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► **To cite this version:**

Fatemeh Pourahmadian, Housseem Haddar. Differential tomography of micromechanical evolution in elastic materials of unknown micro/macrostructure. *SIAM Journal on Imaging Sciences*, 2020, 13 (3), pp.1302-1330. 10.1137/19M1305707. hal-02416504

HAL Id: hal-02416504

<https://inria.hal.science/hal-02416504v1>

Submitted on 17 Dec 2019

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Differential tomography of micromechanical evolution in elastic materials of unknown micro/macrostructure*

Fatemeh Pourahmadian[†] and Housseem Haddar[‡]

Abstract. *Differential evolution indicators* are introduced for 3D spatiotemporal imaging of micromechanical processes in complex materials where progressive variations due to manufacturing and/or aging are housed in a highly scattering background of a-priori unknown or uncertain structure. In this vein, a three-tier imaging platform is established where: (1) the domain is periodically (or continuously) subject to illumination and sensing in an arbitrary configuration; (2) sequential sets of measured data are deployed to distill segment-wise scattering signatures of the domain's internal structure through carefully constructed, non-iterative solutions to the scattering equation; and (3) the resulting solution sequence is then used to rigorously construct an imaging functional carrying appropriate invariance with respect to the unknown stationary components of the background e.g., pre-existing interstitial boundaries and bubbles. This gives birth to differential indicators that specifically recover the 3D support of micromechanical evolution within a network of unknown scatterers. The direct scattering problem is formulated in the frequency domain where the background is comprised of a random distribution of monolithic fragments. The constituents are connected via highly heterogeneous interfaces of unknown elasticity and dissipation which are subject to spatiotemporal evolution. The support of internal boundaries are sequentially illuminated by a set of incident waves and thus-induced scattered fields are captured over a generic observation surface. The performance of the proposed imaging indicator is illustrated through a set of numerical experiments for spatiotemporal reconstruction of progressive damage zones featuring randomly distributed cracks and bubbles.

Key words. differential imaging, micromechanical evolution, complex materials, ultrasonic sensing, waveform tomography

AMS subject classifications. 35R60, 35R30, 35Q74, 65M32

1. Introduction. Fast waveform tomography solutions germane to uncertain (or unknown) environments bear direct relevance to (a) timely detection of degradation in safety-sensitive components [13, 7], and (b) in-situ monitoring of additive manufacturing (AM) processes [16]. In nuclear power plants, for instance, critical components such as reactor and fuel cells are comprised of composite materials whose topology and properties are uncertain at micro-, meso-, and macro-scales as a result of manufacturing and/or aging. The deterioration of these materials due to various chemo-physical mechanisms such as irradiation and thermal cycling are not yet fully understood [30]. These processes, however, spur continuous microstructural evolution leading to an inevitable development of anomalies responsible for the loss of structural integrity and diminished functional performance. Thus, timely detection of deterioration at the microstructure scale and active spatiotemporal tracking of their evolution are paramount for early and robust mitigation of damage in such systems. In advanced manufacturing, one of

*Submitted to the editors 12/05/2019.

Funding: This work was funded by the University of Colorado Boulder through FP's startup and IS-IRT grant.

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38 the main challenges is online evaluation of the AM performance [17, 22], demanding real-time
39 in-situ characterization of components during fabrication [41]. In this vein, a major hindrance
40 is that the scattering signature of evolving regions is often eluded by the footprints of unknown
41 stationary scatterers in a complex specimen. A sensing scheme amenable to highly scattering
42 environments will contribute to (a) better understanding of the manufacturing process and
43 its implications on the quality of the final product, and (ii) optimal design and closed-loop
44 control of the AM processes.

45 Recently, major progress has been made toward developing robust imaging solutions that
46 enable real-time sensing in complex materials [39, 29, 38, 1, 13, 23, 21, 19]. State-of-the-art
47 examples include: ultrasonic surface wave methods [39], nonlinear ultrasound [29], penetrating-
48 radar techniques [1], infrared thermography [13], laser shearography [23], X-ray computed
49 tomography [21] and acoustic tomography imaging [19]. So far, these developments mostly rely
50 upon (a) simplistic characterization of the background disregarding uncertain yet fundamental
51 features (such as interstitial boundaries) across multiple scales [42], (b) significant assumptions
52 on the nature of wave motion [39], (c) partial data inversion deploying only a few signatures
53 of the scattered field measurements [38], and (d) data processing schemes amenable to ad hoc
54 sensing configurations [13]. Such attributes impose a number of limitations, including: (i) high
55 sensitivity to the assumed background structure, (ii) insensitivity to less-understood properties
56 of microstructured materials, (iii) major restrictions in terms of the location of ultrasonic
57 transducers, (iv) limited scalability beyond laboratory applications e.g., for the purpose of
58 in-situ monitoring. Therefore, there is a critical need for the next generation of imaging tools
59 that transcend some of these barriers.

60 Ongoing efforts in this vein are mainly focused on optimization-based approaches to wave-
61 form inversion. These technologies typically incur high computational cost as a crucial obsta-
62 cle to real-time sensing, and high sensitivity to unknown features of the background leading
63 to multiple sets of “optimal” solutions and thus, ambiguity of the results. More recently,
64 approaches to fast waveform tomography [4, 5, 10, 2, 12, 15] have been brought under the
65 spotlight for their capabilities pertinent to imaging in highly scattering media. While this
66 class of inverse solutions generally demand an a-priori characterization of the background for
67 their successful performance, most recent developments including the present study indicate
68 that this requirement could be relaxed in presence of sequential measurements, generating a
69 suit of new imaging modalities that surpass some of the existing limitations.

70 In particular, this study takes advantage of some recent advances in design of sampling
71 methods [36, 2, 20, 35] to develop a non-iterative full-waveform approach for spatiotemporal
72 tracking of progressive variations in complex materials. The idea is to deploy sequential sets
73 of scattered field measurements to rigorously construct an imaging functional endowed with
74 appropriate invariance with respect to (unknown) stationary components of the background
75 e.g., its time-invariant scatterers. The resulting differential indicators uniquely characterize
76 the support of evolution without the need to reconstruct the entire domain across pertinent
77 scales which may be practically insurmountable. In the case of volumetric scatterers, such
78 invariants are furnished via solutions to the so-called interior transmission problem and the
79 relation between two such solutions before and after the evolution [3, 12]. The key observation
80 in developing the imaging functional is that such solutions may be approximated by using
81 sampling type techniques [2, 35].

82 The fundamental challenge which impedes direct extension of these advances to ultrasonic
 83 imaging is the existence of non-volumetric scatterers in solid-state materials e.g., interstitial
 84 boundaries, fractures, and dislocation networks [15, 34]. So that functionals of desired in-
 85 variance may not be established through the analysis of elastodynamic interior transmission
 86 problems. This work aims to address this challenge by studying imaging functionals pertinent
 87 to elastic backgrounds with random interfaces and discontinuities across scales. Our analysis
 88 is based on the boundary integral representation of scattering solutions, enabling rigorous
 89 formulation of invariant quantities critical for establishing differential imaging functionals for
 90 such media. The designed indicators are then synthetically tested and validated in a few
 91 example configurations featuring randomly distributed interfaces and bubbles.

92 This paper is organized as follows. [section 2](#) presents the direct scattering formulation and
 93 admissibility conditions on interstitial boundaries so that the forward problem is wellposed.
 94 [section 3](#) defines the scattering operator and briefly recaps some known results on the prop-
 95 erties of this operator for later reference. The differential evolution indicators are introduced
 96 and analyzed in [section 4](#). [section 5](#) is dedicated to numerical implementation and validation
 97 of this imaging solution.

98 **2. Problem statement.** With reference to [Figure 1](#), consider sequential illumination of
 99 microstructural evolution in an elastic domain at sensing steps $t = \{t_o, t_1, t_2, \dots\}$. The domain
 100 $\mathcal{B} \subset \mathbb{R}^3$ at the outset of sensing $t = t_o$ is comprised of a random distribution of monolithic
 101 fragments of Lipschitz support with mass density ρ and Lamé parameters μ and λ , connected
 102 via perfect or imperfect interfaces $\Gamma_o \subset \mathcal{B}$. The contact condition at the surface of Γ_o is
 103 characterized by a symmetric, complex-valued and heterogeneous interfacial stiffness matrix
 104 $\mathbf{K}_o(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \Gamma_o$. Internal interfaces are subject to variations e.g., driven by chemo-physical
 105 reactions so that at any secondary sensing step $t = t_\kappa > t_o$, the domain features newborn and
 106 evolved interfaces $\Gamma_\kappa \subset \mathcal{B}$ endowed with the contact stiffness $\mathbf{K}_\kappa(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \Gamma_\kappa$.

107 **Assumption 2.1.** *Let us denote by Γ_N^t the support of all traction-free cracks in \mathcal{B} at time t*
 108 *i.e., $\mathbf{K}_t = \mathbf{0}$ on Γ_N^t . In this study, it is assumed that (a) no subset of Γ_N^t constitutes a closed*
 109 *surface, and (b) $\mathcal{B} \setminus \Gamma_N^t$ remains connected.*

110 Let Ω denote the unit sphere centered at the origin. Given a propagation direction $\mathbf{d} \in$
 111 Ω and polarization amplitudes $\mathbf{q}_p, \mathbf{q}_s \in \mathbb{R}^3$ where $\mathbf{q}_p \parallel \mathbf{d}$ and $\mathbf{q}_s \perp \mathbf{d}$, the domain $\mathcal{B}(t)$ is
 112 illuminated at every sensing step via a combination of plane P- and S- waves so that the
 113 incident field takes the form

$$114 \quad (2.1) \quad \mathbf{u}^f(\boldsymbol{\xi}) = \mathbf{q}_p e^{ik_p \boldsymbol{\xi} \cdot \mathbf{d}} + \mathbf{q}_s e^{ik_s \boldsymbol{\xi} \cdot \mathbf{d}}, \quad \mathbf{d} \in \Omega, \boldsymbol{\xi} \in \mathbb{R}^3,$$

115 where k_p and $k_s = k_p \sqrt{(\lambda + 2\mu)/\mu}$ denote the respective wave numbers affiliated with the
 116 “baseline” system shown in [Figure 1](#) (c). At the primary sensing step $t = t_o$, the interaction
 117 of \mathbf{u}^f with the scatterers $\Gamma_o(\boldsymbol{\xi})$ gives rise to the scattered field $\mathbf{v}_o \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \Gamma_o)^3$ solving

$$118 \quad (2.2) \quad \begin{aligned} \nabla \cdot (\mathbf{C} : \nabla \mathbf{v}_o) + \rho \omega^2 \mathbf{v}_o &= \mathbf{0} && \text{in } \mathbb{R}^3 \setminus \Gamma_o, \\ \mathbf{n}_o \cdot \mathbf{C} : \nabla \mathbf{v}_o &= \mathbf{K}_o(\boldsymbol{\xi})[\mathbf{v}_o] - \mathbf{t}_o^f && \text{on } \Gamma_o, \end{aligned}$$

119 where $\omega^2 = k_s^2 \mu / \rho$ is the frequency of excitation; $[\mathbf{v}_o] = [\mathbf{v}_o^+ - \mathbf{v}_o^-]$ is the jump in \mathbf{v}_o across Γ_o ,
 120 hereon referred to as the fracture opening displacement (FOD);

$$121 \quad (2.3) \quad \mathbf{C} = \lambda \mathbf{I}_2 \otimes \mathbf{I}_2 + 2\mu \mathbf{I}_4$$

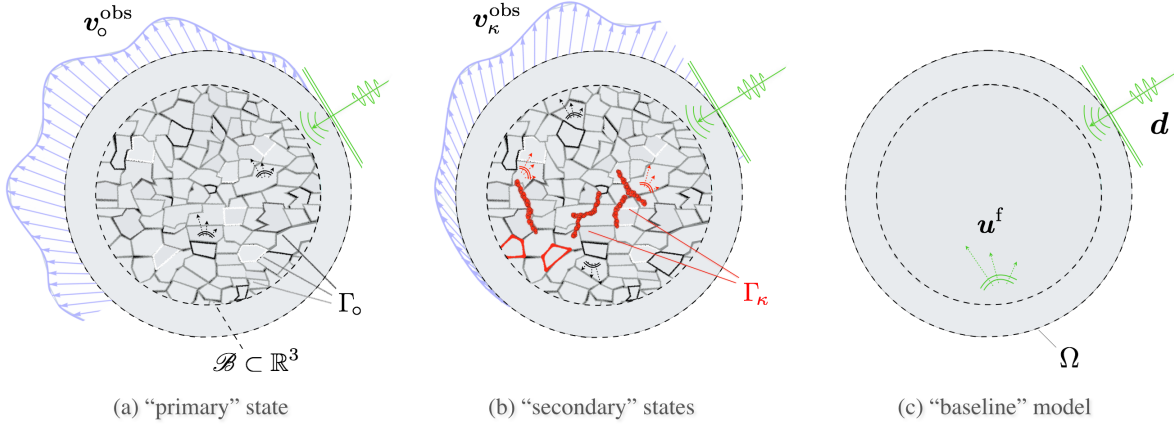


Figure 1. Elastic-wave sensing of microstructural evolution in a background $\mathcal{B} \subset \mathbb{R}^3$ featuring a random network of pre-existing interfaces Γ_0 : (a) primary experiments conducted at $t=t_0$ via a set of P - and S - plane waves propagating in direction $\mathbf{d} \in \Omega$, inducing the incident field \mathbf{u}^f in the baseline system shown in (c); the action of \mathbf{u}^f on Γ_0 results in the scattered field $\mathbf{v}_0^{\text{obs}}$ captured at the far field over the unit sphere of observation angles Ω , (b) secondary experiments performed in a similar setting at a sequence of time steps $t_\kappa = \{t_1, t_2, \dots\}$ – when active chemo-physical processes have created new (or evolved) interfaces Γ_κ in \mathcal{B} , resulting in the scattered field measurements $\mathbf{v}_\kappa^{\text{obs}}$, and (c) baseline model of the system constructed synthetically based on a-priori available knowledge of the domain.

122 signifies the fourth-order elasticity tensor; \mathbf{I}_m ($m=2, 4$) represents the m th-order symmetric
 123 identity tensor; $\mathbf{t}_o^f = \mathbf{n}_o \cdot \mathbf{C} : \nabla \mathbf{u}^f$ is the free-field traction vector; $\mathbf{n}_o = \mathbf{n}_o^-$ is the unit normal
 124 on Γ_0 . At subsequent sensing steps $t = t_\kappa$, the interaction of \mathbf{u}^f with $\Gamma_0 \cup \Gamma_\kappa$ results in the
 125 scattered field $\mathbf{v}_\kappa \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \{\Gamma_0 \cup \Gamma_\kappa\})^3$ satisfying

$$\begin{aligned}
 & \nabla \cdot (\mathbf{C} : \nabla \mathbf{v}_\kappa) + \rho \omega^2 \mathbf{v}_\kappa = \mathbf{0} && \text{in } \mathbb{R}^3 \setminus \{\Gamma_0 \cup \Gamma_\kappa\}, \\
 & \mathbf{n}_\kappa \cdot \mathbf{C} : \nabla \mathbf{v}_\kappa = \mathbf{K}_\kappa(\boldsymbol{\xi}) \llbracket \mathbf{v}_\kappa \rrbracket - \mathbf{t}_\kappa^f && \text{on } \Gamma_\kappa, \\
 & \mathbf{n}_o \cdot \mathbf{C} : \nabla \mathbf{v}_\kappa = \mathbf{K}_o(\boldsymbol{\xi}) \llbracket \mathbf{v}_\kappa \rrbracket - \mathbf{t}_o^f && \text{on } \Gamma_0,
 \end{aligned}
 \tag{2.4}$$

127 where $\llbracket \mathbf{v}_\kappa \rrbracket = [\mathbf{v}_\kappa^+ - \mathbf{v}_\kappa^-]$ denotes FOD across Γ_κ ; $\mathbf{t}_\kappa^f = \mathbf{n}_\kappa \cdot \mathbf{C} : \nabla \mathbf{u}^f$ is the free-field traction
 128 over $\Gamma_\kappa(\boldsymbol{\xi})$, and $\mathbf{n}_\kappa = \mathbf{n}_\kappa^-$ is the unit normal on Γ_κ . Formulations of the direct scattering
 129 problems (2.2) and (2.4) are complete by imposing the Kupradze radiation condition [26] on
 130 \mathbf{v}_o and \mathbf{v}_κ at far field. More specifically, on uniquely decomposing the scattered fields into
 131 irrotational and solenoidal parts as $\mathbf{v}_\alpha = \mathbf{v}_{p_\alpha} \oplus \mathbf{v}_{s_\alpha}$ for $\alpha \in \{o, \kappa\}$ where

$$\mathbf{v}_{p_\alpha} = \frac{1}{k_s^2 - k_p^2} (\Delta + k_s^2) \mathbf{v}_\alpha, \quad \mathbf{v}_{s_\alpha} = \frac{1}{k_p^2 - k_s^2} (\Delta + k_p^2) \mathbf{v}_\alpha, \quad \alpha = \{o, \kappa\},
 \tag{2.5}$$

133 the Kupradze condition can be stated as

$$\frac{\partial \mathbf{v}_{p_\alpha}}{\partial r} - ik_p \mathbf{v}_{p_\alpha} = o(r^{-1}) \quad \text{and} \quad \frac{\partial \mathbf{v}_{s_\alpha}}{\partial r} - ik_s \mathbf{v}_{s_\alpha} = o(r^{-1}) \quad \text{as } r := |\boldsymbol{\xi}| \rightarrow \infty,
 \tag{2.6}$$

135 uniformly with respect to $\hat{\boldsymbol{\xi}} := \boldsymbol{\xi}/r$. The following remarks shine more light on some specific
 136 aspects of the ensuing developments.

137 **Background domain.** Here, our primary knowledge of the system is assumed to be
 138 at the “baseline” level, shown in Figure 1 (c), which is simplistic and mostly limited to

139 idealistic design parameters. However, chemo-physical processes of interest such as early-
 140 stage degradation mostly reside at the micro- and meso- scales, developing in a network
 141 of pre-existing scatterers of similar scale yet uncertain nature. Accordingly, as illustrated
 142 in [Figure 1](#) (a), the background is modeled by an elastic domain endowed with arbitrary
 143 interstitial boundaries of heterogeneous contact condition – spanning from perfectly bonded
 144 to traction-free interfaces. This provides a versatile platform for a range of micromechanical
 145 phenomena e.g., degradation as a generic cloud of (stationary and evolving) micro-scatterers
 146 of random distribution.

