



A note on the connection between product-form Jackson networks and counting lattice walks in the quarter plane

Guy Fayolle

► To cite this version:

Guy Fayolle. A note on the connection between product-form Jackson networks and counting lattice walks in the quarter plane. 2019. hal-02415746v1

HAL Id: hal-02415746

<https://inria.hal.science/hal-02415746v1>

Preprint submitted on 17 Dec 2019 (v1), last revised 6 Jan 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A note on the connection between product-form Jackson networks and counting lattice walks in the quarter plane

December 17, 2019

Guy Fayolle*

1 Introduction

The ultimate goal is to find explicit solutions for counting generating functions (CGF) of some random walks in the quarter plane, starting from the wellknown product-form for the stationary distribution of customers in Jackson's stochastic networks. The first step is to solve a *relatively simple* boundary value problem (BVP) for the queueing system. Then, by using a variational principle based on the continuity with respect to adequate parameters, it is possible to transform some smooth curves in a continuous fashion and to get explicitly formulas for the conformal mappings of Riemann's theorem. This might be a new way of computing the gluing functions appearing in [1, 3], which are a key ingredient in the explicit formulas for CGFs.

2 An illustrative example of walks with steps East, South, North-West: tandem queues

We start by considering the related system consisting of two M/M/1 service stations in tandem, under Poisson (intensity λ) external arrival process and exponential service time distributions with equal service rates μ (this does not change the nature of the results).

Setting $\rho \stackrel{\text{def}}{=} \frac{\lambda}{\mu}$, the stationary distribution for the joint distribution of the number of units in the system satisfies the functional equation

$$Q(x, y)F(x, y) = \left(1 - \frac{y}{x}\right)F(0, y) + \left(1 - \frac{1}{y}\right)F(x, 0), \quad (2.1)$$

with $Q(x, y) = \rho(1 - x) + \left(1 - \frac{y}{x}\right) + \left(1 - \frac{1}{y}\right)$.

For $\rho < 1$, the solution of (2.1) is well-known to have the simple product-form

$$F(x, y) = \frac{(1 - \rho)^2}{(1 - \rho x)(1 - \rho y)}. \quad (2.2)$$

Now, assume we do not know the simple expression (2.2). Then, according the method invented in [1], it is always possible to set a boundary value problem (BVP) of Riemann-Dirichlet-Carleman type for the function $F(0, y)$, which here has a simple formulation. We shall sketch the mains steps.

*SPECFUN Project-Team, Inria, guy.fayolle@inria.fr.

As usual, we first restrict the equation (2.1) to the algebraic curve

$$\{(x, y) \in \mathbb{C}^2 : xyQ(x, y) = 0\}.$$

Solving $xyQ(x, y) = 0$ with respect to y yields

$$y^2 - x(\rho + 2 - \rho x)y + x = 0, \quad (2.3)$$

which has two roots $Y_1(x), Y_2(x)$ (the two branches of the global algebraic function $Y(x)$), such that

$$|Y_1(x)| \leq 1 \leq |Y_2(x)|, \forall |x| = 1, \quad \text{and} \quad |Y_1(x)| \leq |Y_2(x)|, \forall x \in \mathbb{C}.$$

It is also worth noting

$$Y_1(x) \cdot Y_2(x) = x, \quad \forall x \in \mathbb{C}. \quad (2.4)$$

The discriminant of equation (2.3)

$$D(x) = x^2(\rho + 2 - \rho x)^2 - 4x = x(x - 1)(\rho^2 x^2 - \rho(\rho + 4)x + 4)$$

has four real zeros: $x_1 = 0 < x_2 = 1 < x_3 < x_4$, which are the branch points of the algebraic curve $Y(x)$ in the complex plane \mathbb{C}_x .

For $x \in [0, 1]$, we have $Y_1(x) = \overline{Y_2(x)}$, since the two roots of $Q(x, y)$ considered as a polynomial in y are complex conjugate, and $Y(x)$ describes a simple closed contour \mathcal{L} . Indeed \mathcal{L} is the *smallest* component of a quartic curve, which in the \mathbb{C}_y -plane has the following equation, with $y = u + iv$,

$$\rho(u^2 + v^2)^2 - (\rho + 2)(u^2 + v^2) + 2u = 0. \quad (2.5)$$

The contour \mathcal{L} is situated entirely in the right half- \mathbb{C}_y -plane, inside the unit circle, and crosses the real axis at $u = 0$ and $u = 1$, where it has two vertical tangents. The interior domain bounded by \mathcal{L} will be denoted by \mathcal{L}^+ .

Letting x tend successively to the upper and lower edge of the slit (cut) $[0, 1]$, we can eliminate $F(x, 0)$ which is continuous on $[0, 1]$. So, after a quick algebra, setting for short

$$\Phi(y) \stackrel{\text{def}}{=} F(0, y), \quad A(y) \stackrel{\text{def}}{=} \left(\frac{y}{1 - y} \right)^2,$$

we can state the following BVP for $F(0, y)$.

