, \nabla \cdot u_f)_{\Omega_f} \\ \quad + (P_l, v_{f,n})_{\Sigma_l} + \frac{\epsilon K_n^{-1}}{4} (u_{f,n}, v_{f,n})_{\Sigma_l} = 0, \\ (\epsilon K_\tau \nabla_\tau P_l, \nabla_\tau q_l)_{\Sigma_l} - (u_{f,n}, q_l)_{\Sigma_l} = 0, \end{cases} \quad (6)$$

for all  $v_f, q_f, q_l$ , where  $n = n_f$  is the outer normal of the fluid domain  $\Omega_f$ .

### 3 The fluid-structure-poroelastic-contact interaction system

Now, we consider a fluid-structure-contact interaction system with a thin porous layer on the part of the exterior boundary, where contact might take place. The moving boundary of the solid is denoted by  $\Sigma(t)$  and the porous layer by  $\Sigma_l$ . In absence of contact, we have the following system of equations

$$\begin{cases} \rho_f \partial_t u_f - \nabla \cdot \sigma_f(u_f, p_f) = 0 \\ \nabla \cdot u_f = 0 \end{cases} \quad \text{in } \Omega_f(t), \\ \rho_s \partial_t \dot{d} - \nabla \cdot \sigma_s(d) = 0 \quad \text{in } \Omega_s(t), \\ u_f = \dot{d}, \quad \sigma_s n = \sigma_f n \quad \text{in } \Sigma(t),$$

$$\left\{ \begin{array}{l} -\nabla_\tau \cdot (\epsilon K_\tau \nabla_\tau P_l) = u_{l,n}|_{\gamma_f} \\ \sigma_{f,nn} = -P_l - \underbrace{\frac{\epsilon K_n^{-1}}{4} u_{l,n}|_{\gamma_f}}_{\sigma_p} \quad \text{in } \Sigma_l, \\ \tau^T \sigma_f n = 0 \end{array} \right. \quad (7)$$

where, in addition to the quantities introduced above,  $\rho_s$  denotes the solid density,  $d$  stands for the solid displacement and  $\sigma_s$  denotes the tensor of linear elasticity

$$\sigma_s = \frac{\lambda_s}{2} (\nabla d + \nabla d^T) + \frac{\mu_s}{2} \text{tr} (\nabla d + \nabla d^T).$$

In addition, we impose that the solid  $\Omega_s$  can not penetrate into the porous medium  $\Sigma_l$

$$d_n - g \leq 0, \quad \lambda \leq 0, \quad \lambda(d_n - g) = 0 \quad \text{on } \Sigma(t). \quad (8)$$

Here,  $g$  denotes the gap function to  $\Sigma_l$  and  $\lambda$  is a Lagrange multiplier for the no-penetration condition defined by

$$\begin{aligned} \lambda &= \sigma_{s,nn} - \sigma_{f,nn} && \text{on } \Sigma(t) \setminus \Sigma_l, \\ \lambda &= \sigma_{s,nn} - \sigma_p && \text{on } \Sigma(t) \cap \Sigma_l. \end{aligned}$$

The ‘‘switch’’ on the right-hand side occurs, as the solid on one side of  $\Sigma(t)$  couples either to the fluid  $\Omega_f$  or the porous medium  $\Sigma_l$  on the other side of  $\Sigma(t)$ . The conditions (8) can equivalently be written as

$$\lambda = \gamma_C \underbrace{[d_n - g - \gamma_C^{-1} \lambda]}_{P_\gamma} \Big|_+ \quad \text{on } \Sigma(t)$$

for arbitrary  $\gamma_C > 0$ . Using this notation, we can characterise the zone of ‘‘active’’ contact as follows

$$\Sigma_c(t) = \{x \in \Sigma(t) \mid P_\gamma > 0\}.$$

To summarise, we have the following interface conditions:

- Contact condition on  $\Sigma(t)$ :

$$d_n - g \leq 0, \quad \lambda \leq 0, \quad \lambda(d_n - g) = 0 \quad \text{on } \Sigma(t).$$