147 **Anatomy of evolution.** Stress concentration, chemical reaction, and early-stage irra-
 148 diation are common producers of *interfacial* damage at micro- and meso- scales [40, 28].
 149 Thermal cycling, fatigue, and shock-wave loading, however, are mostly responsible for dis-
 150 tributed fracture zones [27]. Thus, active processes in this work are identified, according
 151 to [Figure 1](#) (b), with connected or unconnected sets of heterogeneous *fractures* Γ_κ of interfa-
 152 cial elasticity $\mathbf{K}_\kappa(\boldsymbol{\xi})$.

153 **Illumination frequency.** The proposed differential imaging scheme is rooted in the sam-
 154 pling methods [24, 11, 8] recognized for providing good quality reconstruction of hidden scat-
 155 terers at resolution scales transcending the traditional limits of NDE. Here, the illuminating
 156 wavelength $\lambda_s = 2\pi/k_s$ is considered to be comparable with the characteristic length scale
 157 of the sought-for processes e.g., micro-meso-scale features are probed by micro-meso-scale
 158 waves. It is worth mentioning that for multiscale characterization, multi-frequency illumina-
 159 tion i.e., input signals with appropriate spectral content may be adopted [18].

160 **Dimensional platform.** In what follows, all quantities are rendered *dimensionless* by
 161 taking ρ , μ , and ℓ_o – denoting the minimum length scale affiliated with internal boundaries, as
 162 the respective reference scales for mass density, elastic modulus, and length – which amounts
 163 to setting $\rho = \mu = \ell_o = 1$ [6].

164 **Wellposedness of the sequential direct scattering problems.**

165 **Assumption 2.2.** *In this study, it is assumed that $\mathbf{t}_o^f \in H^{-1/2}(\Gamma_o)^3$ (resp. $\mathbf{t}_\kappa^f \in H^{-1/2}(\Gamma_\kappa)^3$)*
 166 *and that $\mathbf{K}_o \in L^\infty(\Gamma_o)^{3 \times 3}$ (resp. $\mathbf{K}_\kappa \in L^\infty(\Gamma_\kappa)^{3 \times 3}$) is symmetric while satisfying $\bar{\boldsymbol{\theta}} \cdot \mathfrak{S} \mathbf{K}_o(\boldsymbol{\xi}) \cdot \boldsymbol{\theta} \leq$*
 167 *0, $\forall \boldsymbol{\theta} \in \mathbb{C}^3$, $\boldsymbol{\xi} \in \Gamma_o$ (resp. $\bar{\boldsymbol{\theta}} \cdot \mathfrak{S} \mathbf{K}_\kappa(\boldsymbol{\xi}) \cdot \boldsymbol{\theta} \leq 0$, $\forall \boldsymbol{\theta} \in \mathbb{C}^3$, $\boldsymbol{\xi} \in \Gamma_\kappa$).*

168 Under [Assumption 2.1](#) and [Assumption 2.2](#), the direct scattering problems (2.2) and (2.4)
 169 are wellposed. The proof draws from (a) Lemma 3.1 and Theorem 3.2 of [35] and arguments of
 170 unique continuation principles. The proof of [35, Theorem 3.2] directly substantiates that (2.2)
 171 and (2.4) are of Fredholm type, and thus, their wellposedness is certified as soon as the
 172 uniqueness of a solution is guaranteed. To verify the latter, let $\mathbf{t}_o^f = \mathbf{t}_\kappa^f = \mathbf{0}$, then according
 173 to [35, Lemma 3.1], the scattered waveforms $\mathbf{v}_\alpha(\boldsymbol{\xi})$, $\alpha \in \{o, \kappa\}$ vanish at the far field as
 174 $|\boldsymbol{\xi}| \rightarrow \infty$. The argument is then followed for the case of fragmented backgrounds shown
 175 in [Figure 1](#) (a) and (b) where \mathcal{B} is described as a union of simply connected bounded domains
 176 \mathcal{D}_i , $\{i = 1, 2, \dots, N_{\mathcal{D}}\}$ of Lipschitz boundaries denoted by Γ_o (resp. $\Gamma_o \cup \Gamma_\kappa$) at the primary
 177 (resp. secondary) sensing step. On setting $\mathcal{D}_0 = \mathbb{R}^3 \setminus \mathcal{B}$ and $\mathcal{G}_0 = \{0\}$, lets define

$$178 \quad (2.7) \quad \mathcal{G}_j = \left\{ i \mid i \notin \bigcup_{\kappa=0}^{j-1} \mathcal{G}_\kappa \text{ and } \bigcup_{\kappa \in \mathcal{G}_{j-1}} \mathcal{D}_\kappa \cap \mathcal{D}_i = \mathbf{S} \right\}, \quad i = 1, 2, \dots, N_{\mathcal{D}}, \quad j = 1, 2, \dots, N_{\mathcal{G}},$$

179 where \mathbf{S} identifies any piecewise analytic surface in \mathcal{B} . In light of (2.7), the domain may be

180 partitioned into $N_{\mathcal{G}} + 1$ layers $L_j = \cup_{\kappa \in \mathcal{G}_j} \mathcal{D}_{\kappa}$, $j = 0, 1, \dots, N_{\mathcal{G}}$, such that successive application
 181 of the unique continuation theorem and Holmgren's principle in each layer completes the
 182 uniqueness proof. More specifically, starting from $L_0 = \mathbb{R}^3 \setminus \mathcal{B}$ where $\mathbf{v}_{\alpha} = \mathbf{0}$ as $|\boldsymbol{\xi}| \rightarrow \infty$,
 183 the unique continuation theorem is deployed to infer $\mathbf{v}_{\alpha} = \mathbf{0}$ in L_0 . Subsequently, the jump
 184 in displacement $\llbracket \mathbf{v}_{\alpha} \rrbracket(\boldsymbol{\xi})$ vanishes over the interface of $\boldsymbol{\xi} \in L_0 \cap L_1 \subset \Gamma_{\circ} \cup \Gamma_{\kappa}$ according to
 185 the elastic boundary conditions over Γ_{\circ} and Γ_{κ} in (2.2) and (2.4). In this setting, Holmgren's
 186 theorem implies that the scattered field \mathbf{v}_{α} vanishes in an open neighborhood of $L_0 \cap L_1$ which
 187 by virtue of the unique continuation theorem leads to $\mathbf{v}_{\alpha}(\boldsymbol{\xi}) = \mathbf{0}$ in $\boldsymbol{\xi} = L_1$. On repeating this
 188 argument in $L_2, \dots, L_{\mathcal{G}}$, one arrives at $\mathbf{v}_{\alpha}(\boldsymbol{\xi}) = \mathbf{0}$ in $\mathcal{B} \setminus \{\Gamma_{\circ} \cup \Gamma_{\kappa}\}$ which completes the proof
 189 for the uniqueness of a scattering solution in \mathcal{B} , and thus, substantiates the wellposedness of
 190 the forward problem.

191 The scattered waveforms $\mathbf{v}_{\circ} \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \Gamma_{\circ})^3$ and $\mathbf{v}_{\kappa} \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \{\Gamma_{\circ} \cup \Gamma_{\kappa}\})^3$ are sequentially
 192 captured at $t = \{t_{\circ}, t_1, t_2, \dots\}$ in the form of far-field patterns $\mathbf{v}_{\alpha}^{\infty} = \mathbf{v}_{p_{\alpha}}^{\infty} \oplus \mathbf{v}_{s_{\alpha}}^{\infty}$, $\alpha \in \{\circ, \kappa\}$,
 193 according to the asymptotic expansion

$$194 \quad (2.8) \quad \mathbf{v}_{\alpha}(\boldsymbol{\xi}) = -\frac{e^{ik_p r}}{4\pi(\lambda+2\mu)r} \mathbf{v}_{p_{\alpha}}^{\infty}(\hat{\boldsymbol{\xi}}) - \frac{e^{ik_s r}}{4\pi\mu r} \mathbf{v}_{s_{\alpha}}^{\infty}(\hat{\boldsymbol{\xi}}) + O(r^{-2}) \quad \text{as } r := |\boldsymbol{\xi}| \rightarrow \infty,$$

195 where $\mathbf{v}_{p_{\alpha}}^{\infty} \parallel \hat{\boldsymbol{\xi}}$ and $\mathbf{v}_{s_{\alpha}}^{\infty} \perp \hat{\boldsymbol{\xi}}$ denote respectively the far-field patterns affiliated with $\mathbf{v}_{p_{\alpha}}$ and $\mathbf{v}_{s_{\alpha}}$
 196 in (2.5), satisfying the integral representations corresponding to (2.2)-(2.6),

$$197 \quad (2.9) \quad \mathbf{v}_{p_{\alpha}}^{\infty}(\hat{\boldsymbol{\xi}}) = -\int_{\Gamma_{\circ} \cup \Gamma_{\alpha}} \llbracket \mathbf{v}_{\alpha} \rrbracket(\mathbf{y}) \cdot \boldsymbol{\Sigma}_p^{\infty}(\hat{\boldsymbol{\xi}}, \mathbf{y}) \cdot \mathbf{n}_{\alpha}(\mathbf{y}) \, dS_{\mathbf{y}} =$$

$$198 \quad (2.10) \quad -ik_p \hat{\boldsymbol{\xi}} \int_{\Gamma_{\circ} \cup \Gamma_{\alpha}} \left\{ \lambda \llbracket \mathbf{v}_{\alpha} \rrbracket \cdot \mathbf{n}_{\alpha} + 2\mu (\mathbf{n}_{\alpha} \cdot \hat{\boldsymbol{\xi}}) \llbracket \mathbf{v}_{\alpha} \rrbracket \cdot \hat{\boldsymbol{\xi}} \right\} e^{-ik_p \hat{\boldsymbol{\xi}} \cdot \mathbf{y}} \, dS_{\mathbf{y}}, \quad \alpha \in \{\circ, \kappa\}, \quad \hat{\boldsymbol{\xi}} \in \Omega,$$

$$199 \quad \mathbf{v}_{s_{\alpha}}^{\infty}(\hat{\boldsymbol{\xi}}) = -\int_{\Gamma_{\circ} \cup \Gamma_{\alpha}} \llbracket \mathbf{v}_{\alpha} \rrbracket(\mathbf{y}) \cdot \boldsymbol{\Sigma}_s^{\infty}(\hat{\boldsymbol{\xi}}, \mathbf{y}) \cdot \mathbf{n}_{\alpha}(\mathbf{y}) \, dS_{\mathbf{y}} =$$

$$-ik_s \hat{\boldsymbol{\xi}} \times \int_{\Gamma_{\circ} \cup \Gamma_{\alpha}} \left\{ \mu (\llbracket \mathbf{v}_{\alpha} \rrbracket \times \hat{\boldsymbol{\xi}})(\mathbf{n}_{\alpha} \cdot \hat{\boldsymbol{\xi}}) + \mu (\mathbf{n}_{\alpha} \times \hat{\boldsymbol{\xi}})(\llbracket \mathbf{v}_{\alpha} \rrbracket \cdot \hat{\boldsymbol{\xi}}) \right\} e^{-ik_s \hat{\boldsymbol{\xi}} \cdot \mathbf{y}} \, dS_{\mathbf{y}}, \quad \alpha \in \{\circ, \kappa\}, \quad \hat{\boldsymbol{\xi}} \in \Omega.$$

200 Here, $\boldsymbol{\Sigma}_p^{\infty}$ and $\boldsymbol{\Sigma}_s^{\infty}$ respectively indicate the far-field P- and S- patterns of the elastodynamic
 201 fundamental stress tensor $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_s \oplus \boldsymbol{\Sigma}_p$ (see [35, Appendix B]).

202 **3. Anatomy of the inverse scattering solution.** This section introduces key elements of
 203 sequential sensing pertinent to the analysis in section 4.

204 At every sensing step t_{α} , $\alpha \in \{\circ, \kappa\}$, the domain is excited by a set of plane waves identified
 205 with their direction of propagation $\mathbf{d} \in \Omega$ and polarization amplitudes $\mathbf{q} = \mathbf{q}_p \oplus \mathbf{q}_s$, as in (2.1),
 206 and thus-scattered far-field patterns $\mathbf{v}_{\alpha}^{\infty}(\hat{\boldsymbol{\xi}} | \mathbf{d}, \mathbf{q})$ are recorded over a set of observation angles
 207 $\hat{\boldsymbol{\xi}} \in \Omega$ according to (2.8). In this setting, the far field kernel $\mathbf{W}_{\alpha}^{\infty}(\mathbf{d}, \hat{\boldsymbol{\xi}}) \in \mathbb{C}^{6 \times 6}$ is constructed
 208 from far-field data such that

$$209 \quad (3.1) \quad \mathbf{W}_{\alpha}^{\infty}(\mathbf{d}, \hat{\boldsymbol{\xi}}) \cdot \mathbf{q} := \mathbf{v}_{\alpha}^{\infty}(\hat{\boldsymbol{\xi}} | \mathbf{d}, \mathbf{q}), \quad \mathbf{d}, \hat{\boldsymbol{\xi}} \in \Omega, \quad \mathbf{q} \in \mathbb{R}^3.$$

210 Given $\mathbf{W}_{\alpha}^{\infty}$ at any t_{α} , the far field operator $F_{\alpha} : L^2(\Omega)^3 \rightarrow L^2(\Omega)^3$ is defined by

$$211 \quad (3.2) \quad F_{\alpha}(\mathbf{g})(\hat{\boldsymbol{\xi}}) = \int_{\Omega} \mathbf{W}_{\alpha}^{\infty}(\mathbf{d}, \hat{\boldsymbol{\xi}}) \cdot \mathbf{g}(\mathbf{d}) \, dS_{\mathbf{d}}, \quad \alpha \in \{\circ, \kappa\}, \quad \mathbf{g} \in L^2(\Omega)^3, \quad \hat{\boldsymbol{\xi}} \in \Omega,$$

212 Each density function $\mathbf{g} \in L^2(\Omega)^3$ can be uniquely decomposed as $\mathbf{g} = \mathbf{g}_p \oplus \mathbf{g}_s$ where
 213 $\forall \mathbf{d} \in \Omega$, $\mathbf{g}_p(\mathbf{d}) \parallel \mathbf{d}$ and $\mathbf{g}_s(\mathbf{d}) \perp \mathbf{d}$. Then, the far field operator maps a density $\mathbf{g} \in L^2(\Omega)^3$ to
 214 the far-field pattern of $\mathbf{v}_\alpha \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \{\Gamma_o \cup \Gamma_\alpha\})^3$ solving (2.2)-(2.6) when $\mathbf{u}^f = \mathbf{u}_g$ and where

$$215 \quad (3.3) \quad \mathbf{u}_g(\boldsymbol{\xi}) := \int_{\Omega} \mathbf{g}_p(\mathbf{d}) e^{ik_p \mathbf{d} \cdot \boldsymbol{\xi}} dS_{\mathbf{d}} \oplus \int_{\Omega} \mathbf{g}_s(\mathbf{d}) e^{ik_s \mathbf{d} \cdot \boldsymbol{\xi}} dS_{\mathbf{d}}, \quad \boldsymbol{\xi} \in \mathbb{R}^3,$$

216 denotes a Herglotz wavefield of density $\mathbf{g} = \mathbf{g}_p \oplus \mathbf{g}_s$ [14]. At every sensing step $t = t_\alpha$, we define
 217 the Herglotz operator $\mathcal{H}_\alpha: L^2(\Omega)^3 \rightarrow H^{-1/2}(\Gamma_o \cup \Gamma_\alpha)^3$ that maps the incident polarization
 218 densities \mathbf{g} of (3.3) to the free-field traction \mathbf{t}_α^f induced over the scattering interfaces,

$$219 \quad (3.4) \quad \mathcal{H}_\alpha(\mathbf{g}) := \mathbf{n}_\alpha \cdot \mathbf{C} : \nabla \mathbf{u}_g \quad \text{on } \Gamma_o \cup \Gamma_\alpha.$$

220 With reference to (2.9) and (2.10), it is then straightforward to show that the adjoint
 221 Herglotz operator $\mathcal{H}_\alpha^*: \tilde{H}^{1/2}(\Gamma_o \cup \Gamma_\alpha)^3 \rightarrow L^2(\Omega)^3$ takes the form

$$222 \quad (3.5) \quad \mathcal{H}_\alpha^*(\mathbf{a})(\hat{\boldsymbol{\xi}}) = - \int_{\Gamma_o \cup \Gamma_\alpha} \mathbf{a}(\mathbf{y}) \cdot \boldsymbol{\Sigma}^\infty(\hat{\boldsymbol{\xi}}, \mathbf{y}) \cdot \mathbf{n}_\alpha(\mathbf{y}) dS_{\mathbf{y}}, \quad \boldsymbol{\Sigma}^\infty = \boldsymbol{\Sigma}_s^\infty \oplus \boldsymbol{\Sigma}_p^\infty.$$

223 Then, each far field operator F_α possesses the factorization

$$224 \quad (3.6) \quad F_\alpha = \mathcal{H}_\alpha^* T_\alpha \mathcal{H}_\alpha, \quad \alpha \in \{o, \kappa\},$$

225 where the operator $T_\alpha: H^{-1/2}(\Gamma_o \cup \Gamma_\alpha)^3 \rightarrow \tilde{H}^{1/2}(\Gamma_o \cup \Gamma_\alpha)^3$, at $t = t_\alpha$, takes the free field
 226 traction \mathbf{t}_α^f of (3.4) to the scattered FOD $\mathbf{a} := \llbracket \mathbf{v}_\alpha \rrbracket$ across $\Gamma_o \cup \Gamma_\alpha$ via the elastic contact laws
 227 of (2.2) and (2.4),

$$228 \quad (3.7) \quad T_\alpha(\mathbf{t}_\alpha^f)(\boldsymbol{\xi}) := \llbracket \mathbf{v}_\alpha(\boldsymbol{\xi}) \rrbracket, \quad \boldsymbol{\xi} \in \Gamma_o \cup \Gamma_\alpha, \quad \alpha \in \{o, \kappa\},$$

229 where $\mathbf{v}_\alpha \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \{\Gamma_o \cup \Gamma_\alpha\})^3$ solves (2.2)-(2.6) for $\mathbf{u}^f = \mathbf{u}_g$.

