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# On Adaptive Estimation of Bacterial Growth in the Competitive Chemostat

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## Abstract:

In this paper, the problem of state estimation of a bioreactor containing a single substrate and several competing species is studied. This scenario is well-known as a *competition model*, in which multiple species compete for the same nutrient. The goal is to estimate concentrations of the substrate and all species, considering the concentration of total biomass as measurement. To address such a challenge, the estimation scheme relies in the coupling of two estimation techniques: an asymptotic observer, which takes advantage of the model structure being independent of its kinetics, and a finite-time parameter estimation technique, which also drops the usual requirement of persistence of excitation.

*Keywords:* chemostat, biological, robust estimation, finite-time, observer

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## 1. INTRODUCTION

In modern and practical biotechnological and biochemical applications, bioreactors are widely known for allowing experiments involving living micro-organisms, under controlled conditions and that mimics a natural environment. Many applications arise from such experiments, like pharmaceutical production, yeast fermentation, ethanol production, polymerization and many others. A bioreactor might operate in three distinct modes (Singh et al., 2014): batch, fed-batch and continuous. In the first, media is added and the process allowed to proceed until a certain condition is reached, in the second fresh media can be continuously fed into the bioreactor but not removed and, finally, the third comprehends the case in which fresh media might be added and removed proportionally. A well-known example of continuous bioreactor is the *chemostat*.

The issue of monitoring the processes variables inside a bioreactor arises as an important question. In fact, for some applications it is crucial to have real-time information about variables such as concentration of biomass, dissolved products or reactants, gaseous outflows or growth, death and production rates of living organisms. The main difficulties are originated by the lack of available sensors for such variables, their cost, their physical set-up or even their sampling time of measurements.

To overcome such drawbacks, a well-known option is the use of *software sensors* (Bogaerts and Vande Wouwer, 2003). These sensors, as viewed from the control community point of view, consist basically on the design of

state observers. Throughout the years, many studies have been presented in this framework and exploiting the use of different observers, e.g., see survey (Ali et al., 2016).

The major difficulties on observer design for biological processes come from uncertainties in the non-linear models and their observability conditions evaluation, which might not be a trivial task. This latter issue is directly related on how the measurements are available and how well-known the dynamics are.

In this paper, we consider the state estimation problem applied to a bioreactor containing a single nutrient and several competing species by means of an adaptive observer scheme. Regarding such an objective, we couple two estimation techniques: the design of an asymptotic observer, which is appropriate for this application as it is independent on the kinetics rates, and a finite-time (FT) parameter estimator. This FT estimator is presented by Wang et al. (2019), where authors therein applied the *dynamical regressor extension and mixing* (DREM) (Aronovskiy et al., 2017) method and proposed algorithms possessing certain robustness against external perturbances and measurement noises, and do not require the usual condition of excitation persistence. This last feature is of interest to applications like bacterial growth, since trajectories obtained by highly excited inputs might not be possible or without physical meaning in real-life experiments.

The estimation of substrate and biomass concentrations using the total biomass as measurement is poorly addressed in the literature. This is due to observability issues and approaches like high-gain observers fail to offer good

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estimates without a highly-excited and persistent input, which are unrealistic for such an application.

*Structure of the paper:* the problem statement is presented in Section 2. Some preliminary concepts on finite-time stability, numerical differentiation and parameter identification are introduced in Section 3. The state estimation schemes are given in Section 4 and numerical examples illustrate its application in Section 5. Concluding remarks and future directions are discussed in Section 6.