Find a function Φ holomorphic in \mathcal{L}^+ , the limiting values of which on the contour \mathcal{L} are continuous and satisfy the relation

$$\boxed{A(t)\Phi(t) - A(\bar{t})\Phi(\bar{t}) = 0, \quad t \in \mathcal{L}}. \quad (2.6)$$

Remark 2.1. The problem (2.6) is a priori of generalized Dirichlet-Carleman type, *since the function $A(\cdot)$ has a zero at $t = 0$ and a pole at $t = 1$ on the contour*, but is otherwise analytic in \mathcal{L}^+ . Therefore slightly sharper technicalities appear.

Setting

$$B(t) \stackrel{\text{def}}{=} \frac{A(\bar{t})}{A(t)} = \left(1 - \frac{1}{t}\right)^2 / \left(1 - \frac{1}{\bar{t}}\right)^2$$

the BVP (2.6) can be rewritten in the form

$$\boxed{\Phi(t) = B(t)\Phi(\bar{t}), \quad t \in \mathcal{L}}. \quad (2.7)$$

Checking that $B(t)$ is continuous on the contour \mathcal{L} , where it satisfies a Hölder condition, we get a classical Riemann-Hilbert problem, see [2].

In order to solve it, we introduce the conformal mapping $T(z)$ of the closed unit disc $\mathcal{D} = \{z : |z| \leq 1\}$ onto the region $\mathcal{L} \cup \mathcal{L}^+$. By symmetry with respect to the real axis in the respective planes, we can take

$$y = T(z), \quad \bar{y} = \overline{T(z)} = T(\bar{z}) = T(1/z), \quad \forall |z| \leq 1,$$

and we shall also choose $T(-1) = 0$, which implies necessarily $T(1) = 1$.

Introduce the new functions $\Psi(z) \stackrel{\text{def}}{=} \Phi(T(z))$, $C(z) \stackrel{\text{def}}{=} B(T(z))$, and denote by \mathcal{U} the unit circle $|z| = 1$.

Then condition (2.7) can be put into Hilbert's standard form

$$\boxed{\Psi^+(t) = C(t)\Psi^-(t), \quad t \in \mathcal{U}}, \quad (2.8)$$

where $\Psi(\cdot)$ is sought to be sectionally analytic, of finite degree at infinity, and the solution in \mathcal{L}^+ is given by (see e.g. [2])

$$\Psi^+(z) = e^{\Gamma(z)} \quad \text{with} \quad \Gamma(z) = \frac{1}{2i\pi} \int_{\mathcal{U}} \frac{\log[t^{-\chi}C(t)]dt}{t-z}, \quad \forall |z| < 1. \quad (2.9)$$

where χ is the *index*, defined by

$$\chi = \mathcal{I}nd[C]_{\mathcal{U}} \stackrel{\text{def}}{=} \frac{1}{2\pi} [\arg C]_{\mathcal{U}} = \frac{1}{2i\pi} [\log C]_{\mathcal{U}}, \quad (2.10)$$

which represents the variation of the argument of $C(u)$, as u moves along the unit circle \mathcal{U} in the positive direction. Hence, the function $\log[u^{-\chi}C(u)]$ is one-valued on \mathcal{U} .

From the expression of $B(t)$ given in (2.7), it appears that $|C(u)| = 1$ for $|u| = 1$, and $C(u)$ is also continuous at the points $u = 0, 1$. From these remarks, it is not difficult to prove that, for the special model under consideration, $\chi = 0$ [details are omitted].

Starting from the expression

$$C(t) = \left(1 - \frac{1}{T(t)}\right)^2 / \left(1 - \frac{1}{T(1/t)}\right)^2,$$

we can write $\log[C(t)]$ in the following form, which takes into account the singularities of the logarithms at $t = 1$ and $t = -1$ (respectively $y = 1$ and $y = -1$ in the original \mathbb{C}_y plane).

$$\log[C(t)] = 2 \left[\log \frac{1-T(t)}{1-t} - \log \frac{T(t)}{t+1} - \log \frac{1-T(1/t)}{1-t} + \log \frac{T(1/t)}{t+1} \right]. \quad (2.11)$$

Now the computation of $\Gamma(z)$ defined in (2.9) becomes straightforward by using the Cauchy residue theorem for the integrand (2.11). This yields

$$\Psi^+(z) = \left[\frac{(1+z)(1-T(z))}{(1-z)T(z)} \right]^2, \quad \forall |z| \leq 1. \quad (2.12)$$