230 The ensuing analysis requires the following assumption.

231 **Assumption 3.1 (illumination prompts scattering).** *At any sensing step t_α , it is assumed*
 232 *that $\Gamma_o \cup \Gamma_\alpha$ and ω are such that the Herglotz operator $\mathcal{H}_\alpha: L^2(\Omega)^3 \rightarrow H^{-1/2}(\Gamma_o \cup \Gamma_\alpha)^3$ is*
 233 *injective, and thus, its adjoint $\mathcal{H}_\alpha^*: \tilde{H}^{1/2}(\Gamma_o \cup \Gamma_\alpha)^3 \rightarrow L^2(\Omega)^3$ has a dense range.*

234 **Assumption 3.1** is expected to hold true in general for all $\omega > 0$ possibly excluding a discrete
 235 set of values without finite accumulation points. This may be observed by decomposing $\Gamma_o \cup \Gamma_\alpha$
 236 into $M_\alpha \geq 1$ (possibly disjoint) analytic surfaces $\Gamma_m \subset \Gamma_o \cup \Gamma_\alpha$, $m = 1, \dots, M_\alpha$, and identifying
 237 their unique analytic continuation ∂D_m . In this setting, **Assumption 3.1** holds according
 238 to [35, Lemma 5.3] if $\omega > 0$ is not a ‘‘Neumann’’ eigenfrequency of the Navier equation in D_m
 239 that satisfies

$$240 \quad (3.8) \quad \begin{aligned} \nabla \cdot (\mathbf{C} : \nabla \mathbf{u}) + \rho \omega^2 \mathbf{u} &= \mathbf{0} && \text{in } D_m, \\ \mathbf{n} \cdot \mathbf{C} : \nabla \mathbf{u} &= \mathbf{0} && \text{on } \partial D_m. \end{aligned}$$

241 Here, $\mathbf{u} \in H^1(D_m)^3$ indicates the eigen-waveform affiliated to ω . If D_m is bounded, the
 242 real eigenfrequencies of (3.8) form a discrete set according to [26, Chapter VII, Theorem 1.4].

243 Analogous to [35, Lemmas 5.2, 5.6, 5.7], it may be shown that at all sensing steps
 244 t_α : (a) \mathcal{H}_α^* is compact and injective, and (b) T_α is bounded and coercive – i.e., there ex-
 245 ists a constant $c > 0$ independent of φ such that

$$(3.9) \quad |\langle \varphi, T_\alpha(\varphi) \rangle| \geq c \|\varphi\|_{H^{-\frac{1}{2}}(\Gamma_o \cup \Gamma_\alpha)}^2, \quad \forall \varphi \in H^{-1/2}(\Gamma_o \cup \Gamma_\alpha)^3,$$

and has a continuous inverse. Subsequently, according to [35, Lemma 6.6], the far-field operator F_α is injective, compact and under [Assumption 3.1](#) has a dense range. Given (3.2), the operator $F_{\alpha_\#} : L^2(\Omega)^3 \rightarrow L^2(\Omega)^3$ is defined by

$$(3.10) \quad F_{\alpha_\#} := \frac{1}{2}|F_\alpha + F_\alpha^*| + \frac{1}{2i}(F_\alpha - F_\alpha^*).$$

Similar to [35, Theorem 6.3, Lemma 6.4], one may prove the following where the space $\tilde{H}^{1/2}(\Gamma_o \cup \Gamma_\alpha)$ identifies the dual of $H^{-1/2}(\Gamma_o \cup \Gamma_\alpha)$.

Theorem 3.2. *Under [Assumption 2.2](#) and [Assumption 3.1](#), the operator $F_{\alpha_\#}$ is positive and has the following factorization*

$$(3.11) \quad F_{\alpha_\#} = \mathcal{H}_\alpha^* T_{\alpha_\#} \mathcal{H}_\alpha, \quad \alpha \in \{\circ, \kappa\},$$

where the middle operator $T_{\alpha_\#} : H^{-1/2}(\Gamma_o \cup \Gamma_\alpha)^3 \rightarrow \tilde{H}^{1/2}(\Gamma_o \cup \Gamma_\alpha)^3$ is selfadjoint and positively coercive, i.e., there exists a constant $c > 0$ independent of φ so that

$$(3.12) \quad \langle \varphi, T_{\alpha_\#}(\varphi) \rangle_{H^{-1/2}(\Gamma_o \cup \Gamma_\alpha), \tilde{H}^{1/2}(\Gamma_o \cup \Gamma_\alpha)} \geq c \|\varphi\|_{H^{-\frac{1}{2}}(\Gamma_o \cup \Gamma_\alpha)}^2, \quad \forall \varphi \in H^{-1/2}(\Gamma_o \cup \Gamma_\alpha)^3.$$

Moreover, the range of \mathcal{H}_α^* coincides with that of $F_{\alpha_\#}^{1/2}$.

Philosophy of the sampling-based data inversion. With reference to [Figure 1](#), let us define a search volume $\mathcal{B} \subset \mathbb{R}^3$ and a set of trial scatterers $L(\mathbf{x}_o, \mathbf{R}) \subset \mathcal{B}$ such that for every pair $(\mathbf{x}_o, \mathbf{R})$, $L := \mathbf{x}_o + \mathbf{R}L$ specifies a smooth arbitrary-shaped dislocation L at $\mathbf{x}_o \in \mathcal{B}$ whose orientation is identified by a unitary rotation matrix $\mathbf{R} \in U(3)$. In this setting, the far-field pattern $\Phi_L^\infty : \tilde{H}^{1/2}(L) \rightarrow L^2(\Omega)^3$ induced by $L(\mathbf{x}_o, \mathbf{R})$ as a sole scatterer in \mathbb{R}^3 endowed with an admissible FOD profile $\mathbf{a} \in \tilde{H}^{1/2}(L)$ is given by

$$(3.13) \quad \begin{aligned} \Phi_L^\infty(\mathbf{a})(\hat{\xi}) = & - \left(ik_p \hat{\xi} \int_L \left\{ \lambda(\mathbf{a} \cdot \mathbf{n}) + 2\mu(\mathbf{n} \cdot \hat{\xi})(\mathbf{a} \cdot \hat{\xi}) \right\} e^{-ik_p \hat{\xi} \cdot \mathbf{y}} dS_{\mathbf{y}} \right. \\ & \left. \oplus ik_s \hat{\xi} \times \int_L \left\{ \mu(\mathbf{a} \times \hat{\xi})(\mathbf{n} \cdot \hat{\xi}) + \mu(\mathbf{n} \times \hat{\xi})(\mathbf{a} \cdot \hat{\xi}) \right\} e^{-ik_s \hat{\xi} \cdot \mathbf{y}} dS_{\mathbf{y}} \right), \end{aligned}$$

where $\mathbf{n}(\mathbf{y})$ signifies the unit normal at $\mathbf{y} \in L$, and $\hat{\xi} \in \Omega$ is the observation direction. In light of (3.13), one may generate a library of scattering signatures affiliated with a grid of trial pairs $(\mathbf{x}_o, \mathbf{R})$ sampling $\mathcal{B} \times U(3)$.

The underpinning concept of sampling methods [2, 35] is segment-wise reconstruction of $\Gamma_o \cup \Gamma_\alpha$ through a careful implementation of synthetic wavefront shaping at every sensing step t_α , aiming to distill the scattering signature of domain's internal structure segment by segment from waveform data. In this vein, the library of far field patterns Φ_L^∞ , generated on the basis of trial dislocations $L(\mathbf{x}_o, \mathbf{R})$, is deployed to probe the range of the operator F_α (or $F_{\alpha_\#}^{1/2}$ in the factorization method) by solving

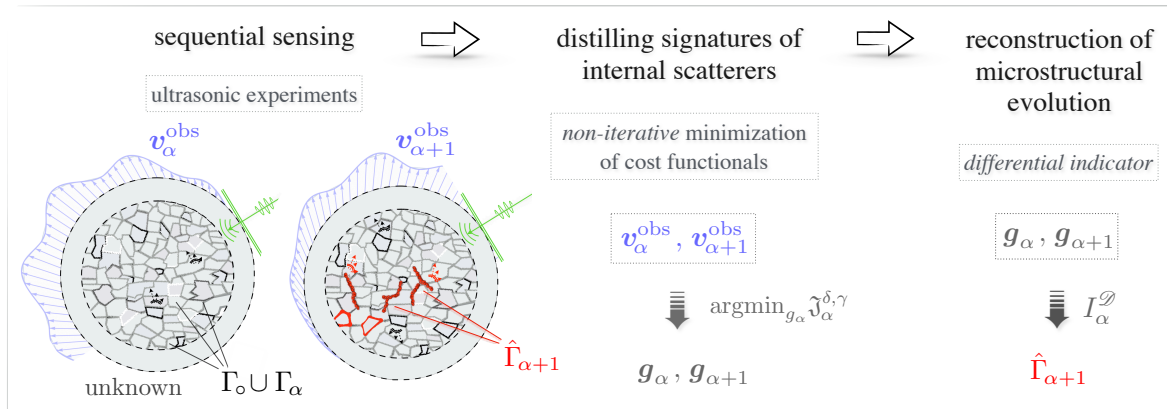
$$(3.14) \quad F_\alpha \mathbf{g} \simeq \Phi_L^\infty, \quad (\text{or } F_{\alpha_\#}^{1/2} \mathbf{g} = \Phi_L^\infty \text{ for the factorization method})$$

for the illumination densities $\mathbf{g}(\mathbf{x}_o, \mathbf{R}) = \mathbf{g}_p \oplus \mathbf{g}_s$. In this setting, the main theorems underlying sampling methods (e.g., the factorization method and generalized linear sampling method)

279 rigorously furnish the distinct behavior of solution $\mathbf{g}(\mathbf{x}_o, \mathbf{R})$ in the vicinity of hidden scatterers,
 280 giving rise to a suit of imaging criteria to reconstruct $\Gamma_o \cup \Gamma_\alpha$. We refer to the following section
 281 for the indicator related to the generalized linear sampling method. For the factorization
 282 method, the equation is solvable if and only if $L \subset \Gamma_o \cup \Gamma_\alpha$.

283 *Remark 3.3 (finite domains).* It should be mentioned that one may also rigorously define
 284 parallel operators pertinent to finite backgrounds that carry similar properties mentioned in
 285 this section, see e.g., [31]. Thus, the ensuing developments directly lend themselves to both
 286 finite and infinite domains, as demonstrated by the numerical experiments in section 5.

287 **4. Differential evolution indicators.** As mentioned earlier, existing sampling approaches
 288 to waveform tomography mostly require an accurate characterization of the background for
 289 their successful performance. This section aims to relax this requirement by introducing
 290 a *three-tier platform* for targeted reconstruction of evolution in elastic backgrounds of a-
 291 priori unknown structure. As illustrated in Figure 2, in this framework: **(1)** the domain is
 292 sequentially subject to illumination and sensing in an arbitrary configuration; **(2)** the resulting
 293 sequence of sensory data ($\mathbf{v}_\alpha^{\text{obs}}, \mathbf{v}_{\alpha+1}^{\text{obs}}$) are deployed to *non-iteratively* compute the associated
 294 source densities i.e., synthetic wavefronts ($\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}$) to distill segment-wise signatures of the
 295 domain's internal structure from scattered field measurements; and **(3)** thus-obtained densities
 296 are then used to selectively reconstruct the support of interfaces $\hat{\Gamma}_{\alpha+1} := \Gamma_{\alpha+1} \setminus \Gamma_\alpha$ born or
 297 evolved between $t \in [t_\alpha, t_{\alpha+1}]$ (or any pairs of sensing steps) within a network of pre-existing
 scatterers $\Gamma_\alpha \cup \Gamma_o$.



298 **Figure 2.** Three-tier approach to differential tomography of microstructural evolution in highly
 299 scattering backgrounds.

300 Construction of an evolution indicator is rooted in minimizing sequences to the cost func-
 301 tional

$$(4.1) \quad \mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}) := \|F_\alpha \mathbf{g} - \Phi_L^\infty\|_{L^2(\Omega)}^2 + \gamma (\mathbf{g}, F_{\alpha\#} \mathbf{g}), \quad \mathbf{g} \in L^2(\Omega)^3, \quad \gamma > 0.$$

On denoting $G_\alpha = \mathcal{H}_\alpha^* T_\alpha$, we also define

$$\mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \psi) = \|G_\alpha \psi - \Phi_L^\infty\|_{L^2(\Omega)}^2 + \gamma (\psi, T_{\alpha\#} \psi), \quad \psi \in H^{-\frac{1}{2}}(\Gamma_o \cup \Gamma_\alpha)^3, \quad \gamma > 0,$$

where

$$\mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \mathcal{H}_\alpha \mathbf{g}) = \mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}).$$

302 In what follows, the strong convergence of germane minimizing sequences is established by
 303 way of the strong convexity of $\mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \cdot)$ on $H^{-\frac{1}{2}}(\Gamma_o \cup \Gamma_\alpha)$. This approach is slightly different
 304 from the related arguments in [3].

305 **Theorem 4.1.** Consider the minimizing sequence $\mathbf{g}^\gamma \in L^2(\Omega)^3$ for $\mathfrak{J}_\alpha^\gamma$ such that

$$306 \quad (4.2) \quad \mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}^\gamma) \leq \mathfrak{j}_\alpha^\gamma(\Phi_L^\infty) + \eta(\gamma), \quad \gamma > 0,$$

307 where $\eta(\gamma)/\gamma \rightarrow 0$ as $\gamma \rightarrow 0$ and

$$308 \quad \mathfrak{j}_\alpha^\gamma(\Phi_L^\infty) := \inf_{\mathbf{g} \in L^2(\Omega)^3} \mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}).$$

309 Then,

$$310 \quad (4.3) \quad \begin{aligned} \Phi_L^\infty \in \text{Range}(G_\alpha) &\Rightarrow \lim_{\gamma \rightarrow 0} (\mathbf{g}^\gamma, F_{\alpha_\#} \mathbf{g}^\gamma) = \lim_{\gamma \rightarrow 0} (\mathcal{H}_\alpha \mathbf{g}^\gamma, T_{\alpha_\#} \mathcal{H}_\alpha \mathbf{g}^\gamma) < \infty, \\ \Phi_L^\infty \notin \text{Range}(G_\alpha) &\Rightarrow \liminf_{\gamma \rightarrow 0} (\mathbf{g}^\gamma, F_{\alpha_\#} \mathbf{g}^\gamma) = \liminf_{\gamma \rightarrow 0} (\mathcal{H}_\alpha \mathbf{g}^\gamma, T_{\alpha_\#} \mathcal{H}_\alpha \mathbf{g}^\gamma) = \infty. \end{aligned}$$

311 Moreover, when $G_\alpha \psi = \Phi_L^\infty$, the sequence $\mathcal{H}_\alpha \mathbf{g}^\gamma$ strongly converges to $\psi \in H^{-\frac{1}{2}}(\Gamma_o \cup \Gamma_\alpha)$
 312 as $\gamma \rightarrow 0$.

313 *Proof.* The limits of (4.3) are directly drawn from [35, Theorem 6.7]. In the case where
 314 $G_\alpha \psi = \Phi_L^\infty$ for some $\psi \in H^{-\frac{1}{2}}(\Gamma_o \cup \Gamma_\alpha)$, the limiting behavior of $\mathcal{H}_\alpha \mathbf{g}^\gamma$ to ψ may be analyzed
 315 by using the strong convexity of $\mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \cdot)$. More specifically, using that $\mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}^\gamma) =$
 316 $\mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \mathcal{H}_\alpha \mathbf{g}^\gamma)$ and that \mathcal{H}_α has dense range, we have

$$317 \quad (4.4) \quad \begin{aligned} \mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}^\gamma) - \eta(\gamma) + \gamma \frac{1}{2} \|\mathcal{H}_\alpha \mathbf{g}^\gamma - \psi\|^2 &\leq \mathfrak{j}_\alpha^\gamma(\Phi_L^\infty) + \gamma \frac{1}{2} \|\mathcal{H}_\alpha \mathbf{g}^\gamma - \psi\|^2 \leq \\ &\leq \mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \frac{1}{2} \mathcal{H}_\alpha \mathbf{g}^\gamma + \frac{1}{2} \psi) + \gamma \frac{1}{2} \|\mathcal{H}_\alpha \mathbf{g}^\gamma - \psi\|^2 \leq \max\{\mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}^\gamma), \mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \psi)\}. \end{aligned}$$

318 Then, in light of

$$319 \quad (4.5) \quad \mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \psi) - \mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}^\gamma) \leq \gamma [\langle \psi, T_{\alpha_\#} \psi \rangle - \langle \mathcal{H}_\alpha \mathbf{g}^\gamma, T_{\alpha_\#} \mathcal{H}_\alpha \mathbf{g}^\gamma \rangle],$$

320 for $\mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \psi) > \mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}^\gamma)$, observe from (4.4) that

$$321 \quad \limsup_{\gamma \rightarrow 0} \|\mathcal{H}_\alpha \mathbf{g}^\gamma - \psi\|^2 = 0,$$

322 which proves the strong convergence of $\mathcal{H}_\alpha \mathbf{g}^\gamma$ to ψ as $\gamma \rightarrow 0$. ■

323 **Noisy data.** Consider the perturbed operators F_α^δ and $F_{\alpha_\#}^\delta$,

$$324 \quad (4.6) \quad \|F_\alpha^\delta - F_\alpha\| \leq \delta, \quad \|F_{\alpha_\#}^\delta - F_{\alpha_\#}\| \leq \delta,$$

325 where $\delta > 0$ is a measure of noise in data and the self-adjoint operator $F_{\alpha_\#}^\delta : L^2(\Omega)^3 \rightarrow L^2(\Omega)^3$
 326 is drawn from F_α^δ via

$$327 \quad (4.7) \quad F_{\alpha_\#}^\delta := \frac{1}{2} |F_\alpha^\delta + F_\alpha^{\delta*}| + \frac{1}{2i} (F_\alpha^\delta - F_\alpha^{\delta*}).$$

329 **Assumption 4.2.** $\forall t_\alpha, F_\alpha^\delta$ and $F_{\alpha_\#}^\delta$ are compact.