## 2. PROBLEM STATEMENT

Consider the following non-linear system describing bacterial growth of  $n$  species inside a chemostat (time-dependence was omitted for readability):

$$\begin{aligned} \frac{dS}{dt} &= (S_{in} - S)D - \sum_{i=1}^n \mu_i(S)x_i \\ \frac{dx_i}{dt} &= (\mu_i(S) - D)x_i, \quad i = 1 \dots n \end{aligned} \quad (1)$$

where  $S$  is the nutrient concentration,  $x_i$  is the concentration of  $i$ -th species,  $S_{in}$  and  $D$  are the control inputs (nutrient inflow concentration and dilution rate, respectively). In this work, we considered that yield rates were suppressed by normalization. Function  $\mu_i(S)$  describes the kinetics of bacterial growth and is given by Monod's law as follows:

$$\mu_i(S) = \mu_{i,max} \frac{S}{\alpha_i + S}, \quad \mu_{i,max} > 0 \quad (2)$$

where the index represents the  $i$ -th species. If this index is omitted, it is assumed that  $\mu$  is a component-wise vector, i.e.,  $\mu = [\mu_1 \dots \mu_n]^T$ .

The aim of this paper is then to estimate  $S$  and  $x_i$ , without knowledge of initial conditions and using  $y = \sum_{i=1}^n x_i + w$  (i.e., the total biomass concentration inside the bioreactor) as measurement. In real experiments, this kind of measurements are easily obtained by optical density methods and we assumed presence of an essentially bounded measurement noise signal  $w$ .

This specific measurement set-up is the main interest of this paper, due to the mathematical complexity that it imposes to the problem.

## 3. PRELIMINARIES

Consider a time-dependent differential equation (Khalil, 2002):

$$\frac{dx(t)}{dt} = f(t, x(t), d(t)), \quad t \geq t_0, \quad t_0 \in \mathbb{R}, \quad (3)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $d(t) \in \mathbb{R}^m$  is the vector of external inputs and  $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$ ;  $f : \mathbb{R}^{n+m+1} \rightarrow \mathbb{R}^n$  is a continuous function with respect to  $x$ ,  $d$  and piecewise continuous with respect to  $t$ ,  $f(t, 0, 0) = 0$  for all  $t \in \mathbb{R}$ . A solution of the system (3) for an initial condition  $x_0 \in \mathbb{R}^n$  at time instant  $t_0 \in \mathbb{R}$  and some  $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$  is denoted as  $X(t, t_0, x_0, d)$ , and we assume that  $f$  ensures definiteness and uniqueness of solutions  $X(t, t_0, x_0, d)$  in forward time at least on some finite time interval  $[t_0, t_0 + T)$ , where  $T > 0$  may be dependent on the initial condition  $x_0$ , the input  $d$  and the initial time  $t_0$ .

As it will be used in the following, let us define some classes of functions. A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}$  if  $\alpha(0) = 0$  and the function is strictly increasing. A function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}$  for each fixed  $t \in \mathbb{R}_+$  and  $\beta(s, \cdot)$  is decreasing and  $\lim_{t \rightarrow +\infty} \beta(s, t) = 0$  for each fixed  $s \in \mathbb{R}_+$ , a function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{GKL}$  if  $\beta(s, 0) \in \mathcal{K}$ ,  $\beta(s, \cdot)$  is decreasing and for each  $s \in \mathbb{R}_+$  there is  $T_s \in \mathbb{R}_+$  such that  $\beta(s, t) = 0$  for all  $t \geq T_s$ .

### 3.1 Robust stability definitions

Consider the following definition of robust stability for (3) with  $d \neq 0$ :

*Definition 1.* The system (3) is said to be

(a) *short-finite-time input-to-state stable (ISS)* with respect to  $(\Omega, T^0, T_f, D)$  if there exist  $\beta \in \mathcal{GKL}$  and  $\gamma \in \mathcal{K}$  such that for all  $x_0 \in \Omega$ , all  $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$  with  $\|d\|_\infty < D$  and  $t_0 \in [-T^0, T^0]$ :

$$|X(t, t_0, x_0, d)| \leq \beta(|x_0|, t - t_0) + \gamma(\|d\|_\infty) \quad \forall t \in [t_0, t_0 + T_f]$$

and  $\beta(|x_0|, T_f) = 0$ ;

(b) *globally short-finite-time ISS* for  $T^0 \geq 0$  if there exist  $\beta \in \mathcal{GKL}$  and  $\gamma \in \mathcal{K}$  such that for any bounded set  $\Omega \subset \mathbb{R}^n$  containing the origin there is  $T_f > 0$  such that the system is short-finite-time ISS with respect to  $(\Omega, T^0, T_f, +\infty)$ .