On the other hand, we have by (2.2)

$$F(0, T(z)) \equiv \Psi^+(z) = \frac{(1-\rho)^2}{1-\rho T(z)} \quad \forall |z| \leq 1,$$

so that, comparing with (2.12), we see that $T(z)$ satisfies a polynomial equation of third degree. It is in fact simpler to write a closed form formula for the inverse mapping $S(y)$, which transforms conformally \mathcal{L}_+ onto the unit disc $|z| \leq 1$. We obtain

$$S(y) = \frac{(1-\rho)y - (1-\rho\sqrt{1-\rho y})}{(1-\rho)y + (1-\rho\sqrt{1-\rho y})}.$$

3 More general two-queue Jackson networks

In Fig. 3.1, we have drawn three generators representing the evolution of two-queue Jackson networks, which means that the invariant measure (assuming ergodicity conditions to be satisfied) has a product-form solution. From left to right, we refer to them as models $M1, M2, M3$. Then, in the context of Section 4 and according to the definition given in [1], the corresponding walks have respective *groups of order* 6, 6, 8.

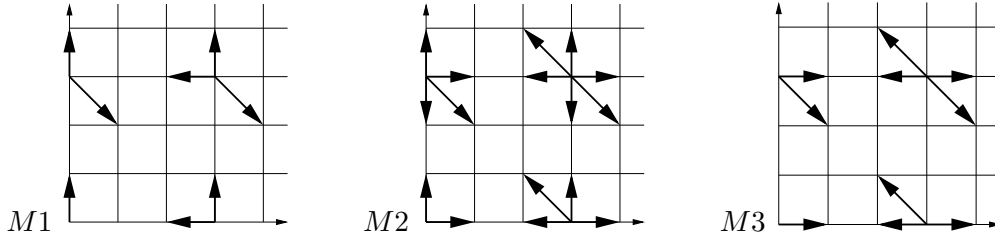


Fig. 3.1: Three examples of coupled queues representing a product-form Jackson network.

3.1 A BVP for the models of figure 3.1

Again we shall view $M2$ as a simple stochastic network of two coupled $M/M/1$ queues. The following choice for the rates of the generator of the underlying Markov process will be made, with obvious notations for the cardinal points, and $0 \leq p_i \leq 1$, $p_i + q_i = 1, i = 1, 2$.

$$\lambda_1 \text{ (E)}, \quad \lambda_2 \text{ (N)}, \quad p_1\mu \text{ (W)}, \quad p_2\mu \text{ (S)}, \quad q_1\mu \text{ (N-W)}, \quad q_2\mu \text{ (S-E)}.$$

Setting $\rho_i = \frac{\lambda_i}{\mu}, i = 1, 2$, it is not difficult to check that the invariant measure satisfies the following functional equation.

$$\begin{cases} R(x, y)G(x, y) = y[x - (p_1 + q_1y)]G(0, y) + x[y - (p_2 + q_2x)]G(x, 0), \\ R(x, y) = xy[\rho_1(1-x) + \rho_2(1-y)] + y[x - (p_1 + q_1y)] + x[y - (p_2 + q_2x)]. \end{cases} \quad (3.1)$$

Then the solution of (3.1) turns out to have the nice product-form

$$G(x, y) = \frac{(1-r_1)(1-r_2)}{(1-r_1x)(1-r_2y)}, \quad (3.2)$$

where

$$r_1 = \frac{\lambda_1 + q_2\lambda_2}{(1-q_1q_2)\mu}, \quad r_2 = \frac{\lambda_2 + q_1\lambda_1}{(1-q_1q_2)\mu},$$

assuming the ergodicity conditions $r_i < 1, i = 1, 2$ are satisfied.

Then, after setting for short $H(y) \stackrel{\text{def}}{=} G(0, y)$, we return to (3.1) to obtain the following BVP in the \mathbb{C}_y -plane (the contour \mathcal{L} being, as before, the convenient component of a quartic curve), which is of the same type as (2.6).

Find $H(\cdot)$ meromorphic in the domain \mathcal{L}^+ and satisfying

$$K(t)H(t) - K(\bar{t})H(\bar{t}) = 0, \quad t \in \mathcal{L}^+, \quad (3.3)$$

where

$$K(t) = \frac{t[X_0(t) - (p_1 + q_1 t)]}{X_0(t)[t - (p_2 + q_2 X_0(t))]},$$

noting that $H(\cdot)$ might have a pole inside $\mathcal{L}^+ \setminus \mathcal{D}$, and the function $X_0(\cdot)$ is the analogous of $Y_0(\cdot)$ defined in (2.4) for the kernel $R(x, y)$.