330 **Theorem 4.3 (noisy data).** For $\mathbf{g} \in L^2(\Omega)^3$, $\gamma > 0$, consider the cost functional

$$331 \quad (4.8) \quad \mathfrak{J}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}) := \|F_\alpha^\delta \mathbf{g} - \Phi_L^\infty\|_{L^2(\Omega)}^2 + \gamma (\mathbf{g}, F_{\alpha_\#}^\delta \mathbf{g}) + \gamma^{1-\chi} \delta \|\mathbf{g}\|_{L^2(\Omega)}^2,$$

332 where $\chi \in]0, 1[$, and $\mathfrak{J}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g})$ admits the minimizer

$$333 \quad (4.9) \quad \mathbf{g}_L^{\delta,\gamma} = \arg \min_{\mathbf{g} \in L^2(\Omega)^3} \mathfrak{J}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}),$$

334 satisfying

$$335 \quad (4.10) \quad \lim_{\gamma \rightarrow 0} \limsup_{\delta \rightarrow 0} \mathfrak{J}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}_L^{\delta,\gamma}) = 0.$$

336 In this setting,

$$337 \quad (4.11) \quad \begin{aligned} \Phi_L^\infty \in \text{Range}(G_\alpha) &\Rightarrow \limsup_{\gamma \rightarrow 0} \limsup_{\delta \rightarrow 0} ((\mathbf{g}_L^{\delta,\gamma}, F_{\alpha_\#}^\delta \mathbf{g}_L^{\delta,\gamma}) + \delta \gamma^{-\chi} \|\mathbf{g}_L^{\delta,\gamma}\|^2) < \infty, \\ \Phi_L^\infty \notin \text{Range}(G_\alpha) &\Rightarrow \liminf_{\gamma \rightarrow 0} \liminf_{\delta \rightarrow 0} ((\mathbf{g}_L^{\delta,\gamma}, F_{\alpha_\#}^\delta \mathbf{g}_L^{\delta,\gamma}) + \delta \gamma^{-\chi} \|\mathbf{g}_L^{\delta,\gamma}\|^2) = \infty. \end{aligned}$$

338 In addition, when $G_\alpha \psi = \Phi_L^\infty$, there holds

$$339 \quad (4.12) \quad \limsup_{\gamma \rightarrow 0} \limsup_{\delta \rightarrow 0} \delta \|\mathbf{g}_L^{\delta,\gamma}\|^2 = 0.$$

340 Also, there exists $\delta_o(\gamma)$ such that $\forall \delta(\gamma) \leq \delta_o(\gamma)$, $\mathcal{H}_\alpha \mathbf{g}_L^{\delta(\gamma),\gamma}$ converges strongly to ψ as $\gamma \rightarrow 0$.

341 *Proof.* The limiting behavior of $\mathfrak{J}_\alpha^{\delta,\gamma}$ in (4.10), and limits of the penalty term in (4.11)
342 are established in [3]. Moreover, given (4.11) for the case where $\Phi_L^\infty \in \text{Range}(G_\alpha)$, (4.12)
343 is self-evident. This relation along with the strong convergence result, when $G_\alpha \psi = \Phi_L^\infty$,
344 constitutes the foundation of differential imaging with noisy operators and may be observed
345 as the following. Define

$$346 \quad (4.13) \quad \begin{aligned} D_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}) &:= \delta^2 \|\mathbf{g}\|^2 + 2\delta \|F_\alpha \mathbf{g} - \Phi_L^\infty\| \|\mathbf{g}\| + \delta \gamma \|\mathbf{g}\|^2, \\ \tilde{D}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}) &:= D_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}) + \delta \gamma^{1-\chi} \|\mathbf{g}\|^2. \end{aligned}$$

347 Then, observe that

$$348 \quad (4.14) \quad \begin{aligned} [\mathfrak{J}_\alpha^{\delta,\gamma} - \mathfrak{J}_\alpha^\gamma](\Phi_L^\infty; \mathbf{g}) &\leq \tilde{D}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}), \\ [\mathfrak{J}_\alpha^\gamma - \mathfrak{J}_\alpha^{\delta,\gamma}](\Phi_L^\infty; \mathbf{g}) &\leq D_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}). \end{aligned}$$

349 Consider $\eta(\gamma)$ and \mathbf{g}^γ as in Theorem 4.1, (4.2), (4.9) and (4.14), then observe that

$$350 \quad (4.15) \quad \begin{aligned} \mathfrak{J}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}_L^{\delta,\gamma}) - \tilde{D}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}^\gamma) - \eta(\gamma) + \gamma \frac{1}{2} \|\mathcal{H}_\alpha \mathbf{g}_L^{\delta,\gamma} - \psi\|^2 &\leq \\ \mathfrak{J}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}^\gamma) - \tilde{D}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}^\gamma) - \eta(\gamma) + \gamma \frac{1}{2} \|\mathcal{H}_\alpha \mathbf{g}_L^{\delta,\gamma} - \psi\|^2 &\leq \\ \mathfrak{J}_\alpha^\gamma(\Phi_L^\infty) + \gamma \frac{1}{2} \|\mathcal{H}_\alpha \mathbf{g}_L^{\delta,\gamma} - \psi\|^2 &\leq \max\{\mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}_L^{\delta,\gamma}), \mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \psi)\}. \end{aligned}$$

which in light of (4.14) results in

$$\gamma \frac{1}{2} \|\mathcal{H}_\alpha \mathbf{g}_L^{\delta,\gamma} - \psi\|^2 \leq \max\{\mathfrak{L}_\alpha^\gamma(\Phi_L^\infty; \psi) - \mathfrak{J}_\alpha^\gamma(\Phi_L^\infty; \mathbf{g}_L^{\delta,\gamma}), 0\} + D_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}_L^{\delta,\gamma}) + \tilde{D}_\alpha^{\delta,\gamma}(\Phi_L^\infty; \mathbf{g}^\gamma) + \eta(\gamma),$$

351 By definition of $D_{\alpha}^{\delta_o(\gamma),\gamma}(\Phi_L^{\infty}, \mathbf{g})$ in (4.13), it is evident that one may choose a sequence $\delta_o(\gamma)$
 352 such that $\forall \delta(\gamma) \leq \delta_o(\gamma)$,

$$353 \quad \lim_{\gamma \rightarrow 0} \frac{\tilde{D}_{\alpha}^{\delta(\gamma),\gamma}(\Phi_L^{\infty}, \mathbf{g}^{\gamma})}{\gamma} = 0.$$

354 In view of (4.11) and (4.12), one may also find $\delta_o(\gamma)$ such that $\forall \delta(\gamma) \leq \delta_o(\gamma)$,

$$355 \quad \lim_{\gamma \rightarrow 0} \frac{D_{\alpha}^{\delta(\gamma),\gamma}(\Phi_L^{\infty}, \mathbf{g}_L^{\delta(\gamma),\gamma})}{\gamma} = 0.$$

356 Consequently, from (4.5), one may conclude that

$$357 \quad \limsup_{\gamma \rightarrow 0} \|\mathcal{H}_{\alpha} \mathbf{g}_L^{\delta(\gamma),\gamma} - \boldsymbol{\psi}\|^2 = 0, \quad \forall \delta(\gamma) \leq \delta_o(\gamma),$$

358 verifying the strong convergence of $\mathcal{H}_{\alpha} \mathbf{g}_L^{\delta,\gamma}$ to $\boldsymbol{\psi}$ as $\gamma \rightarrow 0$. ■

359 **Single-step GLSM imaging criteria.** For future reference, it should be mentioned
 360 that [Theorem 4.3](#) forms the foundation of GLSM imaging indicator [\[35\]](#),

$$361 \quad (4.16) \quad I^{\mathcal{G}_{\sharp}}(L) = \frac{1}{\sqrt{\|(F_{\alpha_{\sharp}}^{\delta})^{\frac{1}{2}} \mathbf{g}_L^{\delta,\gamma}\|^2 + \delta \|\mathbf{g}_L^{\delta,\gamma}\|^2}},$$

362 constructed on the basis of scattered field data $F_{\alpha_{\sharp}}^{\delta}$ captured at a single time step t_{α} . $I^{\mathcal{G}_{\sharp}}(L)$
 363 attains its highest values when the trial dislocation $L(\mathbf{x}_o, \mathbf{R})$ meets the support of hidden
 364 scatterers $\Gamma_o \cup \Gamma_{\alpha}$.

365 *Remark 4.4.* The GLSM indicator $I^{\mathcal{G}_{\sharp}}$ is primarily designed for imaging in elastic back-
 366 grounds whose topology and material properties are precisely identified [\[2, 35, 15\]](#). Such rig-
 367 orous knowledge of the background is not achievable in many practical situations particularly
 368 at micro- and meso- scales. Furthermore, in a fully characterized background domain, it is
 369 shown in [section 5](#) that $I^{\mathcal{G}_{\sharp}}$ loses its resolution in presence of multiple closely spaced scatterers
 370 whose pairwise distances are of the order of a fraction of the illuminating wavelength.

371 **Invariants of scattering solution.** Requiring an exact knowledge of background may
 372 be relaxed by taking advantage of (a) unique attributes of the cost functionals $\mathfrak{J}_{\alpha}^{\gamma}$ (*resp.* $\mathfrak{J}_{\alpha}^{\delta,\gamma}$)
 373 introduced in [Theorem 4.1](#) (*resp.* [Theorem 4.3](#)) – namely, their convex nature and robustness
 374 against noise, and (b) newly established strong convergence of the proposed minimizing se-
 375 quence $\mathbf{g}_L^{\delta,\gamma}$ to a unique minimizer when $\Phi_L^{\infty} \in \text{Range}(G_{\alpha})$. This claim is further motivated
 376 by the following [Theorem 4.5](#), where the relation between any pairs of synthetic wavefronts
 377 $(\mathbf{g}_{\alpha}, \mathbf{g}_{\alpha+1})$, computed in *Tier 2*, is established in terms of their affiliated FODs i.e., fracture
 378 opening displacement profiles ($\llbracket \mathbf{v}_{\alpha} \rrbracket, \llbracket \mathbf{v}_{\alpha+1} \rrbracket$). FODs are directly linked to the penalty terms
 379 in $(\mathfrak{J}_{\alpha}^{\delta,\gamma}, \mathfrak{J}_{\alpha+1}^{\delta,\gamma})$, and thus imaging indicators of the sampling type e.g., $I^{\mathcal{G}_{\sharp}}$ in (4.16). Based on
 380 these developments, [Theorem 4.6](#) and [Theorem 4.7](#) introduce a new class of functionals that
 381 remain systematically invariant with respect to the stationary scatterers $\Gamma_{\alpha} \cup \Gamma_o$ between any
 382 pair of distinct experimental campaigns $t \in [t_{\alpha}, t_{\alpha+1}]$. This leads to the differential evolution
 383 indicators $I_{\alpha}^{\mathcal{D}}, \hat{I}_{\alpha}^{\mathcal{D}}$ (*resp.* $I_{\alpha,\delta}^{\mathcal{D}}, \hat{I}_{\alpha,\delta}^{\mathcal{D}}$) in (4.22) (*resp.* (4.23)) enabling selective reconstruction of
 384 evolution $\hat{\Gamma}_{\alpha+1} = \Gamma_{\alpha+1} \setminus \Gamma_{\alpha}$ within the interval $[t_{\alpha}, t_{\alpha+1}]$ in a complex background without
 385 the need to reconstruct the entire domain i.e., $\Gamma_o \cup \Gamma_{\alpha}$ across pertinent scales, which may be
 386 practically insurmountable.

387 **Theorem 4.5.** Given $(\mathbf{v}_\alpha^\infty, \mathbf{v}_{\alpha+1}^\infty)$, consider sampling the search volume $\mathbf{x}_o \in \mathcal{B}$ by a set
 388 of trial dislocations $L(\mathbf{x}_o, \mathbf{R})$ endowed with an admissible FOD $\mathbf{a}(\boldsymbol{\xi}) \in \tilde{H}^{1/2}(L)^3$. The re-
 389 sulting source densities $(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})(L; \delta, \gamma)$ minimizing $(\mathfrak{J}_\alpha^{\delta, \gamma}, \mathfrak{J}_{\alpha+1}^{\delta, \gamma})$ are deployed to identify
 390 the affiliated Herglotz incidents $(\mathbf{u}_{\mathbf{g}_\alpha}, \mathbf{u}_{\mathbf{g}_{\alpha+1}})$ in (3.3), and thereof, the scattered FOD pro-
 391 files $\llbracket \mathbf{v}_\alpha \rrbracket(\boldsymbol{\xi}) \in \tilde{H}^{1/2}(\Gamma_\alpha \cup \Gamma_o)^3$ and $\llbracket \mathbf{v}_{\alpha+1} \rrbracket(\boldsymbol{\xi}) \in \tilde{H}^{1/2}(\Gamma_{\alpha+1} \cup \Gamma_o)^3$ according to (2.4). Then,
 392 under *Assumption 2.2* and *Assumption 3.1*,

- 393 • If $L \subset \Gamma_\alpha \cup \Gamma_o$ then $\llbracket \mathbf{v}_\alpha \rrbracket = \llbracket \mathbf{v}_{\alpha+1} \rrbracket$ over $\Gamma_\alpha \cup \Gamma_o$.
- 394 • If $L \subset \hat{\Gamma}_{\alpha+1}$ then $\llbracket \mathbf{v}_\alpha \rrbracket \neq \llbracket \mathbf{v}_{\alpha+1} \rrbracket = \mathbf{0}$ over $\Gamma_\alpha \cup \Gamma_o$.

395 *Proof.* Consider the following:

- 396 • If $L \subset \Gamma_\alpha \cup \Gamma_o$, then $\tilde{H}^{1/2}(L)^3 \subset \tilde{H}^{1/2}(\Gamma_\alpha \cup \Gamma_o)^3$ (resp. $\tilde{H}^{1/2}(L)^3 \subset \tilde{H}^{1/2}(\Gamma_{\alpha+1} \cup \Gamma_o)^3$). By
 397 extending the domain of $\mathbf{a} \in \tilde{H}^{1/2}(L)^3$ from L to $\Gamma_\alpha \cup \Gamma_o$ (resp. $\Gamma_{\alpha+1} \cup \Gamma_o$) through zero
 398 padding, one immediately obtains $\Phi_L^\infty \in \text{Range}(\mathcal{H}_\alpha^*)$ (resp. $\Phi_L^\infty \in \text{Range}(\mathcal{H}_{\alpha+1}^*)$)
 399 thanks to (3.5) and (3.13). As a result, $\mathcal{H}_\alpha^* \llbracket \mathbf{v}_\alpha \rrbracket = \Phi_L^\infty$ (resp. $\mathcal{H}_{\alpha+1}^* \llbracket \mathbf{v}_{\alpha+1} \rrbracket = \Phi_L^\infty$)
 400 possesses a unique solution such that $\llbracket \mathbf{v}_\alpha \rrbracket = \mathbf{a}$ over L and $\llbracket \mathbf{v}_\alpha \rrbracket = \mathbf{0}$ on $\Gamma_\alpha \cup \Gamma_o \setminus L$
 401 (resp. $\llbracket \mathbf{v}_{\alpha+1} \rrbracket = \mathbf{a}$ over L and $\llbracket \mathbf{v}_{\alpha+1} \rrbracket = \mathbf{0}$ on $\Gamma_{\alpha+1} \cup \Gamma_o \setminus L$). Therefore, $\llbracket \mathbf{v}_\alpha \rrbracket = \llbracket \mathbf{v}_{\alpha+1} \rrbracket$
 402 over $\Gamma_\alpha \cup \Gamma_o$.
- 403 • If $L \subset \hat{\Gamma}_{\alpha+1}$, then $\Phi_L^\infty \in \text{Range}(\mathcal{H}_{\alpha+1}^*)$ while $\Phi_L^\infty \notin \text{Range}(\mathcal{H}_\alpha^*)$. In this case, according
 404 to the above argument $\mathcal{H}_{\alpha+1}^* \llbracket \mathbf{v}_{\alpha+1} \rrbracket = \Phi_L^\infty$ has a unique solution, and $\llbracket \mathbf{v}_{\alpha+1} \rrbracket = \mathbf{0}$ on
 405 $\Gamma_\alpha \cup \Gamma_o$ as it falls in the zero-padded region (recall that $L \not\subset \Gamma_\alpha \cup \Gamma_o$). On the other
 406 hand, the norm of any approximate solution to $\mathcal{H}_\alpha^* \llbracket \mathbf{v}_\alpha \rrbracket = \Phi_L^\infty$ become unbounded
 407 ($\|\llbracket \mathbf{v}_\alpha \rrbracket\|_{\tilde{H}^{1/2}(\Gamma_\alpha \cup \Gamma_o)^3} \rightarrow \infty$) as $\gamma \rightarrow 0$. To observe this, let us assume to the contrary
 408 that there exists $\mathbf{b} \in \tilde{H}^{1/2}(\Gamma_\alpha \cup \Gamma_o)^3$ such that

$$409 \quad \Phi_L^\infty(\mathbf{a})(\hat{\boldsymbol{\xi}}) = - \left(ik_p \hat{\boldsymbol{\xi}} \int_{\Gamma_\alpha \cup \Gamma_o} \left\{ \lambda(\mathbf{b} \cdot \mathbf{n}) + 2\mu(\mathbf{n} \cdot \hat{\boldsymbol{\xi}})(\mathbf{b} \cdot \hat{\boldsymbol{\xi}}) \right\} e^{-ik_p \hat{\boldsymbol{\xi}} \cdot \mathbf{y}} dS_{\mathbf{y}} \right. \\ \left. \oplus ik_s \hat{\boldsymbol{\xi}} \times \int_{\Gamma_\alpha \cup \Gamma_o} \left\{ \mu(\mathbf{b} \times \hat{\boldsymbol{\xi}})(\mathbf{n} \cdot \hat{\boldsymbol{\xi}}) + \mu(\mathbf{n} \times \hat{\boldsymbol{\xi}})(\mathbf{b} \cdot \hat{\boldsymbol{\xi}}) \right\} e^{-ik_s \hat{\boldsymbol{\xi}} \cdot \mathbf{y}} dS_{\mathbf{y}} \right),$$

410 associated with the layer potential
 (4.17)

$$411 \quad \Phi_{\Gamma_\alpha \cup \Gamma_o}(\boldsymbol{\xi}) = \int_{\Gamma_\alpha \cup \Gamma_o} \mathbf{b}(\mathbf{y}) \cdot \mathbf{T}(\boldsymbol{\xi}, \mathbf{y}) dS_{\mathbf{y}}, \quad \mathbf{T}(\boldsymbol{\xi}, \mathbf{y}) = \mathbf{n}(\mathbf{y}) \cdot \boldsymbol{\Sigma}(\boldsymbol{\xi}, \mathbf{y}), \quad \boldsymbol{\xi} \in \mathcal{B} \setminus \{\Gamma_\alpha \cup \Gamma_o\}.$$

412 On the other hand, owing to Definition of $\Phi_L^\infty(\hat{\boldsymbol{\xi}})$ in (3.13), the potential $\Phi_{\Gamma_\alpha \cup \Gamma_o}(\boldsymbol{\xi})$
 413 should coincide with

$$414 \quad (4.18) \quad \Phi_L(\boldsymbol{\xi}) = \int_L \mathbf{a}(\mathbf{y}) \cdot \mathbf{T}(\boldsymbol{\xi}, \mathbf{y}) dS_{\mathbf{y}}, \quad \boldsymbol{\xi} \in \mathcal{B} \setminus L,$$

415 over $\boldsymbol{\xi} \in \mathcal{B} \setminus (L \cup \Gamma_\alpha \cup \Gamma_o)$. Now, let $\Gamma_\alpha \cup \Gamma_o \not\ni \boldsymbol{\xi}^\circ \in L$ and let \mathcal{B}_ϵ be a small ball centered
 416 at $\boldsymbol{\xi}^\circ$ such that $\mathcal{B}_\epsilon \cap \{\Gamma_\alpha \cup \Gamma_o\} = \emptyset$. In this case $\Phi_{\Gamma_\alpha \cup \Gamma_o}$ is analytic in \mathcal{B}_ϵ , while Φ_L
 417 has a discontinuity across $\mathcal{B}_\epsilon \cap L$ – which by contradiction completes the proof. ■

418 **Theorem 4.6 (invariants of the solution sequence $(\mathbf{g}_\alpha)_{\alpha \in \mathbb{Z}^*}$ for noise-free data).** Define

$$419 \quad (4.19) \quad \Lambda_\alpha(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) := (\mathbf{g}_{\alpha+1} - \mathbf{g}_\alpha, F_{\alpha\sharp}(\mathbf{g}_{\alpha+1} - \mathbf{g}_\alpha)), \\ \Upsilon_\alpha(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) := |(\mathbf{g}_{\alpha+1}, F_{\alpha+1\sharp} \mathbf{g}_{\alpha+1}) - (\mathbf{g}_\alpha, F_{\alpha\sharp} \mathbf{g}_\alpha)|, \quad \mathbf{g}_\alpha, \mathbf{g}_{\alpha+1} \in L^2(\Omega)^3,$$

420 where $(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})(L; \gamma)$ are the constructed minimizers of $(\mathfrak{J}_\alpha^\gamma, \mathfrak{J}_{\alpha+1}^\gamma)$ in (4.1) according to (4.2).