As we can conclude from Definition 1, if the system (3) is short-finite-time ISS, then for  $d = 0$  there exists  $\Xi \subset \mathbb{R}^n$ ,  $\Omega \subset \Xi$  such that the system is short-finite-time stable with respect to  $(\Omega, \Xi, T^0, T_f)$ . The difference of global short-finite-time ISS and a conventional (finite-time or fixed-time) ISS (Hong, 2001; Bernuau et al., 2013) is that in the former case the stability property is considered on a finite interval of time  $[t_0, t_0 + T_f]$  only for  $t_0 \in [-T^0, T^0]$ .

### 3.2 Robust exact differentiator

The design of real-time differentiators is a well-known and studied problem nowadays (see the special issue on differentiators on (IJC, 2018)). The main challenge to such an implementation is sensitivity to noise. Many approaches to design differentiators became popular throughout the years, like high-gain observers and sliding-mode differentiators.

In this section, we use results from Levant (2003) for the design of an arbitrary-order, robust and exact differentiator. These exact differentiators demonstrate a finite-time convergence and also good sensitivity to input noise.

Let an output signal  $f(t) = f_0(t) + w(t)$  to be defined in  $[0, \infty)$  and being corrupted by an unknown but Lebesgue-measurable noise  $w(t)$ , where an unknown base signal  $f_0(t)$  has its  $n$ -th derivative having a known Lipschitz constant  $L > 0$ . The objective is then to have robust and exact estimation of  $f_0, \dot{f}_0 \dots f_0^{(n)}$ . The following recursive scheme offers such an estimate:

$$\begin{cases} \dot{z}_0 = -\lambda_n L^{\frac{1}{n+1}} |z_0 - f(t)|^{\frac{n}{n+1}} \text{sign}(z_0 - f(t)) + z_1 \\ \dot{z}_1 = -\lambda_{n-1} L^{\frac{1}{n}} |z_0 - f(t)|^{\frac{n-1}{n}} \text{sign}(z_0 - f(t)) + z_2 \\ \dots \\ \dot{z}_{n-1} = -\lambda_1 L^{\frac{1}{2}} |z_0 - f(t)|^{\frac{1}{2}} \text{sign}(z_0 - f(t)) + z_n \\ \dot{z}_n = -\lambda_0 L |z_0 - f(t)|, \end{cases} \quad (4)$$

where  $\lambda_i$  are tuning parameters for  $i = 1, \dots, n$ . Although an infinite sequence  $\lambda_i$  can be built, it has been shown that  $\{\lambda_0, \lambda_1\} = \{1.1, 1.5\}$  suffice for the zero- and first-order derivatives.

According to Levant (2003), if the input noise satisfy  $|w| \leq \epsilon$  for all  $t \geq 0$ , then the differentiation ensured accuracy satisfies the following inequality:

*Theorem 2.* Let the input noise satisfy  $|w(t)| \leq \epsilon$  for all  $t \geq 0$ . Then the following inequalities are established in finite-time  $T > 0$ , for some positive constant  $\mu_i$  depending exclusively on the parameters  $\lambda_1, \dots, \lambda_n$  of the differentiator:

$$|z_i(t) - f_0^{(i)}(t)| \leq \mu_i L^{\frac{1}{n+1}} \epsilon^{\frac{n-i+1}{n+1}}, \quad \forall t \geq T, \quad i = 0, 1, \dots, n \quad (5)$$

Also, all solutions of this scheme are Lyapunov stable. For proofs, the reader is invited to refer to Levant (2003).