The next step is to use the above product form. Indeed, by (3.2), the right-hand side of (3.1) yields the following algebraic system

$$\begin{cases} \frac{y[x - (p_1 + q_1 y)]}{1 - r_2 y} + \frac{x[y - (p_2 + q_2 x)]}{1 - r_1 x} = 0, \\ \{(x, y) \in \mathbb{C}^2 : R(x, y) = 0\}, \quad |x|, |y| \leq 1. \end{cases} \quad (3.4)$$

We note first that the BVP (3.3) is equivalent to the BVP

$$(1 - r_2 t)H(t) - (1 - r_2 \bar{t})H(\bar{t}) = 0, \quad t \in \mathcal{L}, \quad (3.5)$$

which in the present situation admits (as expected!) precisely the unique solution $H(y) = \frac{H(0)}{1 - r_2 y}$.

To proceed as in Section 2 to find the conformal mapping $T(z)$ for the model $M2$, it is in fact necessary to solve *directly* the BVP (3.3). This is the matter of an ongoing work.

4 Application to counting walks in the quarter-plane

Let us recall below the basic assumptions.

- Walks start at the origin.
- The set \mathcal{S} of admissible steps is included in the set of the 8 nearest neighbors, i.e., $\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$. By using an extended Kronecker's delta, we shall write

$$\delta_{i,j} = \begin{cases} 1 & \text{if } (i, j) \in \mathcal{S}, \\ 0 & \text{if } (i, j) \notin \mathcal{S}. \end{cases}$$

Let $f(i, j, k) \stackrel{\text{def}}{=}$ the number of paths in \mathbb{Z}_+^2 starting from $(0, 0)$ and ending at (i, j) after k steps. Then

$$F(x, y, s) \stackrel{\text{def}}{=} \sum_{i,j,k \geq 0} f(i, j, k) x^i y^j s^k \quad (4.1)$$

satisfies the functional equation

$$\boxed{K(x, y, s)F(x, y, s) = c(x)F(x, 0, s) + \tilde{c}(y)F(0, y, s) + c_0(x, y, s)}, \quad (4.2)$$

a priori valid in the domain $|x| \leq 1, |y| \leq 1, |s| < 1/|\mathcal{S}|$, where

$$\begin{cases} K(x, y; s) = xy \left[\sum_{(i,j) \in \mathcal{S}} x^i y^j - 1/s \right], \\ c(x) = \sum_{i \leq 1} \delta_{i,-1} x^{i+1}, \quad \tilde{c}(y) = \sum_{j \leq 1} \delta_{-1,j} y^{j+1}, \\ c_0(x, y, s) = -\delta_{-1,-1} F(0, 0, s) - xy/s, \end{cases}$$

As for the special model of Section 2, we have

$$K(x, y, s) = y^2 - x(1/s - x)y + x.$$

which we compare with the kernel given by (2.3), namely

$$xyQ(x, y) = y^2 - x(\rho + 2 - \rho x)y + x.$$

4.1 A varational principle for the curve \mathcal{L} in the east, South and North-West model

The idea is to find a linear homeomorphism $\mathcal{H} : (x, y) \rightarrow (X, Y)$ as simple as possible, transforming $xyQ(x, y)$ into $K(X, Y, z) = 0$, for convenient values of the parameter ρ , so that z remains unchanged (note that we consider z as a parameter).

The change of variables $x = aX, y = bY$, with $ab \neq 0$, in (2.3) gives (after having divided by b^2)

$$Y^2 - \frac{aX(\rho + 2 - aX)}{b}Y + \frac{aX}{b^2} = 0.$$

Hence, taking $a = b^2$, the identification of $\frac{a(\rho + 2 - aX)}{b}$ with $\frac{1}{s} - X$ yields the two following necessary conditions:

$$b(\rho + 2) = \frac{1}{s}, \quad \text{and} \quad b^3\rho = 1.$$

Finally, we can completely specify the sought homeomorphism by

$$x = b^2X, \quad y = bX, \quad \rho = b^{-3}, \quad s = \frac{b^2}{1 + 2b^3}. \quad (4.3)$$

Two facts should be quoted.

- (i) Here b is real positive and, since $\rho < 1$, we have $b > 1$. Moreover, s given by (4.3) is smaller than $1/3$ and thus belongs to the initial domain of definition of (4.1).
- (ii) In the sequel, s will not be necessarily real.

References

- [1] Guy Fayolle, Roudolph Iasnogorodski, and Vadim A. Malyshev. *Random Walks in the Quarter Plane: Algebraic Methods, Boundary Value Problems, Applications to Queueing Systems and Analytic Combinatorics*, volume 40 of *Probability Theory and Stochastic Modelling*. Springer International Publishing, February 2017. *Second edition*, <https://hal.inria.fr/hal-01651919>.
- [2] F.D. Gakhov. *Boundary value problems*. Pergamon Press, 1966.
- [3] K. Raschel. Counting walks in a quadrant: a unified approach via boundary value problems. *Journal of the European Mathematical Society*, 14:749–777, 2012.