421 Then, on defining $\tilde{\Gamma}_{\alpha+1} := \Gamma_{\alpha+1} \cap \{\Gamma_\alpha \cup \Gamma_\circ\}$, it follows that

422 • If $L \subset \Gamma_\alpha \cup \Gamma_\circ \setminus \tilde{\Gamma}_{\alpha+1}$ then

$$423 \quad \lim_{\gamma \rightarrow 0} \Lambda_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) = 0 \text{ and } \lim_{\gamma \rightarrow 0} \Upsilon_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) = 0.$$

424 • If $L \subset \tilde{\Gamma}_{\alpha+1}$ then

$$425 \quad 0 < \lim_{\gamma \rightarrow 0} \Lambda_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) < \infty \text{ and } 0 < \lim_{\gamma \rightarrow 0} \Upsilon_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) < \infty.$$

426 • If $L \subset \hat{\Gamma}_{\alpha+1}$ then

$$427 \quad \lim_{\gamma \rightarrow 0} \Lambda_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) = \infty \text{ and } \lim_{\gamma \rightarrow 0} \Upsilon_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) = \infty.$$

428 *Proof.* If $L \subset \Gamma_\circ \cup \Gamma_\alpha \cup \hat{\Gamma}_{\alpha+1}$, then on recalling that (a) $\mathcal{H}_\alpha^*[\mathbf{v}_\alpha] = \Phi_L^\infty$ (resp. $\mathcal{H}_{\alpha+1}^*[\mathbf{v}_{\alpha+1}] =$
429 Φ_L^∞) from Theorem 4.5, (b) continuity of T_α (resp. $T_{\alpha+1}$) and its inverse T_α^{-1} (resp. $T_{\alpha+1}^{-1}$) as
430 per section 3, and (c) the fact that $\Phi_L^\infty \in \text{Range}(G_\alpha)$ (resp. $\Phi_L^\infty \in \text{Range}(G_{\alpha+1})$) in Theo-
431 rem 4.1, it follows that

$$432 \quad \lim_{\gamma \rightarrow 0} \mathcal{H}_\alpha \mathbf{g}_\alpha(L; \gamma) = T_\alpha^{-1}[\mathbf{v}_\alpha], \text{ (resp. } \lim_{\gamma \rightarrow 0} \mathcal{H}_{\alpha+1} \mathbf{g}_{\alpha+1}(L; \gamma) = T_{\alpha+1}^{-1}[\mathbf{v}_{\alpha+1}]), \quad \mathbf{g}_\alpha, \mathbf{g}_{\alpha+1} \in L^2(\Omega)^3.$$

433 In this setting,

434 • If $L \subset \Gamma_\circ \cup \Gamma_\alpha \setminus \tilde{\Gamma}_{\alpha+1}$, then observe that (a) $\mathbf{K}_{\alpha+1} = \mathbf{K}_\alpha$ on $\Gamma_\alpha \setminus \tilde{\Gamma}_{\alpha+1}$ in the field
435 equations (2.4), and (b) $[\mathbf{v}_{\alpha+1}] = [\mathbf{v}_\alpha]$ on $\Gamma_\alpha \cup \Gamma_\circ$ and $[\mathbf{v}_{\alpha+1}] = \mathbf{0}$ on $\hat{\Gamma}_{\alpha+1}$ as per Theo-
436 rem 4.5. Accordingly, $\mathbf{v}_\alpha = \mathbf{v}_{\alpha+1}$ in $\mathcal{B} \setminus \{\Gamma_\alpha \cup \Gamma_\circ\}$ in light of the respective integral
437 representations, $\forall \boldsymbol{\xi} \in \mathcal{B} \setminus \{\Gamma_\alpha \cup \Gamma_\circ\}$,

$$438 \quad (4.20) \quad \mathbf{v}_\alpha(\boldsymbol{\xi}) = \int_{\Gamma_\alpha \cup \Gamma_\circ} [\mathbf{v}_\alpha](\mathbf{y}) \cdot \mathbf{T}(\boldsymbol{\xi}, \mathbf{y}) \, dS_{\mathbf{y}}, \quad \mathbf{v}_{\alpha+1}(\boldsymbol{\xi}) = \int_{\Gamma_\alpha \cup \Gamma_\circ} [\mathbf{v}_{\alpha+1}](\mathbf{y}) \cdot \mathbf{T}(\boldsymbol{\xi}, \mathbf{y}) \, dS_{\mathbf{y}},$$

439 where $\mathbf{T}(\boldsymbol{\xi}, \mathbf{y})$ is defined in (4.17). As a result, the contact laws in (2.4), governing
440 T_α and T_α^{-1} , read $T_{\alpha+1}^{-1}[\mathbf{v}_{\alpha+1}] = T_\alpha^{-1}[\mathbf{v}_\alpha]$ and $T_{\alpha+1}(\cdot) = T_{\alpha\sharp}(\cdot)$ on $\Gamma_\alpha \cup \Gamma_\circ$. Thus,
441 $\lim_{\gamma \rightarrow 0} \mathcal{H}_\alpha \mathbf{g}_\alpha(L; \gamma) = \lim_{\gamma \rightarrow 0} \mathcal{H}_\alpha \mathbf{g}_{\alpha+1}(L; \gamma) = T_\alpha^{-1}[\mathbf{v}_\alpha]$ on $\Gamma_\alpha \cup \Gamma_\circ$. Invoking (3.11), one
442 concludes

$$443 \quad \lim_{\gamma \rightarrow 0} \Lambda_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) = \lim_{\gamma \rightarrow 0} (\mathcal{H}_\alpha \mathbf{g}_{\alpha+1} - \mathcal{H}_\alpha \mathbf{g}_\alpha, T_{\alpha\sharp}(\mathcal{H}_\alpha \mathbf{g}_{\alpha+1} - \mathcal{H}_\alpha \mathbf{g}_\alpha)) = 0,$$

$$443 \quad \lim_{\gamma \rightarrow 0} \Upsilon_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) = \lim_{\gamma \rightarrow 0} |(\mathcal{H}_\alpha \mathbf{g}_\alpha, (T_{\alpha+1\sharp} - T_{\alpha\sharp}) \mathcal{H}_\alpha \mathbf{g}_\alpha)| = 0.$$

444 • If $L \subset \tilde{\Gamma}_{\alpha+1}$, then $\mathbf{K}_{\alpha+1} \neq \mathbf{K}_\alpha$ on $\tilde{\Gamma}_{\alpha+1}$ while $[\mathbf{v}_{\alpha+1}] = [\mathbf{v}_\alpha] \in \tilde{H}^{1/2}(\Gamma_\alpha \cup \Gamma_\circ)$, and
445 $[\mathbf{v}_{\alpha+1}] = \mathbf{0}$ on $\hat{\Gamma}_{\alpha+1}$. Thus, $\lim_{\gamma \rightarrow 0} \mathcal{H}_\alpha \mathbf{g}_\alpha(L; \gamma) = T_\alpha^{-1}[\mathbf{v}_\alpha]$ and $\lim_{\gamma \rightarrow 0} \mathcal{H}_{\alpha+1} \mathbf{g}_{\alpha+1}(L; \gamma) =$
446 $\lim_{\gamma \rightarrow 0} \mathcal{H}_\alpha \mathbf{g}_{\alpha+1}(L; \gamma) = T_{\alpha+1}^{-1}[\mathbf{v}_{\alpha+1}]$. Note that here $T_\alpha^{-1}[\mathbf{v}_\alpha] \in H^{-1/2}(\Gamma_\alpha \cup \Gamma_\circ)$ and
447 $T_{\alpha+1}^{-1}[\mathbf{v}_{\alpha+1}] \in H^{-1/2}(\Gamma_{\alpha+1} \cup \Gamma_\circ)$ while $T_\alpha^{-1}[\mathbf{v}_\alpha] \neq T_{\alpha+1}^{-1}[\mathbf{v}_{\alpha+1}]$ and $T_{\alpha+1\sharp}(\cdot) \neq T_{\alpha\sharp}(\cdot)$.
448 Therefore,

$$449 \quad \begin{aligned} \lim_{\gamma \rightarrow 0} \Lambda_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) &= ((T_{\alpha+1}^{-1} - T_\alpha^{-1})[\mathbf{v}_\alpha], T_{\alpha\sharp}(T_{\alpha+1}^{-1} - T_\alpha^{-1})[\mathbf{v}_\alpha]) \\ &\leq \|T_{\alpha\sharp}\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_\alpha \cup \Gamma_\circ)} \|(T_{\alpha+1}^{-1} - T_\alpha^{-1})[\mathbf{v}_\alpha]\|_{H^{-\frac{1}{2}}(\Gamma_\alpha \cup \Gamma_\circ)}^2, \end{aligned}$$

$$\begin{aligned}
450 \quad & \lim_{\gamma \rightarrow 0} \Upsilon_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) = |(T_{\alpha+1}^{-1}[\mathbf{v}_\alpha], T_{\alpha+1\#} T_{\alpha+1}^{-1}[\mathbf{v}_\alpha]) - (T_\alpha^{-1}[\mathbf{v}_\alpha], T_{\alpha\#} T_\alpha^{-1}[\mathbf{v}_\alpha])| \\
& \leq (\|T_{\alpha+1\#}\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_\alpha \cup \Gamma_\circ)} \|T_{\alpha+1}^{-1}[\mathbf{v}_\alpha]\|_{H^{-\frac{1}{2}}(\Gamma_\alpha \cup \Gamma_\circ)}^2 + \|T_{\alpha\#}\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_\alpha \cup \Gamma_\circ)} \|T_\alpha^{-1}[\mathbf{v}_\alpha]\|_{H^{-\frac{1}{2}}(\Gamma_\alpha \cup \Gamma_\circ)}^2),
\end{aligned}$$

451 • If $L \subset \hat{\Gamma}_{\alpha+1}$, then $\lim_{\gamma \rightarrow 0} \mathcal{H}_{\alpha+1} \mathbf{g}_{\alpha+1}(L; \gamma) = T_{\alpha+1}^{-1}[\mathbf{v}_{\alpha+1}] \in H^{-1/2}(\Gamma_{\alpha+1} \cup \Gamma_\circ)$ while
452 $\lim_{\gamma \rightarrow 0} \mathcal{H}_\alpha \mathbf{g}_\alpha(L; \gamma) = \infty$. Invoking (4.19) and (3.12), one may observe

$$\begin{aligned}
453 \quad & \Lambda_\alpha(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) = (\mathcal{H}_\alpha \mathbf{g}_{\alpha+1} - \mathcal{H}_\alpha \mathbf{g}_\alpha, T_{\alpha\#}(\mathcal{H}_\alpha \mathbf{g}_{\alpha+1} - \mathcal{H}_\alpha \mathbf{g}_\alpha)) \\
& \geq c \|\mathcal{H}_\alpha \mathbf{g}_{\alpha+1} - \mathcal{H}_\alpha \mathbf{g}_\alpha\|_{H^{-\frac{1}{2}}(\Gamma_\circ \cup \Gamma_\alpha)}^2,
\end{aligned}$$

454 indicating that $\lim_{\gamma \rightarrow 0} \Lambda_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) = \infty$.

455 A similar argument results in $\lim_{\gamma \rightarrow 0} \Upsilon_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \gamma) = \infty$. ■

456 **Theorem 4.7** (invariants of the solution sequence for noisy data). *Define*

$$\begin{aligned}
457 \quad (4.21) \quad & \Lambda_\alpha^\delta(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) := (\mathbf{g}_{\alpha+1} - \mathbf{g}_\alpha, F_{\alpha\#}^\delta(\mathbf{g}_{\alpha+1} - \mathbf{g}_\alpha)) + \delta \|\mathbf{g}_{\alpha+1} - \mathbf{g}_\alpha\|^2, \\
& \Upsilon_\alpha^\delta(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) := |\Lambda_{\alpha+1}^\delta(\mathbf{g}_{\alpha+1}, \mathbf{0}) - \Lambda_\alpha^\delta(\mathbf{g}_\alpha, \mathbf{0})|, \quad \mathbf{g}_\alpha, \mathbf{g}_{\alpha+1} \in L^2(\Omega)^3,
\end{aligned}$$

458 where $(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})(L; \delta, \gamma)$ are the constructed minimizers of $(\mathfrak{J}_\alpha^{\delta, \gamma}, \mathfrak{J}_{\alpha+1}^{\delta, \gamma})$ in (4.8) according
459 to (4.9). Then,

460 • If $L \subset \Gamma_\alpha \cup \Gamma_\circ \setminus \tilde{\Gamma}_{\alpha+1}$ then

$$461 \quad \lim_{\gamma \rightarrow 0} \liminf_{\delta \rightarrow 0} \Lambda_\alpha^\delta[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta, \gamma) = 0 \quad \text{and} \quad \lim_{\gamma \rightarrow 0} \liminf_{\delta \rightarrow 0} \Upsilon_\alpha^\delta[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta, \gamma) = 0.$$

462 • If $L \subset \tilde{\Gamma}_{\alpha+1}$ then

$$463 \quad 0 < \lim_{\gamma \rightarrow 0} \liminf_{\delta \rightarrow 0} \Lambda_\alpha^\delta[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta, \gamma) < \infty \quad \text{and} \quad 0 < \lim_{\gamma \rightarrow 0} \liminf_{\delta \rightarrow 0} \Upsilon_\alpha^\delta[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta, \gamma) < \infty.$$

464 • If $L \subset \hat{\Gamma}_{\alpha+1}$ then

$$465 \quad \lim_{\gamma \rightarrow 0} \liminf_{\delta \rightarrow 0} \Lambda_\alpha^\delta[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta, \gamma) = \infty \quad \text{and} \quad \lim_{\gamma \rightarrow 0} \liminf_{\delta \rightarrow 0} \Upsilon_\alpha^\delta[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta, \gamma) = \infty.$$

466 *Proof.* If $L \subset \Gamma_\alpha \cup \Gamma_\circ$, then Theorem 4.3 and Theorem 4.5 indicate that there exists a
467 sequence $\delta(\gamma)$ such that

$$468 \quad \limsup_{\gamma \rightarrow 0} \|\mathcal{H}_\alpha \mathbf{g}_\alpha^{\delta(\gamma), \gamma} - T_\alpha^{-1}[\mathbf{v}_\alpha]\|^2 = 0, \quad \limsup_{\gamma \rightarrow 0} \|\mathcal{H}_{\alpha+1} \mathbf{g}_{\alpha+1}^{\delta(\gamma), \gamma} - T_{\alpha+1}^{-1}[\mathbf{v}_{\alpha+1}]\|^2 = 0.$$

469 From Theorem 4.6, one may then conclude that for $L \subset \Gamma_\alpha \cup \Gamma_\circ \setminus \tilde{\Gamma}_{\alpha+1}$,

$$470 \quad \lim_{\gamma \rightarrow 0} \Lambda_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta(\gamma), \gamma) = 0, \quad \lim_{\gamma \rightarrow 0} \Upsilon_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta(\gamma), \gamma) = 0,$$

471 and for $L \subset \tilde{\Gamma}_{\alpha+1}$,

$$472 \quad 0 < \lim_{\gamma \rightarrow 0} \Lambda_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta(\gamma), \gamma) < \infty, \quad 0 < \lim_{\gamma \rightarrow 0} \Upsilon_\alpha[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta(\gamma), \gamma) < \infty.$$

473 One may also observe that

$$\begin{aligned} 474 \quad & |[\Lambda_\alpha^\delta - \Lambda_\alpha](\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})| \leq 2\delta(\|\mathbf{g}_\alpha\|^2 + \|\mathbf{g}_{\alpha+1}\|^2), \\ & |[\Upsilon_\alpha^\delta - \Upsilon_\alpha](\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})| \leq 2\delta(\|\mathbf{g}_\alpha\|^2 + \|\mathbf{g}_{\alpha+1}\|^2). \end{aligned}$$