### 3.3 DREM method and robust FT parameter estimation

Consider a usual estimation problem in the static linear regression model (Ljung, 1987) as follows:

$$y(t) = \omega^\top(t)\theta + w, \quad t \in \mathbb{R} \quad (6)$$

where  $x(t) \in \mathbb{R}$ ,  $\theta \in \mathbb{R}^n$  is the vector of unknown constant parameters which are to be estimated,  $\omega : \mathbb{R} \rightarrow \mathbb{R}^n$  is the regressor function (supposedly known and bounded) and  $y(t) \in \mathbb{R}$  is the available measurement signal with measurement noise  $w : \mathbb{R} \rightarrow \mathbb{R}$ . It is well-known that problem (6) has a solution (if  $\omega(t)$  is persistently excited, see Ljung (1987)) given by:

$$\hat{\theta}(t) = \gamma \omega(t) \left( y(t) - \omega(t)^\top \hat{\theta}(t) \right), \quad \gamma > 0 \quad (7)$$

where  $\hat{\theta}$  is the estimate of  $\theta$ .

Following this idea, Aranovskiy et al. (2017) proposed the *dynamic regressor extension and mixing method* (DREM), which basically decouples (6) into  $n$  one-dimensional regression models, allowing each parameter  $\theta_i$ ,  $i = 1 \dots n$ , to be evaluated under another condition than the one of excitation persistence of  $\omega(t)$ .

For that, under assumption that  $\omega \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n)$  and  $w \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ , one designs  $n - 1$  linear operators  $H_j : \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}) \rightarrow \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ . This operator  $H_j$  can be any stable linear time invariant filter or delay, see Wang et al. (2019). As consequence of above assumptions,  $y(t) \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ , one gets the following by the superposition principle:

$$\tilde{y}_j(t) = H_j(y(t)) = \tilde{\omega}_j^\top(t)\theta + \tilde{w}_j(t), \quad j = 1, \dots, n$$

where  $\tilde{y}_j(t) \in \mathbb{R}$  is  $j$ -th operator output,  $\tilde{\omega}_j : \mathbb{R} \rightarrow \mathbb{R}^n$  is the  $j$ -th filtered regressor function and  $\tilde{w}_j$  is the filtered  $j$ -th noise signal. Defining new vector variables:

$$\tilde{Y}(t) = [y(t) \tilde{y}_1(t) \dots \tilde{y}_{n-1}(t)]^\top \in \mathbb{R}^n,$$

$$\tilde{W}(t) = [w(t) \tilde{w}_1(t) \dots \tilde{w}_{n-1}(t)]^\top \in \mathbb{R}^n$$

and a time-varying matrix

$$M(t) = [\omega(t) \tilde{\omega}_1(t) \dots \tilde{\omega}_{n-1}(t)]^\top \in \mathbb{R}^{n \times n}, \quad (8)$$

and rewriting (6) using the above  $n - 1$  regressor models, one has

$$\tilde{Y}(t) = M(t)\theta + \tilde{W}(t).$$

After multiplication of both sides of the above equation by the adjoint matrix of  $M(t)$  (see Wang et al. (2019) for further details), one has that the  $n$  scalar decoupled regressor models are given by

$$Y_i(t) = \phi(t)\theta_i + W_i(t) \quad (9)$$

where  $\phi(t) = \det(M(t))$ ,  $Y(t) = \text{adj}(M(t))\tilde{Y}(t)$  and  $W(t) = \text{adj}(M(t))\tilde{W}(t)$ .

Finally, one can relate equations (9) and the estimation algorithm (7):

$$\dot{\hat{\theta}}_i(t) = \gamma_i \phi(t) \left( Y_i(t) - \phi(t)\hat{\theta}_i(t) \right), \quad \gamma_i > 0. \quad (10)$$

Another estimation algorithm has been proposed in Ríos et al. (2017, 2018):

$$\dot{\hat{\theta}}(t) = \gamma \phi(t) \left[ Y(t) - \phi(t)\hat{\theta}(t) \right]^\alpha, \quad \gamma > 0, \quad \alpha \in [0, 1), \quad (11)$$

where  $[x]^\alpha = |x|^\alpha \text{sign}(x)$ . This algorithm has the following applicability conditions and stability properties:

*Assumption 1.* Let  $\omega \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n)$  and  $w \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ .

*Assumption 2.* A constant  $\bar{\theta} > 0$  is given such that  $\theta \in \Omega = [-\bar{\theta}, \bar{\theta}]^n$ .

*Proposition 3.* Let assumptions 1 and 2 be satisfied, and

$$\int_t^{t+\ell} |\phi(s)|^{1+\alpha} ds \geq v > 0 \quad (12)$$

for all  $t \in [-T_0, T_0 + T]$  and some  $\ell \in (0, T)$ ,  $v > 0$ . Take

$$\gamma \geq \frac{2^{\frac{5-\alpha}{2}} \bar{\theta}^{1-\alpha}}{1-\alpha} \frac{T}{\ell} - 1,$$

then the estimation error  $e(t) = \theta - \hat{\theta}(t)$  dynamics for (11) with  $\hat{\theta}(t_0) = 0$  is short-finite-time ISS with respect to  $(\Omega, T^0, T, +\infty)$ .

For proofs, the reader is invited to refer to Wang et al. (2019).

*Remark 1.* The persistence of excitation in a finite-time interval given by (12) is a sufficient condition for observability of system (6).

## 4. DESIGN OF STATE ESTIMATORS FOR BACTERIAL GROWTH IN A COMPETITIVE ENVIRONMENT

Consider now the stated estimation problem for system (1). In the following, we will propose solutions for the estimation problem using each of the aforementioned measurement signal. Also, for the sake of readability throughout this section, let us introduce the following notation:

$$\varphi_i(t) = e^{\int_0^t \mu_i(S(\tau)) - D(\tau) d\tau}, \quad \hat{\varphi}_i(t) = e^{\int_0^t \mu_i(\hat{S}(\tau)) - D(\tau) d\tau}$$

where the index represents the  $i$ -th species and  $\hat{S}$  is the estimate of  $S$ . If this index is omitted, it is assumed that  $\varphi$  is a component-wise vector, i.e.,  $\varphi = [\varphi_1 \dots \varphi_n]^\top$ .

As it will be often used in the following, recall the time varying solution for each species concentration in (1) as follows:

$$x_i(t) = x_i(0)\varphi_i(t) \quad (13)$$

where  $x_i(0)$  is the initial condition of each state  $x_i$ .

#### 4.1 Using the total biomass as measurement

If compared to other possible measurements (for instance, the substrate or a single biomass concentration), the measurement of the total biomass, i.e.,  $y = \sum_{i=1}^N x_i + w$ , addresses a more difficult task (from a mathematical point of view) to the stated problem. This difficulty comes from the fact that the whole state vector must be estimated and, due to observability issues, a finite-time observer can not be readily designed.

As it can be easily seen in (1), equations for the species concentrations  $x_i(t)$  and the nutrient concentration  $S(t)$  are coupled by the kinetic rates  $\mu_i(S)$ , which describe the nutrient consumption by the  $i$ -th species. Due to this coupling, the first step is to obtain an observer for  $S(t)$  that does not depend on these kinetics (Dochain et al., 1992). This problem has been addressed in Dochain et al. (1992) and this observer can be designed by introducing the change of variables  $z(t) = S(t) + y(t)$ , which admits the dynamics

$$\dot{z} = DS_{in} - Dz$$

and results in the following observer equations:

$$\dot{\hat{z}} = DS_{in} - D\hat{z} \quad (14)$$

and, hence,  $\hat{S} = \hat{z} - y$ .

The following theorem (Dochain et al., 1992) states the convergence condition and properties of observer (14):

*Theorem 4.* : If the following condition is satisfied

$$\lim_{T \rightarrow +\infty} \int_0^T D(s)ds = +\infty$$

then  $\lim_{t \rightarrow \infty} |z - \hat{z}| = 0$ , implying that  $\hat{z} \rightarrow S + y$  as  $t \rightarrow \infty$ .

**Proof:** The proofs are skipped due to page limitations.

Following the error dynamics given by

$$\frac{d}{dt}(z - \hat{z}) = -D(z - \hat{z}) \quad (15)$$

one has that the time evolution of the discrepancy  $S - \hat{S}$  is given by

$$|S - \hat{S}| = |z(0) - \hat{z}(0)|e^{-\int_0^t Dd\tau} \quad (16)$$

where  $\hat{z}(0)$  is the initial condition of the asymptotic observer (14) and  $z(0)$  is supposedly unknown but upper bounded.