- Kinematic coupling on  $\Sigma_{f,si}(t) = \Sigma(t) \setminus \Sigma_l$

$$u_f = \dot{d} \quad \text{on } \Sigma_{f,si}(t).$$

- Dynamic coupling on  $\Sigma(t)$ :

$$\begin{aligned} \sigma_s n &= \lambda n - \sigma_p n = \gamma_C [P_\gamma]_{+n} - \sigma_p n && \text{on } \Sigma(t) \cap \Sigma_l, \\ \sigma_s n &= \lambda n - \sigma_f n = \gamma_C [P_\gamma]_{+n} - \sigma_f n && \text{on } \Sigma(t) \setminus \Sigma_l. \end{aligned}$$

We have the following Nitsche-based variational formulation: Find  $u_f \in \mathcal{V}_f, p_f \in \mathcal{L}_f, d \in \mathcal{V}_s, P_l \in \mathcal{V}_l$  such that

$$\begin{aligned} & (\partial_t u_f, v)_{\Omega_f} + (\partial_t \dot{d}, w)_{\Omega_s} + a_f(u_f, p_f; v, q) + a_s(d, w) \\ & - (\sigma_f n, v - w)_{\Sigma(t) \setminus \Sigma_l} - (u_f - \dot{d}, \sigma_f(v, -q))_{\Sigma(t) \setminus \Sigma_l} + \frac{\gamma_{fsi}}{h} (u_f - \dot{d}, v - w)_{\Sigma(t) \setminus \Sigma_l} \\ & - (\sigma_p, v \cdot n)_{\Sigma_l \setminus \Sigma(t)} - (\sigma_p, w \cdot n)_{\Sigma_l \cap \Sigma(t)} + ([P_\gamma]_+, w \cdot n)_{\Sigma(t)} \\ & + (\epsilon K_\tau \nabla_\tau P_l, \nabla_\tau q_l)_{\Sigma_l} - (u_f, n, q_l)_{\Sigma_l \setminus \Sigma(t)} - (\dot{d}_n, q_l)_{\Sigma_l \cap \Sigma(t)} = 0 \\ & \forall v \in \mathcal{V}_f, q \in \mathcal{L}_f, w \in \mathcal{V}_s, q_l \in \mathcal{V}_l. \end{aligned}$$

The porous stress  $\sigma_p$  is given by

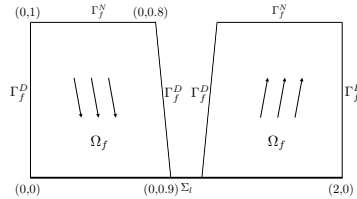
$$\sigma_p = -P_l + \frac{\epsilon K_n^{-1}}{4} u_{l,n}|_{\gamma_f} = \begin{cases} -P_l + \frac{\epsilon K_n^{-1}}{4} u_{f,n} & \text{on } \Sigma_l \setminus \Sigma(t) \\ -P_l + \frac{\epsilon K_n^{-1}}{4} \dot{d}_n & \text{on } \Sigma_l \cap \Sigma(t). \end{cases} \quad (9)$$

## 4 Numerical experiments

Here we will report on some numerical experiments using the above models. First we consider the mixed dimensional Stokes'-Darcy system and then the fluid-structure interaction system with contact and porous layer in the contact zone.

### 4.1 Stokes-Darcy example

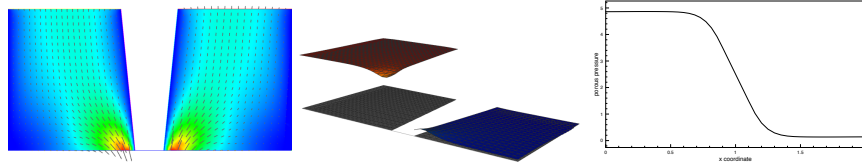
In this example, we consider two disconnected fluid reservoirs, the domain  $\Omega_f$ , connected through a thin-walled porous media located on the bottom wall  $\Sigma_l$ , as



**Fig. 1** Geometrical configuration for the Stokes model with a thin-walled porous medium on the bottom wall.

shown in Figure 1. The physical parameters are  $\mu = 0.03$ ,  $\rho_f = 1$ ,  $\epsilon = 0.01$  and  $K_\tau = K_n = 1$ . We impose a pressure drop across the two parts of the boundary  $\Gamma_f^N$ . The purpose of this example is to illustrate how the porous model is able to connect

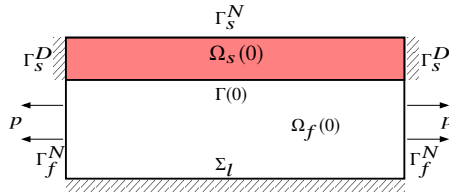
the fluid flow between the two containers. This can be clearly inferred from the results reported in Figure 2, which respectively show a snapshot of the fluid velocity, the elevation of the fluid pressure and the associated porous pressure.