475 Invoking (4.12), one then deduces that

$$\begin{aligned} 476 \quad & \limsup_{\gamma \rightarrow 0} |(\Lambda_\alpha^{\delta(\gamma)} - \Lambda_\alpha)[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta(\gamma), \gamma)| = 0, \\ & \limsup_{\gamma \rightarrow 0} |(\Upsilon_\alpha^{\delta(\gamma)} - \Upsilon_\alpha)[\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}](L; \delta(\gamma), \gamma)| = 0, \end{aligned}$$

477 which completes the proof for the first two parts of the theorem. If $L \subset \hat{\Gamma}_{\alpha+1}$, then based on
478 (3.12), (4.6), (4.3), and (4.11), one may find that

$$\begin{aligned} 479 \quad & \Lambda_\alpha^\delta(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) \geq c_0 \|\mathcal{H}_\alpha \mathbf{g}_{\alpha+1} - \mathcal{H}_\alpha \mathbf{g}_\alpha\|_{H^{-\frac{1}{2}}(\Gamma_\circ \cup \Gamma_\alpha)}^2, \\ & \Upsilon_\alpha^\delta(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) \geq |c_1 \|\mathcal{H}_\alpha \mathbf{g}_\alpha\|_{H^{-\frac{1}{2}}(\Gamma_\circ \cup \Gamma_\alpha)}^2 - c_2 \|\mathcal{H}_{\alpha+1} \mathbf{g}_{\alpha+1}\|_{H^{-\frac{1}{2}}(\Gamma_\circ \cup \Gamma_{\alpha+1})}^2|, \end{aligned}$$

480 where $c_0, c_1, c_2 > 0$ are constants independent of $\mathcal{H}_\alpha \mathbf{g}_\alpha$. The theorem's statement will then
481 follow in light of

$$482 \quad \limsup_{\gamma \rightarrow 0} \limsup_{\delta \rightarrow 0} \|\mathcal{H}_{\alpha+1} \mathbf{g}_{\alpha+1}\|_{H^{-\frac{1}{2}}(\Gamma_\circ \cup \Gamma_{\alpha+1})}^2 < \infty, \quad \liminf_{\gamma \rightarrow 0} \liminf_{\delta \rightarrow 0} \|\mathcal{H}_\alpha \mathbf{g}_\alpha\|_{H^{-\frac{1}{2}}(\Gamma_\circ \cup \Gamma_\alpha)}^2 = \infty. \quad \blacksquare$$

483 Based on Theorems 4.5-7, the differential evolution indicators are introduced in the sequel.

484 **Differential evolution indicators for noise-free data.** Let us introduce the evolution
485 indicator functionals $I_\alpha^\mathcal{D} : L^2(\Omega^3) \times L^2(\Omega^3) \rightarrow \mathbb{R}$ and $\hat{I}_\alpha^\mathcal{D} : L^2(\Omega^3) \times L^2(\Omega^3) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} 486 \quad (4.22) \quad & I_\alpha^\mathcal{D}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) := \frac{1}{\sqrt{\Lambda_{\alpha+1}(\mathbf{0}, \mathbf{g}_{\alpha+1}) [1 + \Lambda_{\alpha+1}(\mathbf{0}, \mathbf{g}_{\alpha+1}) \mathcal{D}_\alpha^{-1}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})]}}, \\ & \hat{I}_\alpha^\mathcal{D}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) := \frac{1}{\sqrt{\Lambda_\alpha(\mathbf{g}_\alpha, \mathbf{0}) + \Lambda_{\alpha+1}(\mathbf{0}, \mathbf{g}_{\alpha+1}) [1 + \Lambda_\alpha(\mathbf{g}_\alpha, \mathbf{0}) \mathcal{D}_\alpha^{-1}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})]}} \end{aligned}$$

487 where $\mathcal{D}_\alpha \in \{\Lambda_\alpha, \Upsilon_\alpha\}$, and $(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})(L; \gamma) \in L^2(\Omega^3) \times L^2(\Omega^3)$ are the constructed minimizers
488 of $(\mathfrak{J}_\alpha^\gamma, \mathfrak{J}_{\alpha+1}^\gamma)$ in (4.1) according to (4.2). Then, it follows that

$$489 \quad \bullet \quad L \subset \tilde{\Gamma}_{\alpha+1} \cup \hat{\Gamma}_{\alpha+1} \iff \lim_{\gamma \rightarrow 0} I_\alpha^\mathcal{D}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})(L; \gamma) > 0.$$

$$490 \quad \bullet \quad L \subset \hat{\Gamma}_{\alpha+1} \iff \lim_{\gamma \rightarrow 0} \hat{I}_\alpha^\mathcal{D}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})(L; \gamma) > 0.$$

491 This may be observed (a) by invoking [Theorem 4.1](#) which reads $L \subset \Gamma_\circ \cup \Gamma_{\alpha+1}$ (resp. $L \subset$
492 $\Gamma_\circ \cup \Gamma_\alpha$) if and only if $\lim_{\gamma \rightarrow 0} \Lambda_{\alpha+1}(\mathbf{0}, \mathbf{g}_{\alpha+1})(L; \gamma) < \infty$ (resp. $\lim_{\gamma \rightarrow 0} \Lambda_\alpha(\mathbf{g}_\alpha, \mathbf{0})(L; \gamma) < \infty$), implying

493 that $\lim_{\gamma \rightarrow 0} I_\alpha^\mathcal{D}(\mathbf{0}, \mathbf{g}_{\alpha+1})(L; \gamma) = 0$ (resp. $\lim_{\gamma \rightarrow 0} \hat{I}_\alpha^\mathcal{D}(\mathbf{g}_\alpha, \mathbf{0})(L; \gamma) = 0$) when $L \subset \mathcal{B} \setminus \{\Gamma_\circ \cup \Gamma_{\alpha+1}\}$

494 (resp. $L \subset \mathcal{B} \setminus \{\Gamma_\circ \cup \Gamma_\alpha\}$), (b) in view of the first statement of [Theorem 4.6](#) which ensures that
495 for $L \subset \Gamma_\circ \cup \Gamma_\alpha \setminus \hat{\Gamma}_{\alpha+1}$, $\lim_{\gamma \rightarrow 0} I_\alpha^\mathcal{D}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})(L; \gamma) = 0$ and $\lim_{\gamma \rightarrow 0} \hat{I}_\alpha^\mathcal{D}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})(L; \gamma) = 0$, and (c)

496 by recalling the second and third statements of [Theorem 4.6](#). In other words, $\hat{I}_\alpha^\mathcal{D}$ illuminates
497 the support of *geometric* evolution between $[t_\alpha, t_{\alpha+1}]$ by achieving its highest values at the
498 loci of newly born interfaces $\hat{\Gamma}_{\alpha+1}$. However, $I_\alpha^\mathcal{D}$ more holistically reconstructs the support of

499 the micromechanical evolution which includes the new interstitial spaces $\hat{\Gamma}_{\alpha+1}$ as well as the
 500 pre-existing interfaces $\tilde{\Gamma}_{\alpha+1}$ whose elastic properties have changed between $[t_\alpha, t_{\alpha+1}]$ e.g. due
 501 to chemical reaction or micro-slip.

502 **Differential evolution indicators for noisy data.** Consider the indicator functionals
 503 $I_{\alpha,\delta}^{\mathcal{D}^\delta} : L^2(\Omega^3) \times L^2(\Omega^3) \rightarrow \mathbb{R}$ and $\hat{I}_{\alpha,\delta}^{\mathcal{D}^\delta} : L^2(\Omega^3) \times L^2(\Omega^3) \rightarrow \mathbb{R}$ such that

$$(4.23) \quad \begin{aligned} I_{\alpha,\delta}^{\mathcal{D}^\delta}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) &:= \frac{1}{\sqrt{\Lambda_{\alpha+1}^\delta(\mathbf{0}, \mathbf{g}_{\alpha+1}) [1 + \Lambda_{\alpha+1}^\delta(\mathbf{0}, \mathbf{g}_{\alpha+1}) \mathcal{D}_{\alpha,\delta}^{-1}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})]}}, \\ \hat{I}_{\alpha,\delta}^{\mathcal{D}^\delta}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1}) &:= \frac{1}{\sqrt{\Lambda_\alpha^\delta(\mathbf{g}_\alpha, \mathbf{0}) + \Lambda_{\alpha+1}^\delta(\mathbf{0}, \mathbf{g}_{\alpha+1}) [1 + \Lambda_\alpha^\delta(\mathbf{g}_\alpha, \mathbf{0}) \mathcal{D}_{\alpha,\delta}^{-1}(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})]}}, \end{aligned}$$

505 where $\mathcal{D}_{\alpha,\delta} \in \{\Lambda_\alpha^\delta, \Upsilon_\alpha^\delta\}$, and $(\mathbf{g}_\alpha, \mathbf{g}_{\alpha+1})(L; \delta, \gamma) = (\mathbf{g}_\alpha^{\delta,\gamma}, \mathbf{g}_{\alpha+1}^{\delta,\gamma}) \in L^2(\Omega^3) \times L^2(\Omega^3)$ are the
 506 constructed minimizers of $(\mathfrak{J}_\alpha^{\delta,\gamma}, \mathfrak{J}_{\alpha+1}^{\delta,\gamma})$ in (4.8) according to (4.9). Then, it follows that

$$\begin{aligned} 507 \quad &\bullet L \subset \tilde{\Gamma}_{\alpha+1} \cup \hat{\Gamma}_{\alpha+1} \iff \lim_{\gamma \rightarrow 0} \liminf_{\delta(\gamma) \rightarrow 0} I_{\alpha,\delta}^{\mathcal{D}^\delta}(\mathbf{g}_\alpha^{\delta,\gamma}, \mathbf{g}_{\alpha+1}^{\delta,\gamma}) > 0. \\ 508 \quad &\bullet L \subset \hat{\Gamma}_{\alpha+1} \iff \lim_{\gamma \rightarrow 0} \liminf_{\delta(\gamma) \rightarrow 0} \hat{I}_{\alpha,\delta}^{\mathcal{D}^\delta}(\mathbf{g}_\alpha^{\delta,\gamma}, \mathbf{g}_{\alpha+1}^{\delta,\gamma}) > 0. \end{aligned}$$

509 This may be established on the basis of (a) [Theorem 4.3](#) which reads $L \subset \Gamma_\circ \cup \Gamma_{\alpha+1}$ (resp.
 510 $L \subset \Gamma_\circ \cup \Gamma_\alpha$) if and only if $\lim_{\gamma \rightarrow 0} \liminf_{\delta(\gamma) \rightarrow 0} \Lambda_{\alpha+1}^\delta(\mathbf{0}, \mathbf{g}_{\alpha+1}^{\delta,\gamma}) < \infty$ (resp. $\lim_{\gamma \rightarrow 0} \liminf_{\delta(\gamma) \rightarrow 0} \Lambda_\alpha^\delta(\mathbf{g}_\alpha^{\delta,\gamma}, \mathbf{0}) < \infty$),
 511 implying that $\lim_{\gamma \rightarrow 0} \liminf_{\delta(\gamma) \rightarrow 0} I_{\alpha,\delta}^{\mathcal{D}^\delta}(\mathbf{0}, \mathbf{g}_{\alpha+1}^{\delta,\gamma}) = 0$ (resp. $\lim_{\gamma \rightarrow 0} \liminf_{\delta(\gamma) \rightarrow 0} \hat{I}_{\alpha,\delta}^{\mathcal{D}^\delta}(\mathbf{g}_\alpha^{\delta,\gamma}, \mathbf{0}) = 0$) when $L \subset$
 512 $\mathcal{B} \setminus \{\Gamma_\circ \cup \Gamma_{\alpha+1}\}$ (resp. $L \subset \mathcal{B} \setminus \{\Gamma_\circ \cup \Gamma_\alpha\}$), (b) first statement of [Theorem 4.7](#) which ensures that
 513 for $L \subset \Gamma_\circ \cup \Gamma_\alpha \setminus \tilde{\Gamma}_{\alpha+1}$, $\lim_{\gamma \rightarrow 0} \liminf_{\delta(\gamma) \rightarrow 0} I_{\alpha,\delta}^{\mathcal{D}^\delta}(\mathbf{g}_\alpha^{\delta,\gamma}, \mathbf{g}_{\alpha+1}^{\delta,\gamma}) = 0$ and $\lim_{\gamma \rightarrow 0} \liminf_{\delta(\gamma) \rightarrow 0} \hat{I}_{\alpha,\delta}^{\mathcal{D}^\delta}(\mathbf{g}_\alpha^{\delta,\gamma}, \mathbf{g}_{\alpha+1}^{\delta,\gamma}) = 0$,
 514 and (c) second and third statements of [Theorem 4.7](#). In other words, $\hat{I}_{\alpha,\delta}^{\mathcal{D}^\delta}$ illuminates the
 515 support of *geometric* evolution between $[t_\alpha, t_{\alpha+1}]$ by achieving its highest values at the loci
 516 of newly born interfaces $\hat{\Gamma}_{\alpha+1}$. However, $I_{\alpha,\delta}^{\mathcal{D}^\delta}$ reconstructs the support of the evolution more
 517 comprehensively including the new interfacial spaces $\hat{\Gamma}_{\alpha+1}$ as well as the pre-existing interfaces
 518 $\tilde{\Gamma}_{\alpha+1}$ whose elastic properties have changed between $[t_\alpha, t_{\alpha+1}]$.

519 **5. Implementation and results.** To illustrate the theoretical developments, this section
 520 examines the performance of differential evolution indicators (4.22) and (4.23) through a set
 521 of numerical experiments and compares the results to those obtained by the generalized linear
 522 sampling method [35]. In what follows the synthetic sensory data, namely the scattered fields
 523 \mathbf{v}_α at sensing steps $t_\alpha = \{t_\circ, t_1, t_2, \dots\}$, are simulated by a computational platform based on
 524 the elastodynamic boundary integral equations, see [33, 32, 9] for details of the computational
 525 method.

526 **5.1. Testing configuration.** Two test setups are considered as illustrated in [Figure 3](#)
 527 and [Figure 4](#) where an elastic plate of dimensions $3 \times 3 \times 0.02$ is endowed with (I) a randomly
 528 cracked damage zone, and (II) a pore zone. The shear modulus, mass density, and Poisson's
 529 ratio of the plate are taken as $\mu = 1$, $\rho = 1$ and $\nu = 0.25$, whereby the shear and compressional
 530 wave speeds read $c_s = 1$ and $c_p = 1.73$. In *Setup I*, shown in [Figure 3](#), the damage zone is
 531 comprised of randomly distributed cracks $\Gamma_1 - \Gamma_{24}$ evolving hidden within the thickness of the
 532 specimen in seven time steps $t_1 - t_7$. A detailed description of scatterers including the center

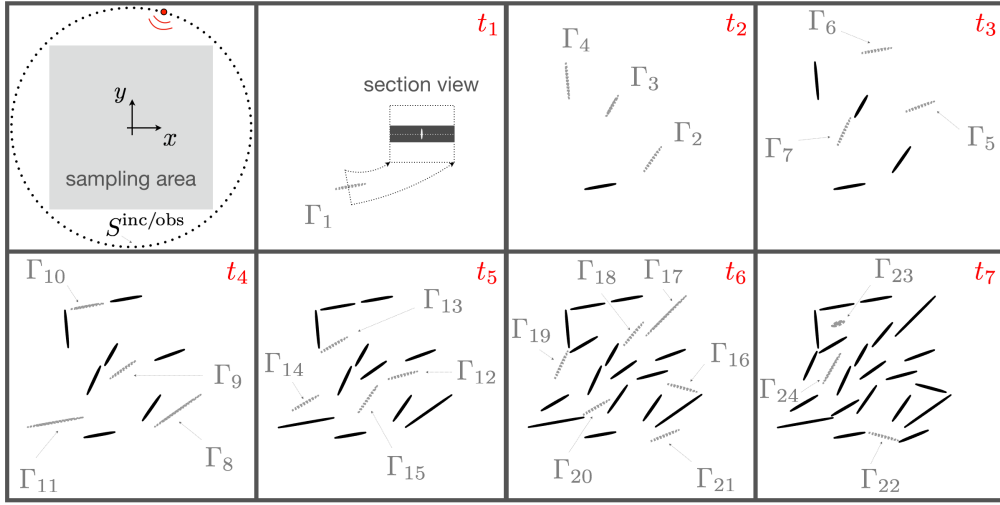


Figure 3. Sensing configuration of synthetic sequential experiments on an elastic plate (top left), featuring a damage zone comprised of randomly distributed cracks $\Gamma_1 - \Gamma_{24}$ evolving in seven time steps ($t_1 - t_7$) within the thickness of the specimen according to the sectional view shown at t_1 .

533 (x_c, y_c) , length ℓ , and orientation ϕ (with respect to x axis) of each crack Γ_κ , $\kappa = \{1, 2, \dots, 24\}$
 534 is provided in Table 1. All fractures in this configuration are traction-free i.e., the interfacial
 535 stiffness $\mathbf{K}(\boldsymbol{\xi}) = \mathbf{0}$ on $\boldsymbol{\xi} \in \bigcup_{\kappa=1}^{24} \Gamma_\kappa$. In Setup II, depicted in Figure 4, a bubble zone is growing
 536 within the plate thickness, comprised of randomly distributed pores $\Pi_1 - \Pi_{21}$ developing in
 537 seven time steps $t_1 - t_7$. A detailed description of the specimen including the center (x_c, y_c)
 538 and radius r of each pore Π_κ , $\kappa = \{1, 2, \dots, 21\}$ is provided in Table 2.

Table 1

Damage zone configuration illustrated in Figure 3: center (x_c, y_c) , length ℓ , and orientation ϕ (with respect to x axis) of cracks Γ_κ , $\kappa = \{1, 2, \dots, 24\}$.