To estimate  $x_i(t)$ , it will be needed to compute an estimate of  $\dot{S}(t)$ . This estimate can be obtained by applying the differentiation scheme given by (4) as follows:

$$\begin{aligned} \dot{z}_0 &= z_1 - \lambda_0 L^{\frac{1}{2}} |z_0 - y|^{1/2} \text{sign}(z_0 - y) \\ \dot{z}_1 &= -\lambda_1 L \text{sign}(z_0 - y) \end{aligned} \quad (17)$$

and hence, according to Levant (2003),  $z_1 \rightarrow \dot{y}$  in a finite time in the noise-free case. Hence, the observation scheme (14) can be rewritten as follows:

$$\dot{\hat{S}} = DS_{in} - D(\hat{S} - y) - z_1 \quad (18)$$

where  $\dot{\hat{S}} \approx \frac{d}{dt}\hat{S}$ .

Knowing both estimates  $S$  and  $\dot{S}$ , we are now able to build an estimation scheme for the unknown initial conditions  $x_i(0)$ . By means of solution (13), one can rewrite the equation for the substrate in (1) as the well-known regressor model (6):

$$\dot{S} + D(S - S_{in}) = \omega^\top \theta \quad (19)$$

where  $\omega(t)$  and  $\theta$  are, respectively, the regressor function and the constant parameter vector are given by

$$\begin{aligned} \omega(t) &= [-\mu_1(S)\varphi_1, \dots, -\mu_n(S)\varphi_n]^\top \\ \theta &= [x_1(0), \dots, x_n(0)]^\top \end{aligned}$$

and hence, we can apply the finite-time estimation methods aforementioned, i.e., DREM and algorithm (11).

So, combining results (17) and (19), one gets

$$\dot{\hat{S}} + D(\hat{S} - S_{in}) = \hat{\omega}^\top \hat{\theta} + w_e \quad (20)$$

where  $\hat{S} = \hat{w} - y$ ,  $\hat{\theta} = [\hat{x}_1(0), \dots, \hat{x}_n(0)]$  is the estimate of the vector of unknown parameters  $\theta$ ,  $\hat{\omega} = -\mu(\hat{S})\hat{\varphi}$  and  $w_e$  is the noise cause by the measurement sensor and the differentiation algorithm, being an essentially bounded function of time due to results of Levant (2003).

In order to well characterize the noise  $w_e$  in (20), let us rewrite (6) as follows:

$$\begin{aligned} 0 &= \dot{S} + D(S - S_{in}) - \omega^\top \theta \\ &= \dot{\hat{S}} + (\dot{S} - \dot{\hat{S}}) + D(\hat{S} - S_{in}) + D(S - \hat{S}) - \omega^\top \theta \end{aligned}$$

As aforementioned,  $\omega = \mu(S)\varphi$  and  $\hat{\omega} = \mu(\hat{S})\hat{\varphi}$ . With simple algebraic manipulations, one can write the following equality:

$$\begin{aligned} \omega &= \mu(\hat{S})\varphi + (\mu(S) - \mu(\hat{S}))\varphi \\ &= \mu(\hat{S})\hat{\varphi} + (\varphi - \hat{\varphi})\mu(\hat{S}) + (\mu(S) - \mu(\hat{S}))\varphi \\ &= \hat{\omega} + w_\omega \end{aligned}$$

Referring back to (20), one may write the following inequality:

$$|w_e| \leq |w_\omega|\theta + |\dot{S} - \dot{\hat{S}}| + D|S - \hat{S}| \quad (21)$$

Finally, in order to apply algorithm (11) in (20), the regressor function  $\phi(t)$  is computed by constructing the time-varying matrix (8) as  $M(t) = [\hat{\omega} \ \tilde{\omega}_1 \ \dots \ \tilde{\omega}_n]^\top$ , where  $\tilde{\omega}_j = H_j \hat{\omega}$ , for  $j = 1 \dots n$  and  $H_j$  being a properly chosen linear filter.