**Fig. 2** Left: Snapshot of the fluid velocity. Middle: Elevation of the fluid pressure. Right: Porous pressure.

### 4.2 Fluid-structure interaction with contact

To test the FSI-contact model, we consider flow in a 2-dimensional pipe, where the upper wall is elastic, see Figure 3. Due to the application of a large pressure  $\bar{P}$  on the left and right boundary, the upper wall is deflected downwards until it reaches the bottom. Note that when contact occurs, the configuration is topologically equivalent to the situation in section 4.1. Shortly before the time of impact we set  $\bar{P}$  to zero, such that contact is released again after a certain time. This model problem is taken from [4], where further details on the configuration and the discretisation can be found. To deal with the topology change in the fluid domain at the impact, we apply a *Fully Eulerian* approach for the FSI problem [2]. In order to obtain a continuous and physically relevant transition from FSI to solid-solid contact, we use the FSI-contact model derived in section 3 and place a thin porous domain  $\Sigma_l$  on the lower boundary.



**Fig. 3** Geometrical configuration for the FSI-contact model. We apply a porous medium model on the (rigid) lower wall, where contact might take place.

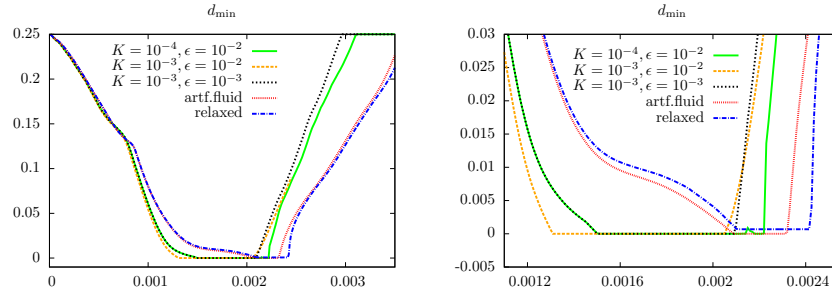
In Figure 4 we compare this model for different parameters  $K = K_\tau = K_n$  and  $\epsilon$  with the approaches for FSI-contact problems introduced in [4] in terms of the minimal distance of the solid to  $\Sigma_p$  over time. In [4] two approaches were presented in order to extend the fluid stresses to the contact region during solid-solid contact, namely a so-called *relaxed* and an *artificial fluid* approach. It was observed that for



the *artificial fluid* approach contact happens earlier, as penetration of the fluid flow into the artificial region is prevented only asymptotically, i.e.  $u_{f,n} \rightarrow 0$  ( $h \rightarrow 0$ ) on  $\Sigma_p$ , in contrast to  $u_{f,n} = 0$  for the relaxed approach. In the model presented here, we have similarly from (7) and  $u_{l,n} = u_{f,n}$  on  $\Sigma_p$

$$u_{f,n} = -\nabla_\tau \cdot (\epsilon K_\tau \partial_\tau P_l) \rightarrow 0 \quad (\epsilon K_\tau \rightarrow 0).$$

For this reason we observe in Figure 4 that the impact happens earlier for a larger value of  $\epsilon K_\tau$ . The time of the release seems to depend also on  $\epsilon K_n^{-1}$ , which appears in the definition of  $\sigma_p$  (9). A detailed investigation of this dependence and the investigation of stability and convergence of the numerical method are subject to future work.



**Fig. 4** Minimal distance of  $\Omega_s$  to the lower wall  $\Sigma_p$  over time. Right: zoom-in around the contact interval. We compare the new approach presented in Section 3 for different parameters with the *artificial fluid* and the *relaxed* contact approach studied in [4].

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