κ	1	2	3	4	5	6	7	8	9	10	11	12
$x_c(\Gamma_\kappa)$	-0.33	0.21	-0.21	-0.68	0.4	-0.05	-0.39	0.49	-0.09	-0.46	-0.8	0.21
$y_c(\Gamma_\kappa)$	-0.62	-0.34	0.22	0.49	0.21	0.8	-0.05	-0.37	0.06	0.72	-0.5	0
$\ell(\Gamma_\kappa)$	1/3	1/3	1/4	2/5	1/3	1/3	1/3	3/5	1/3	2/5	3/5	1/3
$\phi(\Gamma_\kappa)$	$\pi/18$	$11\pi/36$	$\pi/3$	$19\pi/36$	$\pi/9$	$\pi/18$	$13\pi/36$	$7\pi/36$	$7\pi/36$	$\pi/18$	$\pi/18$	$\pi/12$
κ	13	14	15	16	17	18	19	20	21	22	23	24
$x_c(\Gamma_\kappa)$	-0.5	-0.8	-0.15	0.52	0.36	0.01	-0.74	-0.38	0.34	0.02	-0.45	-0.51
$y_c(\Gamma_\kappa)$	0.32	-0.29	-0.25	-0.13	0.62	0.43	0.1	-0.34	-0.63	-0.64	0.55	0.08
$\ell(\Gamma_\kappa)$	1/3	1/3	7/20	1/3	3/5	1/3	1/3	1/3	1/3	1/3	1/7	1/3
$\phi(\Gamma_\kappa)$	$\pi/6$	$\pi/6$	$11\pi/36$	$-\pi/12$	$\pi/4$	$5\pi/18$	$13\pi/36$	$\pi/6$	$11\pi/90$	$-\pi/12$	$\pi/9$	$\pi/3$

539 **5.2. Forward scattering simulations.** Numerical experiments are conducted in seven steps
 540 at $t = \{t_1, t_2, \dots, t_7\}$ when the specimen assumes the associated configurations shown in Fig-
 541 ure 3 and Figure 4 ($t_1 - t_7$). Every sensing step entails in-plane harmonic excitation at a set
 542 of source points residing on the incident grid S^{inc} . The excitation frequency $\omega = 72$ rad/s is

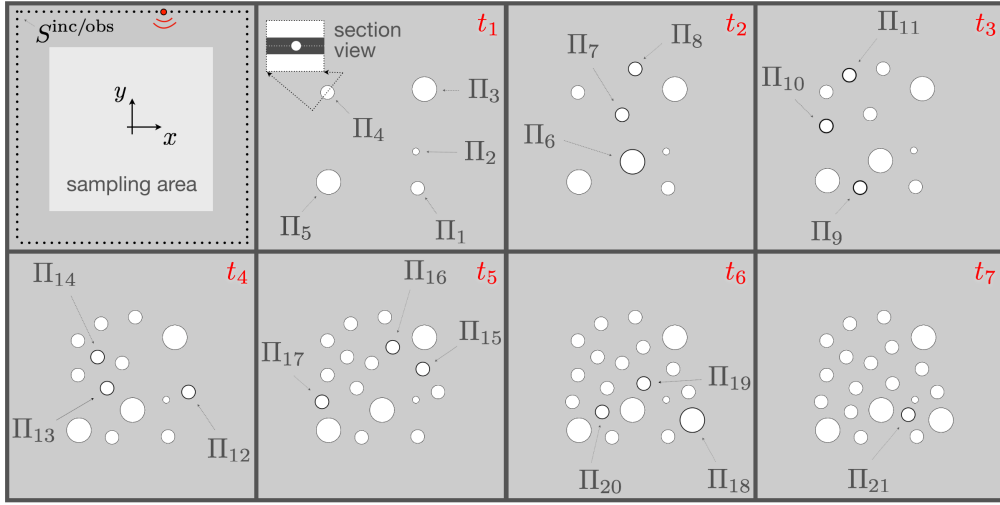


Figure 4. Sensing configuration of synthetic sequential experiments on an elastic plate (top left), featuring a bubble zone comprised of randomly distributed pores $\Pi_1 - \Pi_{21}$ evolving in seven time steps ($t_1 - t_7$) within the thickness of the specimen according to the sectional view shown at t_1 .

Table 2

Bubble zone configuration illustrated in [Figure 4](#): center (x_c, y_c) and radius r of bubbles Π_κ , $\kappa = \{1, 2, \dots, 21\}$.

κ	1	2	3	4	5	6	7	8	9	10	11
$x_c(\Pi_\kappa)$	0.42	0.4	0.51	-0.66	-0.65	0	-0.13	0.03	-0.25	-0.66	-0.38
$y_c(\Pi_\kappa)$	-0.72	-0.27	0.48	0.44	-0.64	-0.4	0.17	0.72	-0.73	0.02	0.64
$r(\Pi_\kappa)$	0.08	0.04	0.15	0.08	0.15	0.15	0.08	0.08	0.08	0.08	0.08
κ	12	13	14	15	16	17	18	19	20	21	
$x_c(\Pi_\kappa)$	0.67	-0.31	-0.42	0.49	0.12	-0.73	0.72	0.13	-0.37	0.32	
$y_c(\Pi_\kappa)$	-0.18	-0.13	0.24	0.09	0.36	-0.3	-0.52	-0.08	-0.42	-0.45	
$r(\Pi_\kappa)$	0.08	0.08	0.08	0.08	0.08	0.08	0.15	0.08	0.08	0.08	

543 set such that the induced shear wavelength λ_s in the specimen is approximately 0.08, giving
 544 a shear-wavelength-to-specimen-thickness ratio of about 4. In this setting, the phase error
 545 committed by the plane-stress approximation for the wave motion is less than 3% [34]. The
 546 incident wave interacts with both pre-existing and newly born scatterers at each t_α giving
 547 rise to the scattered field \mathbf{v}_α , governed by (2.4) in Setup I – whose pattern $\mathbf{v}_\alpha^{\text{obs}}$ over the
 548 observation grid S^{obs} is then computed. It must be mentioned that since the scatterers are
 549 buried within the plate thickness – see the sectional views in [Figure 3](#) and [Figure 4](#), our simu-
 550 lations are performed in three dimensions via an elastodynamic code rooted in the boundary
 551 element method [9, 32]. However, the nontrivial components of the computed scattered fields
 552 lay in the $x - y$ plane, as expected in light of the earlier remarks. To study the sensitivity of
 553 evolution indicators to sensing arrangement, the incident/observation grid in *Setup I* is a circle
 554 of radius 1.45 in the mid-section of the plate, while the support of $S^{\text{inc/obs}}$ in *Setup II* is the
 555 external boundary of the specimen i.e., a square of size 3×3 .

556 **5.3. Data Inversion.** With the preceding data, one may generate the evolution indicator
 557 maps affiliated with (4.23) in three steps, namely by: (1) constructing the discrete scattering

558 operators \mathbf{F}_α and \mathbf{F}_α^δ from synthetic data related to every sensing step t_α , (2) computing the
 559 trial signature patterns Φ_L pertinent to a finite host domain, and (3) evaluating the differential
 560 evolution indicator in the sampling area through careful minimization of the discretized cost
 561 functional (4.8) as elucidated in the sequel.

562 **Step 1: construction of the discrete scattering operator.** For both illumination and
 563 sensing purposes, $S^{\text{inc/obs}}$ is sampled by a uniform grid of N excitation and observation points.
 564 Given the in-plane nature of wave motion, i.e., that the polarization amplitude of excitation
 565 \mathbf{q} and the nontrivial components of associated scattered fields \mathbf{v}_α lay in the $x - y$ plane of
 566 orthonormal bases $(\mathbf{e}_1, \mathbf{e}_2)$, the discretized scattering operator \mathbf{F}_α may be represented by a
 567 $2N \times 2N$ matrix with components

$$568 \quad (5.1) \quad \mathbf{F}_\alpha(2k+1:2k+2, 2j+1:2j+2) = \begin{bmatrix} W_\alpha^{11} & W_\alpha^{12} \\ W_\alpha^{21} & W_\alpha^{22} \end{bmatrix} (\mathbf{x}_j, \boldsymbol{\xi}_k), \quad j, k = 0, \dots, N-1,$$

569 where $W_\alpha^{\ell v}(\mathbf{x}_j, \boldsymbol{\xi}_k)$ ($\ell, v = 1, 2$) is the ℓ^{th} component of the displacement field measured at
 570 $\boldsymbol{\xi}_k \in S^{\text{obs}}$ due to a unit harmonic excitation applied at $\mathbf{x}_j \in S^{\text{inc}}$ along the coordinate
 571 direction v such that

$$572 \quad \tilde{\mathbf{v}}_\alpha(\boldsymbol{\xi}_k) = \begin{bmatrix} \mathbf{v}_\alpha \cdot \mathbf{e}_1 \\ \mathbf{v}_\alpha \cdot \mathbf{e}_2 \end{bmatrix}(\boldsymbol{\xi}_k) = \mathbf{W}_\alpha(\mathbf{x}_j, \boldsymbol{\xi}_k) \times \begin{bmatrix} \mathbf{q} \cdot \mathbf{e}_1 \\ \mathbf{q} \cdot \mathbf{e}_2 \end{bmatrix}(\mathbf{x}_j).$$

573 Unless stated otherwise, we assume $N = 500$.

574 *Noisy data.* To account for the presence of noise in measurements, we consider the per-
 575 turbed operators

$$576 \quad (5.2) \quad \mathbf{F}_\alpha^\delta := (\mathbf{I} + \mathbf{N}_\epsilon) \mathbf{F}_\alpha,$$

577 where \mathbf{I} is the $2N \times 2N$ identity matrix, and \mathbf{N}_ϵ is the noise matrix of commensurate dimension
 578 whose components are uniformly-distributed (complex) random variables in $[-\epsilon, \epsilon]^2$. In what
 579 follows, the measure of noise in data with reference to definition (4.6) is $\delta = \|\mathbf{N}_\epsilon \mathbf{F}_\alpha\| = 0.05$.

580 **Step 2: A physics-based library of trial patterns.** This step aims to construct a suitable
 581 right hand side for the discretized far field equation in bounded and unbounded domains
 582 pertinent to the numerical experiments of this section and analytical developments of section 3,
 583 respectively.

584 *Unbounded domain in \mathbb{R}^3 .* In this case, the trial far-field pattern $\Phi_L^\infty \in L^2(S^2)$ is given
 585 by (3.13) indicating that (a) the right hand side is not only a function of the dislocation
 586 geometry L but also a function of the trial opening displacement profile \mathbf{a} , and (b) computing
 587 Φ_L^∞ generally requires an integration process at every sampling point \mathbf{x}_o . In an unbounded
 588 domain, however, one may dispense with the integration process by considering a sufficiently
 589 localized (trial) FOD such as $\mathbf{a}(\mathbf{y}) = \delta(\mathbf{y} - \mathbf{x}_o) |\mathbf{L}|^{-1} \mathbf{R} \mathbf{n}_o$. In this setting, without loss of
 590 generality, the dislocation support may be interpreted as an infinitesimal crack $L = \mathbf{x}_o + \mathbf{R} \mathbf{L}$
 591 where \mathbf{R} is a unitary rotation matrix, and \mathbf{L} represents a vanishing penny-shaped crack of unit
 592 normal $\mathbf{n}_o := \{0, 0, 1\}$. Thus, on denoting $\mathbf{n} = \mathbf{R} \mathbf{n}_o$, (3.13) may be recast as
 593 (5.3)

$$593 \quad \Phi_L^\infty(\hat{\boldsymbol{\xi}}) = -ik_p \hat{\boldsymbol{\xi}} [\lambda + 2\mu(\mathbf{n} \cdot \hat{\boldsymbol{\xi}})^2] e^{-ik_p \hat{\boldsymbol{\xi}} \cdot \mathbf{x}_o} \oplus -2i\mu k_s \hat{\boldsymbol{\xi}} \times (\mathbf{n} \times \hat{\boldsymbol{\xi}}) (\mathbf{n} \cdot \hat{\boldsymbol{\xi}}) e^{-ik_s \hat{\boldsymbol{\xi}} \cdot \mathbf{x}_o}, \quad \hat{\boldsymbol{\xi}} \in \Omega, \mathbf{x}_o \in \mathbb{R}^3.$$

594 *Bounded domain.* This case corresponds to the numerical experiments of this section
 595 where the background is an elastic plate P of finite dimensions, bearing direct relevance to

596 potential application of differential imaging to additive manufacturing and non-destructive
 597 evaluation where the target domain i.e., real-life specimen is bounded. In this setting, it
 598 is straightforward to rigorously show that the associated patterns Φ_L for a finite domain is
 599 governed by

$$\begin{aligned} \nabla \cdot (\mathbf{C} : \nabla \Phi_L) + \rho \omega^2 \Phi_L &= \mathbf{0} && \text{in } P \setminus L, \\ \mathbf{n} \cdot \mathbf{C} : \nabla \Phi_L &= \mathbf{0} && \text{on } \partial P, \\ \llbracket \Phi_L \rrbracket &= \mathbf{a} && \text{on } L. \end{aligned} \quad (5.4)$$

601 In what follows, the trial signatures $\mathbf{v}_{\mathbf{x}_o, \mathbf{n}}(\boldsymbol{\xi}_k)$ over the observation grid $\boldsymbol{\xi}_k \in S^{\text{obs}}$ are
 602 computed separately for every sampling point $\mathbf{x}_o \in P$ by solving

$$\begin{aligned} \nabla \cdot (\mathbf{C} : \nabla \mathbf{v}_{\mathbf{x}_o, \mathbf{n}}) + \rho \omega^2 \mathbf{v}_{\mathbf{x}_o, \mathbf{n}} &= \mathbf{0} && \text{in } P \setminus L, \\ \mathbf{n} \cdot \mathbf{C} : \nabla \mathbf{v}_{\mathbf{x}_o, \mathbf{n}} &= \delta(\boldsymbol{\xi} - \mathbf{x}_o) |L|^{-1} \mathbf{R} \mathbf{n}_o && \text{on } \mathbf{x}_o + \mathbf{R}L, \\ \mathbf{n} \cdot \mathbf{C} : \nabla \mathbf{v}_{\mathbf{x}_o, \mathbf{n}} &= \mathbf{0} && \text{on } \partial P, \end{aligned} \quad (5.5)$$

604 within the same computational platform mentioned earlier using the boundary element method [9,
 605 32]. On recalling (5.1), note that for every sensing point $\boldsymbol{\xi}_k$, $\mathbf{v}_{\mathbf{x}_o, \mathbf{n}}$ has only two non-trivial
 606 components in the $x - y$ plane, with orthonormal bases $(\mathbf{e}_1, \mathbf{e}_2)$, which are arranged into a
 607 $2N \times 1$ vector as the following

$$\Phi_{\mathbf{x}_o, \mathbf{n}}(2k + 1 : 2k + 2) = \begin{bmatrix} \mathbf{v}_{\mathbf{x}_o, \mathbf{n}} \cdot \mathbf{e}_1 \\ \mathbf{v}_{\mathbf{x}_o, \mathbf{n}} \cdot \mathbf{e}_2 \end{bmatrix}(\boldsymbol{\xi}_k), \quad k = 0, \dots, N - 1. \quad (5.6)$$

609 *Sampling.* With reference to Figure 3 and Figure 4, the search area i.e., the sampling region
 610 is a square $[-0.8, 0.8]^2 \subset P$ probed by a uniform 100×100 grid of sampling points \mathbf{x}_o where
 611 the featured evolution indicator functionals are evaluated, while the unit circle – spanning
 612 possible orientations for trial dislocation L – is sampled by a 72 grid of trial normal directions
 613 $\mathbf{n} = \mathbf{R} \mathbf{n}_o$. Accordingly, the evolution indicator map is constructed through minimizing (4.8)
 614 for a total of $M = 10000 \times 36$ trial pairs $(\mathbf{x}_o, \mathbf{n})$.

615 *Remark 5.1.* It is worth mentioning that the scattering operators \mathbf{F}_α^δ – constructed from
 616 the forward scattering simulations of Step 1 at every sensing step t_α , is independent of any par-
 617 ticular choice of $L(\mathbf{x}_o, \mathbf{n})$, and thus, remain the same for all M variations of $\Phi_{\mathbf{x}_o, \mathbf{n}}$. Moreover,
 618 the right hand side of the scattering equation

$$\mathbf{F}_\alpha^\delta \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta} = \Phi_{\mathbf{x}_o, \mathbf{n}}, \quad (5.7)$$

620 is invariant with respect to the sensing step t_α . Therefore, for computational efficiency, one
 621 may construct a $2N \times M$ matrix that may be interpreted as a library of physically admissible
 622 patterns as the right hand side of (3.14) – encompassing all choices of $L(\mathbf{x}_o, \mathbf{n})$, and solve only
 623 one equation to construct the entire imaging indicator map at every t_α .

624 **Step 3: Differential indicators of evolution.** A critical observation is that the scattering
 625 equation (5.7) is highly ill-posed at all sensing steps in that $\det(\mathbf{F}_\alpha^\delta) = 0$. This problem may
 626 arise from (a) highly nonlinear nature of the inverse problem, (b) limited incident and/or
 627 “viewing” aperture furnished by $S^{\text{inc/obs}}$, and (c) the emergence of local (e.g., interfacial)
 628 scattered waves – propagating in a neighborhood of certain scatterers and boundaries [37]
 629 – whose footprint cannot be sensed on S^{obs} . Accordingly, (5.7) will be solved via a careful

630 regularization process via the cost functional (4.8). In this setting, it is rigorously shown in
 631 section 3 that cost functionals of type $J_\alpha^{\delta,\gamma}$ are convex and their minimizer can be obtained
 632 without iteration. In this vein, the discretized minimizer $\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}$ of (4.8) at each sensing step
 633 t_α is computed by by invoking (5.1), (5.6), and (5.7) via

$$634 \quad (5.8) \quad \left(\mathbf{F}_\alpha^{\delta*} \mathbf{F}_\alpha^\delta + \gamma_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta} (\mathbf{F}_{\alpha_\#}^\delta)^{\frac{1}{2}*} (\mathbf{F}_{\alpha_\#}^\delta)^{\frac{1}{2}} + \delta \gamma_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta} \mathbf{I}_{2N \times 2N} \right) \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta} = \mathbf{F}_\alpha^{\delta*} \Phi_{\mathbf{x}_o, \mathbf{n}},$$

635 where $(\cdot)^*$ is the Hermitian operator; $\mathbf{F}_{\alpha_\#}^\delta$ is evaluated on the basis of definitions (3.10)
 636 and (5.1); and, following [32],

$$637 \quad (5.9) \quad \gamma_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta} := \frac{\eta_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}}{\|\mathbf{F}_\alpha^\delta\| + \delta}.$$

638 Here $\eta_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}$ is a regularization parameter computed via the Morozov discrepancy principle [25].
 639 As a result, $\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}$ is a $2N \times 1$ vector (or $2N \times M$ matrix for all the constructed right hand
 640 sides) identifying the structure of source densities at sensing step t_α . On repeating (5.8) for
 641 all sensing steps i.e., $\alpha = \{0, 1, 2, \dots\}$, one obtains all the arguments needed to construct a the
 642 differential evolution indicator maps.