*Assumption 3.* Suppose that condition (12) holds.

*Assumption 4.* There exists a constant  $R \geq 0$  such that  $|\varphi(t)| \leq R$ , for all  $t \geq 0$ .

*Theorem 5.* Let assumptions 3–4 hold. Then, estimates of concentrations  $\hat{x}_i(t)$  are computed directly by

$$\hat{x}_i(t) = \hat{x}_i(0)\hat{\varphi} \quad (22)$$

and the estimation error is bounded by  $|x(t) - \hat{x}(t)| \leq \sigma(|w_e|)R + \sigma(|w(0) - \hat{w}(0)|)$ , where  $\sigma \in \mathcal{K}$ .

**Proof:** The proofs are skipped due to page limitations.

Although estimates of  $x_i(0)$  are obtained in finite-time, this estimation scheme is limited by the (asymptotic) rate of convergence of observer (14). To guarantee that only good estimates  $\hat{S}$  are used in (20), an interesting solution is to use (16) as an activation switch for such an algorithm. This can be done by setting an accuracy  $|w - \hat{w}| < \epsilon$ , with a tuning parameter  $\epsilon > 0$ . This step requires only an upper bound on  $w(0)$ , which is clearly not a harsh assumption for an experimental set-up.

Finally, estimates of  $x(t)$  will be available after the time for accuracy  $\epsilon$  to be attained on (14) plus the finite-time of convergence of (17) and (20).

## 5. NUMERICAL EXAMPLE

In order to obtain a feasible simulation, results from Mazenc et al. (2008) have been used here in order to properly control the chemostat using periodic trajectories. Consider system (1) with  $N = 2$  and the following kinetic rates for microbial growth:

$$\mu_1(S) = \frac{10S}{1+20S}, \quad \mu_2(S) = \frac{S}{1+S}$$

In the following, we present simulation results of the aforementioned estimation schemes. Also, throughout this section it is assumed that all measurements are corrupted by a noise signal given by  $w(t) = 0.001 \sin(10t)$ . Through numerical simulation, we selected the linear filter  $H_1 = \frac{s+10}{s^2+1}$  in order to apply DREM, since this filter has shown to provide sufficient excitation on regressor function  $\omega$ .

### 5.1 Using the total biomass as measurement

In this application, the first step is to implement the asymptotic observer for  $\hat{S}$ , as given by (14). As we dispose of no information about initial conditions, this observer is initialized with an arbitrary (and sufficiently large) initial condition  $\hat{w}(0) = 3$ . This implementation is straightforward and requires no choice of further parameters. Figure 1 illustrates this implementation.

In order to guarantee the use of good estimates  $S$  in the following, we used criterion (16) with  $\epsilon = 0.05$  while assuming an upper bound  $|w(0)| = 10$ . As it is required by (20), an estimate of  $\hat{S}$  is computed by (17)–(18). Figure 2 illustrates this computation.

Hence, after accuracy  $\epsilon$  is attained on  $\hat{S}$  and  $\hat{S}$  is estimated, we dispose of all information needed in order to implement the estimation scheme (20). Choosing  $T = 20$  and  $\alpha = 0.8$  for algorithm (11), figure 3 illustrates the estimation of  $x_1(0)$  and  $x_2(0)$ . Finally, by means of (22), figure 4 shows the final estimates  $\hat{x}_1(t)$  and  $\hat{x}_2(t)$ .

It is worth noticing that the identified initial conditions  $\hat{x}_i(0)$  relates to the instant where integration of  $\hat{\theta}$  on (11) takes place, i.e., relates to the instant where accuracy  $\epsilon$  is attained on  $\hat{S}$ . Furthermore, note that trajectories of  $x_i(t)$  are realistic for such an application. Hence, it highlights the usefulness of the proposed scheme, since it does not require a highly-excited input in order to properly compute estimates  $\hat{x}_i(t)$ .