643 Next, one may compute the invariant functionals

$$644 \quad (5.10) \quad \begin{aligned} \Lambda^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta}) &= (\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta} - \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{F}_{\alpha_\#}^\delta (\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta} - \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta})) + \delta \|\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta} - \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}\|^2, \\ \Upsilon^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta}) &= |\Lambda^{\alpha+1, \delta}(\mathbf{0}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta}) - \Lambda^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{0})|. \end{aligned}$$

645 Whereby, the differential evolution indicators may be computed as follows
 (5.11)

$$646 \quad \begin{aligned} \mathcal{I}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta}) &= \frac{1}{\sqrt{\Lambda^{\alpha+1, \delta}(\mathbf{0}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta}) [1 + \Lambda^{\alpha+1, \delta}(\mathbf{0}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta}) \mathbf{D}_{\alpha, \delta}^{-1}(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta})]}}, \\ \hat{\mathcal{I}}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta}) &= \frac{1}{\sqrt{\Lambda^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{0}) + \Lambda^{\alpha+1, \delta}(\mathbf{0}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta}) [1 + \Lambda^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{0}) \mathbf{D}_{\alpha, \delta}^{-1}(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta})]}}}, \end{aligned}$$

647 where $\mathbf{D}_{\alpha, \delta} \in \{\Lambda^{\alpha, \delta}, \Upsilon^{\alpha, \delta}\}$. Upon introducing
 (5.12)

$$648 \quad (\mathbf{g}_{\mathbf{x}_o}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o}^{\alpha+1, \delta}) := \operatorname{argmin}_{(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta})} (\mathcal{I}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}), \quad (\hat{\mathbf{g}}_{\mathbf{x}_o}^{\alpha, \delta}, \hat{\mathbf{g}}_{\mathbf{x}_o}^{\alpha+1, \delta}) := \operatorname{argmin}_{(\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha+1, \delta})} (\hat{\mathcal{I}}_{\mathbf{x}_o, \mathbf{n}}^{\alpha, \delta}),$$

649 one obtains the evolution indicator maps
 (5.13)

$$650 \quad \begin{aligned} \mathcal{I}_{\mathbf{x}_o}^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o}^{\alpha+1, \delta}) &= \frac{1}{\sqrt{\Lambda^{\alpha+1, \delta}(\mathbf{0}, \mathbf{g}_{\mathbf{x}_o}^{\alpha+1, \delta}) [1 + \Lambda^{\alpha+1, \delta}(\mathbf{0}, \mathbf{g}_{\mathbf{x}_o}^{\alpha+1, \delta}) \mathbf{D}_{\alpha, \delta}^{-1}(\mathbf{g}_{\mathbf{x}_o}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o}^{\alpha+1, \delta})]}}, \\ \hat{\mathcal{I}}_{\mathbf{x}_o}^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o}^{\alpha+1, \delta}) &= \frac{1}{\sqrt{\Lambda^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o}^{\alpha, \delta}, \mathbf{0}) + \Lambda^{\alpha+1, \delta}(\mathbf{0}, \mathbf{g}_{\mathbf{x}_o}^{\alpha+1, \delta}) [1 + \Lambda^{\alpha, \delta}(\mathbf{g}_{\mathbf{x}_o}^{\alpha, \delta}, \mathbf{0}) \mathbf{D}_{\alpha, \delta}^{-1}(\mathbf{g}_{\mathbf{x}_o}^{\alpha, \delta}, \mathbf{g}_{\mathbf{x}_o}^{\alpha+1, \delta})]}}}, \end{aligned}$$

651 Here, $\mathcal{I}_{\mathbf{x}_o}^{\alpha, \delta}$ and $\hat{\mathcal{I}}_{\mathbf{x}_o}^{\alpha, \delta}$ canvas the support of evolution between two sensing steps α and
 652 $\alpha + 1$. More specifically, $\hat{\mathcal{I}}_{\mathbf{x}_o}^{\alpha, \delta}$ assumes (a) its highest values at those sampling points that
 653 meet the newly developed or evolved scatterers between t_α and $t_{\alpha+1}$, and (b) near zero
 654 values everywhere else including the loci of (unknown) stationary scatterers within $[t_\alpha, t_{\alpha+1}]$
 655 timeframe.

656 *Remark 5.2.* To gain better insight into the effectiveness of the proposed approach, the
 657 reconstructed evolution indicators will be compared to their corresponding GLSM map. With
 658 reference to (4.16), the latter can be computed via

$$659 \quad (5.14) \quad \mathbf{I}_{\mathbf{x}_o}^{\mathcal{G}^{\alpha,\delta}} = \frac{1}{\sqrt{\|(\mathbf{F}_\#^\delta)^{\frac{1}{2}} \mathbf{g}_{\mathbf{x}_o}^{\alpha,\delta}\|^2 + \delta \|\mathbf{g}_{\mathbf{x}_o}^{\alpha,\delta}\|^2}}, \quad \mathbf{g}_{\mathbf{x}_o}^{\alpha,\delta} := \operatorname{argmin}_{\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha,\delta}} \|\mathbf{g}_{\mathbf{x}_o, \mathbf{n}}^{\alpha,\delta}\|_{L^2}^2 .$$

660 **5.4. Simulation results.** The synthetic scattered data deployed to generated the ensuing
 661 results are perturbed by $\delta = 5\%$ white noise. Figure 5 illustrates the full-aperture GLSM
 662 reconstruction of a progressive damage zone using scattered field data computed in seven
 663 sensing steps $t_\alpha = \{t_1, t_2, \dots, t_7\}$. Every panel in Figure 5 demonstrates the distribution of
 664 indicator functional $\mathbf{I}_{\mathbf{x}_o}^{\mathcal{G}^{\alpha,\delta}}$ over the indicated sampling area. Note that the GLSM functional
 665 at every t_α is solely dependent upon the far field data at that step $\mathbf{v}_\alpha^{\text{obs}}$ i.e., this imaging
 666 indicator does not require knowledge of the sequential sensory data ($\mathbf{v}_{\alpha+1}^{\text{obs}}$ or $\mathbf{v}_{\alpha-1}^{\text{obs}}$). Figure 8
 667 shows parallel results pertaining to an evolving bubble zone related to the second set numerical
 668 experiments depicted in Figure 4.

669 Figure 6 illustrates the sequential reconstruction of newborn fractures emerging between
 670 every pair of successive sensing steps $t_\alpha - t_{\alpha+1}$ for $\alpha = \{1, 2, \dots, 6\}$ by way of the proposed
 671 differential evolution indicators. Each panel in Figure 6 demonstrates the distribution of
 672 evolution indicator functional $\mathbf{I}_{\mathbf{x}_o}^{\mathcal{G}^{\alpha,\delta}}$ over the sampling area. According to (5.13), evaluation of
 673 the differential indicators requires pairs of sensory data in the form of ($\mathbf{v}_\alpha^{\text{obs}}, \mathbf{v}_{\alpha+1}^{\text{obs}}$). Figure 9
 674 shows similar plots corresponding to an evolving bubble zone.

675 Figure 7 and Figure 10 compare the performance of existing GLSM criteria with that of
 676 the proposed differential evolution indicators in reconstructing highly scattering damage and
 677 bubble zones where the distance between scatterers is of the order of a fraction of the shear
 678 wavelength. Here, the GLSM maps of damage/bubble zone at t_7 are depicted against that
 679 assembled map of fractures (*resp.* pores) constructed by averaging the differential indicator
 680 maps shown in Figure 6 (*resp.* Figure 9). Note that the pre-existing bubbles shown in Fig-
 681 ure 4 (t_1), and reconstructed in Figure 8 (t_1), do not appear in the evolution indicator map
 682 of Figure 10 (b) depicting the reconstructed evolution of pore zone between $t_1 - t_7$.

683 **6. Conclusion.** A robust framework for waveform tomography of *progressive evolution*
 684 in highly scattering materials of uncertain structure is developed. Our three-tier imaging
 685 platform is inherently non-iterative enabling fast inversion of dense data in support of the
 686 real-time sensing. In addition, this method allows for concurrent reconstruction of multiple
 687 interfacial and volumetric scatterers of arbitrary geometry. This imaging solution enables
 688 targeted characterization of active zones with little sensitivity to the noise in measurements
 689 while remaining systematically agnostic with respect to uncertainties of the host domain.
 690 As a perspective it would be interesting to incorporate a multi-frequency framework that
 691 would enable multiscale characterization of evolution without the need to reconstruct the
 692 entire domain across pertinent scales which may be practically insurmountable. The proposed
 693 imaging solution is formulated in a generic framework whose validity is rigorously established.
 694 As a result, this method may potentially contribute to a wide range of civil, aerospace and
 695 mechanical systems.

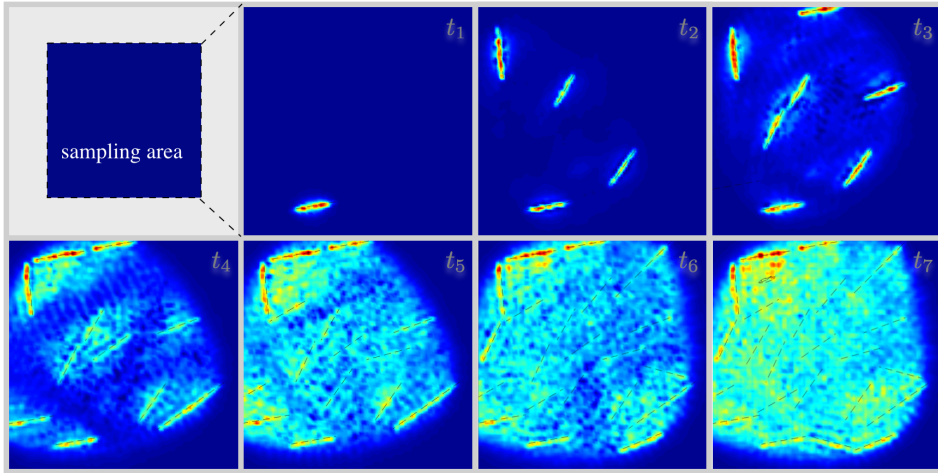


Figure 5. Disjoint reconstruction of an evolving damage zone via the existing indicator $\mathbf{I}_{\mathbf{x}_0}^{\mathcal{G}^{\alpha, \delta}}$ at time steps $t_\alpha = \{t_1, t_2, \dots, t_7\}$. Each panel is a GLSM map constructed via (5.14) on the basis of scattered field data $\mathbf{v}_\alpha^{\text{obs}}$ exclusively captured at the indicated sensing step t_α .

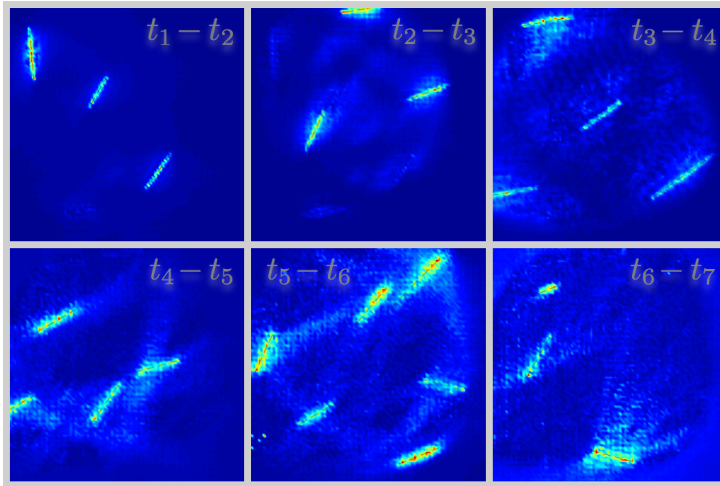


Figure 6. Differential reconstruction of newborn fractures, via the proposed indicator $\mathbf{I}_{\mathbf{x}_0}^{\mathcal{D}^{\alpha, \delta}}$, within every sensing sequence $t_\alpha - t_{\alpha+1}$ for $\alpha = \{1, 2, \dots, 6\}$. Each panel is a Differential Evolution Indicator map constructed via (5.13) on the basis of sequential scattered field data $(\mathbf{v}_\alpha^{\text{obs}}, \mathbf{v}_{\alpha+1}^{\text{obs}})$, captured at the indicated sensing steps $t_\alpha, t_{\alpha+1}$.

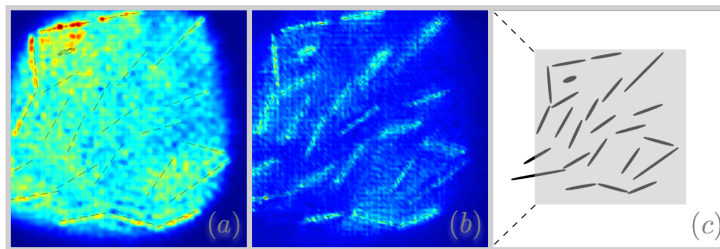


Figure 7. GLSM vs. Differential Evolution Indicators: (a) GLSM map of damage zone within the sampling area at t_7 , (b) assembled map of fractures constructed by averaging the differential indicator maps shown in Figure 6, and (c) true anatomy of the damage zone where the shaded area highlights the sampling region.

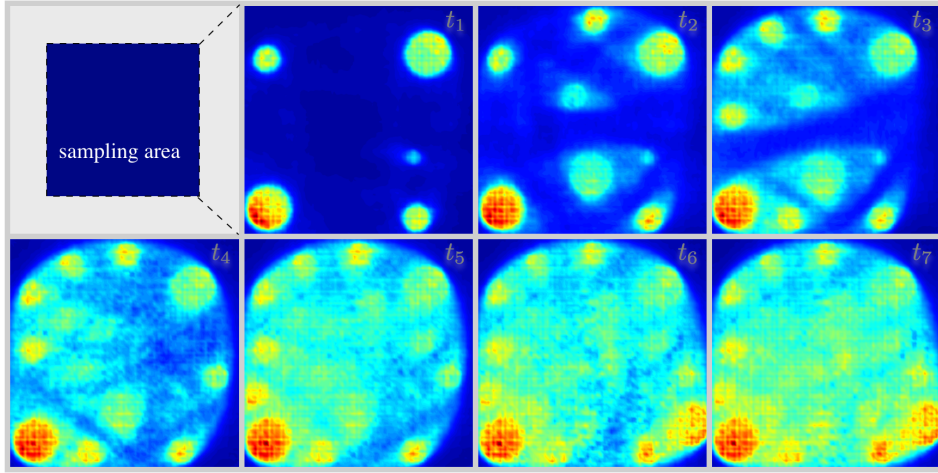


Figure 8. Disjoint reconstruction of an evolving bubble zone via the existing indicator $\mathbf{I}_{\mathbf{x}_o}^{\mathcal{G},\delta}$ at time steps $t_\alpha = \{t_1, t_2, \dots, t_7\}$. Each panel is a GLSM map constructed via (5.14) on the basis of scattered field data \mathbf{v}_α^{obs} exclusively captured at the indicated sensing step t_α .

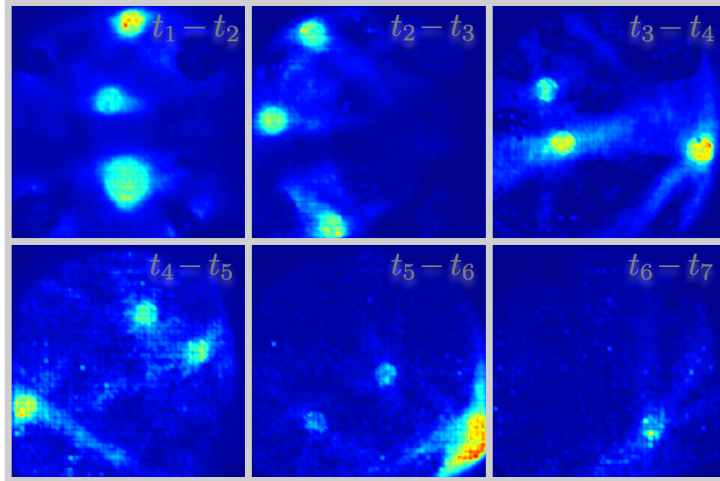


Figure 9. Differential reconstruction of newborn pores, via the proposed indicator $\mathbf{I}_{\mathbf{x}_o}^{\mathcal{D},\delta}$, within every sensing sequence $t_\alpha - t_{\alpha+1}$ for $\alpha = \{1, 2, \dots, 6\}$. Each panel is a Differential Evolution Indicator map constructed via (5.13) on the basis of sequential scattered field data $(\mathbf{v}_\alpha^{obs}, \mathbf{v}_{\alpha+1}^{obs})$, captured at the indicated sensing steps $t_\alpha, t_{\alpha+1}$.

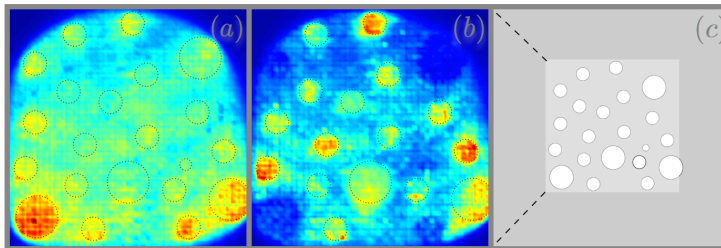


Figure 10. GLSM vs. Differential Evolution Indicators: (a) GLSM map of bubble zone within the sampling area at t_7 , (b) assembled map of pores constructed by averaging the differential indicator maps shown in Figure 9, and (c) true anatomy of the pore zone where the shaded area highlights the sampling region.

696 **7. Acknowledgements.** The corresponding author kindly acknowledges the support pro-
 697 vided by the University of Colorado Boulder through the Imaging Science seed grants and
 698 FP's startup. This work utilized the RMACC Summit supercomputer, which is supported
 699 by the National Science Foundation (awards ACI-1532235 and ACI-1532236), the University
 700 of Colorado Boulder, and Colorado State University. The Summit supercomputer is a joint
 701 effort of the University of Colorado Boulder and Colorado State University.

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