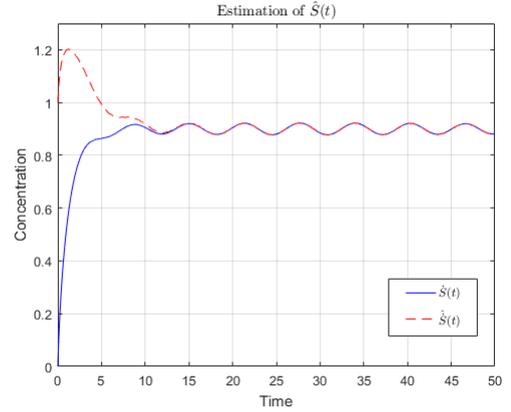


Fig. 1. Estimation of  $S(t)$

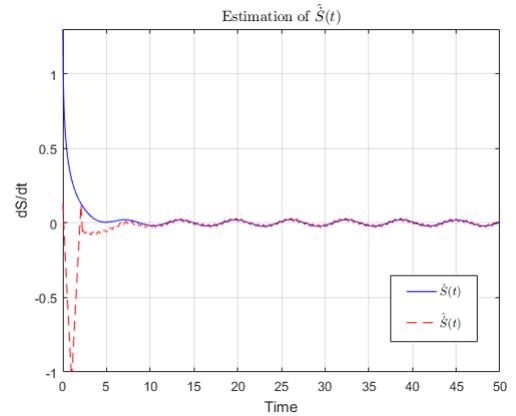


Fig. 2. Estimation of  $\dot{S}(t)$

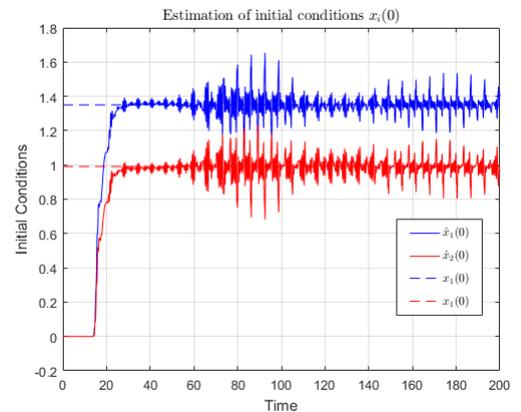


Fig. 3. Estimation of  $x_1(0)$  and  $x_2(0)$

## 6. CONCLUSION

In this paper we further explored the problem of estimation of bacterial growth on bioreactors. The proposed scheme aims at providing estimates of all unknown variables (e.g., the substrate and biomass concentration) and in a finite-time whenever possible. In fact, using measurement  $y = \sum_{i=0}^N x_i$ , the estimation time is delayed by an asymptotic convergence of the estimate  $\hat{S}(t)$ . The main key in this scheme is the coupling of different estimation approaches, like sliding-mode exact differentiators and a finite-time parameter estimation technique. It is worth

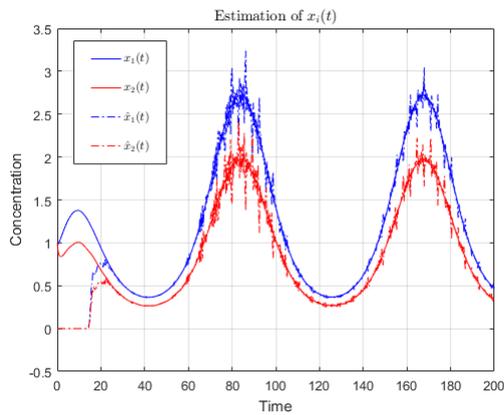


Fig. 4. Estimation of  $x_i(t)$

noticing that, although common in many estimation techniques, this approach does not require persistence of excitation, making it very interesting for such an application with realistic trajectories.

As an object of future research, an interesting direction is to investigate the sensitivity to noise of such an application. As it can be easily seen, even though stability is still attainable and the error remains bounded, the performance of the estimation depreciates in presence of a weak measurement noise.

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